Chapter 2
The Reverse Directional Distance Function

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Abstract The aim of any Data Envelopment Analysis (DEA) inefficiency model is to calculate the efficient projection of each unit belonging to a certain finite sample. The reverse directional distance function (*RDDF*) is a new tool developed in this chapter that allows us to express any known DEA inefficiency model as a directional distance function (*DDF*). Hence, given a certain DEA inefficiency model, its *RDDF* is a specific *DDF* that truly reproduces the functioning of the considered DEA model. Automatically, all the interesting properties that apply to any *DDF* are directly transferable to the considered DEA model through its *RDDF*. Hence, the *RDDF* enlarges the set of properties exhibited by any DEA model. For instance, given any DEA inefficiency model, its economic inefficiency—in any of its three possible versions—, can be easily defined and decomposed as the sum of technical inefficiency and allocative inefficiency thanks to the *RDDF*. We further propose to transform any non-strong *DDF* into a strong *DDF*, i.e., into a *DDF* that projects all the units onto the strongly efficient frontier. This constitutes another indication of the transference capacity of the *RDDF*, because its strong version constitutes in itself a strong version of the original DEA model considered. We further propose to search for alternative projections so as to minimize profit inefficiency, and add an appendix showing how to search for multiple optimal solutions in additive-type models.

Keywords Data envelopment analysis · Reverse directional distance function · Economic inefficiency decomposition · Strong DEA models

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2.1 Introduction

This "Introduction" comprises a description of the subsequent sections as well as a revision of the previous related literature. In Sect. 2.2, we start showing how to transform any DEA inefficiency model into an equivalent *DDF*, called *RDDF*. As mentioned before, the advantages of the *RDDF* is that it has, as a *DDF*, interesting properties, which are inherited by the DEA model where it comes from.

Any time we consider a strong DEA model, i.e., a model whose projections belong to the strongly efficient frontier, the corresponding *RDDF* is also a strong *DDF*. Otherwise, we propose in Sect. 2.3 a method for transforming a weak *DDF*, that is, a *DDF* whose set of projections does not belong to the strong frontier, into a strong *DDF*. Our method identifies the set of strongly efficient projections that defines the new strong *DDF*. In particular, if we want to transform a weak DEA inefficiency measure into a strong one, we always have the option of working with its associated *RDDF*. Consequently, this methodology solves the general problem of transforming any weak DEA inefficiency measure into a strong DEA inefficiency measure. We are only aware of a previous paper that transforms two specific weak DEA efficiency measures into strong DEA efficiency measures (see Asmild and Pastor 2010). In order to derive a comprehensive inefficiency measure associated to the generated strong *DDF*, all we have to do is to consider, at each unit being rated, a strong directional vector that is comparable with the original directional vector associated to the weak *DDF*. We close Sect. 2.3 by making a simple proposal that basically pursues the notion that the two directional vectors at each point have the same Euclidian length.

Section 2.4 extends the findings of Sect. 2.3 to any DEA inefficiency model, *M*, through the corresponding *RDDF*. If it happens that its *RDDF* is a weak *DDF*, the tools developed in Sect. 2.3 are directly applied in order to generate a strong RDDF. This strong *RDDF* is associated to *M* and offers a comprehensive inefficiency measure for it. Consequently, we have solved the problem of associating to any DEA non-comprehensive inefficiency model a *DDF* comprehensive inefficiency model. Moreover, we apply this result to generate, for the first time, comprehensive radial inefficiency models.

Overall inefficiency measurement and decomposition are important for firms facing a world of changing prices since the resultant loss has implications on managers' decision making. In standard microeconomic theory, the economic behavior of a DMU (Decision Making Unit) is usually characterized by cost minimization, revenue maximization, or profit maximization. In particular, if profit maximization is assumed, the DMU faces exogenously determined market output and input prices, and we may assume that the objective of each DMU is to choose the output combination that yields the maximum profit efficiency. In this sense, profit efficiency indicates how close the actual profit of the evaluated DMU approaches the maximum feasible profit. Additionally, in the Farrell (1957) tradition, overall efficiency has usually been decomposed into the product of two components, technical efficiency and allocative efficiency, as a way to understand what needs to be done to enhance the performance of the assessed unit.

Chronologically, the empirical estimation of technologies from a dataset began in the area of economics with the application of regression analysis and Ordinary Least Squares (OLS) to estimate a parametrically specified 'average' production function (see, e.g., Cobb and Douglas 1928). Later, Farrell (1957) was the first in showing, for a single output and multiple inputs, how to estimate an isoquant enveloping all the observations. Farrell also showed how to decompose cost efficiency into technical and allocative efficiencies. In his paper one can find the first practical implementation of the Debreu coefficient of resource utilization (Debreu 1951) and the Shephard input distance function (Shephard 1953). Farrell's paper inspired other authors to continue this line of research by either a non-parametric piece-wise linear technology or a parametric function. The first possibility was taken up by Charnes et al. (1978) and Banker et al. (1984) resulting in the development of DEA radial models, closely related to the Shephard distance functions; while the latter approach was taken up at the same time by Aigner et al. (1977), Battesse and Corra (1977) and Meeusen and van den Broeck (1977), subsequently resulting in the development of the stochastic frontier models.

As previously mentioned, the decomposition proposed by Farrell was inspired on the work of Shephard, in the sense that the technical efficiency component is really the inverse of the Shephard input distance function. Indeed, Shephard (1953) also defined an output-oriented distance function and established several dual relationships. Much later, Färe and Primont (1995) developed a dual, but not natural, correspondence between Shephard's distance functions and the profit function. In recent years there has been extensive interest in the duality theory and distance functions as can be easily checked. If one defines an optimization problem with respect to quantities, then a dual problem can be defined with respect to (shadow) prices that has the same value. This approach is of great interest for microeconomics both for understanding the mathematics and for clarifying the economics. Chronologically speaking, Luenberger (1992a, b) and later Chambers et al. (1996, 1998) and Briec and Lesourd (1999), have produced a series of papers in this field. Specifically, Luenberger (1992a, b) introduced the concept of benefit function[1] as a representation of the amount that an individual is willing to trade, in terms of a specific reference commodity bundle. Luenberger also defined a so-called shortage function, which basically measures the distance in the direction of a vector from a production plan (DMU) to the boundary of the production possibility set. In other words, the shortage function measures the amount by which a specific unit is short of reaching the frontier of the production possibility set. Some years later, Chambers et al. (1996, 1998) redefined the benefit function and the shortage function as inefficiency measures, introducing to this end new distance functions, the so called *DDFs*. They showed how the *DDFs* encompass, among others, the Shephard input and output distance functions. And they also derived a dual

[1]Briec and Garderes (2004) have tried to generalize the Luenberger benefit function.

correspondence between the directional distance functions and the profit function that, in their opinion, generalized all previous dual relationships. A few years later, Briec and Lesourd (1999) introduced the so-called Hölder metric distance functions intending to relate the concept of efficiency and the notion of distance in topology. Along these lines, they proved that the profit function can be derived from the Hölder metric distance functions and that these distance functions can be recovered from the profit function.

In contrast to the parametric literature on efficiency, where the measurement of technical efficiency in the context of multiple-outputs is based on a few measures in practice, basically the Shephard input and output distance functions and the directional distance functions, the first years of life of DEA saw the introduction of a bunch of different technical efficiency/inefficiency measures, such as the Russell input and output measures of technical efficiency and their graph extension, the Russell Graph Measure of technical efficiency (see Färe et al. 1985), as well as the additive model (Charnes et al. 1985), followed, several years later by the Range-Adjusted Measure (Cooper et al. 1999), the Enhanced Russell Graph Measure (Pastor et al. 1999) re-baptized as the Slacks-Based Measure (Tone 2001), or the Bounded Adjusted Measure (Cooper et al. 2011a), to name but a few. This short list shows that there is a wide array of tools available for estimating technical inefficiency in the non-parametric world.

On the other hand, most of the classical results and applications in microeconomics related to the measurement and decomposition of overall inefficiency, in terms of technical and allocative inefficiency, are based on the notion of distance function[2] and duality theory. A distance function behaves, in fact, as a technical inefficiency measure when an observation belonging to the corresponding technology is evaluated, with a meaning of 'distance' from the assessed interior point to the boundary of the production possibility set. Also, the distance functions have dual relationships with well-known support functions in microeconomics, as the profit function or the cost and revenue functions, depending on the suppositions that we are willing to assume with respect to the firms´ behavior. In a non-parametric framework, the use of typical parametric tools, such as the Shephard distance functions or the directional distance function, is possible, because their duality relationships with classical support functions were proved for production possibility sets fulfilling general axioms (e.g. convexity) and, in particular, they can be applied to non-parametric polyhedral technologies. Nonetheless, the majority of attempts for estimating overall efficiency have overlooked the concept of distance function, a fact that contrasts significantly with the traditional view of economics of production, where both this concept and duality are the cornerstones of the applied theory. In this respect, some researchers have tried to use additive-type models in DEA for measuring not only technical inefficiency but also profit inefficiency without resorting directly to the notion of distance function (Cooper et al. 2011b and

[2]We would like to remark that moving from the Shephard distance functions to the directional distance functions entails moving from the inverse of efficiency measures to inefficiency measures.

Aparicio et al. 2013). A similar treatment has been applied to Russell oriented-measures (Aparicio et al. 2015), without being able to derive duality results for the corresponding non-oriented measure known as the Enhanced Russell Graph measure (Pastor et al. 1999). However, DEA is a field where there are other alternative efficiency measures and where it seems possible to introduce new ones. Therefore, defining an appropriate methodology to measure and decompose overall inefficiency with whatever DEA measure is something necessary. We accomplish this task in Sects. 2.5–2.7, resorting to the definition of a new concept, the *RDDF*.

We close this chapter by searching, in Sect. 2.8 and within a DEA framework, for an alternative projection for each unit that minimizes profit inefficiency. The new alternative projection has to dominate the point being rated, which guarantees that technical inefficiency can also be evaluated. Finally, we present our conclusions in Sect. 2.9, and add an Appendix for searching for alternative optimal solutions, taking any additive type model as reference.

2.2 Associating a *RDDF* Inefficiency Measure to Any Known DEA Inefficiency Measure

First of all, let us introduce the definition of the traditional *DDF* within a DEA framework. That means that the production possibility set is generated based on a finite sample of units to be rated, and that the inefficiency associated to each unit is obtained by solving a linear program. From now on, we will further assume variable returns to scale (VRS), which guarantees that the three economic functions we are going to consider later on are well-defined.

Let us consider a sample of n units to be rated. Unit $j \in \{1, 2, \ldots, n\}$ uses a specific amount of m inputs, $x_j = (x_{1j}, \ldots, x_{mj}) \in R_+^m$, to produce a certain amount of s outputs $y_j = (y_{1j}, \ldots, y_{sj}) \in R_+^s$. As usual, let us denote the unit to be rated as (x_0, y_0).[3] The production possibility set generated by the finite sample of units is

$$T = \left\{ (x, y) \in R_+^{m+s} : \sum_{j=1}^{n} \lambda_j x_{ij} \le x_i, \forall i, \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_r, \forall r, \lambda_j \ge 0, \forall j, \sum_{j=1}^{n} \lambda_j = 1 \right\},$$

while the efficient frontier of T is defined as[4]

[3]The condition that inputs and outputs need to be non-negative can be relaxed provided the considered *DDF* is translation invariant (see Aparicio et al. 2016).

[4]The efficient frontier, $\partial(T)$, or simply the frontier of T, comprises the weak-efficient frontier. $\partial^W(T)$, and the strong-efficient frontier, or subset of all the Pareto efficient points. See Färe et al. (1985) for the definition of $\partial^W(T)$.

$$\partial(T) := \{(x, y) \in T : \hat{x} \leq x, \hat{y} \geq y \quad and \quad (x, y) \neq (\hat{x}, \hat{y}) \Rightarrow (\hat{x}, \hat{y}) \notin T\}.$$

Each DDF (Chambers et al. 1996, 1998) is identified by specifying a directional vector $g = (-g_x, g_y) \neq 0_{m+s},\ g_x \in R^m_+,\ g_y \in R^s_+$. In order to measure the inefficiency associated to a specific unit of the sample, the DDF projects the unit onto the weakly efficient frontier of the technology along the positive semi-ray defined by vector g. Additionally, g may be constant, i.e. g is the same vector for all units, or may be variable, i.e. it is a specific vector for each unit. In the latter case and for unit (x_0, y_0), we write g_0 instead of g. By definition, the projection of unit (x_0, y_0) onto the efficient frontier is the intersection of the semi-ray $\{(x_0, y_0) + \beta_0(-g_{0x}, g_{0y}),\ \beta_0 \geq 0\}$ with the efficient frontier. The specific value of scalar β_0 that identifies the point of intersection is the inefficiency value measured by the *DDF* associated to point (x_0, y_0), obtained as the optimal solution, β_0^*, of the next linear program, which corresponds to a generic *DDF* working under VRS.[5]

$$\begin{aligned}
\vec{D}(X_0, y_0; g_{0x}, g_{0y}) = Max \quad & \beta_0 \\
s.t \quad & \\
& \sum_{j=1}^{n} \lambda_{j0} X_{ij} \leq X_{i0} - \beta_0 g_{i0x}, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_{j0} y_{rj} \geq y_{r0} + \beta_0 g_{r0y}, \quad r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_{j0} = 1, \\
& \lambda_{j0} \geq 0, \quad j = 1, \ldots, n
\end{aligned} \tag{2.1}$$

It is well known that $\beta_0^* = 0$ identifies the unit being rated as efficient[6], while $\beta_0^* > 0$ identifies the unit being rated as inefficient.

[5] The linear program we are working with corresponds to the "envelopment form" associated to a *DDF*. Its linear dual is known as the "multiplier form" of the *DDF*. The "envelopment form" deals with units and evaluates their efficient projections, working in the $m + s$ dimensional space where each coordinate corresponds to an input or to an output, while the "multiplier form" identifies the supporting hyperplane of each efficient projection and works also in an $m + s$ dimensional space where each coordinate corresponds to the shadow price of an input or of an output. In this chapter we will only consider "envelopment forms".

[6] An efficient unit is a unit that belongs to the efficient frontier. Any efficient unit may be strongly efficient or, alternatively, weakly efficient. The subset of the efficient frontier of strongly efficient points is called the strongly efficient frontier. The directional vector may also be specified as $(g = g_x, g_y)$. What matters is that g_x appears preceded by a minus sign in (2.1)

Let us now assume that we have obtained the projection[7] of any point of our sample by means of a specific DEA inefficiency model, identified as M. As explained in Footnote 7, if M is single-valued, the projection of any point is unique. Although the method we are going to propose is valid for any DEA model, the most usual case is that inefficiency model M is a linear programming model or a programming model that can be linearized through an appropriate change of variables[8] (see, e.g., Pastor et al. 1999).

If (x_0, y_0) denotes the point being rated, let us denote as $\left(x_0^M, y_0^M\right)$ its efficient projection. Moreover, let us denote as Π^M the set of efficient projections obtained through M and associated to the sample of points being rated.[9] Let us further denote as $\tau_0^M := TI^M(x_0, y_0)$ the technical inefficiency evaluated by means of model M. It is well known that $\tau_0^M \geq 0$. Moreover, if (x_0, y_0) belongs to the efficient frontier, then $\tau_0^M = 0$.

Definition 1 Associated to DEA model M and to the evaluated set of efficient projections, Π^M, we define the *reverse directional distance function model,* $RDDF^{M,\Pi^M}$, by specifying the directional vector g_0^{M,Π^M} at point (x_0, y_0) as follows[10]:

$$g_0^{M,\Pi^M} := \left\{ \begin{array}{lll} \frac{1}{\tau_0^M}\left(x_0 - x_0^M, y_0^M - y_0\right) \geq 0_{m+s}, & if & \left(x_0 - x_0^M, y_0^M - y_0\right) \neq 0 \, and \, \tau_0^M > 0 \\ (1_m, 1_s), & if & \tau_0^M = 0 \end{array} \right\}. \tag{2.2}$$

As usual, the technical inefficiency associated to the directional distance function $RDDF^{M,\Pi^M}$ is denoted as $\vec{D}\left(x_0, y_0; g_{0x}^{M,\Pi^M}, g_{0y}^{M,\Pi^M}\right)$. The definition of g_0^{M,Π^M} distinguishes between two kinds of points, just as model M does: the points that get an

[7]Usually DEA researchers and practitioners are satisfied computing a unique projection for each point, although in many DEA models multiple projections can be identified. Here we accommodate our findings to this tradition and work initially with a single projection for each point. Nevertheless, in Sect. 2.4, Example 4.2, we consider an input-oriented additive model with multiple projections. DEA models with a unique projection for each point can be baptized as "single-value DEA models", as opposed to "multiple-value DEA models".

[8]If M is not a DEA inefficiency model but a DEA efficiency model, we can always conveniently modify its objective function so as to get an inefficiency model (see, e.g., Aparicio et al. 2015). The novel loss distance function (Pastor et al. 2012) embraces all well-known DEA models as they are or with minor changes in its objective function, and constitutes the widest known family of DEA inefficiency models. Since the loss distance function considers the multiplier form of each DEA model we will not go into further details.

[9]Observe that we obtain a single projection for each point through the corresponding linear program. This fact does not exclude the possible existence of alternative optimal projections, whose study is introduced only for additive type models in Appendix 1.

[10]The name "reverse" directional distance function (*RDDF*) is a consequence of how we define the associated *DDF*. Usually, for defining a *DDF*, we need to know, at each point, the corresponding directional vector and, based on it, we determine its efficient projection. In our new proposed approach, we do it the other way round, i.e., we know beforehand the projection of each point being rated and, based on it, we derive the corresponding directional vector at that point.

inefficiency score $\tau_0^M > 0$ and the rest of the points, for which $\tau_0^M = 0$. In any case, the second subset of points that satisfy $\tau_0^M = 0$ corresponds to all the points whose projection through model M belongs to the frontier. It is clear that for any point of this second subset, the proposed fix directional vector $(1_m, 1_s)$ of $RDDF^{M,\Pi^M}$ will assign an inefficiency equal to 0, just as model M does.

As a direct consequence of the last definition the next statements holds.

Proposition 1

(a) *$RDDF^{M,\Pi^M}$ has exactly the same projections as M.*[11] *As a consequence, $RDDF^{M,\Pi^M}$ inherits the same frontier and the same returns to scale characteristics as M.*
(b) *The* technical inefficiency *associated to any unit (x_0, y_0) through $RDDF^{M,\Pi^M}$ is exactly the same as the technical inefficiency evaluated through M, τ_0^M.*

Proof

(a) Trivial.
(b) We have two cases. First, if $\tau_0^M = 0$ then (x_0, y_0) belongs to the efficient frontier and, by (2.2), $g_0^{M,\Pi^M} = (1_m, 1_s) > 0_{m+s}$. Therefore, $\vec{D}\left(x_0, y_0; g_{0x}^{M,\Pi^M}, g_{0y}^{M,\Pi^M}\right) = 0$. Second, if $\tau_0^M > 0$, by Lemma 2.2(c) in Chambers et al. (1998) we have that $\vec{D}\left(x_0, y_0; g_{0x}^{M,\Pi^M}, g_{0y}^{M,\Pi^M}\right) = \tau_0^M \vec{D}\left(x_0, y_0; \left(x_0 - x_0^M\right), \left(y_0^M - y_0\right)\right) = \tau_0^M$, as a consequence of being $\vec{D}\left(x_0, y_0; \left(x_0 - x_0^M\right), \left(y_0^M - y_0\right)\right) = 1$. ∎

As a direct consequence of Proposition 1(a), we get the next Corollary.

Corollary 1.1 *If model M projects all units onto the strong (weak) efficient frontier, so does $RDDF^{M,\Pi^M}$.*

Corollary 1.1 shows an easy way to generate DDFs with all their projections onto the strongly efficient frontier,[12] something that seldom occurs in the framework of the usual directional distance functions.

The next result is straightforward and adds consistency to our proposal.

[11] We would like to remind the reader that the definition of the *RDDF* is based on the evaluation of a single projection for each of the points being rated. In case one or more points have multiple possible projections, the change of the projection of any of the considered points gives rise to a different associated *RDDF*. This is the reason for writing $RDDF^{M,\Pi^M}$, making explicit through the super-indexes that *RDDF* depends not only on model M but on the set of computed projections Π^M.

[12] This happens exactly when M delivers a strong inefficiency measure, a measure whose projections belong to the strongly efficient frontier. A well-known example is the weighted additive model (Lovell and Pastor 1995) that will be considered in the examples of the next three sections.

Corollary 1.2 *If model M corresponds to a DDF, then M and $RDDF^M$ collapse together, with the possible exception of the directional vectors associated to points being rated where* $\tau_0^M = 0$.

Proof According to expression (2.2) it is obvious that at any point being rated where $\tau_0^M > 0$, both *M* and $RDDF^M$ are exactly the same *DDF*. Moreover, it is also obvious that at any point being rated where $\tau_0^M = 0$, the *DDF* directional vector associated to that point may be different from $(1_m, 1_s)$, which is the fixed directional vector that expression (2.2) assigns to that point in the definition of $RDDF^M$. ■

The next section takes advantage of the above introduced $RDDF^{M,\Pi^M}$ allowing us to transform any "weak *DDF*", i.e., any *DDF* whose set of efficient projections does not belong to the strongly efficient frontier, into a closely related "strong *DDF*", that is, a *DDF* whose set of efficient projections belongs to the strongly efficient frontier. In many real life applications, non-dominated peers are preferred over dominated ones, which is precisely the difference between strongly efficient projections and non-strongly efficient ones. This is the main reason for introducing Sect. 2.3.[13] This was also the objective of Fukuyama and Weber (2009), where a modified directional distance function was defined. As Pastor and Aparicio (2010) pointed out, this modified *DDF* really coincides with a specific weighted additive measure.

2.3 Generating a Strong DDF Based on a Weak DDF

We know in advance that all the projections associated to any *DDF* definitely belong to the efficient frontier and, sometimes, they all belong to the strongly efficient frontier. Let us start showing this last unusual case by means of an example.

Example 3.1 Analyzing a Strong *DDF*

Let us consider, in the one input—one output space, the next set of units to be rated U1(2,4), U2(4,8), U3(6,2), U4(3,4) and U5(10,8). The corresponding directional vectors are: (4,1) for U1, (8,2) for U2, (4,2) for U3, (4,1) for U4 and (1,0) for U5. As usual, we assume a VRS technology. After performing the first stage we get the corresponding set of projections (see Table 2.1). The results of Table 2.1 suggest that U1 and U2 are strongly efficient units.[14] U3 is an inefficient unit that belongs to

[13] Although we focus here on *DDF* inefficiency measures, it is easy to associate a strong *DDF* to any DEA inefficiency measure by resorting, as we do here, to the *RDDF*, as explained in more detail in Sect. 2.4.

[14] This fact can be corroborated by means of a two-dimensional graphical display.

Table 2.1 Results associated with Example 3.1

Unit: (x_0, y_0)	Directional vector	Inefficiency β_0^*	Projection: $\left(x_0^{p1}, y_0^{p1}\right)$	Input slack: s^{-*}	Output slack: s^{+*}
U1(2,4)	(4,1)	0	(2,4) = U1	0	0
U2(4,8)	(8,2)	0	(4,8) = U2	0	0
U3(6,2)	(4,2)	1	(2,4) = U3 + 1(−4,2) = U1	0	0
U4(3,4)	(4,1)	2/9	(19/9,38/9) = U4 + 2/9 (−4,1) = 17/18U1 + 1/18U2	0	0
U5(10,8)	(1,0)	6	(4,8) = U5 + 6(−1,0) = U2	0	0

the interior of the production possibility set and whose projection (2,4) is the strongly efficient point U1.

U4 is also an interior point whose *DDF* projection is a strongly efficient point that is a linear convex combination of the two strongly efficient units:

$$U4 + \frac{2}{9}(-4{,}1) = \left(3 - \frac{8}{9}, 4 + \frac{2}{9}\right) = \left(\frac{19}{9}, \frac{38}{9}\right) = \frac{17}{18}U1 + \frac{1}{18}U2.$$

Finally, U5 is itself a weakly efficient point dominated by its projection U2:

$$U5 + 6(-1{,}0) = (10{,}8) - (6{,}0) = (4{,}8) = U2.$$

In this simple example, we have been able to identify each of the two strongly efficient units, U1 or U2. In this particular case a convex combination of U1 and U2 is also a strongly efficient point, as for example the projection of U4, which means that all the located projections belong to the strongly efficient frontier.

Let us now consider the most frequent case, i.e., a weak *DDF*, where at least one of the projections of the sample of points being rated does not belong to the strongly efficient frontier. Let us first introduce a procedure for classifying the projection of any of the weak *DDF* inefficient points as strongly efficient or as not strongly efficient. In the second case, the procedure we are going to introduce is able to identify a new strongly efficient point that dominates the initial weakly efficient projection.

2.3.1 Converting a Weak DDF into a Strong DDF

Some years ago Asmild and Pastor (2010) designed a two stage procedure for getting strongly efficient projections in two particular cases: the multi-directional analysis measure (MEA) of Bogetoft and Hougaard (1999), and the range directional measure (RDM) of Silva-Portela et al. (2004). Although both are efficiency measures, the same reasoning can be applied to inefficiency measures, such as *DDFs*. We are going to replicate the procedure here for analyzing any *DDF*

(Chambers et al. 1998). Basically the first stage will get the projections of the considered *DDF* model, and the second stage will project the obtained projection onto the strongly efficient frontier, with the help of the additive model (Banker et al. 1984), formulated as follows:

$$\begin{aligned} Add(x_0, y_0) = Max_{s^-, s^+, \lambda} \quad & \sum_{i=1}^{m} s_{i0}^- + \sum_{r=1}^{s} s_{r0}^+ \\ & s.t. \\ & \sum_{j=1}^{n} \lambda_j x_{ij} = x_{i0} - s_{i0}^-, \quad i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} = y_{r0} + s_{r0}^+, \quad r = 1, \ldots, s \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n \\ & s_{i0}^- \geq 0, \quad i = 1, \ldots, m, \quad s_{r0}^+ \geq 0, \quad r = 1, \ldots, s \end{aligned} \tag{2.3}$$

Additive model (2.3) has the next advantage over other used DEA models, such as radial models or DDF models: it always achieves a strongly efficient projection for any point being rated. Since the additive model identifies an L_1-path towards the frontier connecting the point being rated and its projection, and the length of this path is obtained as the sum of all the optimal slack values that appear in the objective function, named as $Add(x_0, y_0)$ in model (2.3), it is straightforward to enunciate the next statement: "(x_0, y_0), the point being rated, is a strongly efficient point if, and only if, $Add(x_0, y_0) = 0$".[15]

Now, let us go back to *DDF* model (2.1). We can reformulate it very easily by taking the following action: add a single slack variable to each inequality transforming it into an equality as follows.

[15]The subset of strongly efficient point of the sample being rated are denoted as *E*. There are many other strongly efficient points of the production possibility set that do not belong to *E*. Under VRS, only convex linear combinations of points of *E* are potential candidates. Again, in order to check if one of these points is strongly efficient or not, the easiest way is to resort to the additive model [3] and analyze the mentioned convex linear combination of points of *E*. Only if the optimal objective value is 0, or, equivalently, all the optimal slack values are 0, the point being rated is strongly efficient.

$$\vec{D}\left(x_0, y_0; g_{0x}, g_{0y}\right) = Max \quad \beta_0$$
$$s.t$$
$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j x_{ij} + s_{i0}^{-} = x_{i0} - \beta_0 g_{i0x}, \quad i = 1, \ldots, m \\ &\sum_{j=1}^{n} \lambda_j y_{rj} - s_{r0}^{+} = y_{r0} + \beta_0 g_{r0y}, \quad r = 1, \ldots, s \\ &\sum_{j=1}^{n} \lambda_j = 1, \\ &\lambda_j \geq 0, \qquad\qquad j \in E \end{aligned} \tag{2.4}$$

We are searching for strongly efficient projections. Therefore, we assume that the projection identified through (2.4) is no longer $\left(x_{01} - \beta_0^* g_{01x}, \ldots, x_{0m} - \beta_0^* g_{0mx}, y_{01} + \beta_0^* g_{01y}, \ldots, y_{0s} + \beta_0^* g_{0sy}\right)$ as in model (2.1), but $\sum_{j=1}^{n} \lambda_{j0}^* \left(x_j, y_j\right)$, a point that satisfies

$$\sum_{j=1}^{n} \lambda_j^* \left(x_j, y_j\right) + \left(s_0^{-*}, -s_0^{+*}\right) = \left(x_0, y_0\right) + \beta_0^* \left(-g_{0x}, g_{0y}\right). \tag{2.5}$$

This constitutes the basic difference between model (2.4) and (2.1). As a direct consequence of (2.5), it is clear that $\sum_{j=1}^{n} \lambda_{j0}^* \left(x_j, y_j\right)$ dominates, or is equal, to $\left(x_0, y_0\right) + \beta_0^* \left(-g_{0x}, g_{0y}\right)$, and, consequently, is closer to the strongly efficient frontier. Our proposed second stage, based, as said before, on the additive model, checks if the first stage projection $\sum_{j=1}^{n} \lambda_{j0}^* \left(x_j, y_j\right)$ is itself a strongly efficient point or, alternatively, finds a strongly efficient point for replacing it.

As explained before, we are going to design a *two stage process,* which combines a given *DDF*, which offers us a first stage projection for each point of the sample, with a second stage *additive model*, that projects each first stage projection onto the strongly efficient frontier, ending up with a second stage strongly efficient projection. Relating each inefficient point with its final second stage projection gives rise to a comprehensive *DDF* inefficiency measure that combines all the detected inefficiencies into a single number.

Second Stage Analysis: Identifying the Strongly Efficient Projections

In order to classify the first stage projection $\left(x_0^{p1}, y_0^{p1}\right)$ of $\left(x_0, y_0\right)$ obtained through model (2.4), and as a direct consequence of (2.5) we can write the next equivalent expression:

$$\sum_{j=1}^{n} \lambda_j^* (x_j, y_j) = (x_0, y_0) + \beta_0^* (-g_{0x}, g_{0y}) + \left(-s_0^{-*}, s_0^{+*}\right). \tag{2.6}$$

Take additive model (2.3) and evaluate point $\sum_{j=1}^{n} \lambda_j^* (x_j, y_j)$. The result can be written as

$$\sum_{j=1}^{n} \lambda_j^* (x_j, y_j) = \sum_{j=1}^{n} \hat{\lambda}_j (x_j, y_j) + \left(\hat{s}_0^-, -\hat{s}_0^+\right), \tag{2.7}$$

where $\left(\hat{s}_0^-, \hat{s}_0^+\right)$ is the optimal solution of (2.3) linked to $\hat{\lambda}$. We know that $\sum_{j=1}^{n} \hat{\lambda}_j (x_j, y_j)$ is a strongly efficient point identified through our additive second stage projection and named alternatively as $\left(x_0^{p2}, y_0^{p2}\right)$. Combining (2.6) with (2.7) we can relate our initial inefficient point (x_0, y_0) with our second stage projection, obtaining the next relationship:[16]

$$\begin{aligned} \left(x_0^{p2}, y_0^{p2}\right) &= \sum_{j=1}^{n} \hat{\lambda}_j (x_j, y_j) = \sum_{j=1}^{n} \lambda_j^* (x_j, y_j) - \left(\hat{s}_0^-, -\hat{s}_0^+\right) \\ &= (x_0, y_0) + \beta_0^* (-g_{0x}, g_{0y}) + \left(-s_0^-, s_0^+\right) - \left(\hat{s}_0^-, -\hat{s}_0^+\right) \\ &= (x_0, y_0) + \beta_0^* (-g_{0x}, g_{0y}) + \left(-\left(s_0^- + \hat{s}_0^-\right), s_0^+ + \hat{s}_0^+\right). \end{aligned} \tag{2.8}$$

In summary, the calculated second stage strongly efficient projection is reached after performing a directional vector movement, $\beta_0^* (-g_{0x}, g_{0y})$, followed by a non-directional L_1-movement,[17] $\left(-\left(s_0^- + \hat{s}_0^-\right), s_0^+ + \hat{s}_0^+\right)$. The compound movement towards the strongly efficient frontier is given by $\beta_0^* (-g_{0x}, g_{0y}) + \left(-\left(s_0^- + \hat{s}_0^-\right), s_0^+ + \hat{s}_0^+\right)$, where the minus and plus signs indicates that in order to reach the strongly efficient frontier we need to reduce inputs and to increase outputs. Let us show an example before assigning a sensible inefficient value to each new strongly efficient projection.

Example 3.2 Part 1. Getting Strong Projections for a Weak *DDF*

Let us consider in the two-input one-output space the next five units: U1(4,2,4), U2(4,4,7), U3 (4,6,9), U4(4,3,2), U5(10,3,1) and U6(8,5,20/3). Let us assume that the considered *DDF* has a unique constant directional vector $g = (2,0,0)$. The first

---

[16] Even if $\left(s_0^-, s_0^+\right) = 0_{m+s}$, it is not assured that our first stage projection, $\sum_{j=1}^{n} \lambda_j^* (x_j, y_j)$, is itself a strongly efficient point. What is additionally needed is that the second-stage slacks, $\left(\hat{s}_0^-, \hat{s}_0^+\right)$, are equal to zero. Hence, the second stage is always needed.

[17] This is the typical movement associated with the additive model; the L_1-distance is also known as the Manhattan metric. Moreover, although the considered second stage could have alternative optimal solutions, only one of them is being considering here. Hence we can simplify the notation for describing the associated *RDDF*.

Table 2.2 Results associated with Example 3.2, Part 1

Unit: (x_0, y_0)	Dir. vector	Inefficiency β_0^*	Projection: $\left(x_0^{p1}, y_0^{p1}\right)$	Input 1 slack: $\hat{s}_1^{-*}$	Input 2 slack: $\hat{s}_2^{-*}$	Output slack: $\hat{s}^{+*}$
U1(4,2,4)	(2,0,0)	0	U1	0	0	0
U2(4,4,7)	(2,0,0)	0	U2	0	0	0
U3(4,6,9)	(2,0,0)	0	U3	0	0	0
U4(4,3,2)	(2,0,0)	0	U4	0	0	0
U5(10,3,1)	(2,0,0)	3	U4	0	0	1
U6(8,5,20/3)	(2,0,0)	2	$\frac{14}{15}U2 + \frac{1}{15}U4 = \left(4, \frac{59}{15}, \frac{20}{3}\right)$	0	$\frac{16}{15}$	0

stage projections are reported in Table 2.2. (All our linear programs have been solved resorting to Excel-Solver.) Our *DDF* model projects the first four units onto themselves, which means that, in each case, the directional inefficiency is $\beta^* = 0$ and the three optimal slack values detected by model (2.4) are equal to 0. The projection of U5 is point U4, getting an optimal directional inefficiency equal to 3 ($\beta_{U5}^* = 3$), and optimal slack values equal to (0,0,1). Finally, the optimal value of *DDF* model (2.4) for U6 equals $\beta_{U6}^* = 2$ while the three optimal slack values are $\left(0, \frac{16}{15}, 0\right)$ and its projection is point $\frac{14}{15}U2 + \frac{1}{15}U4$. Hence the first stage projection of U5 is point U4 (4,3,2), and that corresponding to U6 is point (4,59/15,20/3). We further need to determine how the additive model (2.3) rates the six first stage projections (see Table 2.3).

Table 2.3 Results associated with Example 3.2, Part 2

Unit: $\left(x_0^{p1}, y_0^{p1}\right)$	Inefficiency $s_1^{-*} + s_2^{-*} + s^{+*}$	Projection: $\left(x_0^{p2}, y_0^{p2}\right)$	Input1 slack: $\hat{s}_1^{-*}$	Input2 slack: $\hat{s}_2^{-*}$	Output slack: $\hat{s}^{+*}$
U1(4,2,4)	0	U1	0	0	0
U2(4,4,7)	0	U2	0	0	0
U3(4,6,9)	0	U3	0	0	0
U4(4,3,2)	0	0.5(U1 + U2) = $\left(4, 3, \frac{11}{2}\right)$	0	0	3.5
U4(4,3,2)	3	0.5(U1 + U2)	0	0	3.5
$\frac{14}{15}U2 + \frac{1}{15}U4 = \left(4, \frac{59}{15}, \frac{20}{3}\right)$	2	$\frac{1}{30}U1 + \frac{29}{30}U2 = \left(4, \frac{59}{15}, \frac{69}{10}\right)$	0	0	$\frac{7}{30}$

The result is that U1, U2 and U3 are rated as strongly efficient points, while the second stage projection of U4 is point 0.5(U1 + U2) = (4,3,11/2), with second stage optimal slacks equal to (0,0,7/2). Hence the final projection of U5 is also point

0.5(U1 + U2), with total inefficiencies equal to 3(2,0,0) +(0,0,7/2) = (6,0,7/2). Finally, the first stage projection of U6 has 1/30(U1 + 29U2) = (4,59/15,69/10) as second stage projection, with second stage optimal slacks equal to (0,0,7/30). Hence the total slacks associated to U6 are 2(2,0,0) +(0,16/15,0) +(0,0,7/30) = (4,32/30,7/15). These results are reported in Table 2.3. In this last case, the presence of second stage slacks indicates that the first stage projection of U6 is not a strongly efficient point.

2.3.2 *Measuring the Comprehensive Inefficiencies of the Derived Strong* DDF

So far we have devised a procedure that generates, after a second stage analysis, a strong *DDF*, whose final outcomes are strongly efficient projections for each unit being rated. This constitutes a simple way of transforming a weak *DDF*, that is, an inefficiency measure that accounts only for directional inefficiencies, into a strong *DDF*, that is, a comprehensive inefficiency measure, that accounts for all types of inefficiencies, both directional and non-directional. In order to complete our proposal, we need to measure the global inefficiency associated to each point. Since, at each point, the inefficiency associated to the strongly efficient projection we are seeking should be comparable to and greater than the inefficiency associated to the initial weakly efficient projection, β_0^*, we propose to normalize the "strong" directional vector, $\left(x_0 - x_0^{p2}, y_0^{p2} - y_0\right)$, obtained through the two stage procedure, with respect to the directional vector considered at the first stage, $\left(x_0 - x_0^{p1}, y_0^{p1} - y_0\right)$. The length of the strong directional vector $\left(x_0 - x_0^{p2}, y_0^{p2} - y_0\right)$ is always greater or equal than the length of the initial directional vector $\left(x_0 - x_0^{p1}, y_0^{p1} - y_0\right)$, because it holds that $\left(x_0 - x_0^{p2}, y_0^{p2} - y_0\right) = \left(x_0 - x_0^{p1}, y_0^{p1} - y_0\right) + \left(x_0^{p1} - x_0^{p2}, y_0^{p2} - y_0^{p1}\right)$ with $x_0^{p2} \leq x_0^{p1}$ and $y_0^{p2} \geq y_0^{p1}$. If $\|g_0\|_2$ denotes the Euclidean norm of Stage I directional vector g_0 and, as said before, we want to normalize the strong directional vector $\left(x_0 - x_0^{p2}, y_0^{p2} - y_0\right)$ so as to get the same "length", all we have to do is to divide it by its actual length $\left\|\left(x_0 - x_0^{p2}, y_0^{p2} - y_0\right)\right\|_2$ and to multiply it by $\|g_0\|_2$. As a consequence, the associated strong inefficiency value β_0^{*p2} satisfies

$$\beta_0^{*p2} := \frac{\left\|\left(x_0 - x_0^{p2}, y_0^{p2} - y_0\right)\right\|_2}{\|g_0\|_2}, \tag{2.9}$$

which, as explained above, is at least as big as $\beta_0^* = \frac{\left\| \left(x_0 - x_0^{p1}, y_0^{p1} - y_0 \right) \right\|_2}{\|g_0\|_2}$.[18]

Example 3.2 Part 2. Estimating the Inefficiencies of the Derived Strong *DDF*.

Let us consider again the data and the results of Example 3.2. In short, at the first stage, the considered *DDF* rated the first four units as directional efficient. At the second stage, only the first three units remained truly efficient—U1(4,2,4), U2(4,4,7), U3 (4,6,9)—, while the fourth one, U4(4,3,2), the fifth one U5(10,3,1) and the sixth one U6(8,5,20/3) were rated as inefficient and achieved second stage projections different from the first stage. Moreover, the strongly efficient projection of U4 and U5, after performing the second stage, was point (4,3,11/2). Finally, the second stage strongly efficient projection of U6 was point (4, 59/15, 69/10). Hence, the three directional vectors that connect U4, U5 and U6 with their final projections that were calculated at the bottom of Example 3.2, were $\left(0,0,\frac{7}{2}\right)$, $\left(-6,0,\frac{7}{2}\right)$, and $\left(-4,-\frac{16}{15},\frac{7}{30}\right)$. It is easy to verify that their corresponding Euclidean norms are 3.5, 6.946 and 4.146, respectively. In order to standardize the three new evaluated directional vectors with respect to the unique directional vector of Stage I, vector (2,0,0), we need to shorten them to reduce their length to 2, which means dividing each of them by its length and, at the same time, multiplying each of them by 2. Since the original connecting vector detects always an inefficiency equal to 1, the new shortened reverse directional vector will exhibit an inefficiency value of $\frac{3.5}{2} = 1.75$ for U4, $\frac{6.946}{2} = 3.473$ for U5, and $\frac{4.146}{2} = 2.073$, for U6. These new comprehensive inefficiencies are higher than the directional inefficiency reported in Table 2.2 (0, 3, and 2, respectively), and correspond to the inefficiency value expression contained in (2.9). Hence, the new strong *DDF* detects an inefficiency higher than the weak *DDF* for the three considered inefficiency units, which is reasonable because the new *DDF* accounts for all types of inefficiencies, the directional inefficiency as detected by the weak *DDF* and the inefficiencies detected by the slacks, which are basically non-directional and are quite often greater than 0. The interesting point is that all these inefficiencies can be combined and measured through a new strong *DDF* that, obviously, avoids slacks, and offers a single number that measures all types of inefficiencies.

2.4 Deriving DEA Comprehensive Inefficiency Measures. the Case of the Comprehensive Radial Models

Thanks to the exercise performed in Sect. 2.3 it is easy to build a comprehensive inefficiency measure based on any weak DEA inefficiency model. In fact, any model M gives rise to the corresponding $RDDF^{M,\Pi^M}$. Considering this $RDDF$ as the

[18]Observe that each component of the vector in the numerator equals the corresponding component of the vector in the denominator times β_0^*.

DDF of the first stage and applying to it the findings of Sect. 2.3, namely the two stage procedure, we end up creating a new strong *RDDF*, which is exactly the solution we are looking for. Based on the results of Sect. 2.3 it is straightforward to formulate the directional vector that relates model M with its strong *RDDF*, a new expression that substitutes the expression given by (2.2). Let us name the new directional vector as $\hat{g}_0^M$. Now the distinction is not based on the original directional inefficiency values $\tau_0^M \geq 0$, but on the comprehensive inefficiency values obtained after performing the second stage, denoted as T_0^M. It is obvious that $\mathrm{T}_0^M \geq \tau_0^M$. Let us observe that only the strongly efficient points will get $\mathrm{T}_0^M = 0$, which means that any non-strongly efficient point will get $\mathrm{T}_0^M > 0$. Hence, new expression (2.2) will be based on E, the subset of strongly efficient points, and not on τ_0^M. The new expression that gives the directional vector associated to each point (x_0, y_0) being rated and that relates model M with the corresponding derived strong *RDDF* is:

$$\hat{g}_0^M := \left\{ \begin{array}{ll} \frac{1}{\mathrm{T}_0^M}\left(x_0 - x_0^{p2}, y_0^{p2} - y_0\right) \geq 0_{m+s}, & \textit{if } (x_0, y_0) \notin E \\ (1_m, 1_s), & \textit{if } (x_0, y_0) \in E \end{array} \right\} \tag{2.10}$$

Let us observe that the strong *RDDF* represents the comprehensive inefficiency model associated to model M. Hence, our procedure is a general procedure for generating comprehensive inefficiency models based on any DEA inefficiency model.[19]

In particular, we are ready to define comprehensive Radial Models. These particular models are even easier to deal with, because we know how to formulate them as DDFs and, therefore, Sect. 2.3 gives us the solution directly. The constant returns to scale (CRS) Radial Models were the first defined DEA models and, in honor of their authors, they are known as the CCR models (Charnes et al. 1978). Later on, the VRS Radial Models were introduced (Banker et al. 1984) and, for the same mentioned reason, they are also known by the acronym BCC. We will focus our attention on the latter ones. There are two BCC models: the BCC input-oriented and the BCC output-oriented models. The first one is formulated as follows.[20]

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[19]We would like to point out that a referee has pointed out that our procedure opens a new research avenue for ranking units resorting to appropriate *DDFs*, by selecting appropriate directional vectors.

[20]As already mentioned in Footnote 5, we will only consider "the envelopment form" of any BCC model and will not even mention its linear dual program known as "the multiplier form", which, in this case, can also be expressed as a linear fractional program called "the ratio form".

$$
\begin{aligned}
BCC_{EFF}^{IO}(x_0, y_0) = Min \quad & \theta_0 \\
& s.t. \\
& \sum_{j \in E} \lambda_j x_{ij} \le \theta_0 x_{i0}, \quad i = 1, \ldots, m \\
& \sum_{j \in E} \lambda_j y_{rj} \ge y_{r0}, \quad r = 1, \ldots, s \\
& \sum_{j \in E} \lambda_j = 1, \\
& \lambda_j \ge 0, \quad j = 1, \ldots, n
\end{aligned}
\tag{2.11}
$$

This model is a DEA efficiency model, and θ_0^* is known as the input efficiency score of unit (x_0, y_0). Being aware that θ_0^* is a quantity in between 0 and 1, making the change of variable $\theta_0 = 1 - \beta_0$ we just move from an efficiency model to an inefficiency model. Since minimizing θ_0 is equivalent to maximizing $-\theta_0 = \beta_0 - 1$, the last model can be reformulated as the next DDF model,[21] provided we add slack variables in the same way as we did in (2.4) for model (2.1).

$$
\begin{aligned}
BCC_{INEFF}^{IO}(x_0, y_0; x_0, 0_s) \quad & = Max \quad \beta_0 \\
& s.t. \quad \sum_{j \in E} \lambda_j x_{ij} + s_{i0}^- = x_{i0} - \beta_0 x_{i0}, \quad i = 1, \ldots, m \\
& \sum_{j \in E} \lambda_j y_{rj} - s_{r0}^+ = y_{r0}, \quad r = 1, \ldots, s \\
& \sum_{j \in E} \lambda_j = 1, \\
& \lambda_j \ge 0, \qquad j = 1, \ldots, n \\
& s_{i0}^-, s_{r0}^+ \ge 0, \quad i = 1, \ldots, m, \\
& \qquad\qquad r = 1, \ldots, s
\end{aligned}
\tag{2.12}
$$

It is obvious that $\beta_0^* = 1 - \theta_0^* \ge 0$ because $0 \le \theta_0^* \le 1$. Moreover, $\beta_0^* \le 1$. Since the directional vector $(x_0, 0_s)$ is unable to change outputs, it is input-oriented. Moreover, since the change in inputs is guided by x_0 we pursue a proportional—or radial—reduction in inputs. It is well established that this model does not necessarily project all the inefficient points onto the strongly efficient frontier, or, in other words, depending on the sample of units being rated, BCC_{INEFF}^{IO} can be a weak *DDF*. Applying to it the two stage procedure developed in Sect. 2.3 it is easy to end

[21] Strictly speaking, model $BCC_{INEFF}^{IO}(x_0, y_0; x_0, 0_s)$ has as objective function $\beta_0 - 1$ instead of β_0, but its expression as a *DDF* requires the proposed related objective function.

up with a comprehensive strong *DDF*, as well as with the corresponding derived strong inefficiency scores. Although the initial weak inefficiency scores are always less than or equal to 1, the final strong inefficiency scores may be greater than 1.

Example 4.1 Deriving a Comprehensive BCC Input-Oriented Model

Let us consider in the two-input one-output space the next five units: U1(4,2,40), U2(4,4,90), U3(4,6,120), U4(40,30,4) and U5(4,4,80). Applying model (2.12) we ensure that the BCC input-oriented inefficiency model rates the first three units as efficient, assigning each of them an optimal inefficiency of $\beta^* = 0$ and all optimal slack values at level 0. Furthermore, model (2.12) assigns to U4 an optimal inefficiency value $\beta^*_{U4} = 0.9$, and optimal slack values equal to (0,1,36). Consequently, the first stage projection of U4 is point (40,30,4) − 0.9(40,30,0) + (0, −1,36) = (4,2,40), which is exactly U1. Finally, U5 is projected onto 0.5 (U1 + U3) = U5, which means that $\beta^* = 0$ and all the optimal slacks are at level 0. We need to perform Stage 2 since we are not sure which of the projections are strongly efficient. First we check, by means of the additive model, if U1, U2, and U3 are strongly efficient units. The answer is yes, and, as a consequence, the first stage projections of U1, U2, U3, and U4 are also its second stage projections. Hence, U4 will maintain its directional inefficiency value as well as its slack non-directional inefficiency value. Both types of inefficiencies will be jointly considered when evaluating the corresponding strong reverse directional vector. Finally, U5 gets a new second stage projection, precisely U2, with the only non-zero slack on the output side and equal to 10. Again, the strong reverse directional vector associated to the second stage projections are, according to (2.10), vector (1,1;1) for U1, U2 and U3, with associated inefficiency equal to 0. The reverse directional vector for U4 is obtained as U1 − U4 = (4,2,40) − (40,30,4) = (−36, −28, 36), while the associated to U5 is U2 − U5 = (4,4,90) − (4,4,80) = (0,0,10). In the last two cases the associated comprehensive inefficiency is obtained as explained before. The Euclidian length of (−36, −28, 36) is 58.103 and the corresponding to (0,0,10) is 10. Since, associated to the first stage projection the used directional vectors were (40,30,0) for U4 and (4,4,0) for U5, their lengths were 50 and 5.657. Therefore, the comprehensive inefficiency of the strong *RDDF* are $\frac{58.103}{50} = 1.162 > 1$ for U4 and $\frac{10}{5.697} = 1.755 > 1$ for U5.

This example shows that non-radial inefficiencies may be relevant enough so as to get an inefficiency score bigger than 1, and that the original input orientation has been lost because the final strong directional vector at U4 has all positive components, which means that it modifies all inputs and outputs in order to reach the desired strong projection. Curiously enough, for U5, its final strong directional vector is output oriented.

Even in the simplest one input-one output space the second stage may be necessary, as shown in the next example.

Example 4.2 The BCC Input-Oriented Model with just Two Dimensions.

In the one input—one output space let us consider the next sample of 6 units: (2,2), (4,2), (2,10), (4,14), (5,14) and (12,12). If we draw a picture, see Fig. 2.1, and apply the BCC input-oriented inefficiency model, as given by (2.12), we get the results listed in Table 2.4.

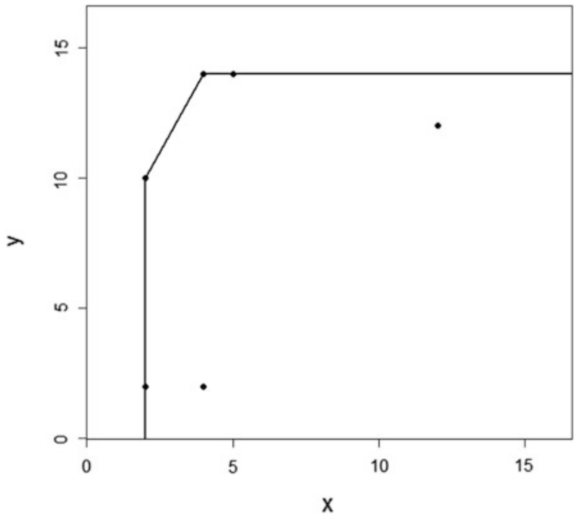

Fig. 2.1 Figure associated with Example 4.2

Table 2.4 Results associated with Example 4.2, First Stage

Unit	β_0^*	$\sum_{j\in E} \lambda_j^* (x_j, y_j)$	Input slack	Output slack	Unit being rated is input-radial
U1(2,2)	0	U1	0	0	Efficient
U2(4,2)	1	U1	0	0	Inefficient
U3(2,10)	0	U3	0	0	Efficient
U4(4,14)	0	U4	0	0	Efficient
U5(8,14)	2	U4	0	0	Inefficient
U6(12,12)	4.5	0.5(U3 + U4) = (3,12)	0	0	Inefficient

Let us observe that the first stage does classify each unit as input-radial efficient or inefficient—see last column of Table 2.4—but is unable to tell us if the radial efficient units belong to the strongly efficient frontier or not. Moreover, it is also unable to tell us if the input-radial inefficient units belong to the interior of the production possibility set or to the weak part of the efficient frontier. For getting this information we need to perform our second stage analysis, whose results are listed in Table 2.5.

Table 2.5 Combined results associated with Example 4.2, Second Stage

Unit	β_0^*	$\sum_{j \in E} \lambda_j^* (x_j, y_j)$	Input slack	Output slack	Unit being rated belongs to the
U1(2,2)	0	U3	0	8	Weak frontier
U2(4,2)	1	U3	0	8	Interior
U3(2,10)	0	U3	0	0	Strong frontier
U4(4,14)	0	U4	0	0	Strong frontier
U5(8,14)	2	U4	0	0	Weak frontier
U6(12,12)	4.5	0.5(U3 + U4) = (3,12)	0	0	Interior

Our second stage analysis shows, in this case, that the first stage projections of units U1 and U2 onto U1 are not strongly efficient, simply because U1 is not a strongly efficient point but a weak one.[22] The presence of a non-zero output slack associated to U1 reveals its nature (see second row of Table 2.5). Moreover, we are now able to classify each point and locate it on the strong frontier, on the weak part of the frontier or on the interior of the production possibility set (see last column of Table 2.5), without the help of any graphical display, which, by the way, are unavailable for any input-output space with dimensions greater than 3.

The BCC output-oriented model admits a similar treatment. Its original formulation follows.

$$
\begin{aligned}
BCC_{EFF}^{OO}(x_0, y_0) = Max \quad & \Phi_0 \\
s.t. \quad & \\
& \sum_{j \in E} \lambda_j x_{ij} \le x_{i0}, \quad i = 1, \ldots, m \\
& \sum_{j \in E} \lambda_j y_{rj} \ge \Phi_0 y_{r0}, \quad r = 1, \ldots, s \\
& \sum_{j \in E} \lambda_j = 1, \quad \lambda_j \ge 0, \quad j = 1, \ldots, n
\end{aligned}
\tag{2.13}
$$

The last model is a DEA efficiency model, and Φ_0^* is known as the output efficiency score of unit (x_0, y_0). Being aware that Φ_0^* is a quantity greater than or equal to 1, making the change of variable $\Phi_0 = 1 + \beta_0$ we just move from an efficiency model to an inefficiency model. Since maximizing Φ_0 is equivalent to maximizing $\beta_0 + 1$, the last model can be reformulated as the next *DDF* model,

[22]As early as in 1979, Charnes, Cooper and Rhodes were aware of this problem in relation to the CCR model (Charnes et al. 1978 and 1979). They published a mathematical solution to it some years later (Charnes and Cooper, 1984), based on the seminal paper of Charnes (1952). The functioning of the BCC model is completely similar.

provided we add slack variables to the restrictions in the same way as we did in model (2.12).

$$
\begin{aligned}
BCC_{INEFF}^{OO}(x_0, y_0; 0_m, y_0) = Max \quad & \beta_0 \\
s.t. \quad & \sum_{j\in E} \lambda_j x_{ij} + s_{i0}^- = x_{i0}, \quad i = 1, \ldots, m \\
& \sum_{j\in E} \lambda_j y_{rj} - s_{r0}^+ = y_{r0} + \beta_0 y_{r0}, \quad r = 1, \ldots, s \\
& \sum_{j\in E} \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n \\
& s_{i0}^-, s_{r0}^+ \geq 0, \quad i = 1, \ldots, m, \; r = 1, \ldots, s
\end{aligned}
\tag{2.14}
$$

It is obvious that $\beta_0^* \geq 0$ because $\Phi_0^* \geq 1$. Since the directional vector $(0_m, y_0)$ is unable to change inputs, it is output-oriented. Moreover, since the change in outputs is guided by y_0 we pursue a proportional—or radial—augmentation of outputs. It is well established that this model does not necessarily project all the inefficient points onto the strongly efficient frontier, or, in other words, depending on the sample of units being rated, BCC_{INEFF}^{OO} can be a weak *DDF*. By applying the two stage procedure developed in Sect. 2.3 it is easy to end up with a comprehensive strong *DDF*, as well as with the corresponding derived strong inefficiency scores. Even in the simplest one input-one output space, the second stage could be needed, just as for the input-oriented version.

The next three sections are devoted to the three common economic inefficiency measures: cost inefficiency, revenue inefficiency and profit inefficiency. The choice depends on the way firms solve technical inefficiencies, by either reducing inputs, or expanding outputs, or both. Let us revise each of the three possibilities. In any of them we need to know the corresponding set of market prices, i.e., q_i, the unitary cost of input $i, i = 1, \ldots, m$, for the first case, p_r, the unitary price of output $r, r = 1, \ldots, s$, for the second case, or both, for the third case. To simplify notation, we denote the m-vector of input market prices by q, and the s-vector of output market prices by p.

2.5 Evaluating and Decomposing Cost Inefficiency Through the Associated *RDDF* Measure

Since, according to Sect. 2.2, $RDDF^{M,\Pi^M}$ has exactly the same behavior as M, in terms of detecting exactly the same technical inefficiency while keeping the same projection for each of the units under scrutiny, we can use the well-known Fenchel–Mahler inequality, developed for directional distance functions by Chambers et al.

(1998), and applied it to the *RDDF*.[23] We will assume in this section that M is an input-oriented model, i.e., for each unit being rated we are interested in reducing inputs as much as possible while maintaining the output levels. The strategy used for reducing inputs is determined by model M. As a direct consequence of Proposition 2.1 we can enunciate the next result.

Corollary 1.3 *If M is an input-oriented model, then its reverse-*DDF *is also an input-oriented model.*

Given an m-vector of unitary input costs, q, as well as a fix level of outputs, y_0, and assuming that T represents the production possibility set, the *cost function* is defined as

$$C(y_0, q) = \inf\{qx : (x, y_0) \in T\}. \tag{2.15}$$

In general, if T satisfies certain mathematical conditions, the infimum is reachable and we can switch from infimum to minimum (see Ray 2004). In particular, this happens for a DEA technology under VRS.

In order to evaluate $C(y_0, q)$ we assume, as before, that T is generated through a finite set of points $\{(x_j, y_j), j = 1, \ldots, n, x_j \geq 0, y_j \geq 0\}$. In this case we need to solve the next linear program.

$$\begin{aligned}
C(y_0, q) = \underset{\lambda, x}{Min} \quad & \sum_{i=1}^{m} q_i x_i \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{r0}, \quad r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n \\
& x_i \geq 0, \quad i = 1, \ldots, m
\end{aligned} \tag{2.16}$$

Before considering the cost inefficiency decomposition through the corresponding Fenchel–Mahler inequality associated to the *RDDF* associated to model M, we need to define the cost inefficiency at point (x_0, y_0), after introducing the concept of cost deviation.

The *cost deviation* at point (x_0, y_0) is simply the difference between the cost at that point and the cost function—or minimum cost—at market input-prices q:

[23]Since model M is given, its associated *RDDF* can be either a weak *DDF* or a strong *DDF*, depending on the nature of M.

$$CD(x_0, y_0) := qx_0 - C(y_0, q) = \sum_{i=1}^{m} q_i x_{i0} - C(y_0, q). \tag{2.17}$$

For the sake of brevity, we write CD_0 for $CD(x_0, y_0)$.

Regarding the *DDF* and its existing dual relationship, we want to point out that it is possible to relate a term of normalized cost deviation, called cost inefficiency,CI_0, with the technical inefficiency detected by a given input-oriented DDF through the next inequality:

$$CI_0 := \frac{CD_0}{qg_{0x}} \geq \vec{D}(x_0, y_0; g_{0x}, 0). \tag{2.18}$$

Expression (2.18) is precisely the aforementioned Fenchel-Mahler inequality valid for any input-oriented *DDF* that allows cost inefficiency to be decomposed into the sum of technical inefficiency and an additive residual term identified as allocative inefficiency.

Next, resorting to the Fenchel-Mahler inequality associated to $RDDF^{M,\Pi^M}$ we get the next inequality,

$$CI_0^{M,\Pi^M} := \frac{CD_0^{M,\Pi^M}}{qg_{0x}^{M,\Pi^M}} \geq \tau_0^M. \tag{2.19}$$

The left hand-side term or cost inefficiency, CI_0^{M,Π^M}, satisfies a desirable index number property: it is homogeneous of degree 0 in prices, which makes CI_0^{M,Π^M} invariant to the currency units of the market input prices. As first pointed out by Nerlove (1965), CD_0^{M,Π^M}, the numerator of CI_0^{M,Π^M}, is homogeneous of degree 1 in prices and, consequently, the cost deviation cannot be considered as an appropriate economic measure. Going back to the last inequality and defining the *allocative inefficiency* as the corresponding additive residual, we get the next equality:

$$CI_0^{M,\Pi^M} = \tau_0^M + AI_0^{M,\Pi^M}. \tag{2.20}$$

In words, at point (x_0, y_0) cost inefficiency is decomposed into the sum of technical inefficiency and allocative inefficiency.

Example 5.1 The Input-Oriented Additive Model with Single Projections

Let us consider again model (2.3), known as the additive model (Charnes et al. 1985). It is a particular case of the weighted additive model (Lovell and Pastor 1995), where the objective function is simply a non-negative weighted sum of all the input and output slacks, with at least one positive weight. The additive model is a weighted additive model with all the weights equal to 1. Its input-oriented version is obtained by setting the weights attached to the output slacks equal to 0 in the objective function. Its formulation is as follows.

$$\begin{aligned}
Add_{IO}(x_0, y_0) = Max_{s^-,\lambda} \quad & \sum_{i=1}^{m} s_{i0}^- \\
& s.t. \\
& \sum_{j\in E} \lambda_j x_{ij} = x_{i0} - s_{i0}^-, \quad i = 1, \ldots, m \\
& \sum_{j\in E} \lambda_j y_{rj} = y_{r0} + s_{r0}^+, \quad r = 1, \ldots, s \\
& \sum_{j\in E} \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n \\
& s_{i0}^- \geq 0, \quad i = 1, \ldots, m
\end{aligned} \tag{2.21}$$

Let us consider a sample of three units to be rated, defined as (4,7;1), (5,5;1) and (13,7;1). They belong to the "two input—one output" space. Since all the units have the same level of output we may represent them on the 2-dimensional input-plane. Moreover, focusing on the two inputs it is easy to realize that the first two units are efficient while the third one is clearly inefficient, because it is dominated by any of the other two efficient units.

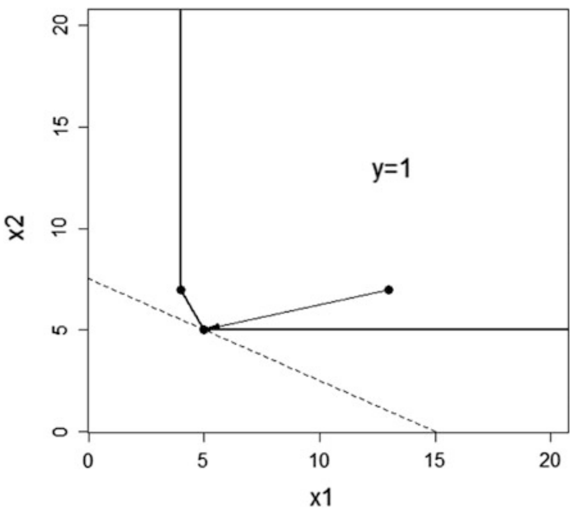

Fig. 2.2 Figure associated with Example 5.1

The L_1 projection of (13,7) onto (4,7) follows the L_1-path that connects both points, and whose components are (−9,0). Hence, the length of this L_1-path is 9 (=9 + 0). Alternatively, the L_1 projection of the inefficient unit onto (5,5) is given through the vector (−8,−2), which corresponds to a L_1-path of length 10 (=8 + 2). The maximum length is 10, which means that program (2.21) identifies (5,5) as the unique optimal projection[24] of unit (13,7), with an associated technical inefficiency, τ_0^M equal to 10. Since the projections are unique for all the three units in our example, we may write g_{0x}^M for g_{0x}^{M,Π^M}, as well as CI_0^M for CI_0^{M,Π^M} and AI_0^M for AI_0^{M,Π^M}.

Let us further assume that the unitary market input-prices are 1, for the first input, and 2, for the second. The minimum cost evaluated through program (2.16) tells us that $C(y_0,(q_1,q_2)) = C(1,(1,2)) = 15$, and that it is achieved at $\left(x_1^M, x_2^M\right) = (5,5)$. In Fig. 2.2 we have drawn the iso-cost line $x_1 + 2x_2 = 15$, which indeed passes through point (5,5). Now, since the obtained M-projection of point (x_0, y_0) = (13,7;1) is the strongly efficient point (5,5;1) and τ_0^M = 10, we get $g_{0x}^M = \left(\frac{8}{10}, \frac{2}{10}\right)$. In this example, $RDDF^M$ is perfectly defined knowing expression (2.2), that is, knowing that the directional vector at the only inefficient point (13,7;1) is $g_0^M = \left(\frac{8}{10}, \frac{2}{10}; 0\right)$. Evaluating the cost deviation at the unique inefficient point we obtain

$$CD(x_0, y_0) := \sum_{i=1}^{m} q_i x_{i0} - C(y_0, q) = (1,2)\cdot(13,7) - 15 = 13 + 14 - 15 = 12.$$

Consequently, the value of CI_0 is $\frac{CD_0}{q g_{0x}^M} = \frac{12}{1\cdot\frac{8}{10} + 2\cdot\frac{2}{10}} = \frac{12\cdot 10}{12} = 10$, which equals τ_0^M = 10. Hence, $AI_0^M = CI_0^M - \tau_0^M = 10 - 10 = 0$.

The projection (5,5) of the inefficient unit (13,7) is, as said before, where the minimum cost is achieved for the considered market input-prices. Obviously this cost minimizing point has a cost deviation of 0, and also a cost inefficiency of 0, since $CI(5,5) = \frac{CD(5,5)}{(1,2)\cdot(1,1)} = \frac{0}{3} = 0$. Moreover, since (5,5) is efficient, its technical inefficiency is 0. Hence, its allocative inefficiency is also 0.

The previous result shows that point (13,7;1) is projected onto the minimum cost point (5,5;1), and has also an allocative inefficiency equal to 0. The next question springs to mind: is there any relationship between the allocative inefficiency of point (x_0, y_0) and the cost deviation of its efficient projection? The next proposition shows that the suggested relationship exists, and proposes an alternative way for evaluating the cost allocative inefficiency of point (x_0, y_0) based on its projection.

[24]Although in this simple example the projection is unique, it is straightforward to devise alternative easy examples for the input-oriented additive model where an inefficient unit may have two, or more, different projections. Our example gives rise to a single-value additive model, as opposed to a multiple-value additive model.

Proposition 2 *Let* $\left(x_0^M, y_0^M\right)$ *denote the projection of* (x_0, y_0) *obtained through model M. Let us assume that* $y_0^M = y_0$. *Then, the cost allocative inefficiency associated to point* (x_0, y_0) *and obtained through* $RDDF^{M,\Pi^M}$ *is*

$$AI_0^{M,\Pi^M} = \frac{CD^{M,\Pi^M}\left(x_0^M, y_0^M\right)}{q \cdot g_{0x}^{M,\Pi^M}} = \frac{\sum_{i=1}^m q_i x_{0i}^M - C\left(y_0^M, q\right)}{q \cdot g_{0x}^{M,\Pi^M}}. \tag{2.22}$$

In particular, $AI_0^{M,\Pi^M} = 0$ *if, and only if,* $CD\left(x_0^M, y_0^M\right) = 0$.

Proof If $(x_0, y_0) = \left(x_0^M, y_0^M\right)$, then $\tau_0^M = 0$, and (2.22) is a direct consequence of (2.20). Consequently, let us assume that (x_0, y_0) is an inefficient point. According to equalities (2.19) and (2.20), $AI_0^{M,\Pi^M} = CI_0^{M,\Pi^M} - \tau_0^M = \frac{CD_0}{qg_{0x}^{M,\Pi^M}} - \tau_0^M = \frac{CD_0 - \tau_0^M qg_{0x}^{M,\Pi^M}}{qg_{0x}^{M,\Pi^M}}$. Hence, taking into account expression (2.22), all we need to prove is that $CD\left(x_0^M, y_0^M\right) = CD_0 - \tau_0^M qg_{0x}^{M,\Pi^M}$, or, equivalently, that $\sum_{i=1}^m q_i x_{i0}^M - C\left(y_0^M, q\right) = \left\{\sum_{i=1}^m q_i x_{i0} - C(y_0, q)\right\} - \tau_0^M qg_{0x}^{M,\Pi^M}$. According to expression (2.2) the equality $qx_0^M = q\left(x_0 - \tau_0^M g_{0x}^{M,\Pi^M}\right)$ holds. Therefore, the previous expression can be reduced to $C\left(y_0^M, q\right) = C(y_0, q)$ which trivially holds because we are assuming that $y_0^M = y_0$. ■

Proposition 2 shows that the factor to be used for normalizing the cost deviation associated to the efficient projection so as to obtain the allocative inefficiency of the inefficient point is exactly the normalization factor associated to the inefficient point. Moreover, it seems an acceptable and intuitive property that, when the projection is a cost minimizing point, the allocative inefficiency associated to the point being rated is 0.

Example 5.2 The Input-Oriented Additive Model with Multiple Projections

Let us consider again, in the two input—one output space, a sample of three units to be rated, defined as (3,7;1), (5,5;1) and (13,7;1). In comparison to Example 5.1 we have only slightly changed the first unit, from (4,7;1) to (3,7;1). This change does not affect the efficiency status of the three units, but does affect the L_1-distance from (13,7) to the first efficient unit (3,7), which has increased and takes exactly the same value, 10, as the distance from (13,7) to the second efficient unit, (5,5). Now the two efficient units are optimal projections for the unique inefficient unit, or, in other words, (13,7) has multiple optimal projections. The preferable option can be a function of a second criterion. For instance, if the market input-prices are again q = (1,2) we might prefer to select the projection that generates a lower allocative inefficiency, or, according to Proposition 2, the projection were the lowest cost deviation is achieved. Since, the new considered efficient point (3,7) has an input-cost of $1 \cdot 3 + 2 \cdot 7 = 17$, higher than the already evaluated input-cost of point (5,5), which equals 15 and corresponds to the value of the cost function—see Example 4.1—, then the cost deviation of the new obtained projection, $CD(3,7) = 17 - 15 = 2$, is greater than the cost deviation of the old one, $CD(5,5) = 0$. Hence, our final choice is to select point (5,5) as our preferred projection, according to our aforementioned second criterion, because its allocative inefficiency is the most convenient one.

In this simple example we have directly considered the two possible alternative optimal projections as points of our sample. When solving a real life problem, with many units and a higher number of inputs and outputs, a specific search for identifying at each inefficient unit of the sample possible alternative optimal projections needs to be developed. This task is accomplished in Appendix 1.

2.6 Evaluating and Decomposing Revenue Inefficiency

Given an s-vector of unitary output prices, $p \geq 0_s$, and being T the production possibility set, the *revenue function* is defined for a fix level of inputs, x_0, as follows.

$$R(x_0, p) = \sup\{py \colon (x_0, y) \in T\}. \tag{2.23}$$

In the case of T being a DEA technology, the supremum in (2.23) may be equivalently changed by maximum.

In order to evaluate $R(x_0, p)$ we assume that the set $\{p_r, r = 1, \ldots, s\}$ of non-negative output prices is known and that, within a DEA framework, T is generated through a finite set of n points $\{(x_j, y_j), j = 1, \ldots, n, x_j \geq 0, y_j \geq 0\}$. In this case we only need to solve the next linear program:

$$\begin{aligned} R(x_0, p) = \underset{\lambda, y}{Max} \quad & \sum_{r=1}^{s} p_r y_r \\ s.t. \quad & \\ & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{i0}, \quad i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r, \quad r = 1, \ldots, s \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n \\ & y_r \geq 0, \quad r = 1, \ldots, s \end{aligned} \tag{2.24}$$

Before considering the revenue inefficiency decomposition associated to $RDDF^{M,\Pi^M}$ through the corresponding Fenchel–Mahler inequality, we need to define the revenue deviation at point (x_0, y_0).

The *revenue deviation* at point (x_0, y_0) is simply the difference between the revenue function and the revenue at that point, given market output-prices p:

$$RD(x_0, y_0) := R(x_0, p) - py_0. \tag{2.25}$$

For the sake of brevity, we write RD_0 for $RD(x_0, y_0)$.

As for the *DDF* and thanks to its dual relationships, it is possible to link, at point (x_0, y_0), a normalized term of revenue deviation, called *revenue inefficiency*, with the optimal value of any output-oriented directional distance function, as follows:

$$RI_0 := \frac{RD_0}{pg_{0y}} \geq \vec{D}\left(x_0, y_0; 0, g_{0y}\right). \tag{2.26}$$

Expression (2.26) is the Fenchel-Mahler inequality associated with any *DDF* that allows decomposing revenue inefficiency into the sum of technical inefficiency and allocative inefficiency.

In our particular case, and for the $RDDF^{M,\Pi^M}$, we get the inequality

$$RI_0^{M,\Pi^M} := \frac{RD_0^{M,\Pi^M}}{pg_{0y}^{M,\Pi^M}} \geq \tau_0^{M,\Pi^M}. \tag{2.27}$$

The left hand-side term is the normalized revenue deviation, also called *revenue inefficiency, RI_0*, which, as CI_0, satisfies that it is homogeneous of degree 0 in prices, which makes RI_0 invariant to the currency units associated to the market output prices. Going back to the last inequality and defining the revenue allocative inefficiency as the corresponding residual, we get the next equality:

$$RI_0^{M,\Pi^M} = \tau_0^{M,\Pi^M} + AI_0^{M,\Pi^M}. \tag{2.28}$$

In words, at point (x_0, y_0), and thanks to the associated $RDDF^{M,\Pi^M}$, normalized revenue inefficiency is decomposed into the sum of technical inefficiency and allocative inefficiency.

Example 6.1 The Output-Oriented Additive Model with Single Projections

Let us now consider as model M the output-oriented version of the additive model, whose objective function only includes output slacks. Its formulation follows.

$$
\begin{aligned}
Add_O(x_0, y_0) = Max_{s^+,\lambda} \quad & \sum_{r=1}^{s} s_{r0}^+ \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{i0}, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} = y_{r0} + s_{r0}^+, \quad r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \ge 0, \quad j = 1, \ldots, n \\
& s_{r0}^+ \ge 0, \quad r = 1, \ldots, s
\end{aligned}
\tag{2.29}
$$

Let us consider now in the "one input—two output space" the next sample of units to be rated: $\{(1;3,5), (1;5,2); (1;2,1)\}$. Since all the units have the same level of input we may represent them on the 2-dimensional output plane. Moreover, focusing on the two outputs it is easy to realize that the first two units are efficient while the third one is inefficient because it is dominated by any of the two efficient units.

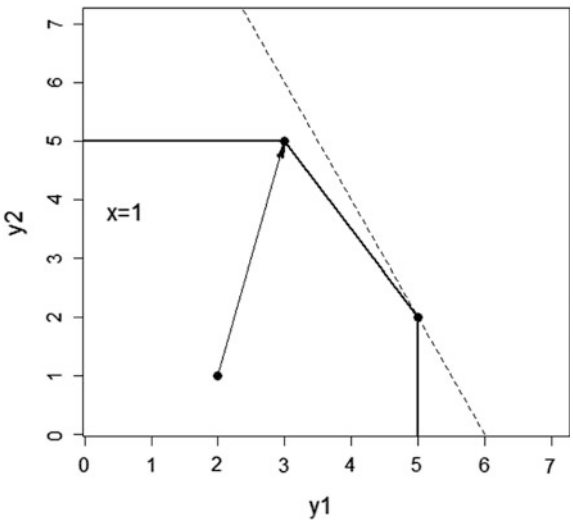

Fig. 2.3 Figure associated with Example 6.1

The L_1 projection of (2,1) onto (3,5) follows the L_1-path that connects both points, and whose components are $(1,4)$. Hence, the length of this L_1-path is 5 (=1 +4). Alternatively, the L_1 projection of the inefficient unit onto (5,2) is given through the vector (3,1) which corresponds to a L_1-path of length 4 (=3+1). The maximum length is 5, which means that program (2.29) will identify (3,5) as the unique optimal projection with an associated technical inefficiency, τ_0^M, as given by the optimal value of the objective function, equal to 5. Let us assume that the market output-prices are given as p = (2,1). The maximum revenue—or revenue function—, as evaluated through program (2.24), tell us that $R(1,(2,1)) = 12$, and that it is achieved at point (5,2), as shown in Fig. 2.3, where we have drawn the iso-revenue line $2y_1 + y_2 = 12$. Now, since the obtained M-projection of point (x_0, y_0) = (1;2,1) is the efficient point (1;3,5) and τ_0^M = 5, we get $g_{y0} = \left(\frac{1}{5}, \frac{4}{5}\right)$. Evaluating the revenue deviation at the inefficient point we obtain $RD(x_0, y_0) := R(x_0, p) - \sum_{r=1}^{s} p_r y_{0r} = 12 - (2,1)\cdot(2,1) = 12 - 5 = 7$. Consequently, the value of RI_0 is $\frac{RD_0}{pg_{0y}^M} = \frac{7}{2\cdot\frac{1}{5} + 1\cdot\frac{4}{5}} = \frac{7\cdot 5}{6} = \frac{35}{6}$, which is greater than $\tau_0^M = 5$. Hence, $AI_0^M = RI_0^M - \tau_0^M = \frac{35}{6} - 5 = \frac{5}{6}$.

It would be interesting, as we did when evaluating cost allocative inefficiency, to connect the revenue deviation of the corresponding efficient projection with the revenue allocative inefficiency of the point being rated. The next proposition gives us the clue.

Proposition 3 *Let $\left(x_0^M, y_0^M\right)$ denote the projection of (x_0, y_0) obtained through model M. Let us assume that $x_0^M = x_0$. Then, the revenue allocative inefficiency associated to point (x_0, y_0) can be obtained as*

$$AI_0^{M,\Pi^M} = \frac{RD_0^{M,\Pi^M}\left(x_0^M, y_0^M\right)}{p\cdot g_{0y}^{M,\Pi^M}} = \frac{R(x_0^M, p) - \sum_{r=1}^{s} p_r y_{0r}^M}{p\cdot g_{0y}^{M,\Pi^M}}. \tag{2.30}$$

In particular, $AI_0^{M,\Pi^M} = 0$ if, and only if, $RD_0^{M,\Pi^M}\left(x_0^M, y_0^M\right) = 0$.

Proof The proof is similar to the proof of Proposition 2 and is left to the reader. ■

2.7 Evaluating and Decomposing Profit Inefficiency

The profit function requires that both market input costs and market output revenues are specified, by knowing the corresponding market unitary prices. As usual, let us denote by $q \geq 0_m$ the market input-prices and by $p \geq 0_s$ the market output-prices. The *profit function* is defined as follows.

$$\Pi(q,p) = \sup\{py - qx : (x,y) \in T\}. \tag{2.31}$$

Under the hypothesis of working with a DEA production possibility set, the supremum in (2.31) is reachable and we switch from supremum to maximum.

Within a DEA framework, the linear program to be solved in order to calculate the profit function is the next one.

$$\begin{aligned} \Pi(q,p) = \underset{\lambda,x,y}{Max} & \sum_{r=1}^{s} p_r y_r - \sum_{i=1}^{m} q_i x_i \\ & s.t. \\ & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, \quad i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r, \quad r = 1, \ldots, s \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n \\ & x_i \geq 0, y_r \geq 0 \quad i = 1, \ldots, m, \ r = 1, \ldots, s \end{aligned} \tag{2.32}$$

As usual, (2.32) is a VRS model. In fact, considering a CRS model could be seen as meaningless from an entrepreneur's point of view when the aim is to measure profit inefficiency, because the CRS assumption implies always either unbounded profit or zero maximal profit. Nevertheless, it would be possible to use another alternative hypothesis on the production possibility set as NIRS (Non-Increasing Returns to Scale), where the constraint $\sum_{j=1}^{n} \lambda_j = 1$ in (2.32) would be substituted by $\sum_{j=1}^{n} \lambda_j \leq 1$.

Before considering the profit inefficiency decomposition through the corresponding Fenchel–Mahler inequality we need to define the profit deviation at point (x_0, y_0).

The *profit deviation* at point (x_0, y_0) is simply the deviation between the profit function and the profit at that point, given market prices (q, p):

$$\Pi D(x_0, y_0) := \Pi(q,p) - (py_0 - qx_0). \tag{2.33}$$

For the sake of brevity, we write ΠD_0 for $\Pi D(x_0, y_0)$.

Additionally, it is possible to relate a normalized term of profit deviation, called *profit inefficiency*, with the inefficiency detected by the directional distance function as follows:

$$\frac{\Pi D_0}{pg_{0y} + qg_{0x}} \geq \vec{D}\left(x_0, y_0; g_{0x}, g_{0y}\right). \tag{2.34}$$

Resorting to the Fenchel–Mahler inequality associated to $RDDF^M$ we get the inequality

$$\Pi I_0^{M,\Pi^M} := \frac{\Pi D_0^{M,\Pi^M}}{pg_{0y}^{M,\Pi^M} + qg_{0x}^{M,\Pi^M}} \geq \tau_0^M. \tag{2.35}$$

The left hand-side term is the profit inefficiency, ΠI_0, which, as CI_0 and RI_0, satisfies that it is homogeneous of degree 0 in prices, which makes ΠI_0 invariant to the currency units for the market output and input prices. Going back to the last inequality and defining the profit allocative inefficiency as the corresponding residual, we get the next equality:

$$\Pi I_0^{M,\Pi^M} = \tau_0^M + AI_0^{M,\Pi^M}. \tag{2.36}$$

In other words, at point (x_0, y_0), profit inefficiency is decomposed into the sum of technical inefficiency and allocative inefficiency.

Example 7.1 The Additive Model with Single Projections

Let us consider model M as the additive model, formulated before and identified as model (2.3). Let us further consider in the "one input—one output space" the next sample of units to be rated: $\{(2; 2), (6; 10); (8; 6)\}$. We represent them directly on a plane. Moreover, it is easy to realize that the first two units are efficient while the third one is not, because it is clearly dominated by (6;10).

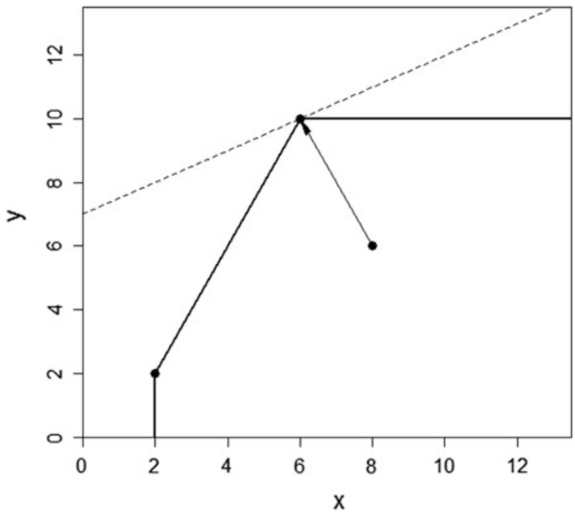

Fig. 2.4 Figure associated with Example 7.1

Graphically, there are several candidates as projection of the inefficient unit (8;6): the point (6;10) or certain convex linear combinations of the two efficient points that dominates point (8;6), such as point (4;6), the middle point in between (2;2) and (6;10). Additive model *M* selects the one that is as far as possible, using as measure the L_1-distance. The projection is point (6;10) with a L_1-path length equal to 6 (= (8 − 6) + 10 − 6). This length is exactly the technical inefficiency, τ_0^M, detected by linear program (2.3). Let us assume that the market prices are given as (*q*,*p*) = (2, 4). The maximum profit as evaluated through program (2.32) tell us that $\Pi(2{,}4) = 40 - 12 = 28$, and that it is achieved at point (6;10), as shown in Fig. 2.4, where we have drawn the iso-profit line $4y - 2x = 28$. Now, since the obtained *M*-projection of point (x_0, y_0) = (8;6) is the efficient point (6;10) and τ_0^M = 6, we get $g_0 = \left(\frac{2}{6}, \frac{4}{6}\right) = \left(\frac{1}{3}, \frac{2}{3}\right)$. Evaluating the profit deviation at the inefficient point we obtain $\Pi D(x_0, y_0) := \Pi(2{,}4) - \left(\sum_{r=1}^{s} p_r y_{r0} - \sum_{i=1}^{m} q_i x_{i0}\right) = 28 - (4 \cdot 6 - 2 \cdot 8) = 28 - 8 = 20$. Consequently, the value of ΠI_0 is $\frac{\Pi D_0}{q g_{0x}^M + p g_{0y}^M} = \frac{20}{2 \cdot \frac{1}{3} + 4 \cdot \frac{2}{3}} = \frac{20}{\frac{10}{3}} = 6$, which is equal to τ_0^M = 6. Hence, $AI_0^M = \Pi I_0^M - \tau_0^M = 6 - 6 = 0$.

Once again, it would be interesting if we could relate the profit deviation of the efficient projection with the allocative inefficiency of the point being rated. The next proposition gives us the answer.

Proposition 4 *Let* $\left(x_0^M, y_0^M\right)$ *denote the projection of* (x_0, y_0) *obtained through model M. Then, the allocative inefficiency associated to point* (x_0, y_0) *can be obtained as*

$$AI_0^{M,\Pi^M} = \frac{\Pi D_0^{M,\Pi^M}\left(x_0^M, y_0^M\right)}{q \cdot g_{0x}^{M,\Pi^M} + p \cdot g_{0y}^{M,\Pi^M}} = \frac{\Pi(q,p) - \left(\sum_{r=1}^{s} p_r y_{r0}^M - \sum_{i=1}^{m} q_i x_{i0}^M\right)}{q \cdot g_{0x}^{M,\Pi^M} + p \cdot g_{0y}^{M,\Pi^M}}. \quad (2.37)$$

In particular, $AI_0^{M,\Pi^M} = 0$ *if, and only if,* $\Pi D_0^{M,\Pi^M}\left(x_0^M, y_0^M\right) = 0$.

The proof is similar to the proof of Proposition 2 and is left again to the reader. The key of the proof is that profit at the efficient projection is equal to profit at the inefficient point plus the technical inefficiency times the normalization factor of the inefficient point.

2.8 Identifying, for Each Inefficient Unit, a Projection that Minimizes Its *RDDF* Profit Inefficiency

For a specific inefficient unit, (x_0, y_0), the considered DEA single-value model *M* generates a specific efficient projection, $\left(x_0^M, y_0^M\right)$. Resorting to $RDDF^M$, we have been able to measure and decompose its economic inefficiency. Let us focus our attention on profit inefficiency, knowing that a completely similar treatment can be developed for cost or revenue inefficiency. The question that we want to tackle is the following: is it possible to identify a different projection with better—or lower—profit inefficiency? Let us refer to this new projection as $\left(x_0^*, y_0^*\right)$. Since our additional aim is to maintain the introduced profit decomposition through the *RDDF*, we will not accept the possibility of increasing some inputs and decreasing some outputs, as Zofío et al. (2013) did.

Being our aim to reduce profit inefficiency as much as possible, we have devised the following strategy. First of all, let us observe that according to expression (2.35) profit inefficiency equals $\frac{\Pi D_0^M}{p g_{0y}^M + q g_{0x}^M}$. Hence, it is a ratio whose numerator, the profit deviation of point (x_0, y_0), is fixed, and, consequently, if we want to reduce profit inefficiency, the only action we can take is to enlarge its denominator. Therefore, what we would like to do is to search for an efficient projection $\left(x_0^*, y_0^*\right)$ that maximizes this denominator. According to expression (2.2), and for any inefficient point (x_0, y_0), we know that[25]

$$q \cdot g_{0x}^* + p \cdot g_{0y}^* = \frac{1}{\tau_0^{*M}} \left[q \cdot \left(x_0 - x_0^*\right) + p \cdot \left(y_0^* - y_0\right) \right]. \tag{2.38}$$

Hence, if we maximize $q \cdot \left(x_0 - x_0^*\right) + p \cdot \left(y_0^* - y_0\right)$ we are maximizing $\tau_0^{*M}\left(q \cdot g_{0x}^* + p \cdot g_{0y}^*\right)$. This is not exactly what we want to do, but it is a useful proxy that will help us to achieve our goal. In fact, assuming that $\tau_0^{*M} \leq \tau_0^M$,[26] we obtain the next chain of inequalities: $q \cdot g_{0x}^M + p \cdot g_{0y}^M = \frac{1}{\tau_0^M} \left[q \cdot \left(x_0 - x_0^M\right) + p \cdot \left(y_0^M - y_0\right) \right]$ $\leq \frac{1}{\tau_0^{*M}} \left[q \cdot \left(x_0 - x_0^M\right) + p \cdot \left(y_0^M - y_0\right) \right] \leq \frac{1}{\tau_0^{*M}} \left[q \cdot \left(x_0 - x_0^*\right) + p \cdot \left(y_0^* - y_0\right) \right] = q \cdot g_{0x}^*$ $+ p \cdot g_{0y}^*$, where the last inequality is true because $\left(x_0^*, y_0^*\right)$ is the efficient point that dominates $\left(x_0, y_0\right)$ and, at the same time, maximizes $q \cdot \left(x_0 - x_0^*\right) + p \cdot \left(y_0^* - y_0\right)$.

Consequently, let us maximize expression $q \cdot \left(x_0 - x_0^*\right) + p \cdot \left(y_0^* - y_0\right)$, or, equivalently, $\left(p \cdot y_0^* - q \cdot x_0^*\right) - \left(p \cdot y_0 - q \cdot x_0\right)$. Since the last parenthesis is a fixed number, we simply need to maximize $\left(p \cdot y_0^* - q \cdot x_0^*\right)$, i.e., the profit achieved at the new efficient projection. Let us consider a linear program whose objective function

[25]The presence of the asterisk means that we are considering a new projection obtained through a specific optimization program and not through model M. Nonetheless, model M is used for determining the technical inefficiency associated to this new projection, which justifies the used notation.

[26]The assumption is valid for DEA models that reach their projections by maximizing a certain "distance", such as the weighted additive model.

maximizes profit at the efficient projection and whose restrictions guarantee that the obtained projection belongs to T. At this point it is worth noticing that a few years ago, the idea of getting a reference benchmark with the highest possible profit was already suggested by Zofio et al. (2013, page 263, Footnote 3).

$$
\begin{aligned}
&Max_{s^-,s^+,\lambda} \sum_{r=1}^{s} p_r\left(y_{r0}+s_{r0}^+\right) - \sum_{i=1}^{m} q_i\left(x_{i0}-s_{i0}^-\right)\\
&\quad s.t.\\
&\quad \sum_{j\in E}\lambda_j x_{ij} = x_{i0}-s_{i0}^-, \quad i=1,\ldots,m\\
&\quad \sum_{j\in E}\lambda_j y_{rj} = y_{r0}+s_{r0}^+, \quad r=1,\ldots,s\\
&\quad \sum_{j\in E}\lambda_j = 1,\\
&\quad \lambda_j \geq 0, \quad j\in E\\
&\quad s_{i0}^- \geq 0, \quad i=1,\ldots,m\\
&\quad s_{r0}^+ \geq 0, \quad r=1,\ldots,s
\end{aligned}
\tag{2.39}
$$

Program (2.39) identifies $\left(x_0^*, y_0^*\right)$ as a strongly efficient point that maximizes $\left(p\cdot y_0^* - q\cdot x_0^*\right)$[27] and, at the same time, dominates (x_0, y_0). The used notation together with program (2.39) indicate that the last identified efficient point is valid for any non-oriented M model.[28] The differences will appear when we evaluate the technical inefficiency associated to that strongly efficient point through model M. For instance, if model M is the additive model, the associated τ_0^{*M} is the length of the L_1-path connecting (x_0, y_0) with $\left(x_0^*, y_0^*\right)$, and it is very likely that $\tau_0^{*M} < \tau_0^M$. Let us illustrate these findings with an easy numerical example.

Example 8.1 Maximizing the *RDDF* Profit Inefficiency of the Additive Model

Let us consider the next sample of points on the XY plane:$\{(2,2), (3,4), (5,7), (11,9), (8,3)\}$. Resorting to the additive model it is easy to check that all the points are strongly efficient except the last one. In fact, point (8,3) is dominated by point (3,4) as well as by point (5,7). Let us further assume that the market prices are $q = 1$ and $p = 11$. Figure 2.5 below shows the graphical representation of the sample of points as well as of the corresponding VRS efficient frontier. We have also drawn the line associated to maximum profit, $11y - x = 88$.

[27]In DEA literature it is known that maximum profit is achieved in at least a strongly efficient point.

[28]The other two possibilities are that model M is input-oriented or output-oriented. In the first case, program [39] needs to be adjusted just by deleting the output-slacks, and symmetrically for the output-oriented case.

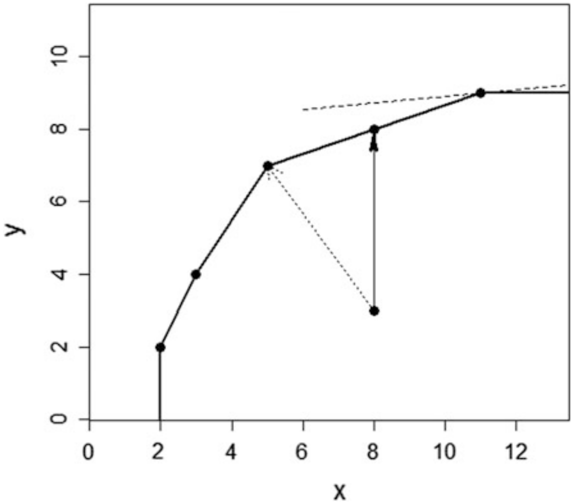

Fig. 2.5 Figure associated with Example 8.1

For the only inefficient point, $(x_0, y_0) = (8{,}3)$, we solve linear program (2.39) and obtain the profit maximizing projection $\left(x_0^*, y_0^*\right) = (8{,}8)$, that belongs to the facet defined by efficient points (5,7) and (11,9). The profit at point (8,8) is 80 (=11 · 8 – 1 · 8), the profit at point (8,3) is 25 (=33 − 8), while the optimal profit, $\Pi(1{,}2) = 88$, is achieved at point (11,9). Let us now consider that model M is the additive model. Clearly, the technical inefficiency associated to the new obtained projection equals 5 (length of the L_1-path connecting (8,3) with (8,8)). Moreover, $q \cdot g_{0x}^{*M} + p \cdot g_{0y}^{*M} = 1 \cdot 0 + 11 \cdot 5 = 55$. Consequently, according to expression (2.35), the normalized profit inefficiency at point (8,3) is $\frac{88-25}{55} = \frac{63}{55} = 1\frac{8}{55}$.

In order to evaluate the gain obtained by applying the new proposed strategy, let us compare the last result with the result derived directly by applying the additive model and evaluating the associated normalized profit inefficiency through the $RDDF^M$. Since the additive projection of point (8,3) is point (5,7), the corresponding τ_0^M equals 7(=(8 − 5) + (7 − 3)) which means that $\left(g_{0x}^M, g_{0y}^M\right) = \frac{1}{\tau_0^M}\left(x_0 - x_0^M, y_0^M - y_0\right) = \left(\frac{3}{7}, \frac{4}{7}\right)$, with a normalization factor value equal to $1 \cdot \frac{3}{7} + 11 \cdot \frac{4}{7} = \frac{47}{7}$, which gives a normalized profit inefficiency value equal to $\frac{63}{\frac{47}{7}} = \frac{63 \cdot 7}{47} = \frac{441}{47} = 9\frac{8}{47}$, clearly much bigger than $1\frac{8}{55}$.

2.9 Conclusions

The introduction, in Sect. 2.2, of the *RDDF* associated to DEA inefficiency model M, denoted as $RDDF^{M,\Pi^M}$, has allowed us, for the first time, to express any DEA model as a *DDF*. The key idea is that the *RDDF* maintains exactly the same projections as the original DEA model. However, if the original DEA model has multiple projections for at least one inefficient unit, we are able to define as many *RDDFs* as combinations of single projections we can perform. For the case of the weighted additive model we have included, in Appendix 1, a new method for identifying different projections at each inefficient point. Which projection to choose will depend on secondary criteria that will guide our final selection.

Our new introduced tool is also relevant for transforming a weak *DDF* into a comprehensive measure or strong *DDF*. Moreover, any DEA inefficiency measure can also be transformed into its comprehensive version, by applying the *RDDF* technique twice. In particular, we have shown how to transform a radial model into a comprehensive one that is likely to lose its radial type of projection, as well as its one-sided orientation.

The first introduced (multiplicative) decomposition of economic efficiency, cost efficiency in particular, was due to Farrell, designed specifically for radial models. In the nineties Chambers, Chung and Färe proposed an additive decomposition of economic inefficiency, cost, revenue or profit, based on the *DDF*. So far, the subsequent proposed approaches for estimating and decomposing economic inefficiency have been all additive in nature and have emerged during the last lustrum. As explained in Sect. 2.1, the models that have captured the attention of the researchers in DEA were the weighted additive model, the output-oriented weighted additive model, and the two Russell oriented models. The new *RDDF* introduced in this chapter is responsible for defining the cost, revenue or profit inefficiency of any DEA inefficiency measure as well as their additive decomposition into its technical and allocative components. Hence the proposed solution constitutes a unified DEA approach that benefits from the known Fenchel–Mahler inequality established for *DDFs*.

Finally, two additional issues have been considered and solved. First, a linear programming procedure has been devised for identifying a new projection for each inefficient unit where profit inefficiency is minimized. And secondly, in Appendix 1, we have shown how to generate alternative optimal solutions in connection with additive type models.

Appendix 1

How to Search for Alternative Optimal Solutions When Using a Weighted Additive Model

The additive model has been introduced in Sect. 2.3, while the weighted additive model (Lovell and Pastor, 1995) has been described at the beginning of Example 4.1. Just as a reminder, the weighted additive model has the same set of restrictions as the additive model but differs in its objective function. In fact, while the objective function of the additive model is the sum of input-slacks and output-slacks, the weighted additive model considers as objective function a weighted sum of input-slacks and output slacks, where the attached weights are all non-negative and at least one of them must be positive. Since any particular weight can be 0, the input-oriented or output-oriented weighted additive models are particular cases of weighted additive models. Moreover, weighted additive models are VRS models, because the convexity constraint is one of its restrictions. It is easy to consider CRS weighted additive models just by deleting from the set of restrictions the mentioned constraint, or even to consider non-increasing returns to scale (NIRS) or non-decreasing returns to scale (NDRS) additive models, by changing slightly the convexity constraint, transforming the equality into an inequality (≤ 1 for NIRS or ≥ 1 for NDRS). Here, as in the rest of the chapter, we will deal exclusively with VRS models, but it is not difficult to derive the corresponding conclusions for non-VRS weighted additive models.

Assume that, for a given inefficient unit (x_0, y_0) we have obtained a first optimal projection identified as $\left(x_0^{p1}, y_0^{p1}\right)$ through the weighted additive model. This projection is always a strongly efficient point. Now we want to search for the existence of alternative optimal solutions, that is, alternative optimal slack values. Let us denote as $w_i^-, i = 1, \ldots, m$ the weights associated to the input-slacks and as $w_r^+, r = 1, \ldots, s$ the weights associated to the output slacks in the objective function. Then, the optimal value of the objective function equals $\sum_{i=1}^m w_i^- \left(x_i^{p1} - x_{i0}\right) + \sum_{r=1}^s w_r^+ \left(y_r^{p1} - y_{r0}\right) = \sum_{i=1}^m w_i^- s_{i0}^{-*} + \sum_{r=1}^s w_r^+ s_{r0}^{+*}$, which is a fixed number, let us say v^*. Knowing v^*, we are able to generate as much as $2m + 2s$ optimal solutions through the procedure proposed next.

Procedure Consider the following linear program, which is equivalent to (2.3) except for the presence of non-negative weights in the objective function.

$$
\begin{aligned}
WAdd(x_0, y_0) = Max_{s^-, s^+, \lambda} \quad & \sum_{i=1}^m w_i^- s_{i0}^- + \sum_{r=1}^s w_r^+ s_{r0}^+ \\
& s.t. \\
& \sum_{j=1}^n \lambda_j x_{ij} = x_{i0} - s_{i0}^-, \quad i = 1, \ldots, m \\
& \sum_{j=1}^n \lambda_j y_{rj} = y_{r0} + s_{r0}^+, \quad r = 1, \ldots, s \\
& \sum_{j=1}^n \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n, \\
& s_{i0}^- \geq 0, \quad i = 1, \ldots, m, \quad s_{r0}^+ \geq 0, \quad r = 1, \ldots, s
\end{aligned}
\tag{2.40}
$$

As said before, and to simplify the notation, we write $v^* := WAdd(x_0, y_0)$.

In order to search for alternative optimal solutions, we further solve the next pair of linear programs for each input slack, $s_{k0}^-, k \in \{1,\ldots,m\}$ and each output slack, $s_{l0}^+, l \in \{1,\ldots,s\}$:

$$\begin{aligned}
Max_{s^-,\lambda} \quad & s_{k0}^-(or\, s_{l0}^+) \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_j x_{ij} = x_{i0} - s_{i0}^-, \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} = y_{r0} + s_{r0}^+, \quad r = 1,\ldots,s \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \sum_{i=1}^{m} w_i^- s_{i0}^- + \sum_{r=1}^{s} w_r^+ s_{r0}^+ = v^* \\
& \lambda_j \geq 0, \quad j = 1,\ldots,n, \\
& s_{i0}^- \geq 0, \quad i = 1,\ldots,m, \; s_{r0}^+ \geq 0, \; r = 1,\ldots,s \\
& s_{i0}^- \geq 0, \quad i = 1,\ldots,m, \; s_{r0}^+ \geq 0, \; r = 1,\ldots,s,
\end{aligned} \tag{2.42}$$

and

$$\begin{aligned}
Min_{s^-,\lambda} \quad & s_{k0}^-(or \quad s_{l0}^+) \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_j x_{ij} = x_{i0} - s_{i0}^-, \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} = y_{r0} + s_{r0}^+, \quad r = 1,\ldots,s \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \sum_{i=1}^{m} w_i^- s_{i0}^- + \sum_{r=1}^{s} w_r^+ s_{r0}^+ = v^* \\
& \lambda_j \geq 0, \quad j = 1,\ldots,n, \\
& s_{i0}^- \geq 0, \quad i = 1,\ldots,m, \; s_{r0}^+ \geq 0, \; r = 1,\ldots,s,
\end{aligned} \tag{2.43}$$

which means that we propose to solve $2m + 2s$ linear programs. Any of these programs searches for a possibly alternative optimal projection due to the addition of the last linear restriction $\sum_{i=1}^{m} w_i^- s_{i0}^- + \sum_{r=1}^{s} w_r^+ s_{r0}^+ = v^*$.

Example A.1.2 Searching for Alternative Optimal Solutions

Let us consider in the two input—one output space the next sample of units: U1 (1,6;3), U2(2,6;4), U3(6,2;4), U4(6,1;3) and U5(8,7;2). Let us work with the additive model, which is a weighted additive model where all weights equal 1. It is easy to check that the first four units are extreme strongly efficient and that U5 is inefficient. The projection we get resorting to Excel-Solver is point U3, with an optimal value equal to 9.

Now we start searching for alternative optimal solutions, with $v^* = 9$, by considering each of the maximizing models gathered in (2.42). For the objective function s_{15}^- we get U1 as a new alternative optimal solution, while for the objective function s_{25}^- we also get U4 as a new solution. Finally, for s_5^+ as the objective function we once more get U3 as solution. Going over to the minimizing models included in (2.43), and starting again with s_{15}^- as the objective function we get U3 as the optimal solution. For the objective function s_{25}^- we get U1 as the solution and last, for the objective function s_5^+ we get also U1 as the solution. Hence our procedure has identified U1, U3 and U4 as alternative optimal solutions, but has not been able to identify the remaining one, U2. This is a very easy example. In practice it is difficult that optimal solutions are single points. In general, solving linear programs as proposed in (2.42) and (2.43) is the sensible way we propose. A final point is worth mentioning. Under VRS, once we have identified three different optimal solutions, all the convex combinations of them are also optimal, which means that we have generated a non-finite number of optimal solutions.

How to Search for Alternative Optimal Solutions When Using a DEA Model with the Same Restrictions as the Additive Model

The last proposed method designed for the weighted additive model can be extended to any other DEA model with the same set of restrictions as the additive model as long as the last added restriction is linear or can be linearized. This happens, for instance, with the "slack-based measure" (Tone, 2001), which is equivalent to the "enhanced Russell graph measure" (Pastor et al. 1999), whose objective function is not linear but fractional. In this case the added non-linear restriction is $\frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{i0}}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{r0}}} = v^*$, that can be linearized just by transposing its left hand-side denominator. The same happens with the translation invariant measure proposed by Sharp et al. (2007).

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