

Preface to Second Edition

This second edition continues to serve primarily as a text for a lively two-quarter or one-semester course in probability theory for students from diverse disciplines, including mathematics and statistics. Exercises have been added and reorganized (i) for reinforcement in the use of techniques, and (ii) to complement some results. Sufficient material has been added so that in its entirety the book may also be used for a two-semester course in basic probability theory. The authors have reorganized material to make Chapters III–XIII as self-contained as possible. This will aid instructors of one semester (or two quarter) courses in picking and choosing some material, while omitting some other. Material from a former chapter on Laplace transforms has been redistributed to parts of the text where it is used.

The early introduction of conditional expectation and conditional probability maintains the pedagogic innovation of the first edition. This enables the student to quickly move to the fundamentally important notions of modern probability besides independence, namely, *martingale dependence* and *Markov dependence*, where new theory and examples have been added to the text. The former includes *Doob's upcrossing inequality*, the *submartingale convergence theorem*, and *reverse martingales*, and the (reverse) *martingale proof of the strong law of large numbers*, while retaining important earlier approaches such as those of Kolmogorov, Etemadi, and of Marcinkiewicz–Zygmund.

A theorem of *Polya* is added to the chapter on weak convergence to show that the convergence to the normal distribution function in the central limit theorem is uniform.

The *Cramér–Chernoff large deviation theory* in Chapter V is sharpened by the addition of a large deviation theorem of *Bahadur and Ranga Rao* using the *Berry–Esseen* convergence rate in the central limit theorem. Also added in Chapter V is a *concentration of measure* type inequality due to *Hoeffding*. The proof of the aforementioned *Berry–Esseen bound* is deferred to Chapter VI on Fourier series and Fourier transform. The *Chung–Fuchs transience/recurrence criteria* for random walk based on Fourier analysis is a new addition to the text.

Special examples of Markov processes such as *Brownian motion*, and *random walks* appear throughout the text to illustrate applications of (i) martingale theory

and stopping times in computations of certain important probabilities. A culmination of the theory developed in the text occurs in Chapters [XI](#) and [XII](#) on Brownian motion. This continues to rank among the primary goals attainable for a course based on the text.

General *Markov dependent* sequences and their convergence to equilibrium is the subject matter of the entirely new Chapter [XIII](#). Illustrative examples are provided, including some of historical importance to the development of the kinetic theory of matter in physics due to Boltzmann, Einstein, and Smoluchowski. The treatment centers on describing a prototypical framework, namely *Doeblin's theorem*, for existence and convergence to a unique *invariant probability* for Markov processes, together with illustrative examples for students with diverse interests ranging from mathematics and statistics to contemporary mathematical finance or biology. Examples include *iterated random maps*, the *Ehrenfest model*, and products of *random matrices*. The *Ornstein–Uhlenbeck process* is shown to be obtained as the unique solution to a *stochastic differential equation*, namely the *Langevin equation*, using *Picard iteration*. This provides students with a glimpse into the broad scope and utility of the probability that they have learned, while motivating continued study of stochastic processes.

Complete references to authors of books cited in footnotes are provided in a closing list of references. This also includes other textbook resources covering the same topics and/or further applications.

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Preface to First Edition

In 1937, A.N. Kolmogorov introduced a measure-theoretic mathematical framework for probability theory in response to David Hilbert's Sixth Problem. This text provides the basic elements of probability within this framework. It may be used for a one-semester course in probability, or as a reference to prerequisite material in a course on stochastic processes. Our pedagogical view is that the subsequent applications to stochastic processes provide a continued opportunity to motivate and reinforce these important mathematical foundations. The book is best suited for students with some prior, or at least concurrent, exposure to measure theory and analysis. But it also provides a fairly detailed overview, with proofs given in appendices, of the measure theory and analysis used.

The selection of material presented in this text grew out of our effort to provide a self-contained reference to foundational material that would facilitate a companion treatise on stochastic processes that *Theory and Applications of Stochastic Processes* we have been developing.¹ While there are many excellent textbooks available that provide the probability background for various continued studies of stochastic processes, the present treatment was designed with this as an explicit goal. This led to some unique features from the perspective of the ordering and selection of material.

We begin with Chapter I on various measure-theoretic concepts and results required for the proper mathematical formulation of a probability space, random maps, distributions, and expected values. Standard results from measure theory are motivated and explained with detailed proofs left to an appendix.

Chapter II is devoted to two of the most fundamental concepts in probability theory: independence and conditional expectation (and/or conditional probability). This continues to build upon, reinforce, and motivate basic ideas from real analysis and measure theory that are regularly employed in probability theory, such as Carathéodory constructions, the Radon–Nikodym theorem, and the Fubini–Tonelli

¹Bhattacharya, R. and E. Waymire (2007): *Theory and Applications of Stochastic Processes*, Springer-Verlag, Graduate Texts in Mathematics.

theorem. A careful proof of the Markov property is given for discrete-parameter random walks on \mathbb{R}^k to illustrate conditional probability calculations in some generality.

Chapter III provides some basic elements of martingale theory that have evolved to occupy a significant foundational role in probability theory. In particular, optional stopping and maximal inequalities are cornerstone elements. This chapter provides sufficient martingale background, for example, to take up a course in stochastic differential equations developed in a chapter of our text on stochastic processes. A more comprehensive treatment of martingale theory is deferred to stochastic processes with further applications there as well.

The various laws of large numbers and elements of large deviation theory are developed in Chapter IV. This includes the classical 0–1 laws of Kolmogorov and Hewitt–Savage. Some emphasis is given to size-biasing in large deviation calculations which are of contemporary interest.

Chapter V analyzes in detail the topology of weak convergence of probabilities defined on metric spaces, culminating in the notion of tightness and a proof of Prohorov’s theorem.

The characteristic function is introduced in Chapter VI via a first principles development of Fourier series and the Fourier transform. In addition to the operational calculus and inversion theorem, Herglotz’s theorem, Bochner’s theorem, and the Cramér–Lévy continuity theorem are given. Probabilistic applications include the Chung–Fuchs criterion for recurrence of random walks on \mathbb{R}^k , and the classical central limit theorem for i.i.d. random vectors with finite second moments. The law of rare events (i.e., Poisson approximation to binomial) is also included as a simple illustration of the continuity theorem, although simple direct calculations are also possible.

In Chapter VII, central limit theorems of Lindeberg and Lyapounov are derived. Although there is some mention of stable and infinitely divisible laws, the full treatment of infinite divisibility and Lévy–Kinchine representation is more properly deferred to a study of stochastic processes with independent increments.

The Laplace transform is developed in Chapter VIII with Karamata’s Tauberian theorem as the main goal. This includes a heavy dose of exponential size-biasing techniques to go from probabilistic considerations to general Radon measures. The standard operational calculus for the Laplace transform is developed along the way.

Random series of independent summands are treated in Chapter IX. This includes the mean square summability criterion and Kolmogorov’s three series criteria based on Kolmogorov’s maximal inequality. An alternative proof to that presented in Chapter IV for Kolmogorov’s strong law of large numbers is given, together with the Marcinkiewicz and Zygmund extension, based on these criteria and Kronecker’s lemma. The equivalence of a.s. convergence, convergence in probability, and convergence in distribution for series of independent summands is also included.

In Chapter X, Kolmogorov’s consistency conditions lead to the construction of probability measures on the Cartesian product of infinitely many spaces. Applications include a construction of Gaussian random fields and discrete-parameter Markov

processes. The deficiency of Kolmogorov's construction of a model for Brownian motion is described, and the Lévy–Ciesielski “wavelet” construction is provided.

Basic properties of Brownian motion are taken up in Chapter XI. Included are various rescalings and time inversion properties, together with the fine-scale structure embodied in the law of the iterated logarithm for Brownian motion.

In Chapter XII many of the basic notions introduced in the text are tied together via further considerations of Brownian motion. In particular, this chapter revisits conditional probabilities in terms of the Markov and strong Markov properties for Brownian motion, stopping times, and the optional stopping and/or sampling theorems for Brownian motion and related martingales, and leads to weak convergence of rescaled random walks with finite second moments to Brownian motion, i.e., Donsker's invariance principle or the functional central limit theorem, via the Skorokhod embedding theorem.

The text is concluded with a historical overview, Chapter XIII, on Brownian motion and its fundamental role in applications to physics, financial mathematics, and partial differential equations, which inspired its creation.

Most of the material in this book has been used by us in graduate probability courses taught at the University of Arizona, Indiana University, and Oregon State University. The authors are grateful to Virginia Jones for superb word processing skills that went into the preparation of this text. Also, two Oregon State University graduate students, Jorge Ramirez and David Wing, did an outstanding job in uncovering and reporting various bugs in earlier drafts of this text. Thanks go to the editorial staff at Springer and anonymous referees for their insightful remarks.

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NOTE: Some of the first edition chapter numbers have changed in the second edition. First edition Chapter VII was moved to Chapter [IV](#), and the material in the first edition Chapter VIII has been redistributed into other chapters. An entirely new Chapter [XIII](#) was added to the second edition.



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