Chapter 1
Introduction

Abstract This introduction gives an overview of some fields of mathematical logic and underlying principles (ω-rule, $L_{\omega_1\omega}$, nonstandard analysis, admissible sets, modal logics) that are used in the rest of the book. Particularly, it provides motivation for various applications of infinitary means in obtaining the presented results. A comparison between the mainstream approach to mathematical theory of probability based on Kolmogorov's axioms and probability logics is given. Finally, the organization of the book is presented.

1.1 What Is this Book About: Consequence Relations and Other Logical Issues

A significant part of mathematical logic explores consequence relations, i.e., derivations of some formulas from other formulas. In that business, it is assumed that mathematical logic should be as reliable as possible. In the first place, it means that a precise definition of a symbolic language in which formulas are formed should be given. Furthermore, semantics should be associated to the language, giving the meaning to building blocks of formulas: atomic formulas, logical connectives, and quantifiers. One can, as suitable instruments, introduce the notions of models and satisfiability relations so that a formula A is a semantical consequence of a (possibly empty) set of formulas T if A is satisfied (in a model, or, alternatively, in a world from a model) whenever all formulas from T are satisfied (in that model, or in that world). Simultaneously, inferences can be studied by means of an axiom system (consisting of axioms and inference rules) where the notion of proof should be determined yielding the notion of syntactical consequence.

A bridge which connects those semantical and syntactical approaches can be established by the soundness and completeness theorems. The usual forms of those theorems are:

- the weak (or simple) completeness: a formula is consistent iff it is satisfiable (i.e., a formula is valid iff it is provable), or

Z. Ognjanović et al., *Probability Logics*,
DOI 10.1007/978-3-319-47012-2_1

- the strong (or extended) completeness: a set of formulas is consistent iff it is satisfiable (a formula is a syntactical consequence of a set of formulas iff it is a semantical consequence of that set).

While the former statement follows trivially from the latter, the opposite direction is not straightforward. In classical propositional and first-order logics these theorems are equivalent, thanks to a significant property formulated as:

- the compactness theorem: a set of formulas is satisfiable iff every finite subset of it is satisfiable.

But, there are logics where compactness fails which complicates their analysis.

In our approach to probability logics, we extend the classical (intuitionistic, temporal, …), propositional or first-order calculus with expressions that speak about probability, while formulas remain true or false. Thus, one is able to make statements of the form (in our notation) $P_{\geq s}\alpha$ with the intended meaning "the probability of α is at least s". Such probability operators behave like modal operators and the corresponding semantics consists of special types of Kripke models (possible worlds) with addition of probability measures defined over the worlds. We will explain in details in Sect. 3.3 that for probability logics compactness generally does not hold, and discuss some consequences of that property. For example, it is possible to construct sets of formulas that are unsatisfiable and consistent[1] with respect to finitary axiomatizations (for the notion of finitary axiom systems see Appendix 1.1.2). That can be a good reason for a logician to investigate possibilities to overcome the mentioned obstacle. On the other hand, from the point of view of applications, one can argue that, since propositional probability logics are generally decidable, all we need is an efficient implementation of a decision procedure which could solve real problems. However, as we know, propositional logic is of rather limited expressivity and in many (even real life) situations first order logic is a must. It was proved that the sets of valid formulas in probabilistic extensions of first-order logic are not recursively enumerable, so that no complete finitary axiomatization is possible at all (see Chap. 4). Hence, there are no finitary tools that allow us to adequately model reasoning in this framework. We believe that this is not only of theoretical interest, which has motivated us to investigate alternative model-theoretic and proof-theoretic methods appropriate for providing strongly complete axiomatizations for the studied systems. The main part of this book is devoted to those issues.

As one of the distinctive characteristics of our approach in exploring relationship between logic and probability,[2] we have used different aspects of infiniteness which has proved to be a powerful tool in this endeavor. At the same time, we will try to accomplish it with tools as weak as possible, i.e., to limit the use of infinitary means: we generally use countable object languages and finite formulas, while only proofs are allowed to be infinite.

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[1]Contradiction cannot be deduced from the set of formulas.

[2]Actually, in Chap. 2 we will present some evidences about common roots of these two important branches of mathematics.

Other important problems which will be addressed in the book are related to decidability and complexity of probability logics. We will also describe our attempts to develop heuristically-based methods for the probability logic satisfiability problem, PSAT.

The main contribution of our work presented in this book concerns development of a new technique for proving strong completeness for non-compact probability logics which combines Henkin style procedures for classical and modal logics and which works with infinitary proofs. This method enabled us to solve some open problems, e.g., strong completeness for real-valued probabilities in the propositional and first-order framework and for polynomial weight formulas (see the Chaps. 3, 4, 5, 7). It was also applied to other non-compact logics, for example to linear and branching discrete time logics [3, 4, 31, 37, 39, 46], and logics with probability functions with partially ordered ranges, etc.

1.2 Finiteness Versus Infiniteness

Standard courses of mathematical logic, usually encompassing classical propositional and first-order logic, assume that axiom systems are finitary. Such a system is presented by a finite list of axiom schemas and inference rules (each rule with a finite number of hypothesis and one conclusion). It might create an impression that all axiom systems are finitary in the above sense. Nevertheless, infiniteness can play an important role and significantly expand expressive power of formal systems. It can be traced back to an extremely important period of development of mathematical logic, i.e., to 1930s.[3] These years brought many significant results in mathematical logic and, what we call today, theoretical computer science. One of the most prominent among them, the first Gödel's incompleteness theorem [10], says that for any consistent first order formal system, expressive enough to represent finite proofs about natural numbers, there is no recursive (finitary) complete axiomatization. It suggests that some kind of infiniteness should be involved into formal systems to study the standard model of arithmetics. Indeed, several such approaches were introduced before 1940.

The seminal work of Gerhard Gentzen [9] showed that, by associating ordinals to derivations, the consistency of the first-order arithmetic is provable in a theory with the principle of transfinite induction up to the infinite ordinal ε_0.

In his Ph.D. Thesis [60] Alan Turing considered a formal system T_0 powerful enough to represent arithmetics, and a sequence of logical theories (each theory T_{i+1} obtained from the preceding one by adding the assertion about consistency of T_i, $T_\omega = \cup T_i$, and further iterated into the transfinite). He asked whether one of the logics indexed with denumerable ordinals is complete with respect to statements true in the standard model of natural numbers. Although Turing established that $T_{\omega+1}$ proves an important subclass of true formulas (all valid Π_1 sentences, i.e., sentences

[3] It is pointed out in Chap. 2 that already Leibnitz discussed infinitary proofs.

of the form $(\forall x)A(x)$, where A is a recursive predicate), later on it was showed in [7] that this progression is not complete (already for true $\forall\exists$ sentences).

Finally (and more relevant to this text), some well-known logicians (Tarski, Hilbert, Carnap) introduced ω-rule to overcome the limitations of finitary formal systems of arithmetic [24, 59].

In Gödel's analysis of undecidability, the role of recursive (Δ_0) and recursively enumerable (Σ_1) sets (arithmetical predicates, formulas) is important. Informally speaking, a Δ_0-formula (or a bounded formula) is a formula whose all quantifiers are bounded, while a Σ_1-formula is, up to equivalence, in the form of a block of existential quantifiers applied on a Δ_0-formula, i.e., if α is a Δ_0-formula, then $\exists x_1 \ldots \exists x_n \alpha$ is a Σ_1-formula. More precisely, a Δ_0-formula is inductively defined as follows:

- Any quantifier free formula is a Δ_0-formula;
- Boolean combination of Δ_0-formulas is a Δ_0-formula;
- If α is a Δ_0-formula, then $\forall x(x \leq t \to \alpha)$ and $\exists x(x \leq t \wedge \alpha)$ are Δ_0 formulas.

In investigation presented in this book, we will be using different manifestations of infinity:

- infinitary proofs,
- infinitary formulas,
- infinitary ranges of probability functions with an infinitary property (σ-additivity),
- ranges of probability functions containing infinitely small values,[4] and
- admissible sets,

but also, where possible, their finitary counterparts will be discussed.

1.2.1 ω-rule

The basic form of this rule in the language of arithmetic $\{+, \cdot, S, 0\}$ is

- from $A(0), A(1), A(2) \ldots$, infer $(\forall x)A(x)$

where $1 = S0$, $2 = SS0$, …, are numerals. When one adds this rule to a usual axiom system of arithmetics (*PA* or Robinson arithmetic *Q*), a complete logic allowing proofs of infinite length is obtained [8, 58]. More recently, some versions of ω-rule (with the additional assumption that proofs of all premises $A(n)$ are recursive) suitable for effective implementation in automated deduction environments have been considered [1].

In axiom systems presented in this book several inference rules with infinite number of premisses and one conclusion, related to different aspects of probability, will be used.

[4] Infinitesimals.

1.2.2 Infinitary Languages

Probably the simplest infinitary logic is $L_{\omega_1\omega}$ which admits at most countable conjunctions and disjunctions, and finite blocks of quantifiers [25].

Note that the increased expressivity enables formal syntactical description of any countable first-order structure. For instance, the additive group $\langle \mathbb{Z}, + \rangle$ of integers can be formally coded by the following $L_{\omega_1\omega}$-sentence:

$$\phi_{\mathbb{Z}} \Leftrightarrow_{\text{def}} \forall x \left(\bigvee_{n\in\mathbb{Z}} x = c_n \right) \wedge \bigwedge_{n\neq m} c_n \neq c_m \wedge \bigwedge_{n,m\in\mathbb{Z}} c_n * c_m = c_{n+m}.$$

The underlying first-order language $L_{\mathbb{Z}}$ contains one binary function symbol $*$ and countably many constants $\{c_n \ : \ n \in \mathbb{Z}\}$. It is easy to see that an $L_{\mathbb{Z}}$-structure $\langle M, *_M \rangle$ is a model of $\phi_{\mathbb{Z}}$ iff it is isomorphic to the group $\langle \mathbb{Z}, + \rangle$.

However, the increased expressiveness comes with the price: the compactness theorem is not true for the $L_{\omega_1\omega}$. Indeed, using the same language $L_{\mathbb{Z}}$ as in the previous example, the following set of $L_{\mathbb{Z}}$-sentences

$$\left\{ \bigvee_{n\in\mathbb{Z}\setminus\{0\}} c_n = c_0 \right\} \cup \{c_n \neq c_0 \ : \ n \in \mathbb{Z} \setminus \{0\}\}$$

is finitely satisfiable, but it is not satisfiable.

As a formal theory, $L_{\omega_1\omega}$ extends classical first-order logic in the following way:

- $L_{\omega_1\omega}$ admits infinitary formulas,
- $L_{\omega_1\omega}$ has three additional axioms:
  - $\bigwedge_{i\in\mathbb{N}} \alpha_i \rightarrow \alpha_k$, for every $k \in \mathbb{N}$
  - $\neg \bigwedge_{i\in\mathbb{N}} \alpha_i \leftrightarrow \bigvee_{i\in\mathbb{N}} \neg\alpha_i$
  - $\neg \bigvee_{i\in\mathbb{N}} \alpha_i \leftrightarrow \bigwedge_{i\in\mathbb{N}} \neg\alpha_i$,
- $L_{\omega_1\omega}$ has an additional infinitary inference rule
  - From $\{\beta \rightarrow \alpha_i : i \in \mathbb{N}\}$ infer $\beta \rightarrow \bigwedge_{i\in\mathbb{N}} \alpha_i$.

For $L_{\omega_1\omega}$ strong completeness fails, and only weak completeness can be proved.

In Chap. 5 a fragment of $L_{\omega_1\omega}$ will be used in characterization of probability functions with arbitrary finite ranges.

1.2.3 Hyperfinite Numbers and Infinitesimals

The nonstandard analysis was introduced by Abraham Robinson (1918–1974) in 1961 [57]. He successfully applied the compactness theorem in order to perform the so-called rational reconstruction of the Leibnitz's differential and integral calculus. The key feature of Robinson's theory was consistent foundation of infinitesimals and hyperfinite numbers.

Suppose that S is an arbitrary set. A superstructure on S is the set

$$V(S) = V_\omega(S) = \bigcup_{n\in\omega} V_n(S),$$

where $V_0(S) = S$ and $V_{n+1}(S) = \mathbb{P}(V_n(S))$. If $S = \emptyset$, then $V(S) = V_\omega = HF$, i.e., $V(\emptyset)$ coincides with the set HF of hereditary finite[5] sets. For the nonstandard analysis the most interesting case is $S \subseteq \mathbb{R}$. Anyhow, S should be large enough to include all relevant objects within the scope of the underlying problem.

A nonstandard universe on S is a pair $\langle {}^*V(S), *\rangle$, where ${}^*V(S)$ is a proper superset of the standard universe $V(S)$ and $*$ is so-called lifting function $* : V(S) \longrightarrow {}^*V(S)$ such that

$${}^*s =_{\text{def}} *(s) = s$$

for all $s \in S$.

A set $X \in V({}^*S)$ is:

- internal, iff there is $A \in V(S)$ such that $X \in {}^*A$;
- external, iff it is not internal;
- standard, iff $X = {}^*A$ for some $A \in V(S)$.

For example, ${}^*\mathbb{N}$ is a standard set,[6] $\sin(Hx)$ is an internal set for any $H \in {}^*\mathbb{N} \setminus \mathbb{N}$, while S and $\mathbb{N}$ are external sets.

In particular, elements of the set ${}^*\mathbb{N}\setminus\mathbb{N}$ are called hyperfinite numbers. An internal set A is called hyperfinite iff there are a hyperfinite number H and an internal bijection $f : H \longrightarrow A$.

So, the notion of a hyperfinite set is a direct generalization of the notion of the finite set. Of special significance for applications of nonstandard analysis in probability theory and probability logic is the so-called hypertime interval

$$T =_{\text{def}} \left\{ \frac{n}{H} \ : \ n \leq H \text{ and } n \in {}^*\mathbb{N} \right\}.$$

Note that, in terms of the nonstandard universe, T is a hyperfinite set since it has H elements. However, T is not only an infinite set, but its cardinality is equal to continuum since there is a bijection between T and the real unit interval $[0, 1]$. This

[5] A recursive definition of HF goes as follows:

- $\emptyset \in HF$;
- $X \in HF$ iff X is finite and all its elements are also hereditary finite.

By axiom of regularity, there is no sequence of sets $\langle x_n \ : \ n \in \omega\rangle$ such that $x_{n+1} \in x_n$ for all n, so our definition is correct. In particular, $\emptyset$ is the simplest hereditary finite set.

[6] It is also a proper superset of $\mathbb{N}$, provided the usual restriction $\mathbb{N} \notin S$.

fact was used by Peter Loeb to define the Loeb measure and to establish a natural correspondence between the counting measure and the Lebesgue measure (so called Loeb construction or Loeb process) [30]. Thus, the notion of hyperfinite is a bridge between discrete and continuous.

An infinitesimal is any $\varepsilon \in {}^*\mathbb{R}$ such that

$$-\frac{1}{n} < \varepsilon < \frac{1}{n}$$

for all $n \in \mathbb{N} \setminus \{0\}$. For example, if H is a hyperfinite number, then $\frac{1}{H}$ is a positive infinitesimal.

Some of the key features of the nonstandard analysis are listed below:

- **Internal definition principle**. A set $X \in V({}^*S)$ is internal iff

  $$X = \{x \ : \ x \in A \text{ and } \alpha(x, A_1, \ldots, A_n)\},$$

  where α is a Δ_0-formula and $A, A_1, \ldots, A_n$ are internal sets;
- **Standard definition principle**. A set $X \in V({}^*S)$ is internal iff

  $$X = \{x \ : \ x \in A \text{ and } \alpha(x, A_1, \ldots, A_n)\},$$

  where α is a Δ_0-formula and $A, A_1, \ldots, A_n$ are standard sets;
- ω_1**-saturatedness**. If $\{A_n \ : \ n \in \mathbb{N}\}$ is a countable descending family of internal nonempty sets (i.e., $A_{n+1} \subseteq A_n$ for all n), then $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$;
- **Congruence**. For any $A \in S$, $x \in A$ iff ${}^*x \in {}^*A$. Similarly, ${}^*(A \cup B) = {}^*A \cup {}^*B$, ${}^*(A \times B) = {}^*A \times {}^*B$, ${}^*(A \setminus B) = {}^*A \setminus {}^*B$ etc.;
- **Overspill**. If A is an internal set and $\mathbb{N} \cap A$ is infinite, then A contains at least one hyperfinite number;
- **Underspill**. If an internal set A contains arbitrary small hyperfinite numbers (i.e., for all hyperfinite $H \in A$ exists a hyperfinite $K \in A$ such that $K < H$), then $A \cap \mathbb{N} \neq \emptyset$.

Nonstandard notions and techniques are used in the Chaps. 5 and 6 to obtain a complete axiomatization and to prove decidability of a logic with approximate conditional probabilities.

1.2.4 Admissible Sets

The theory of admissible sets was introduced by Kenneth Jon Barwise (1942–2000) [2] in order to provide a minimal formal framework for the study of recursion theory. The notion of finiteness is generalized by so-called admissible countability or $\mathbb{A}$-finiteness for the given admissible set $\mathbb{A}$.

Admissible set theory is a fragment of Zermelo–Fraenkel set theory with the following axioms:

- **Extensionality**. $A = B$ iff they have the same elements;
- **Empty set**. $\emptyset =_{\text{def}} \{x : x \neq x\}$ is a set;
- **Pair**. If A and B are sets, then $\{A, B\} =_{\text{def}} \{x : x = A \vee x = B\}$ is also a set;
- **Union**. If A is a set, then $\bigcup A =_{\text{def}} \{x : (\exists a \in A) x \in a\}$ is also a set;
- Δ_0**-separation**. If A is a set and α is a Δ_0-formula, then $\{x : x \in A \wedge \alpha\}$ is also a set;
- Δ_0**-collection.** Suppose that $\alpha(x, y)$ is a Δ_0-formula such that for any set X there is a set Y such that $\alpha(X, Y)$ holds. Then, for any set A there is a set B such that $(\forall a \in A)(\exists b \in B)\alpha(a, b)$ is true;
- **Regularity**. The membership relation $\in$ is regular, i.e., each set has $\in$-minimal element. More precisely, for any set A exists $a \in A$ such that $a \cap A = \emptyset$;
- **Infinity**.[7] There exists set A such that $\emptyset \in A$ and $a \cup \{a\} \in A$ for all $a \in A$.

The most notable difference between the admissible set theory and ZFC is the absence of axioms of choice and the powerset axiom. Hence, the admissible set theory cannot be used for the study of infinitary combinatorics due to the fact that one cannot establish the hierarchy of infinite cardinals. It can be shown that certain important mathematical concepts, such as ordered pair and Cartesian product, can be coded by means of the admissible set theory.

An admissible set is any set $\mathbb{A}$ such that the pair $\langle \mathbb{A}, \in \rangle$ is a model of the admissible set theory. For the study and development of probability logic, the most important example of the admissible set is the set HC of all hereditary countable sets. Similarly to the set HF of hereditary finite sets, the set HC is inductively defined as follows:

- $HF \subseteq HC$;
- $X \in HC$ iff X is at most countable and $x \in HC$ for all $x \in X$.

As before, the axiom of regularity provides the correctness of the above definition.

The main technical aspect of the set HC of all hereditary countable sets is the fact that the admissible fragment $L_{\mathbb{A}}$ of the infinitary logic $L_{\omega_1\omega}$ can be effectively coded in HC by means of the admissible set theory. For example, suppose that $F = \{\alpha_i : i \in I\}$ is a countable admissible set of formulas and that $f : F \longrightarrow HC$ is an admissible coding of F. If $k \in HC$ is a Gödel number (effective or recursive code) of a conjunction, then $\langle k, f \rangle \in HC$ is a Gödel number of the infinitary $L_{\mathbb{A}}$-formula $\bigwedge_{i \in I} \alpha_i$.

In other words, recursive infinitary logical constructions (formula formations, proofs, completion technique) can be represented as sets and set operations in the admissible set theory.

In particular, the elements of an admissible set $\mathbb{A}$ are called $\mathbb{A}$-finite. The most important technical tool of the admissible set theory is the Barwise compactness theorem that connects consistency with $\mathbb{A}$-finitness:

[7] This axiom is optional, i.e., some authors do not include it in the system.

Barwise compactness theorem. Suppose that $\mathbb{A}$ is a countable admissible set and that T is a Σ_1-definable[8] set of $L_{\mathbb{A}}$-sentences. Then, T is satisfiable iff each $\mathbb{A}$-finite subset of T is satisfiable.

An admissible fragment of a probabilistic counterpart of $L_{\omega_1\omega}$ is constructed in Chap. 5 to completely axiomatize probability functions with arbitrary finite ranges.

1.2.5 Ranges of Probability Functions

For our basic logics, in the Chaps. 3 and 4, we develop completion and decidability techniques wrt. the standard real-valued probability functions. However, real-valued probabilities are proved to be inadequate to model different types of uncertainty, as it is the case in default reasoning. For this purpose we consider other kinds of probability functions with various ranges:

- the finite set $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$,
- the unit interval of rational numbers $[0, 1]_{\mathbb{Q}}$, or some other recursive subsets of $[0, 1]$,
- the unit interval of Hardy field $[0, 1]_{\mathbb{Q}(\varepsilon)}$,
- some partially ordered countable commutative monoid with the least element 0, e.g., $[0, 1]_{\mathbb{Q}} \times [0, 1]_{\mathbb{Q}}$, and
- a closed ball in the field $\mathbb{Q}_p$ of p-adic numbers.

As expected, different types of ranges impose numerous challenges in axiomatizations. In this book we provide appropriate methodology to resolve those issues.

1.3 Modal Logics

Motivated by paradoxes of material implication (see Sect. 2.5.4), development of modal logics at first evolved in a pure syntactical framework. Clarence Irving Lewis (1883–1964) published a number of papers since 1912 and proposed several formal systems to axiomatize strict implication[9] understood as "it is impossible that the antecedent is true, while the consequent is false", or equivalently as "it is necessary that if the antecedent is true, then so is the consequent" [13, 29]. There are numerous modal logics, but the most studied between them are so-called normal modal logics. The simplest normal modal logic, denoted K, is axiomatized using the axiom schemata:

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[8] There is a Σ_1-formula α such that $\langle \mathbb{A}, \in \rangle \models T = \{x : \alpha\}$.

[9] In modern notation the formal language of modal logics extends the classical propositional language with the unary necessity operator $\Box$. Then, the strict implication is written as $\Box(\alpha \to \beta)$.

1. all substitutional instances of the classical propositional tautologies, and
2. (Axiom K) $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$,

and the inference rules:

1. Modus ponens, and
2. (Necessitation) if α, then $\Box\alpha$.

Other normal modal systems extend K with additional axioms that determine properties of the modal operator $\Box$.

Today the most widely accepted semantics for modal logics was proposed in the late 1950s by Saul Kripke (1940) [28]. The semantics is based on the idea of possible worlds equipped with a relation which represents visibility or accessibility between worlds. A Kripke model for propositional modal logics is a structure $\mathbf{M} = \langle W, R, v \rangle$ such that:

- W is a nonempty set of objects called worlds,
- $R \subset W \times W$ is an accessibility relation between worlds,
- $v : W \times \phi \rightarrow \{true, false\}$ provides for each world $w \in W$ a two-valued valuation of the set ϕ of primitive propositions,

while a formula α is satisfied in a world w (denoted $w \models \alpha$):

- if $\alpha \in \phi$, $w \models \alpha$ iff $v(w)(\alpha) = true$,
- if $\alpha = \neg\beta$, $w \models \alpha$ iff $w \not\models \beta$,
- if $\alpha = \beta \wedge \gamma$, $w \models \alpha$ iff $w \models \beta$ and $w \models \gamma$, and
- if $\alpha = \Box\beta$, $w \models \alpha$ iff for every $u \in W$, if wRu, then $u \models \beta$.

Note that, since the truth value of $\Box\alpha$ in a world w depends on R, i.e., on worlds accessible from w, modal logics are not truth-functional. Modal models without particular requirements for R characterize the system K. For stronger systems, additional axioms correspond to particular properties of R, for example:

- (T) $\Box\alpha \rightarrow \alpha$ corresponds to reflexivity,
- (4) $\Box\alpha \rightarrow \Box\Box\alpha$ corresponds to transitivity,
- (B) $\alpha \rightarrow \Box\neg\Box\neg\alpha$, etc.

The operator $\Box$ can be interpreted in many ways:

- temporal: $\Box\alpha$ is read "α always holds" [50],
- epistemic: $\Box\alpha$ is read "an agent knows α" [12],
- proof-theoretical: $\Box\alpha$ is read "α is provable" [11], etc.,

which is of great importance in applications. Therefore, modal logics are today accepted as formal bases for many systems in computer science and artificial intelligence.

One of the consequences of similarities between Kripke modal models and probability models (see the Definitions 3.2 and 4.1; instead of accessibility relations those models involve probability spaces) is that probability operators are not truth-functional. Since the semantics of $\Box$ is given using universal quantification over possible worlds, probability operators can be seen as a sort of softening of the necessity operator which gives additional expressivity and inspires possible mixing of the modal and probability languages.

1.4 Kolmogorov's Axiomatization of Probability and Probability Logics

Although there are several other proposals, the axiomatization of probability based on measure-theoretic notions given by Andrei Nikolaevich Kolmogorov (1903–1987) (see Sect. 2.6.4) is generally accepted as a standard. One can legitimately ask whether it is a logic, or what is its relationship with probability logics. To clarify that questions, one should be aware of the methodology which is used in mathematical logic. As we emphasize at the beginning of this Chapter and in Appendix, mathematical logic distinguishes between:

- syntax and semantics,
- object language and meta language, and
- object level and meta-level of reasoning.

While ordinary mathematicians often do not recognize these levels and mix them into one, the primary interest of mathematical logic is to formulate and prove (at the meta-level) statements *about* syntactical and semantical notions from the object level of reasoning (e.g., object-level theorems, valid formulas and so on). So, this methodological difference forces that many questions that are in the focus of probability logics (consequence relations, completeness, compactness, decidability, complexity, etc.) are not of huge importance in probability theory.[10] For instance, we do not expect that a probabilist would be too much interested whether optimal bounds of probabilities for consequences of some uncertain premisses are effectively computable.

In that sense, we do not consider Kolmogorov's axiomatization as a logic. Kolmogorov's axiomatization is used as a basis for semantics for some of the probability logics presented in this book, but, other approaches to probability are also studied: non-real-valued probabilities, probabilities with partially ordered ranges, coherent probabilities, etc.

Finally, we would like to point out that investigations in the field of probability logics can be useful in proving new theorems about probability: e.g., Keisler in [26, 27] proves existence theorems for some stochastic differential equations which are not proved by classical methods.

[10] And vice versa—probability logics do not carefully study some of the issues in probability theory.

1.5 An Overview of the Book

We present a number of probability logics. The main differences between the logics are:

- some of the logics are infinitary, while the others are finitary,
- the corresponding languages contain different kinds of probabilistic operators, both for unconditional and conditional probability,
- some of the logics are propositional, while the others are based on first-order logic,
- for most of the logics we start from classical logic, but in some cases the basic logic can be intuitionistic or temporal,
- in some of the logics iterations of probabilistic operators are not allowed,
- for some of the logics, restrictions of the following kinds are used: only probability measures with fixed finite range are allowed in models; ranges of probability functions are rational numbers, or complex numbers, or p-adic numbers, or domains of monoids; only one probability measure on sets of possible worlds is allowed in a model; the measures are allowed to be finitely additive.

For all these logics we give the corresponding axiomatizations, prove completeness, and discuss their decidability. More precisely, we consider the following logics:

- LPP_1 (L for logic, the first P for propositional, and the second P for probability), probability logic which starts from classical propositional logic, with iterations of the probability operators and real-valued probability functions [38, 42, 52],
- $LPP_1^{\mathrm{Fr}(n)}$ and LPP_1^S propositional probability logics with probability functions restricted to have ranges $\{0, 1/n, \dots, (n-1)/n, 1\}$ and S, respectively [38, 40, 42, 52],
- $LPP_1^{A,\omega_1,\mathrm{Fin}}$, propositional probability logic with probability functions restricted to have arbitrary finite ranges [5, 52],
- LPP_1^{LTL}, probability logic similar to LPP_1, but the basic logic is discrete linear time logic LTL [37–39], and LPP_1^{BTL}, propositional discrete probabilistic branching time logic [3, 46],
- LPP_2, $LPP_2^{\mathrm{Fr}(n)}$, $LPP_2^{A,\omega_1,\mathrm{Fin}}$, and LPP_2^S, probability logics without iterations of the probability operators [38, 42, 51, 52, 52],
- $LPP_{2,P,Q,O}$, probability logic which extends LPP_2 by having a new kind of probabilistic operators of the form Q_F, with the intended meaning "the probability belongs to the set F" [18, 41],
- $LPP_{2,\preceq}$ and $LPP_{2,\preceq}^{\mathrm{Fr}(n)}$, probability logics similar to LPP_2 and $LPP_2^{\mathrm{Fr}(n)}$, but allowing reasoning about qualitative probabilities [45, 47],
- LPP_2^I, probability logics similar to LPP_2, but the basic logic is propositional intuitionistic logic [32–34],
- $LFOP_1$, $LFOP_1^{\mathrm{Fr}(n)}$, $LFOP_1^{A,\omega_1,\mathrm{Fin}}$, $LFOP_1^S$ and $LFOP_2$, the first-order counterparts of the above logics [42, 53],
- several Kolmogorov's style-conditional probability propositional and first-order logic, with or without iterations of the probability operators, with real valued

probability functions, or probability functions with the range $[0, 1]_{\mathbb{Q}(\varepsilon)}$ that can express approximate probabilities [14, 35, 36, 43, 44, 54–56],
- $LPCP_2^{\mathrm{Chr}}$, propositional conditional probability logic, which corresponds to de Finetti's view on coherent conditional probabilities [14, 15],
- LPG_2, $LCOMP_B$, $LCOMP_S$, $CPL_{\mathbb{Z}_{\mathbb{Q}_p}}$, $CPL^{fin}_{\mathbb{Q}_p}$, propositional probability logics with monoid-valued (complex-valued, p-adic-valued) probability functions [16, 17, 19–23],
- LWF and PWF, propositional probability logics with linear and polynomial weight formulas (introduced in [6]) with $^*\mathbb{R}$-valued probability functions [47–49].

The parts of the book can be described as follows.

Chapter 2 introduces readers to a fascinating story about interactions between mathematical logic and probability which is full of great ideas and discoveries. We will particularly try to emphasize topics that motivated our research.

The key concepts (syntax and semantics, an infinitary axiomatization, the corresponding strong completeness, decidability and complexity) of LPP_2 are presented in Chap. 3. As the semantics, we introduce a class of models that combine properties of Kripke models and probabilities defined on sets of possible worlds. We consider the class of so-called measurable models (which means that all sets of possible worlds definable by classical formulas are measurable) and some of its subclasses: in the first case all subsets of worlds are measurable, then probabilities are required to be σ-additive, while models in the last subclass satisfy that only empty set has the zero probability. The proposed axiomatization is infinitary, i.e., there is an inference rule with countably many premisses and one conclusion:

- From $A \to P_{\geq s-\frac{1}{k}}\alpha$, for every integer $k \geq \frac{1}{s}$, and $s > 0$ infer $A \to P_{\geq s}\alpha$.

The rule corresponds to the following property of real numbers: if the probability is arbitrary close to s, it is at least s. Thus, proofs with countably many formulas are allowed. We give full details of the proof of strong completeness, so that it can be, with the corresponding modifications, used as a template for the other completeness proofs presented in the book. Decidability of PSAT, the satisfiability problem for LPP_2, is proved by a reduction to linear programming problem. Since the related linear systems can be of exponential sizes, we describe some heuristic approaches to the probabilistic satisfiability problem.

Chapter 4 investigates the first-order probability logic $LFOP_1$ which allows iterations of probability operators, so that it is possible to formalize reasoning about higher order probabilities. Since validity is not even recursively enumerable in that first-order framework, the presented infinitary axiom system, obtained by adding the probability axioms and inference rules (introduced in Chap. 3) to the classical axiomatization, is a reasonable tool for formalization of the logic. We also discuss relationship between $LFOP_1$ and modal logics by analyzing some properties of first-order modal models (constant domains, rigidness of terms) from the perspective of probability logics. Then we prove (un)decidability of (some fragments of) $LFOP_1$. Finally, a logic which combines temporal and probability reasoning is introduced.

Chapter 5 covers various probability logics, i.e., variants of LPP_2 and $LFOP_1$ obtained by putting restrictions on (or by extending) ranges of probability functions and/or on the used formal languages. We consider new types of probabilistic operators:

- the conditional probability operators $CP_{\geq s}$, $CP_{\leq s}$,
- the probability operators that express imprecise probabilities $P_{\approx s}$, $CP_{\approx s}$,
- the qualitative probability operator $\preceq$,
- the probability operators of the form Q_F with the intended meaning "the probability belongs to the set F".

and alternative ranges of probability functions: finite, countable, with infinitesimals, or partially ordered. We consider inference rules that help us to syntactically define ranges of probability functions. Also, we describe the logic LPP_2^I which extends propositional intuitionistic logic.

Chapter 6 deals with applications. In the case of $LPCP_2^{[0,1]_{\mathbb{Q}(\varepsilon)},\approx}$, the range is the unit interval of a recursive non-Archimedean field which makes it possible to express statements about approximate probabilities: $CP_{\approx s}(\alpha, \beta)$ which means "the conditional probability of α given β is approximately s". Formulas of the form $CP_{\approx 1}(\alpha, \beta)$ can be used to model defaults, i.e., expressions of the form "if β, then generally α". So, we relate $LPCP_2^{[0,1]_{\mathbb{Q}(\varepsilon)},\approx}$ with the well-known system P which forms the core of default reasoning. We also discuss other applications to, for example, reasoning about evidence, modeling of the process of human thinking based on p-adic numbers, etc.

A limited number of related papers published between the 1980s and 2010s are discussed in Chap. 7.

Finally, Appendix provides an overview of some general concepts in mathematical logic (formal systems, syntax, semantics, axiom systems, proofs, completeness, etc.) and probability theory ((σ-) algebras, (σ-) additive measures, the usual and coherent concepts of probability, etc.) which could help less experienced readers to follow the rest of the text.

Each chapter ends with the list of relevant references.

The book concludes with an index.

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Probability Logics
Probability-Based Formalization of Uncertain Reasoning
Ognjanović, Z.; Rašković, M.; Marković, Z.
2016, XI, 215 p. 5 illus., Hardcover
ISBN: 978-3-319-47011-5