
Preface

Over the past fifteen years, motivated by regularity problems in evolution equations, there has been tremendous progress in the analysis of Banach space-valued functions and processes. For so-called UMD spaces in particular, central areas of harmonic analysis, such as the theory of Fourier multipliers and singular integrals, were extended to operator-valued kernels acting on Bochner spaces, and basic estimates of stochastic analysis, including the Itô isometry and the Burkholder–Davis–Gundy inequalities, were generalised to Banach space-valued processes.

As it was long known that extensions of such sophisticated scalar-valued estimates are not possible for all Banach spaces, these results depended on essential progress in the geometry of Banach spaces during the 70s and 80s. The theory of Burkholder and Bourgain on UMD spaces became the foundation on which the recent theory we wish to report on was built; just as important are results of Kwapien, Maurey, and Pisier on type and cotype, since they link the structure of the Banach space to estimates for random sums which replace to some extent the fundamental orthogonality relations in Hilbert spaces.

For most classical Banach spaces, the UMD, type and cotype properties are readily available and therefore the results of vector-valued Analysis can be applied to many situations of interest in the theory of partial differential equations; they have already proved their value by providing sharp regularity estimates for parabolic problems. Our aim is to give a detailed and careful presentation of these topics that is useful not only as a reference book but can be used also selectively as a basis for advanced courses and seminars.

This project ranges over a broad spectrum of Analysis and includes Banach space theory, operator theory, harmonic analysis and stochastic analysis. For this reason we have divided it into three parts. The present volume develops the theory of Bochner integration, Banach space-valued martingales and UMD spaces, and culminates in a treatment of the Hilbert transform, Littlewood–Paley theory and the vector-valued Mihlin multiplier theorem.

Volume II will present a thorough study of the basic randomisation techniques and the operator-theoretic aspects of the theory, such as R -

boundedness, vector-valued square functions and radonifying operators, as well as a detailed treatment of the relevant probabilistic Banach space notions such as type, cotype, K -convexity and properties related to contraction principles. These techniques will allow us to present the theory of H^∞ -functional calculus for sectorial operators and work out the main examples. This sets the stage for our final aim, a presentation of the theory of singular integral operators with operator-valued kernels and its applications to maximal regularity for deterministic and stochastic parabolic evolution equations, which will be the subject matter of Volume III.

The central theme in all volumes is the identification of the Banach spaces to which the key estimates of classical harmonic and stochastic analysis can be extended as those with the fundamental UMD property. The very definition behind this abbreviation is the unconditionality of martingale differences, a primarily probabilistic notion, and a number of different characterisations are formulated in purely probabilistic terms. However, this same property is also equivalent to the boundedness of the vector-valued Hilbert transform, the Littlewood–Paley inequality for vector-valued Fourier integrals, and several other estimates in the realm of classical harmonic analysis.

Each of these aspects of UMD spaces makes a substantial body of theory in its own right, and one could certainly produce respectable treatments of large parts of this material with a “clean” probabilistic or analytic flavour. However, rather than striving for such “purity”, our aim is to emphasise the rich connections between the two worlds and the unity of the subject. For example, while martingales are traditionally regarded as a topic in Probability, we define and discuss them on σ -finite measure spaces from the beginning, so that they are immediately applicable to Analysis on the Euclidean space $\mathbb{R}^d$ without the need of auxiliary truncations or decompositions into probability spaces. Moreover, it is important to observe that even if we (or the reader) wanted to concentrate on the analytic side of UMD spaces only, we could hardly present a complete picture without an occasional reference to the probabilistic notions, at least at the present state of knowledge. For instance, although we know that both the Hilbert transform boundedness and the Littlewood–Paley inequality are equivalent to the UMD property, and therefore to each other, the only known way of proving the equivalence of these two analytic notions passes through the probabilistic UMD. There are numerous other such examples, and new frontiers of the theory have shown over and over again that it is the probabilistic definition of UMD spaces that lies at the centre and connects everything together.

So much said about the unity of Analysis and Probability (in Banach spaces), we should acknowledge the existence of a third side of the triangle, which is barely touched by the present treatise, namely: Geometry (of Banach spaces). Our choice of topics is not meant in any way to downplay the importance of this huge topic, both in its own right and in relation to analytic and

probabilistic questions, but rather to admit our limits and to leave the proper account of the geometric connections for other treatments.

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This book can be studied in a variety of ways and for different motivations. The principal, but not the only, audience that we have in mind consists of researchers who need and use Analysis in Banach spaces as a tool for studying other problems, in particular the regularity of evolution equations mentioned above. Until now, the contents of this extensive and powerful toolbox have been mostly scattered around in research papers, or in some cases monographs addressed to readerships with a rather different background from our focus, and we feel that collecting this diverse body of material into a unified and accessible presentation fills a gap in the existing literature. Indeed, we regard ourselves as part of this audience, and we have written the kind of book that we would have liked to have for ourselves when working through this theory for the first time.

Aside from this, parts of the book may also offer an interesting angle to the classical analysis of scalar-valued functions, which is certainly covered as a special case, and seldom required as a prerequisite or used as a building block for the Banach space-valued theory. For a classical harmonic analyst, the approach that we take, say, to the L^p -boundedness of the Hilbert transform, is possibly exotic, but not necessarily substantially more difficult than more traditional treatments in the scalar-valued case.

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There are a couple of technical features of the book worth mentioning. Most of the time, we are quite explicit with the constants appearing in our estimates, and we especially try to keep track of the dependence on the main parameters involved. Thus, rather than saying that a particular bound “only depends on the UMD constant $\beta_{p,X}$ ”, we prefer to write out, say, $(\beta_{p,X})^2$, or whichever function of $\beta_{p,X}$ appears from the calculation. We often go to the extent of writing, say, “2000” instead of “ c , where c is a numerical constant”, although we also might write “2000” instead of “1764”, when there is no reason to believe that the latter constant, although given by a particular computation, would be anywhere close to optimal. Indeed, except for a few select places, we make no claim that our explicit constants cannot be improved; however, in many places, we have made an effort to present the best (order of) bounds currently available by the existing methods. We hope that making this explicit documentation might spur some interest towards research on such quantitative issues.

We also pay more attention than many texts to the impact of the underlying scalar field (real or complex) on the results under consideration. While this is largely irrelevant for many questions, it does play a role in some others,

and we try to be quite explicit in pointing out the differences when they do occur, hopefully without insisting too much on this point when they do not.

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This project was initiated in Delft and Karlsruhe already in 2008. Critical to its eventual progress was the possibility of intensive joint working periods in the serenity provided by the Banach Center in Będlewo (2012), Mathematisches Forschungsinstitut Oberwolfach (2013), Stiftsgut Keysermühle in Klingenmünster (2014 and 2015) and Hotel 't Paviljoen in Rhenen (2015). All four of us also met twice in Helsinki (2014 and 2016), and a number of additional working sessions were held by subgroups of the author team. One of us (J.v.N.) wishes to thank Marta Sanz-Solé for her hospitality during a sabbatical leave at the University of Barcelona in 2013.

Preliminary versions of parts of the material were presented in advanced courses and lecture series at various international venues and in seminars at our departments, and we would like to thank the students and colleagues who attended these events for feedback that shaped and improved the final form of the text. Special thanks go to Jamil Abreu, Alex Amenta, Markus Antoni, Sonja Cox, Chiara Gallarati, Fabian Hornung, Luca Hornung, Marcel de Jeu, Marcel Kreuter, Nick Lindemulder, Emiel Lorist, Bas Nieraeth, Jan Rozendaal, Jonas Teuwen, and Ivan Yaroslavtsev who did detailed reading of portions of this book. Needless to say, we take full responsibility for any remaining errors. A list with errata will be maintained on our personal websites. We wish to thank Klaas Pieter Hart for L^AT_EX support.

During the writing of this book, we have benefited from external funding by the European Research Council (ERC Starting Grant “AnProb” to T.H.), the Academy of Finland (grants 130166 and 133264 to T.H., and the Centre of Excellence in Analysis and Dynamics, of which T.H. is a member), the Netherlands Organisation for Scientific research (NWO) (VIDI grant 639.032.201 and VICI grant 639.033.604 to J.v.N. and VENI grant 639.031.930 and VIDI grant 639.032.427 to M.V.), the Volkswagenstiftung (grant I/78593 to L.W.) and the Deutsche Forschungsgemeinschaft (grant We 2847/1-2 to L.W.). We also wish to thank the Banach Center in Będlewo and Mathematisches Forschungsinstitut Oberwolfach for allowing us to spend two highly productive weeks in both wonderful locations.

Delft, Helsinki and Karlsruhe,
November 14, 2016.

Analysis in Banach Spaces

Volume I: Martingales and Littlewood-Paley Theory

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2016, XVII, 614 p. 3 illus., Hardcover

ISBN: 978-3-319-48519-5