Chapter 2
Stationary Measures

In this preliminary chapter, we first state general properties of a Markov operator P on a Borel space X. We study the P-invariant probability measures ν on X, and we prove the ergodicity of the associated *forward* dynamical system when ν is ergodic.

We focus then on the Markov–Feller operators, and in particular on the Markov–Feller operator P_μ associated to a random walk. For this operator P_μ and for the P_μ-invariant probability measures ν, which are also called μ*-stationary*, we explain the construction of the *backward* dynamical system and prove its ergodicity, when ν is ergodic.

In the following chapters, this space X will be a projective space or a flag variety and the Markov–Feller operator P will be the operator P_μ associated to a probability measure μ on the group G of automorphisms of X.

2.1 Markov Operators

We begin with some general facts about Markov operators P and the probability measures ν they preserve (Lemma 2.3). We will give equivalent definitions for the ergodicity of ν (Proposition 2.8). A key tool in order to prove the equivalence of these definitions is the adjoint Markov operator P^* (Lemma 2.4).

2.1.1 Markov Chains on Standard Borel Spaces

Let $(X, \mathscr{X})$ be a *standard Borel space*. By a *Markov chain* on X, we mean a Borel map $x \mapsto P_x$ from X to the space of Borel probability measures on X. This space X will sometimes be called the *state space* of the Markov chain. For any bounded Borel function φ on X and any x in X, we set

$$P\varphi(x) = \int_X \varphi \, \mathrm{d}P_x$$

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and we say P is the *Markov operator* associated to the Markov chain. A function φ is said to be *P-invariant* if $P\varphi = \varphi$.

Let us recall the construction of the *Markov probability measures* $\mathbb{P}_x$ associated to P on the space Ω of *forward trajectories*. We set $\Omega = X^{\mathbb{N}}$ and we equip it with the product σ-algebra $\mathscr{B} = \mathscr{X}^{\otimes\mathbb{N}}$. An element ω in Ω will be written as a sequence $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$. For any x in X, there exists a unique Borel probability measure $\mathbb{P}_x$ on Ω such that, for any bounded Borel functions $\varphi_0, \ldots, \varphi_n$ on X, one has

$$\int_\Omega \varphi_0(\omega_0)\cdots\varphi_n(\omega_n)\,\mathrm{d}\mathbb{P}_x(\omega) = (\varphi_0 P(\ldots(\varphi_{n-1}P(\varphi_n))\ldots))(x).$$

In other words, $\mathbb{P}_x$ is implicitly defined by $\mathbb{P}_x = \delta_x \otimes (\int_X \mathbb{P}_y\,\mathrm{d}P_x(y))$. We say $\mathbb{P}_x$ is the Markov measure associated to P and x (see Neveu's book [91, Chap. 3] for more details).

A probability measure ν on $(X, \mathscr{X})$ is said to be *P-invariant* if for every bounded Borel function φ on X, one has $\nu(P\varphi) = \nu(\varphi)$.

2.1.2 Measure-Preserving Markov Operators

Let $(X, \mathscr{X}, \nu)$ be a probability space and let P be an operator on the Banach space $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$ of (equivalence classes of) bounded measurable complex-valued functions on X. The operator P is called a *contraction* if $\|P\| \leq 1$. The operator P is called *non-negative* if for every non-negative function $\varphi \in \mathrm{L}^\infty(X, \nu)$, the image $P\varphi$ is also non-negative. The operator P is called a *measure-preserving Markov operator* on $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$ if it is a non-negative contraction such that $P\mathbf{1} = \mathbf{1}$ and, for every function $\varphi \in \mathrm{L}^\infty(X, \nu)$, one has $\int_X P\varphi\,\mathrm{d}\nu = \int_X \varphi\,\mathrm{d}\nu$.

If $(X, \mathscr{X})$ is a standard Borel space, P a Markov chain on $(X, \mathscr{X})$ and ν is a P-invariant probability measure, then P defines a measure-preserving Markov operator on $(X, \mathscr{X}, \nu)$. Conversely if $(X, \mathscr{X}, \nu)$ is a Lebesgue probability space, then every measure-preserving Markov operator on $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$ comes from a Markov chain on a set of full measure in X.

Let us again assume $(X, \mathscr{X}, \nu)$ is any probability space and P is a general measure-preserving Markov operator on $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$. We shall prove that P may be extended, for any $1 \leq p < \infty$, as a continuous contraction on the space $\mathrm{L}^p(X, \mathscr{X}, \nu)$ of functions φ for which $|\varphi|^p$ is integrable. This will follow from an elementary extension of Jensen's inequality:

Lemma 2.1 *Let P be a measure-preserving Markov operator on* $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$ *and $\theta : \mathbb{C} \to \mathbb{R}$ be a convex function. Then, for any φ in* $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$, *one has*

$$\theta(P\varphi) \leq P(\theta(\varphi)).$$

Proof Pick φ in $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$. By standard arguments about convex functions, there exists a sequence τ_n of affine functions $\mathbb{C} \to \mathbb{R}$ such that, for every z in $\mathbb{C}$, one has $\theta(z) = \sup_n \tau_n(z)$. Now, using successively the fact that P is non-negative and the equality $P\mathbf{1} = \mathbf{1}$, we get, for ν-almost every x in X, for any n in $\mathbb{N}$,

$$P\theta(\varphi)(x) \geq P\tau_n(\varphi)(x) = \tau_n(P\varphi(x)).$$

Thus $P\theta(\varphi)(x) \geq \theta(P\varphi(x))$ and we are done. □

Corollary 2.2 *Let P be a measure-preserving Markov operator on* $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$. *Then, for every $1 \leq p < \infty$, the operator P extends as a continuous contraction on* $\mathrm{L}^p(X, \mathscr{X}, \nu)$.

Proof By Lemma 2.1, one has $|P\varphi|^p \leq P\,|\varphi|^p$, for any φ in $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$, hence, since P is measure-preserving,

$$\|P\varphi\|_p = (\textstyle\int_X |P\varphi|^p \,\mathrm{d}\nu)^{1/p} \leq (\textstyle\int_X P\,|\varphi|^p \,\mathrm{d}\nu)^{1/p} = \|\varphi\|_p\,,$$

which completes the proof. □

An $\mathscr{X}$-measurable subset $E \subset X$ is called ν-almost P-invariant if its characteristic function $\mathbf{1}_E$ is P-invariant as an element of $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$.

The following lemma tells us that every P-invariant function is a limit of linear combinations of P-invariant subsets.

Lemma 2.3 *Let P be a measure-preserving Markov operator on* $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$. *Then, for any $1 \leq p \leq \infty$, the vector subspace generated by the characteristic functions of ν-almost everywhere P-invariant subsets is dense in the space* $\mathrm{L}^p(X, \mathscr{X}, \nu)^P$ *of P-invariant functions.*

Proof It suffices to prove the result for functions with real values. Let φ be a real function in $\mathrm{L}^1(X, \mathscr{X}, \nu)^P$. First note that the function $\varphi_+ := \max(\varphi, 0)$ is also P-invariant. Indeed, since P is non-negative, we have

$$P\varphi_+ \geq \max(P\varphi, 0) = \varphi_+.$$

Combining this inequality with the equality $\int_X P\varphi_+ \,\mathrm{d}\nu = \int_X \varphi_+ \,\mathrm{d}\nu$, we get

$$P\varphi_+ = \varphi_+ \text{ in } \mathrm{L}^1(X, \mathscr{X}, \nu).$$

Now, we claim that the characteristic function $\mathbf{1}_{\{\varphi>0\}}$ is also P-invariant. Indeed, this function is the limit in $\mathrm{L}^1(X, \mathscr{X}, \nu)$ of the functions $\min(1, n\,\varphi_+)$ and, by Corollary 2.2, P is continuous in $\mathrm{L}^1(X, \mathscr{X}, \nu)$. As a consequence, for $a < b$, the characteristic function $\mathbf{1}_{\{a<\varphi\leq b\}}$ is also P-invariant. The result follows, since every real φ in $\mathrm{L}^p(X, \mathscr{X}, \nu)$ is the limit in $\mathrm{L}^p(X, \mathscr{X}, \nu)$

$$\varphi = \lim_{n\to\infty} \textstyle\sum_{-n^2 \leq k \leq n^2} \frac{k}{n} \mathbf{1}_{\{k/n<\varphi\leq(k+1)/n\}}.$$ □

In the following lemma, we define the *adjoint operator* P^* of P and we check that P and P^* have the same invariant functions:

Lemma 2.4 *Let P be a measure-preserving Markov operator on* $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$.

(a) *There exists a unique measure-preserving Markov operator P^* on* $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$, *called the adjoint operator of P, such that, for every $\varphi, \varphi' \in \mathrm{L}^\infty(X, \mathscr{X}, \nu)$, one has*

$$\int_X P\varphi\, \varphi'\, \mathrm{d}\nu = \int_X \varphi\, P^*\varphi'\, \mathrm{d}\nu. \tag{2.1}$$

(b) *A function φ in* $\mathrm{L}^1(X, \mathscr{X}, \nu)$ *is P-invariant if and only if it is P^*-invariant.*

Proof (a) By Lemma 2.1.2, P extends as a continuous operator of $\mathrm{L}^1(X, \mathscr{X}, \nu)$. Let P^* be the adjoint operator to P on $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$, viewed as the dual space of $\mathrm{L}^1(X, \mathscr{X}, \nu)$, so that (2.1) holds and let us check that P^* is a measure-preserving Markov operator.

Since P is a contraction, so is P^*. Since P is non-negative, for any $\varphi, \varphi' \geq 0$ in $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$, one has

$$\int_X \varphi\, P^*\varphi'\, \mathrm{d}\nu = \int_X P\varphi\, \varphi'\, \mathrm{d}\nu \geq 0,$$

so that $P^*\varphi' \geq 0$ and P^* is non-negative.

Finally, since P is measure-preserving, for any φ in $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$, one has

$$\int_X \varphi\, \mathrm{d}\nu = \int_X P\varphi\, \mathrm{d}\nu = \int_X \varphi(P^*\mathbf{1})\, \mathrm{d}\nu,$$

that is, $P^*\mathbf{1} = \mathbf{1}$. In the same way,

$$\int_X P^*\varphi\, \mathrm{d}\nu = \int_X \varphi(P\mathbf{1})\, \mathrm{d}\nu = \int_X \varphi\, \mathrm{d}\nu,$$

that is, P^* is measure-preserving, which was to be shown.

(b) We first check the direct implication when φ is a characteristic function $\varphi = \mathbf{1}_E$, where E is a ν-almost surely P-invariant measurable subset of X. According to (2.1) with $\varphi = \varphi' = \mathbf{1}_E$ and to the bounds $0 \leq P^*\mathbf{1}_E \leq 1$ the function $P^*\mathbf{1}_E$ is equal to 1 on E. Since $\int_X P^*\mathbf{1}_E\, \mathrm{d}\nu = \nu(E)$, we get $P^*\mathbf{1}_E = \mathbf{1}_E$. Now, by Corollary 2.2, P^* acts continuously on $\mathrm{L}^1(X, \mathscr{X}, \nu)$ and, by Lemma 2.3, the characteristic functions of ν-almost surely P-invariant measurable subsets span a dense subset of $\mathrm{L}^1(X, \mathscr{X}, \nu)^P$, so that if φ is P-invariant in $\mathrm{L}^1(X, \mathscr{X}, \nu)$, one has $P^*\varphi = \varphi$. This proves the direct implication. The converse implication follows since $P^{**} = P$. □

Remark 2.5 The definition of the adjoint operator of a Markov operator depends on the measure. For example, let $X = \{0, 1\}^{\mathbb{N}}$ be the set of sequences of 0's and 1's, equipped with the natural σ-algebra, and P be the Markov operator associated to the shift map, that is, for every x in X, the measure P_x is the Dirac mass at Tx, where $(Tx)_k = x_{k+1}$. Fix $0 < p < 1$ and let ν be the Bernoulli measure with parameter p, that is, $\nu = (p\delta_0 + (1-p)\delta_1)^{\otimes \mathbb{N}}$. Then, one checks that ν is P-invariant and, for any φ in $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$, one has $P^*\varphi(x) = p\varphi(0x) + (1-p)\varphi(1x)$, for ν-almost any x in X. This formula depends on p.

2.1.3 Ergodicity of Markov Operators

We again let $(X, \mathscr{X})$ be a standard Borel space, P be a Markov chain on $(X, \mathscr{X})$ and ν be a P-invariant probability measure. We shall give equivalent definitions for ergodicity. First let us describe the functions which are ν-almost surely P-invariant.

Lemma 2.6 *Let $(X, \mathscr{X})$ be a standard Borel space, P be a Markov operator on X and ν be a P-invariant probability measure. Then, every ν-almost surely P-invariant bounded Borel function φ is equal ν-almost everywhere to a P-invariant bounded Borel function ψ.*

Proof Let φ be a bounded Borel function such that $P\varphi = \varphi$ in $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$. For x in X, we set

$$\varphi_\infty(x) = \liminf_{n\to\infty} P^n\varphi(x).$$

By Fatou's lemma, we have $P\varphi_\infty \leq \varphi_\infty$. We set, for any x in X,

$$\psi(x) = \lim_{n\to\infty} P^n\varphi_\infty(x).$$

By the monotone convergence theorem, we have $P\psi = \psi$.

Now, since φ is P-invariant in $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$, there exists a Borel subset E of X with $\nu(E) = 1$ such that, for any x in E, for any $n \geq 0$, one has $P^n\varphi(x) = \varphi(x)$, hence $\varphi_\infty(x) = \varphi(x)$. In particular, φ_∞ is P-invariant in $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$ and there exists a Borel subset F of X with $\nu(F) = 1$ such that, for any x in F, for any $n \geq 0$, one has $P^n\varphi_\infty(x) = \varphi_\infty(x)$, hence $\psi(x) = \varphi_\infty(x)$. We are done, since $\psi = \varphi$ on $E \cap F$. $\square$

Remark 2.7 Here is a subtle point in the definition of ν-almost P-invariant subsets: there may exist ν-almost P-invariant subsets E of X which are not ν-almost everywhere equal to an invariant subset. For example, let X be a triple $\{a, b, c\}$ and P be the Markov operator such that

$$P_a = \tfrac{1}{2}(\delta_b + \delta_c)\ , \quad P_b = \delta_b \text{ and } \ P_c = \delta_c.$$

The measure $\nu := \frac{1}{2}(\delta_b + \delta_c)$ is P-invariant and the set $E := \{b\}$ is ν-almost P-invariant. Indeed, the characteristic function $\varphi := \mathbf{1}_E$ is ν-almost everywhere equal to the function $\psi := \frac{1}{2}\mathbf{1}_{\{a\}} + \mathbf{1}_{\{b\}}$ which is ν-almost P-invariant. One cannot choose ψ to be a characteristic function since the only P-invariant subsets of X are $\emptyset$ and X.

We can now give five equivalent definitions of ergodicity:

Proposition 2.8 *Let $(X, \mathscr{X})$ be a standard Borel space, P be a Markov operator on X and ν be a P-invariant Borel probability measure. The following are equivalent:*

(i) *Every P-invariant bounded Borel function is constant ν-almost everywhere.*
(ii) *Every P-invariant element in $\mathrm{L}^1(X, \mathscr{X}, \nu)$ is constant.*
(iii) *Every P-invariant element in $\mathrm{L}^\infty(X, \mathscr{X}, \nu)$ is constant.*
(iv) *Every ν-almost P-invariant Borel subset of X has measure 0 or 1.*
(v) *ν is extremal in the convex set of P-invariant Borel probability measures.*

In this case ν is said to be P-ergodic.

Proof The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are immediate and their converse (iv) $\Rightarrow$ (ii) follows from Lemma 2.3. The implication (i) $\Rightarrow$ (iii) is a consequence of Lemma 2.6 and its converse (iii) $\Rightarrow$ (i) is immediate.

Let us prove (ii) $\Rightarrow$ (v). Let P^* be the adjoint of P with respect to ν as in Lemma 2.4. Assume that ν is a convex combination $t\nu_1 + (1-t)\nu_2$, where ν_1 and ν_2 are P-invariant Borel probability measures and where $0 < t < 1$. For $i = 1, 2$, the measure ν_i is absolutely continuous with respect to ν and hence can be written as $\varphi_i \nu$, where the function φ_i belongs to $\mathrm{L}^1(X, \mathscr{X}, \nu)$ and has integral 1. Since ν_i is P-invariant, one has $P^*\varphi_i = \varphi_i$. Again by Lemma 2.4(b), one has $P\varphi_i = \varphi_i$, hence by assumption, $\varphi_i = 1$ ν-almost everywhere, that is, $\nu_i = \nu$, which was to be shown.

Finally, let us prove (v) $\Rightarrow$ (iv). If $E \in \mathscr{X}$ is a ν-almost P-invariant subset of X, by Lemma 2.4(b), one has $P^*\mathbf{1}_E = \mathbf{1}_E$, hence the Borel measures $\nu_{|E}$ and $\nu_{|E^c}$ are P-invariant. Since ν is extremal, we get $\nu(E) = 0$ or $\nu(E^c) = 0$, as required. □

2.2 Ergodicity and the Forward Dynamical System

In this section we introduce the dynamical system on the space of forward trajectories of a Markov chain, and we interpret the P-ergodicity of a measure as an ergodicity property of this dynamical system.

Let P be a Markov chain on a standard Borel space $(X, \mathscr{X})$. The *forward dynamical system* $(\Omega, \mathscr{B}, T)$ is the dynamical system on the space of forward trajectories given by

$$T : \Omega \to \Omega\ ;\ (\omega_0, \omega_1, \dots) \mapsto (\omega_1, \omega_2, \dots)$$

For any Borel probability measure ν on X we set $\mathbb{P}_\nu$ for the probability measure on $(\Omega, \mathscr{B})$

$$\mathbb{P}_\nu := \textstyle\int_X \mathbb{P}_x \, \mathrm{d}\nu(x)$$

and $\mathbb{E}_\nu$ for the corresponding expectation operator.

The following proposition interprets the P-invariance and the P-ergodicity of ν as an invariance property and an ergodicity property of the measured forward dynamical system $(\Omega, \mathscr{B}, T, \mathbb{P}_\nu)$.

Proposition 2.9 *Let ν be a Borel probability measure on X.*

(a) *Then ν is P-invariant if and only if $\mathbb{P}_\nu$ is T-invariant.*

(b) *In this case, ν is P-ergodic if and only if $\mathbb{P}_\nu$ is T-ergodic.*

Proof We denote by $\mathscr{X}_0 \subset \mathscr{B}$ the sub-σ-algebra generated by ω_0. More generally, we denote by $\mathscr{X}_n \subset \mathscr{B}$ the sub-σ-algebra generated by $\omega_0, \dots, \omega_n$. By construction of the measures $\mathbb{P}_x$, $x \in X$, and $\mathbb{P}_\nu$, for any bounded Borel function ψ on Ω, the conditional expectation of ψ is given by the formula, for $\mathbb{P}_\nu$-almost all ω in Ω,

$$\mathbb{E}_\nu(\psi \mid \mathscr{X}_n)(\omega) = \int_\Omega \psi(\omega_0, \dots, \omega_{n-1}, \omega'_0, \omega'_1, \dots)\, \mathrm{d}\mathbb{P}_{\omega_n}(\omega'). \tag{2.2}$$

Hence, in particular,

$$\mathbb{E}_\nu(\psi \circ T^n \mid \mathscr{X}_n) = \mathbb{E}_\nu(\psi \mid \mathscr{X}_0) \circ T^n. \tag{2.3}$$

(a) If ψ is a bounded Borel function on Ω, we let φ denote the bounded Borel function on X given by, for every x in X,

$$\varphi(x) = \int_\Omega \psi(\omega)\, \mathrm{d}\mathbb{P}_x(\omega).$$

In other words, $\varphi(x)$ is the expected value of the function ψ for the trajectories of the Markov chain starting at x. The map $\psi \mapsto \varphi$ is onto and, we have, for ν-almost any ω in Ω,

$$\mathbb{E}_\nu(\psi \mid \mathscr{X}_0)(\omega) = \varphi(\omega_0) \text{ and } \mathbb{E}_\nu(\psi \circ T \mid \mathscr{X}_0)(\omega) = P\varphi(\omega_0).$$

Thus, we get

$$\mathbb{E}_\nu(\psi) = \nu(\varphi) \text{ and } \mathbb{E}_\nu(\psi \circ T) = \nu(P\varphi),$$

whence the result.

(b) We assume first that ν is P-ergodic and we want to prove that any T-invariant bounded Borel function ψ on Ω is constant.

We still set, for any x in X, $\varphi(x) = \int_\Omega \psi(\omega)\, \mathrm{d}\mathbb{P}_x(\omega)$. We get

$$P\varphi(x) = \int_X \int_\Omega \psi(\omega)\, \mathrm{d}\mathbb{P}_y(\omega)\, \mathrm{d}P_x(y) = \int_\Omega \psi(T\omega)\, \mathrm{d}\mathbb{P}_x(\omega) = \varphi(x).$$

Thus, φ is constant ν-almost everywhere and we may assume that $\varphi = 0$. Now, since the σ-algebra $\mathscr{B}$ is spanned by the increasing union of the σ-algebras $\mathscr{X}_n$, $n \geq 0$, ψ is the limit in $\mathrm{L}^1(\Omega, \mathbb{P}_\nu)$ of the functions $\mathbb{E}_\nu(\psi \mid \mathscr{X}_n)$. One computes

$$\mathbb{E}_\nu(\psi \mid \mathscr{X}_n) = \mathbb{E}_\nu(\psi \circ T^n \mid \mathscr{X}_n) = \mathbb{E}_\nu(\psi \mid \mathscr{X}_0) \circ T^n = 0.$$

Hence $\psi = 0$ as required.

Conversely, we assume that $\mathbb{P}_\nu$ is T-ergodic and we want to prove that any P-invariant bounded Borel function φ on X is constant ν-almost everywhere. Indeed, let us set, for any $n \geq 0$ and ω in Ω,

$$\psi_n(\omega) = \varphi(\omega_n).$$

By construction, for any $n \geq 1$, for $\mathbb{P}_\nu$-almost any ω, one has

$$\mathbb{E}_\nu(\psi_n \mid \mathscr{X}_{n-1})(\omega) = P\varphi(\omega_{n-1}) = \varphi(\omega_{n-1}) = \psi_{n-1}(\omega),$$

that is, the sequence ψ_n is a uniformly bounded martingale. By Doob's martingale convergence theorem A.3, it converges almost everywhere to a function ψ in $\mathrm{L}^\infty(\Omega, \mathbb{P}_\nu)$. By construction, one has, for $\mathbb{P}_\nu$-almost every ω,

$$\psi(T\omega) = \lim_{n\to\infty} \varphi(\omega_{n+1}) = \psi(\omega)$$

and ψ is constant $\mathbb{P}_\nu$-almost everywhere. Since, for $\mathbb{P}_\nu$-almost every ω, one has

$$\varphi(\omega_0) = \psi_0(\omega) = \mathbb{E}_\nu(\psi \mid \mathscr{X}_0)(\omega),$$

the function φ is constant ν-almost everywhere, as required. □

2.3 Markov–Feller Operators

We define Markov–Feller operators: they are the analogues, in the theory of Markov operators, of continuous transformations in the theory of classical dynamical systems.

When X is a compact space, a *Markov–Feller operator* on X is a nonnegative operator P on the space of continuous functions on X such that $P\mathbf{1} = \mathbf{1}$. In other terms, a Markov–Feller operator is a Markov chain on X such that the map $x \mapsto P_x$ is continuous, when the space $\mathscr{P}(X)$ of Borel probability measures of X is equipped with the weak-$*$ topology.

The following lemma reduces the study of P-invariant measures to the study of those that are ergodic.

Lemma 2.10 *Let P be a Markov–Feller chain on a compact metric space X. Then there exist P-invariant Borel probability measures on X. In the dual space of $\mathscr{C}^0(X)$, equipped with the weak-$*$ topology, the set of P-invariant Borel probability measures is the closed convex hull of the set of ergodic P-invariant probability measures.*

Proof Since X is a compact space, the space $\mathscr{M}(X)$ of complex *Borel measures* on X is the dual space of the space $\mathscr{C}^0(X)$ of continuous functions on X. We endow it with the weak-$*$ topology. The subset $\mathscr{P}(X)$ of Borel probability measures on X is then a compact subset of X.

We use Markov–Kakutani's argument: we start from any point x in X and consider the sequence of probability measures on X

$$\nu_n : \varphi \mapsto \tfrac{1}{n}(\varphi(x) + P\varphi(x) + \cdots + P^{n-1}\varphi(x)).$$

Since the set $\mathscr{P}(X)$ is compact, ν_n admits a cluster point ν_∞ in the weak-$*$ topology. Passing to the limit in the equalities, with φ in $\mathscr{C}^0(X)$,

$$\nu_n(P\varphi) - \nu_n(\varphi) = \tfrac{1}{n}(P^n\varphi(x) - \varphi(x)),$$

one gets

$$\nu_\infty(P\varphi) = \nu_\infty(\varphi).$$

Hence the probability measure ν_∞ is P-invariant.

Finally, by Proposition 2.8, a P-invariant Borel probability measure is P-ergodic if and only if it is extremal. The last part of the lemma now follows from the Krein–Millman Theorem. □

A Markov–Feller operator P is said to be *uniquely ergodic* if it admits a unique P-invariant Borel probability measure. As a corollary of the proof of the previous lemma, we get a nice interpretation of unique ergodicity.

Corollary 2.11 *Let P be a Markov–Feller operator on the compact metric space X. The following are equivalent*:

(i) *P is uniquely ergodic.*
(ii) *There exists a Borel probability measure ν on X such that, for any continuous function φ, one has*

$$\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1} P^k\varphi \xrightarrow[n\to\infty]{} \int_X \varphi\, \mathrm{d}\nu$$

uniformly.

Proof (ii) $\Rightarrow$ (i) Let ν' be a P-invariant Borel probability measure on X. By the dominated convergence theorem, we have, for any continuous function φ,

$$\int_X \varphi\, \mathrm{d}\nu' = \int_X (\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1} P^k\varphi)\, \mathrm{d}\nu' \xrightarrow[n\to\infty]{} \int_X \varphi\, \mathrm{d}\nu.$$

(i) $\Rightarrow$ (ii) Let x_n be a sequence in X. Reasoning as in the proof of Lemma 2.10, we get that any limit point of the sequence of measures $\nu_n := \frac{1}{n}\sum_{k=0}^{n-1}(P^*)^k\delta_{x_n}$ is P-invariant. Hence this sequence ν_n converges to ν. □

2.4 Stationary Measures and the Forward Dynamical System

In this section, we give an alternative construction of the forward dynamical system associated to the action of a probability measure μ on a compact space X.

We recall that a *semigroup* is a set G endowed with an associative multiplication law $G \times G \to G$ and containing a neutral element. For instance, for any set X, the set $\mathscr{F}(X, X)$ of maps from X to X is a semigroup with respect to the composition of maps. A *morphism* of semigroups $\rho : G \to H$ is a map sending the neutral element of G to the neutral element of H and such that, for any g, g' in G, $\rho(gg') = \rho(g)\rho(g')$. An *action* of G on a space X is a morphism from G to $\mathscr{F}(X, X)$.

A *topological semigroup* is a semigroup G endowed with a topology such that the multiplication is continuous. For instance, when X is a compact space, the semigroup $\mathscr{C}^0(X, X)$ of continuous transformations of X endowed with the topology of uniform convergence is a topological semigroup. A *continuous action* of G on X is a continuous morphism of semigroups $G \to \mathscr{C}^0(X, X)$.

Let G be a second countable locally compact semigroup and X be a compact metrizable topological space on which G acts continuously. We denote by $\mathscr{G}$ the Borel σ-algebra of G and by $\mathscr{X}$ the Borel σ-algebra of X.

Let μ be a Borel probability measure on G, we denote by Γ_μ the smallest closed subsemigroup of G such that $\mu(\Gamma_\mu) = 1$. For any Borel probability measure ν on X, let $\mu * \nu$ denote the probability measure on X which is the image of the product measure $\mu \otimes \nu$ on $G \times X$ under the action map, that is,

$$\mu * \nu = \textstyle\int_G g_* \nu \, \mathrm{d}\mu(g).$$

The Borel probability measure ν is said to be *μ-stationary* if

$$\mu * \nu = \nu.$$

If this is the case, it is said to be *μ-ergodic* if it cannot be written as a proper convex combination of two different μ-stationary Borel probability measures.

For instance any Γ_μ-invariant probability measure is μ-stationary. The converse is not true in general but Lemma 2.12 tells us that it is true when X is finite.

Lemma 2.12 *When X is a finite set, any μ-stationary probability measure ν on X is Γ_μ-invariant.*

Proof We can assume that G is finite, equal to Γ_μ and that ν is ergodic. Let $S_\mu \subset G$ be the support of μ and $S_\nu \subset X$ be the support of ν. Stationarity of ν means that

$$\nu(\{x\}) = \textstyle\sum_{g \in S_\mu} \mu(\{g\}) \nu(g^{-1}\{x\}) \tag{2.4}$$

for every x in X. In particular one has the equality $S_\mu S_\nu = S_\nu$. Hence by replacing X with S_ν, we can also assume, with no loss of generality, that $X = S_\nu$ and that $S_\mu X = X$. Let X_0 be the set of points x in X such that $\nu(\{x\})$ is minimal.

Equality (2.4) implies that, for all x in X_0 and g in S_μ, one has

$$\nu(\{x\}) = \nu(g^{-1}\{x\}).$$

This means that ν is Γ_μ-invariant. □

We introduce the one-sided *Bernoulli shift* $(B, \mathscr{B}, \beta, T)$ with *alphabet* $(G, \mathscr{G}, \mu)$, that is, $B = G^{\mathbb{N}^*}$, where $\mathbb{N}^*$ is the set of positive integers, $\mathscr{B}$ is the product σ-algebra $\mathscr{G}^{\otimes\mathbb{N}^*}$, β is the product measure $\mu^{\otimes\mathbb{N}^*}$, and T is the shift map given, by

$$Tb = (b_2, \ldots, b_{n+1}, \ldots) \text{ for } b = (b_1, \ldots, b_n, \ldots) \in B.$$

We now construct the *forward dynamical system* on $B \times X$. We equip $B \times X$ with the σ-algebra $\mathscr{B} \otimes \mathscr{X}$ of Borel subsets and we introduce the skew-product transformation

$$T^X : (b, x) \mapsto (Tb, b_1 x).$$

We identify the σ-algebra $\mathscr{X}$ of Borel subsets of X with the sub-σ-algebra of Borel subsets of $B \times X$ which do not depend on the first coordinate.

For any x in X, set

$$P_{\mu,x} = \mu * \delta_x.$$

One easily check that this defines a Markov–Feller operator P_μ on X.

We explain now how the forward dynamical system on $B \times X$ is related to the forward dynamical system (Ω, T) of the Markov operator $P = P_\mu$ that we introduced in Sect. 2.2. For any x in X, the associated Markov measure $\mathbb{P}_{\mu,x}$ on Ω is the image of the measure $\beta = \mu^{\otimes\mathbb{N}^*}$ on $B = G^{\mathbb{N}^*}$ under the map

$$(b_k)_{k\geq 1} \mapsto (b_k \cdots b_1 x)_{k\geq 0}. \tag{2.5}$$

If ν is a Borel probability measure on X, then ν is μ-stationary if and only if it is P_μ-invariant and, in this case, the measure $\mathbb{P}_\nu$ on Ω is the image of $\beta \otimes \nu$ under the map

$$(b, x) \mapsto (b_k \cdots b_1 x)_{k\geq 0},$$

which intertwines the maps T^X and T. By Proposition 2.8, ν is μ-ergodic if and only if it is P_μ-ergodic.

Remark 2.13 In general, the map $(b, x) \mapsto (b_k \cdots b_1 x)_{k\geq 0}$ is not a Borel isomorphism between $B \times X$ and Ω since non-trivial elements of G may have fixed points in X. Nevertheless, we have the following analogue of Proposition 2.9.

Proposition 2.14 *Let ν be a Borel probability measure on X.*

(a) *Then ν is μ-stationary if and only if $\beta \otimes \nu$ is T^X-invariant.*
(b) *In this case, ν is μ-ergodic if and only if $\beta \otimes \nu$ is T^X-ergodic.*

Proof The proof follows the same lines as for the proof of Proposition 2.9. □

Remark 2.15 There may exist a T^X-invariant Borel probability measure on $B \times X$ whose image by the projection on the first factor is equal to β but which is not of the form $\beta \otimes \nu$ for some μ-stationary Borel probability measure ν on X. For example, let G be the free group on two generators g and h, X be the Gromov boundary of G, i.e. the set of reduced one-sided infinite words in $g^{\pm}$ and $h^{\pm}$ and μ be the probability measure $\mu = \frac{1}{2}(\delta_g + \delta_h)$. For β-almost every b in B, b is a reduced word, that is, b may be seen as an element x_b of X. By construction, one has $x_{Tb} = b_1 x_b$. Hence, the image of β by the graph map $b \mapsto (b, x_b)$ on $B \times X$ is T^X-invariant. It is clearly not a product measure. In fact, this image measure is an example of the measures invariant under the backward dynamical system that we will construct below.

Lemma 2.16 *Given μ, there exists a μ-stationary Borel probability measure on the compact space X.*

Proof This is a special case of Lemma 2.10. □

2.5 The Limit Measures and the Backward Dynamical System

For every μ-stationary probability measure on X, we construct in this section an equivariant measurable family of probability measures ν_b on X indexed by the Bernoulli shift and called the limit measures. We will use this family in order to construct the dynamical system of backward trajectories.

We keep the notations of Sect. 2.4. In particular, G is a second countable locally compact semigroup, μ is a Borel probability measure on G, $(B, \mathscr{B}, \beta, T)$ is the associated one-sided Bernoulli shift, the semigroup G acts continuously on the compact metrizable topological space X and ν is a μ-stationary Borel probability measure on X.

Here is the construction of the *limit measures*.

Lemma 2.17 *There exists a Borel map $b \mapsto \nu_b$ from B to $\mathscr{P}(X)$ such that, for β-almost any b in B, one has $(b_1 \cdots b_n)_* \nu \xrightarrow[n\to\infty]{} \nu_b$.*

Remark 2.18 In this lemma, the compactness assumption on X can be removed (see [13, Lemma 3.2]).

Proof The main tool is Doob's martingale theorem. Let, for any n in $\mathbb{N}$, $\mathscr{B}_n$ be the sub-σ-algebra of $\mathscr{B}$ spanned by the coordinate functions with indices p, $1 \le p \le n$. If ν is a μ-stationary Borel probability measure on X, one checks that, for any bounded Borel function φ on X, the sequence of functions

$$f_n : b \mapsto \int_X \varphi(b_1 \cdots b_n x)\, \mathrm{d}\nu(x)$$

on B is a uniformly bounded martingale with respect to the filtration $(\mathscr{B}_n)_{n\in\mathbb{N}}$: for β-almost all b in B and all $n \ge 0$, one has

$$\mathbb{E}(f_{n+1} \mid \mathscr{B}_n)(b) = f_n(b).$$

By applying Doob's martingale convergence theorem (Theorem A.3) to a countable dense subset D of functions $\varphi \in \mathscr{C}^0(X)$, we deduce that, for b in a subset $B' \subset B$ with $\beta(B') = 1$, for all φ in D, the limit

$$\nu_b(\varphi) := \lim_{n\to\infty} (b_1 \cdots b_n)_* \nu\,(\varphi)$$

exists. Hence, by approximation, this limit exists for all φ in $\mathscr{C}^0(X)$, i.e. the limit $\nu_b = \lim_{n\to\infty} (b_1 \cdots b_n)_* \nu$ exists for all b in B'. □

The following lemma tells us that the stationary measure ν can be recovered from its limit measures ν_b by a simple averaging, and that these limit measures satisfy a nice equivariant property.

Lemma 2.19 *One has $\nu = \int_B \nu_b \, \mathrm{d}\beta(b)$ and, one has $\nu_b = (b_1)_* \nu_{Tb}$, for β-almost any b in B.*

Proof Let φ belong to $\mathscr{C}^0(X)$. As ν is μ-stationary, for any n in $\mathbb{N}$, one has

$$\int_X \varphi \, \mathrm{d}\nu = \int_B f_n(b) \, \mathrm{d}\beta(b).$$

Passing to the limit, the first equality follows by the dominated convergence theorem.

The second assertion follows directly from the definition of ν_b. □

Remark 2.20 Conversely, according to [13, Lemma 3.2], if $b \mapsto \nu_b$ is a Borel map from B to $\mathscr{P}(X)$ such that for β-almost any b in B, one has $\nu_b = (b_1)_* \nu_{Tb}$, then the Borel probability measure $\nu := \int_B \nu_b \, \mathrm{d}\beta(b)$ on X is μ-stationary and, for β-almost any b in B, ν_b is equal to the limit probability measure $\lim_{n\to\infty}(b_1 \cdots b_n)_* \nu$.

We will also need an enhanced version of Lemma 2.17.

Lemma 2.21 *For any m in $\mathbb{N}$, for $\beta \otimes \mu^{*m}$-almost any (b, g) in $B \times G$, one has $(b_1 \cdots b_n g)_* \nu \xrightarrow[n\to\infty]{} \nu_b$.*

Proof Let φ be in $\mathscr{C}^0(X)$ and set Φ to be the function on G

$$\Phi : h \mapsto \int_X \varphi(hx) \, \mathrm{d}\nu(x).$$

Since ν is μ-stationary, one has the equality, for n in $\mathbb{N}$ and h in G,

$$\int_G \Phi(hg) \, \mathrm{d}\mu^{*m}(g) = \Phi(h). \tag{2.6}$$

For g in G, we set f_n^g to be the function on B

$$f_n^g : b \mapsto \Phi(b_1 \cdots b_n g).$$

By Lemma 2.17, since $\mathscr{C}^0(X)$ is separable, it suffices to check that, for μ^{*m}-almost any g in G, the sequence of functions $f_n^g(b) - f_n(b)$ on B converges for β-almost all b towards 0. For any n in $\mathbb{N}$, using (2.6), we compute the integral

$$\begin{aligned} I_n &= \int_G \int_B |f_n^g(b) - f_n(b)|^2 \, \mathrm{d}\beta(b) \, \mathrm{d}\mu^{*m}(g) \\ &= \int_G \int_G |\Phi(hg) - \Phi(h)|^2 \, d\mu^{*m}(g) \, d\mu^{*n}(h) = J_{n+m} - J_n, \end{aligned}$$

where $J_n := \int_G \Phi(h)^2 \, \mathrm{d}\mu^{*n}(h)$. Since J_n is bounded by $\|\varphi\|_\infty$, one gets $\sum_{n=0}^{\infty} I_n < \infty$, and, for $\beta \otimes \mu^{*m}$-almost any (b, g) in $B \times G$,

$$\sum_{n=0}^{\infty} |f_n^g(b) - f_n(b)|^2 < \infty,$$

hence $f_n^g(b) - f_n(b)$ goes to zero as $n \to \infty$, whence the result. □

In order to appreciate the strength of the previous lemmas, we deduce the following corollary which is a reformulation of the classical Choquet–Deny Theorem in [33]. We recall that Γ_μ is the smallest closed subsemigroup of G such that $\mu(\Gamma_\mu) = 1$.

Corollary 2.22 *When G is abelian, every μ-stationary probability measure ν on X is Γ_μ-invariant.*

Proof Since G is abelian, by Lemmas 2.17 and 2.21, for μ-almost every g in G and β-almost every b in B, one has the equality $\nu_b = g_*\nu_b$. Hence, averaging this equality over B and using Lemma 2.19, one gets the equality $\nu = g_*\nu$ for μ-almost every g in G. Now, the result follows, since the stabilizer of ν in G is a closed subsemigroup containing the support of μ. □

We now construct, when G is a group, the *backward dynamical system* on $B \times X$, or dynamical system of *backward trajectories*. We recall that $(B, \mathscr{B}, \beta, T)$ is the one-sided Bernoulli shift with alphabet $(G, \mathscr{G}, \mu)$. We equip the space $B^X := B \times X$ with the σ-algebra $\mathscr{B}^X := \mathscr{B} \otimes \mathscr{X}$ of Borel subsets and we introduce the *skew-product* transformation

$$T^{\vee X} : (b, x) \mapsto (Tb, b_1^{-1}x)$$

and the Borel probability measure β^X on B^X given by

$$\beta^X := \int_B \delta_b \otimes \nu_b \, \mathrm{d}\beta(b).$$

The following proposition is an analog of Proposition 2.9. It interprets the P-ergodicity of ν as the ergodicity of the backward dynamical system $(B^X, \mathscr{B}^X, T^X, \beta^X)$.

Proposition 2.23 *Let G be a second countable locally compact group acting continuously on a compact metrizable topological space X, and ν be a μ-stationary Borel probability measure on X.*

(a) *Then the probability measure β^X on B^X is $T^{\vee X}$-invariant.*
(b) *The measure β^X is $T^{\vee X}$-ergodic if and only if ν is μ-ergodic.*

Proof (a) This follows from the following calculation which uses Lemma 2.19

$$\begin{aligned}\int_{B^X} \varphi(T^{\vee X}(b,x))\,\mathrm{d}\beta^X(b,x) &= \int_B \int_X \varphi(Tb, b_1^{-1}x)\,\mathrm{d}\nu_b(x)\,\mathrm{d}\beta(b)\\ &= \int_B \int_X \varphi(Tb, x)\,\mathrm{d}\nu_{Tb}(x)\,\mathrm{d}\beta(b)\\ &= \int_B \int_X \varphi(b, x)\,\mathrm{d}\nu_b(x)\,\mathrm{d}\beta(b)\\ &= \int_{B^X} \varphi(b,x)\,\mathrm{d}\beta^X(b,x),\end{aligned}$$

where $\varphi : B^X \to \mathbb{R}_+$ is a $(\mathscr{B}\otimes\mathscr{X})$-measurable function.

(b) First, assume β^X is $T^{\vee X}$-ergodic and let ν be equal to a convex combination $t\nu_1 + (1-t)\nu_2$ of μ-stationary probability measures with $0 < t < 1$. We get, for β-almost any b in B,

$$\nu_b = t\nu_{1,b} + (1-t)\nu_{2,b},$$

hence

$$\beta^X = t\beta_1^X + (1-t)\beta_2^X,$$

where, for $i = 1, 2$, β_i^X is constructed from ν_i. Since β^X is $T^{\vee X}$-ergodic, we have $\beta_1^X = \beta_2^X = \beta^X$ and therefore, by projecting on X, $\nu = \nu_1 = \nu_2$. By Proposition 2.8, ν is μ-ergodic.

Conversely, assume now ν is μ-ergodic and let us prove that β^X is $T^{\vee X}$-ergodic. This can be seen as an immediate consequence of the ergodicity of the forward dynamical system thanks to the ideas that will be introduced in Sect. 2.6 below. But we can also give a direct, more computational proof.

Let θ be a $T^{\vee X}$-invariant bounded Borel function on B^X. We want to prove that this function θ is β^X-almost surely constant. Let φ be any bounded Borel function on X and set

$$\rho(\varphi) = \int_{B^X} \varphi(x)\theta(b,x)\,\mathrm{d}\beta^X(b,x).$$

We first claim that the complex measure ρ on X is μ-stationary. This follow from the following calculation, with φ as above,

$$\begin{aligned}\int_G \int_X \varphi(gx)\,\mathrm{d}\rho(x)\,\mathrm{d}\mu(g) &= \int_G \int_B \int_X \varphi(gx)\theta(b',x)\,\mathrm{d}\nu_{b'}(x)\,\mathrm{d}\beta(b')\,\mathrm{d}\mu(g)\\ &= \int_B \int_X \varphi(b_1x)\theta(Tb,x)\,\mathrm{d}\nu_{Tb}(x)\,\mathrm{d}\beta(b)\\ &= \int_B \int_X \varphi(y)\theta(b,y)\,\mathrm{d}\nu_b(y)\,\mathrm{d}\beta(b) = \int_X \varphi\,\mathrm{d}\rho.\end{aligned}$$

We prove now that the measure ρ is absolutely continuous with respect to ν. Indeed, if φ is a non-negative Borel function on X such that $\int_X \varphi \, d\nu = 0$, we have, for β-almost any b in B, $\int_X \varphi \, d\nu_b = 0$ hence $\varphi = 0$ on a set of ν_b-full measure and $\int_X \varphi \, d\rho = 0$. That is, ρ is absolutely continuous with respect to ν.

By Proposition 2.8, as ν is μ-ergodic, ρ is a multiple of ν. It remains to prove the implication

$$\rho = 0 \Rightarrow \theta = 0.$$

Assume that $\rho = 0$. Let $n \geq 0$ and φ, ψ be bounded Borel functions on X and on G^n respectively. We calculate

$$\begin{aligned}
&\int_{B^X} \psi(b_1, \ldots, b_n)\varphi(b_n^{-1} \cdots b_1^{-1} x)\theta(b, x) \, d\beta^X(b, x) \\
&= \int_{B^X} \psi(b_1, \ldots, b_n)\varphi(b_n^{-1} \cdots b_1^{-1} x)\theta(T^n b, b_n^{-1} \cdots b_1^{-1} x) \, d\beta^X(b, x) \\
&= \int_B \int_X \psi(b_1, \ldots, b_n)\varphi(y)\theta(T^n b, y) \, d((b_n^{-1} \cdots b_1^{-1})_* \nu_b)(y) \, d\beta(b) \\
&= \int_{G^n} \int_B \int_X \psi(b_1, \ldots, b_n)\varphi(y)\theta(b', y) \, d\nu_{b'}(y) \, d\beta(b') \, d\mu^{\otimes n}(b_1, \ldots, b_n) \\
&= \mu^{\otimes n}(\psi) \, \rho(\varphi) \; = \; 0.
\end{aligned}$$

Since the map

$$G^n \times X \to G^n \times X, (g_1, \ldots, g_n, x) \mapsto (g_1, \ldots, g_n, g_n^{-1} \cdots g_1^{-1} x)$$

is a homeomorphism, we get, for any bounded Borel function ψ on $G^n \times X$,

$$\int_{B^X} \psi(g_1, \ldots, g_n, x)\theta(b, x) \, d\beta^X(b, x) = 0.$$

This proves that $\theta = 0$, β^X-almost everywhere. □

2.6 The Two-Sided Fibered Dynamical System

We explain in this section how the forward and the backward dynamical systems are related. Indeed, both occur as factors of the space of biinfinite trajectories either equipped with the shift transformation or its inverse.

We keep the notations of Proposition 2.23. We denote by $(\widetilde{B}, \widetilde{\mathscr{B}}, \widetilde{\beta}, \widetilde{T})$ the *two-sided Bernoulli shift* with alphabet $(G, \mathscr{G}, \mu)$, that is, $\widetilde{B}$ is the product space $G^{\mathbb{Z}}$, $\widetilde{\mathscr{B}}$ is the product σ-algebra $\mathscr{G}^{\otimes \mathbb{Z}}$, $\widetilde{\beta}$ is the product measure $\mu^{\otimes \mathbb{Z}}$, and $\widetilde{T}$ is the shift map given by

$$\widetilde{T} b = (\ldots, b_{n+1}, \ldots) \text{ for } b = (\ldots, b_n, \ldots) \in \widetilde{B}.$$

This dynamical system is invertible and the probability measure $\widetilde{\beta}$ is $\widetilde{T}$-invariant.

For $\widetilde{\beta}$-almost every b in $\widetilde{B}$, we define

$$b_+ := (b_1, b_2, \ldots) \in B \text{ and } b_- := (b_0, b_{-1}, b_{-2}, \ldots) \in B.$$

The map $b \mapsto b_+$ realizes the two-sided Bernoulli shift $(\widetilde{B}, \widetilde{\beta}, \widetilde{T})$ as the natural invertible extension of the one-sided Bernoulli shift (B, β, T). Similarly, the map $b \mapsto b_-$ realizes the inverse $(\widetilde{B}, \widetilde{\beta}, \widetilde{T}^{-1})$ of the two-sided Bernoulli shift as the natural invertible extension of the one-sided Bernoulli shift (B, β, T).

We now construct the *two-sided fibered dynamical system* on the space $\widetilde{B} \times X$ that we heuristically consider as the space of biinfinite trajectories. We endow this space with the σ-algebra $\widetilde{\mathscr{B}} \otimes \mathscr{X}$ of Borel subsets and we introduce the skew-product transformation

$$\widetilde{T}^X : (b, x) \mapsto (\widetilde{T}b, b_1 x)$$

and the Borel probability measure $\widetilde{\beta}^X$ on $B \times X$ defined by

$$\widetilde{\beta}^X := \textstyle\int_{\widetilde{B}} \delta_b \otimes \nu_{b_-} \, \mathrm{d}\widetilde{\beta}(b).$$

This dynamical system is invertible and the probability measure $\widetilde{\beta}^X$ is $\widetilde{T}$-invariant.

The map $(b, x) \mapsto (b_+, x)$ realizes the two-sided dynamical system $(\widetilde{B}^X, \widetilde{\beta}^X, \widetilde{T}^X)$ as the natural invertible extension of the forward dynamical system $(B^X, \beta \otimes \nu, T^X)$. Similarly, the map $(b, x) \mapsto (b_-, x)$ realizes the inverse $(\widetilde{B}^X, \widetilde{\beta}^X, (\widetilde{T}^X)^{-1})$ of the two-sided dynamical system as the backward dynamical system $(B^X, \beta^X, T^{\vee X})$. Since the natural invertible extension of an ergodic probability preserving dynamical system is also ergodic, and since the inverse of an ergodic transformation is also ergodic, this discussion gives a direct proof of the equivalences

$$\beta \otimes \nu \text{ is } T^X\text{-ergodic} \Leftrightarrow \widetilde{\beta}^X \text{ is } \widetilde{T}^X\text{-ergodic} \Leftrightarrow \beta^X \text{ is } T^{\vee X}\text{-ergodic}$$

and explains how Propositions 2.9 and 2.23 are related.

2.7 Proximal Stationary Measures

In this section, we introduce the property of μ-proximality for stationary measures. This proximality property will be satisfied by the stationary measures on projective spaces in Sect. 4.2 and by the stationary measures on the flag varieties in Sect. 10.1.

Let G be a second countable locally compact semigroup acting continuously on a compact metrizable topological space X, Say that a μ-stationary Borel probability measure ν on X is *μ-proximal* if, for β-almost any b in B, the Borel probability measure ν_b is a Dirac mass. An important example of a proximal stationary probability measure will be given in Proposition 10.1.

More generally, given a morphism $s : G \to F$ onto a finite group F, we define a *fibration over F* of X as a G-equivariant continuous map $X \to F$. We say that X is *fibered over F* if it is equipped with such a fibration. In this case, we say that ν is *μ-proximal over F* if, for β-almost any b in B, the Borel probability measure ν_b is a uniform average of $|F|$ Dirac masses and its image in F is the normalized counting measure on F. This definition will be used in Sect. 5.3, and an important example of such a situation will be given in Proposition 10.2.

We will apply the following lemma to the embedding of a flag variety in a product of projective spaces in order to prove Proposition 10.1.

Lemma 2.24 *Let $X, X_1, \dots, X_k$ be compact metrizable topological spaces, all of them equipped with a continuous action of a second countable locally compact semigroup G and let $\pi : X \to X_1 \times \cdots \times X_k$ be a continuous injective G-equivariant map. Suppose, for any $1 \le i \le k$, there exists a unique μ-stationary Borel probability measure ν_i on X_i and ν_i is μ-proximal. Then, there exists a unique μ-stationary Borel probability measure on X and it is μ-proximal.*

Proof For any $1 \le i \le k$, since the probability measures ν_i is μ-proximal, there exists a Borel map $\xi_i : B \to X_i$ such that, for β-almost any b in B, one has $(\nu_i)_b = \delta_{\xi_i(b)}$. Set $\pi_i : X \to X_i$ to be the projection map on the factor X_i and set $\xi = (\xi_1, \dots, \xi_k)$. Let ν be a μ-stationary Borel probability measure on X. Since, for any $1 \le i \le k$, the Borel probability measure $(\pi_i)_*\nu$ is μ-stationary, by uniqueness, one has $(\pi_i)_*\nu = \nu_i$ and, for β-almost any b in B, $(\pi_i)_*\nu_b = \delta_{\xi_i(b)}$, so that $\pi_*\nu_b = \delta_{\xi(b)}$. Hence ν is μ-proximal, and, for β-almost any b in B, one has $\xi(b) \in \pi(X)$ and $\pi_*\nu = \xi_*\beta$, whence the result. $\square$

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