Reversible Nets of Polyhedra

Jin Akiyama[1], Stefan Langerman[2(✉)], and Kiyoko Matsunaga[1]

[1] Tokyo University of Science, 1-3 Kagurazaka, Shinjuku, Tokyo 162-8601, Japan
ja@jin-akiyama.com, matsunaga@mathlab-jp.com
[2] Université Libre de Bruxelles, Brussels, Belgium
stefan.langerman@ulb.ac.be

Abstract. An example of reversible (or hinge inside-out transformable) figures is the Dudeney's Haberdasher's puzzle in which an equilateral triangle is dissected into four pieces, then hinged like a chain, and then is transformed into a square by rotating the hinged pieces. Furthermore, the entire boundary of each figure goes into the inside of the other figure and becomes the dissection lines of the other figure. Many intriguing results on reversibilities of figures have been found in prior research, but most of them are results on polygons. This paper generalizes those results to a wider range of general connected figures. It is shown that two nets obtained by cutting the surface of an arbitrary convex polyhedron along non-intersecting dissection trees are reversible. Moreover, a condition for two nets of an isotetrahedron to be both reversible and tessellative is given.

1 Introduction

A pair of hinged figures P and Q (see Fig. 1) is said to be *reversible* (or *hinge inside-out transformable*) if P and Q satisfy the following conditions:

1. There exists a dissection of P into a finite number of pieces, $P_1, P_2, P_3, \ldots, P_n$. A set of dissection lines or curves forms a tree. Such a tree is called a *dissection tree*.
2. Pieces $P_1, P_2, P_3, \ldots, P_n$ can be joined by $n-1$ hinges located on the perimeter of P like a chain.
3. If one of the end-pieces of the chain is fixed and rotated, then the remaining pieces form Q when rotated clockwise and P when rotated counterclockwise.
4. The entire boundary of P goes into the inside of Q and the entire boundary of Q is composed exactly of the edges of the dissection tree of P.

The theory of hinged dissections and reversibilities of figures has a long history and the book by Frederickson [11] contains many interesting results. On the other hand, Abbott et al. [1] proved that every pair of polygons P and Q with the same area is hinge transformable if we don't require the reversible condition. When imposing the reversible condition, hinge transformable figures have some remarkable properties which were studied in [3–6,9,12].

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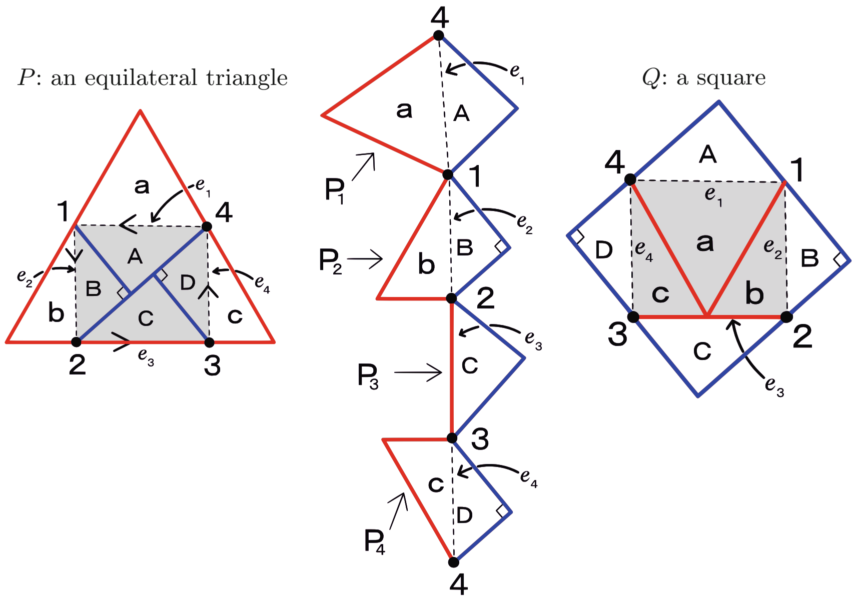

Fig. 1. Reversible transformation between P and Q.

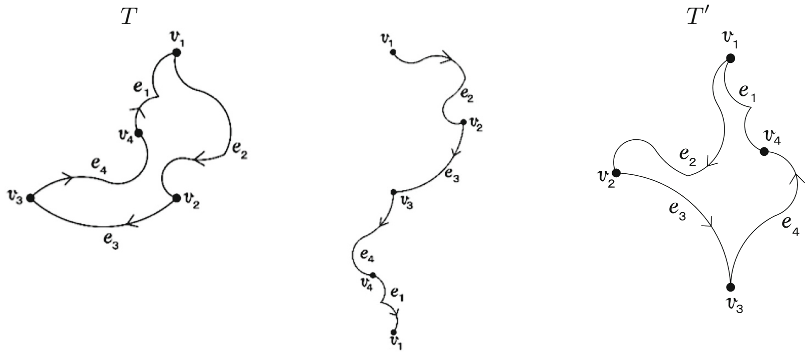

Fig. 2. T and one of its conjugate regions.

Let T be a closed plane region whose perimeter consists of n curved (or straight line) *segments* $e_1, e_2, \ldots, e_n$ and let these lines be labeled in clockwise order. Let T' be a closed region surrounded by the same segments $e_1, e_2, \ldots, e_n$ but in counterclockwise order. We then say that T' is a *conjugate region* of T (Fig. 2).

Let P be a plane figure. A region T with n vertices $v_1, \ldots, v_n$ and with n perimeter parts $e_1, \ldots, e_n$ is called an *inscribed region* of P if all vertices v_i $(i = 1, \ldots, n)$ are located on the perimeter of P and $T \subseteq P$.

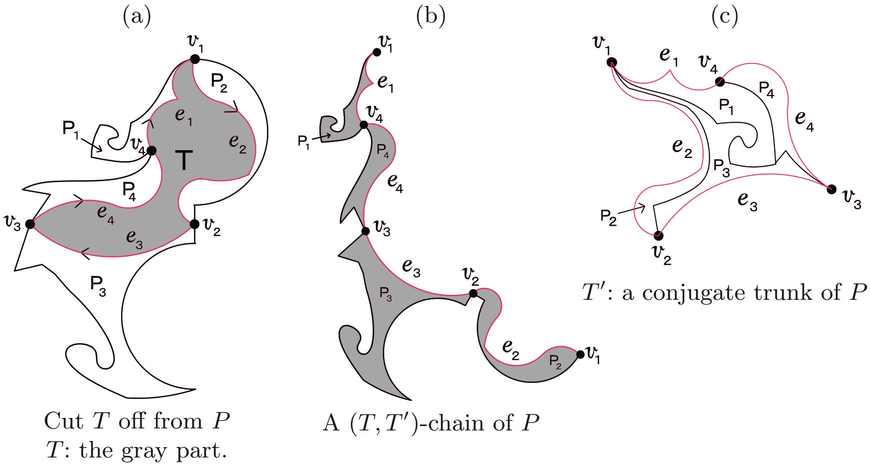

Fig. 3. A trunk T of P, a (T, T')-chain of P and a conjugate trunk T' of P.

A *trunk* of P is a special kind of inscribed region T of P. First, cut out an inscribed region T from P (Fig. 3(a)). Let e_i $(i = 1, \ldots, n)$ be the perimeter part of T joining two vertices v_{i-1} and v_i of T, where $v_0 = v_n$. Denote by P_i the piece located outside of T that contains the perimeter part e_i. Some P_i may be empty (or just a part e_i). Then, hinge each pair of pieces P_i and P_{i+1} at their common vertex v_i $(1 \leq i \leq n-1)$; this results in a chain of pieces P_i $(i = 1, 2, \ldots, n)$ of P (Fig. 3(b)). The chain and T are called (T, T')-chain of P, and *trunk of* P, respectively, if an appropriate rotation of the chain forms T' which is one of the conjugate regions of T with all pieces P_i packed inside T' without overlaps or gaps. The chain T' is called a *conjugate trunk* of P (Fig. 3(c)).

Suppose that a figure P has a trunk T and a conjugate trunk T'; and a figure Q has a trunk T' and a conjugate trunk T. We then have two chains, a (T, T')-chain of P and a (T', T)-chain of Q (Fig. 4).

Combine a (T, T')-chain of P with a (T', T)-chain of Q such that each segment of the perimeter, e_i, has a piece P'_i of P on one side (right side) and a piece Q_i of Q on the other side (left side). The chain obtained in this manner is called a *double chain of* (P, Q) (Fig. 5).

We say that a piece of a double chain is *empty* if that piece consists of only a perimeter part e_i. If a double chain has an empty piece, then we distinguish one side of that edge from the other side so that it satisfies the conditions for reversibility. If one of the end-pieces (Say P_1 and Q_1 in Fig. 5) of the double chain of (P, Q) is fixed and the remaining pieces are rotated clockwise or counterclockwise, then figure P and figure Q are obtained respectively (Fig. 5). The following result is obtained from [3].

Theorem 1 (Reversible Transformations Between Figures). *Let P be a figure with trunk T and conjugate trunk T', and let Q have trunk T' and conjugate trunk T. Then P is reversible to Q.*

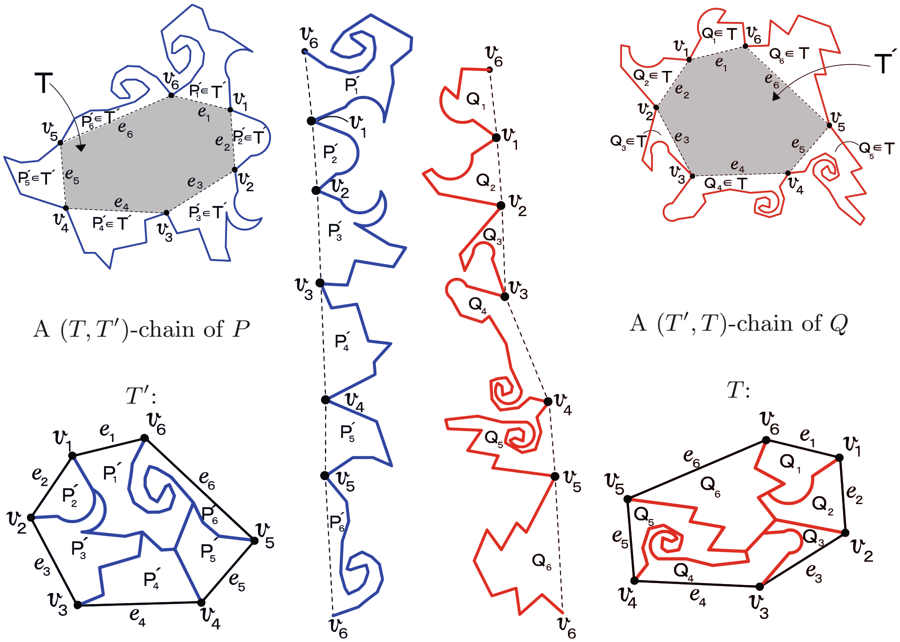

Fig. 4. A (T, T')-chain of P and a (T', T)-chain of Q.

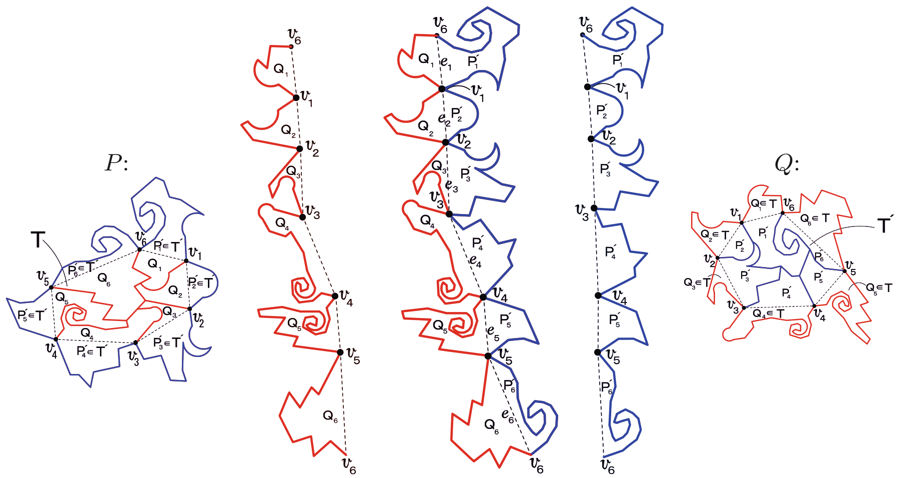

Fig. 5. A double chain of (P, Q).

Remarks

1. In Theorem 1, figure P which is the union of T and n pieces P_i' of the conjugate trunk T' reversibly transforms into figure Q which is the union of T' and n pieces of T.
2. Harberdasher's puzzle by H. Dudeney is also one such reversible pair. In this puzzle, the figures P and Q are an equilateral triangle and a square, respectively. The trunk T and conjugate trunk T' are the identical parallelogram T (the gray part in Fig. 1).

2 Reversible Nets of Polyhedra

A *dissection tree* D of a polyhedron P is a tree drawn on the surface of P that spans all vertices of P. Cutting the surface of P along D results in a *net* of P. Notice that nets of some polyhedron P may have self-overlapping parts (Fig. 6). We allow such cases when discussing reversible transformation of nets.

Theorem 2. *Let P be a polyhedron with n vertices $v_1, \ldots, v_n$ and let D_i $(i = 1, 2)$ be dissection trees on the surface of P. Denote by N_i $(i = 1, 2)$ the nets of P obtained by cutting P along D_i $(i = 1, 2)$, respectively. If D_1 and D_2 don't properly cross, then the pair of nets N_1 and N_2 is reversible, and has a double chain composed of n pieces.*

Proof. Suppose that dissection trees D_1 (the red tree) and D_2 (the green tree) on the surface of P do not properly cross (Fig. 7(a)). Then there exists a closed Jordan curve on the surface of P, which separates the surface of P into two pieces, one containing D_1, the other containing D_2. Let C be an arbitrary such curve (Fig. 7(b)). We call C a separating cycle. The net N_1, obtained by cutting P along D_1, contains an inscribed closed region T whose boundary is C (Fig. 8(a)). On the other hand, a net N_2 which is obtained by cutting P along D_2 contains an inscribed conjugate region T' whose boundary is the opposite side of C (Fig. 8(c)). Hence, a net N_1 has a trunk T and a conjugate trunk T', and a net N_2 has a trunk T' and a conjugate trunk T. By Theorem 1 this pair of N_1 and N_2 is reversible (Fig. 8(b)). □

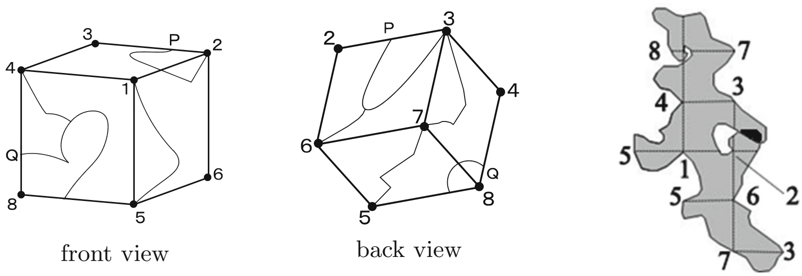

Fig. 6. A net of a cube with self-overlapping part (the overlap is the black part).

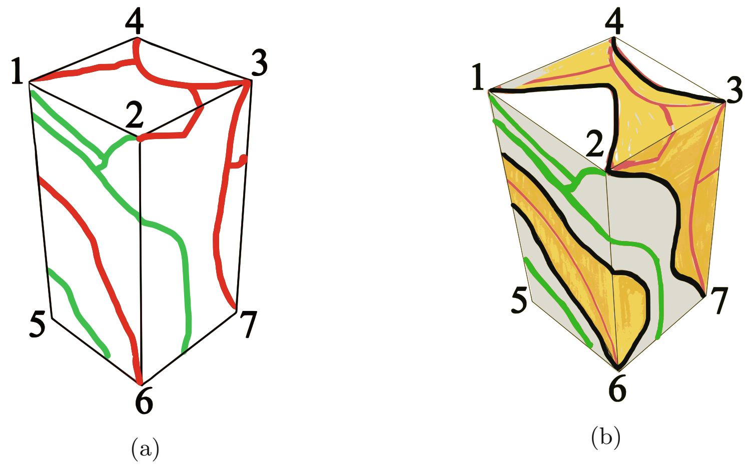

Fig. 7. A polyhedron P with dissection trees D_1 (red tree) and D_2 (green tree), a separating cycle C (black cycle). (Color figure online)

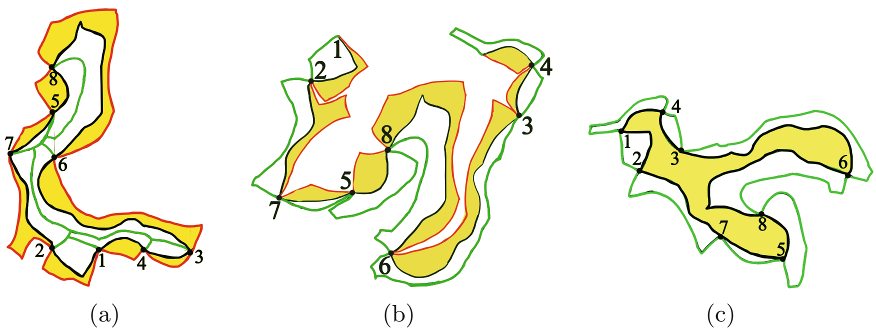

Fig. 8. Nets N_1 and N_2 obtained by cutting the surface of P along D_1 and D_2, respectively.

Theorem 3. *For any net N_1 of a polyhedron P with n vertices, there exist infinitely many nets N_2 of P such that N_1 is reversible to N_2.*

Proof. Any net N of P has a one-to-one correspondence with a dissection tree D on the surface of P. Let the dissection tree of N_i be D_i $(i = 1, 2)$, respectively (Fig. 9(a)). The perimeter of N_i can be decomposed into several parts in which each is congruent to an edge of D_i. Moreover, a vertex with degree k on D_i appears k times on the perimeter of N_i. These duplicated vertices of v_i are labeled as $v_i', v_i'', \ldots$.

Choose an arbitrary vertex v_k among $v_k, v_k', v_k'', \ldots$ on N_1 as a representative and denote it by v_k^*, where $k = 1, 2, \ldots, n$. Since N_1 is connected, it is possible to draw infinitely many arbitrary spanning trees D_2, each of which connects v_k^*

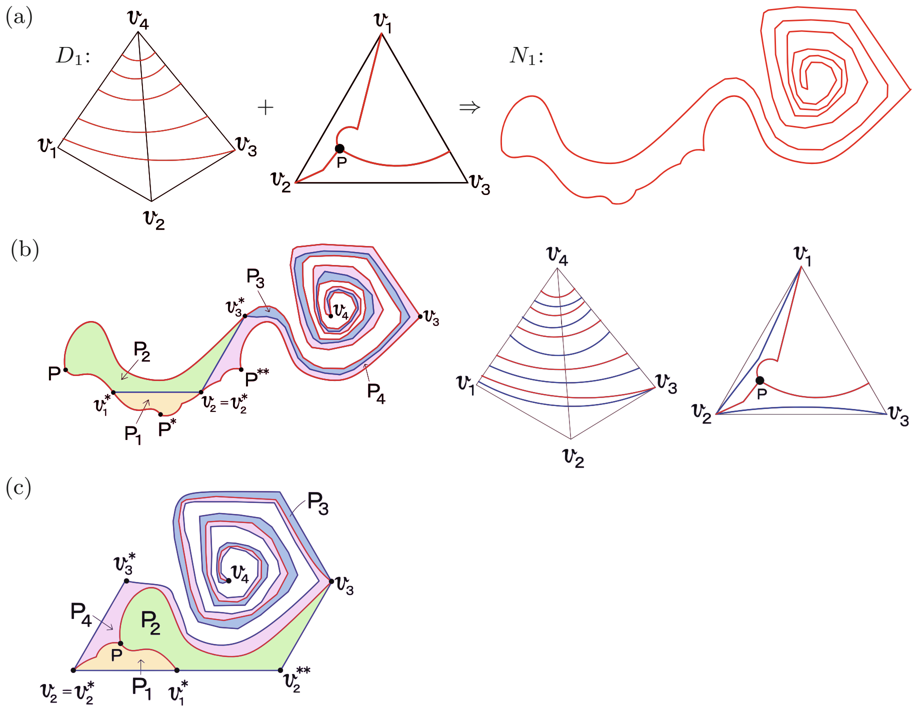

Fig. 9. A swirl net of a regular tetrahedron.

($k = 1, 2, \ldots, n$) inside N_1 (Fig. 9(b)). Then, any such D_2 doesn't intersect D_1. (Fig. 9(c)). As in Theorem 2, dissect N_1 along D_2 into n pieces $P_1, \ldots, P_n$, and then connect them in sequence using $n-1$ hinges on the perimeter of N_1 to form a chain. Fix one of the end-pieces of the chain and rotate the remaining pieces then forming net N_2 which is obtained by cutting P along D_2 (Fig. 9 (d)). □

Corollary 1 (Envelope magic [7]). *Let E be an arbitrary doubly covered polygon (dihedron) and let D_1 and D_2, be dissection trees of E. If dissection tree D_1 doesn't properly cross dissection tree D_2, then a pair of nets N_1 and N_2 obtained by cutting the surface of E along D_1 and D_2 is reversible (Fig. 10).*

The previous two theorems show that it is always possible to dissect any polyhedron P into two nets that are reversible, however, as mentioned in the beginning of this section, those nets may sometimes self-overlap when embedded in the plane. One may then ask whether a convex polyhedron P always has a pair of reversible non self-overlapping nets. The following theorem answers in the positive.

Theorem 4. *For any convex polyhedron P, there exists an infinity of pairs of non self-overlapping nets of P that are reversible.*

Proof. Choose an arbitrary point s on the surface of P, but not on a vertex. The *cut locus* of s is the set of all points t on the surface of P such that the

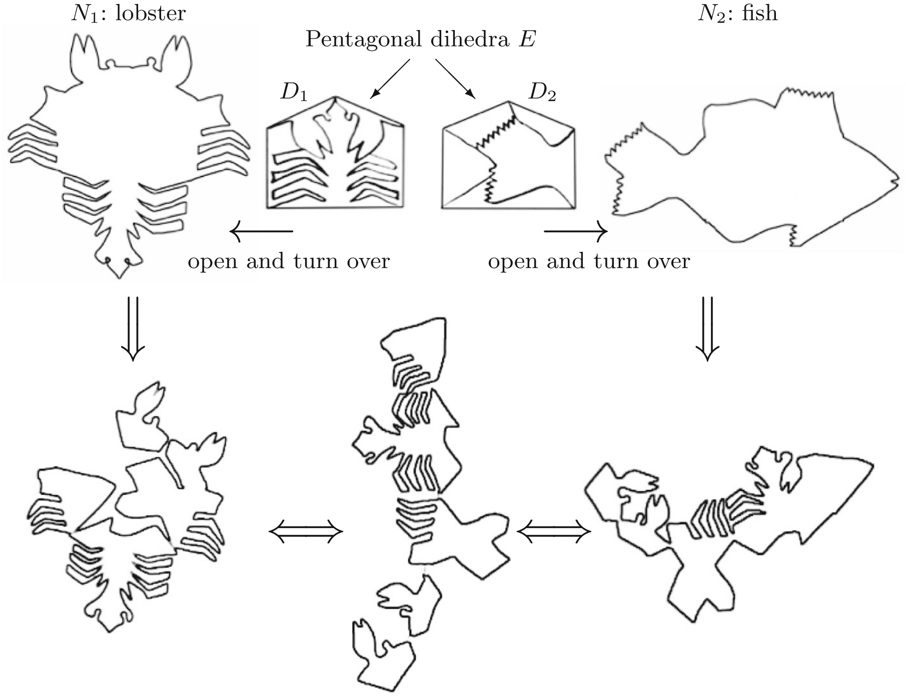

Fig. 10. A lobster transforms into a fish; The separating cycle C is the hem of a pentagonal dihedron.

shortest path from s to t is not unique. It is well known that the cut locus of s is a tree that spans all vertices of P. Cutting P along the cut locus produces the *source unfolding*, which does not overlap [10]. Let D_1 be the cut locus from s, and N_1 the corresponding non self-overlapping net. The net N_1 is a star-shaped polygon, and the shortest path from s to any point t in P unfolds to a straight line segment contained in N_1. The dissection tree D_2 is constructed by cutting P along the shortest path from s to every vertex of P. The net N_2 thus produced is a *star unfolding* and also does not overlap [8]. Note also that the shortest path from s to any vertex of P, when cutting the source tree D_1, unfolds to a straight line segment from s to the corresponding vertex on N_1. Therefore D_1 and D_2 do not properly intersect (In fact D_1 and D_2 may coincide but not properly cross. In order to avoid this, it suffices to choose s not on the cut locus of any vertex of P.) By Theorem 2, N_1 and N_2 are reversible. □

3 Reversibility and Tessellability for Nets of an Isotetrahedron

A tetrahedron T is called an *isotetrahedron* if all faces of T are congruent. Note that there are infinitely many non-similar isotetrahedra. Every net of an

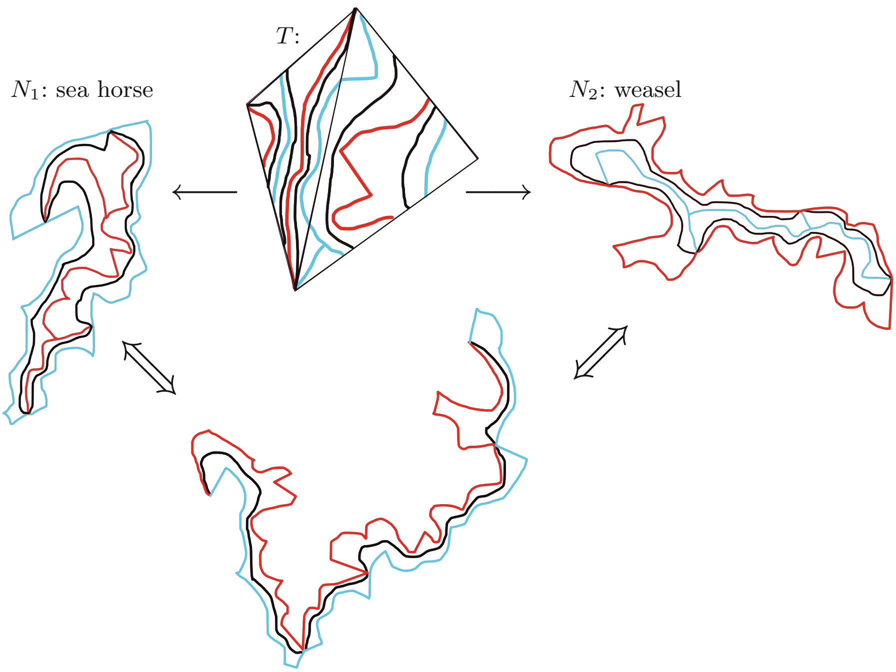

Fig. 11. sea horse ⇔ weasel

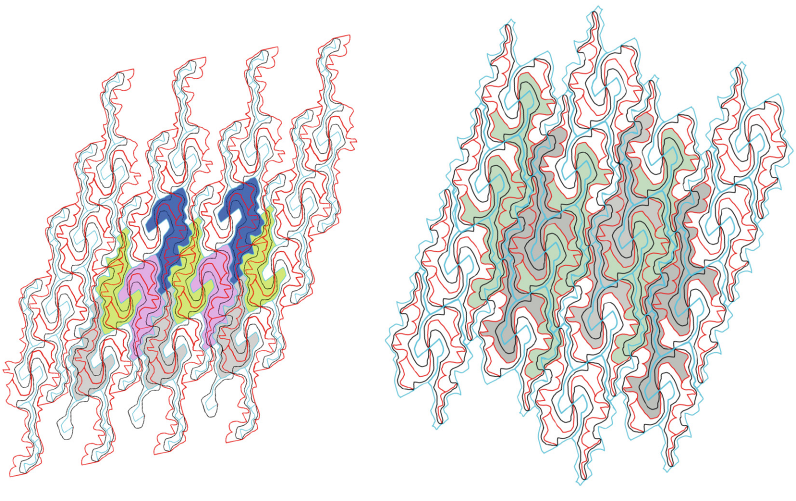

Fig. 12. Tiling by sea horse and weasel

isotetrahedron tiles the plane [2]. Moreover, all nets of isotetrahedron can be topologically classified into five types [3]. By Theorems 2 and 3, the following theorem is obtained:

Theorem 5. *Let D_1 be an arbitrary dissection tree of an isotetrahedron T. Then there exists a dissection tree D_2 of T which doesn't intersect D_1. The pair of nets N_1 and N_2 obtained by cutting along D_1 and D_2 is reversible, and each N_i $(i = 1, 2)$ tiles the plane.*

Proof. By Theorem 3, there exists a D_2 for any D_1. Let four vertices of T be v_k $(k = 1, 2, 3, 4)$. Draw both D_1 and D_2 on two Ts. Cut T along D_1, and the net N_1 inscribing D_2 is obtained. On the other hand, cut T along D_2, and the net N_2 inscribing D_1 is obtained (Fig. 11). As in Theorem 2, dissect N_1 along D_2 (or dissect N_2 along D_1) into four pieces P_1, P_2, P_3 and P_4, and join then in sequence by three hinges on the perimeter of N_1 like a chain. Fix one of the end pieces of the chain and rotate the remaining pieces, then they form the net N_2 which is obtained by cutting T along D_2. Since each of N_1 and N_2 is a net of an isotetrahedron, then both N_1 and N_2 are tessellative figures (Figs. 12 and 13). □

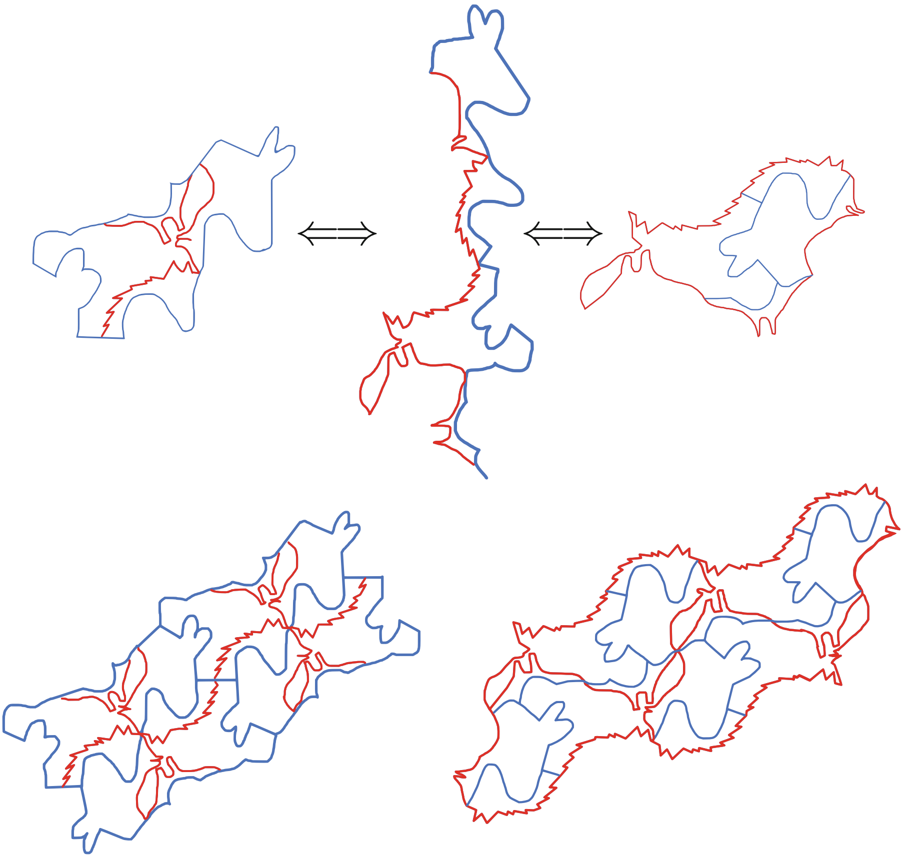

Fig. 13. donkey ⇔ fox

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