Chapter 2
Criteria on Periodic Stabilization in Infinite Dimensional Cases

Abstract This chapter studies the linear periodic feedback stabilization (LPFS, for short) for a class of evolution equations in the framework of Chap. 1. We restrict controls values only in a subspace Z of U, which might be of finite dimension.

Keywords Periodic Equations · Stabilization · Criteria · Infinite Dimension

Defnition 2.1 Equation (1.1) is said to be LPFS with respect to a subspace Z of U if there is a T-periodic $K(\cdot) \in L^\infty\big(\mathbb{R}^+; \mathscr{L}(H, Z)\big)$ so that the Eq. (1.104) is exponentially stable. Any such a $K(\cdot)$ is called an LPFS law for Eq. (1.1) with respect to Z. Write

$$\mathscr{U}^{FS} \triangleq \big\{Z \;\big|\; Z \text{ is a subspace of } U \text{ s.t. Eq. (1.1) is LPFS w.r.t. } Z\big\}. \tag{2.1}$$

We will provide three criteria for judging whether a subspace Z belongs to $\mathscr{U}^{FS}$. They are related with the following subjects: the attainable subspace of (1.1), which will be introduced in (2.2); the unstable subspace H_1 of (1.1) with the null control, which was defined in (1.21); the periodic map associated to (1.1) with the null control, which was defined in (1.12); and two unique continuation properties for the dual equations of (1.1) on different time horizons $[0, T]$ and $[0, n_0T]$ (where n_0 was defined by (1.17)), which will be introduced in (2.57) and (2.58) respectively. We also show that if $U \in \mathscr{U}^{FS}$, then there is a finite dimensional subspace Z in $\mathscr{U}^{FS}$. Hence, Eq. (1.1) is LPFS if and only if it is LPFS with respect to a finite dimensional subspace Z of U. This might help us to design feedback laws numerically.

2.1 Attainable Subspaces

This section is devoted to studies of attainable subspaces w.r.t. a subspace $Z \subset U$. Let

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$$\mathscr{A}_k^Z \triangleq \left\{ \int_0^{kT} \Phi(kT,s)B(s)u(s)\mathrm{d}s \;\middle|\; u(\cdot) \in L^2(\mathbb{R}^+;Z) \right\} \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.2}$$

It is called the kT-attainable subspace of Eq. (1.1) w.r.t. Z. Recall (1.12), (1.19) and (1.20). We simply write

$$H_1 \triangleq H_1(0),\; H_2 \triangleq H_2(0),\; \mathbb{P} \triangleq \mathbb{P}(0) \;\text{ and }\; \mathscr{P} \triangleq \mathscr{P}(0). \tag{2.3}$$

Let

$$\hat{\mathscr{A}}_k^Z = \mathbb{P}\mathscr{A}_k^Z, \qquad k \in \mathbb{N}. \tag{2.4}$$

These subspaces play important roles in our studies of LPFS.

Lemma 2.1 *Let $[A(\cdot), B(\cdot)]$ be a T-periodic pair satisfying the conditions $(\mathscr{H}_1)$-$(\mathscr{H}_2)$. Assume that Q and R satisfy (1.88) and $Q \gg 0$. Let $h \in H$. Then Problem $(LQ)_{0,h}^{\infty}$ (defined by (1.54) with $\hat{T} = T$, $M = 0$ and $U = Z$) satisfies the FCC at h if*

$$\mathbb{P}\big(\mathscr{P}^k h\big) \in \hat{\mathscr{A}}_k^Z \;\text{ for some }\; k \in \{0,1,2,\dots\}. \tag{2.5}$$

Proof Suppose that $h \in H$ satisfies (2.5) for some $k \in \mathbb{N}$. Then there is $u \in L^2(0,\infty;Z)$ so that

$$\mathbb{P}\big(\mathscr{P}^k h\big) = \mathbb{P}\int_0^{kT} \Phi(kT,s)B(s)u(s)\mathrm{d}s,$$

from which, it follows that $\mathbb{P}y(kT;0,h,-u) = 0$. Let $\hat{u} = -\chi_{[0,kT]}u$. Then $\hat{u}$ and $y(kT;0,h,\hat{u})$ are in $L^2(t,\infty;Z)$ and H_2 respectively. These, together with (f) of Proposition 1.4, yield that

$$\begin{aligned} &\|y(s;0,h,\hat{u})\| = \|y(s;kT,y(kT;0,h,\hat{u}),\hat{u})\| \\ &= \|y(s;kT,y(kT;0,h,\hat{u}),0)\| = \|\Phi(s,kT)y(kT;0,h,\hat{u})\| \\ &\le C_{\bar{\rho}}e^{-\rho(s-kT)}\|y(kT;0,h,\hat{u})\| \;\text{ for all }\; s \ge kT. \end{aligned}$$

From this, one can easily verify that $J_{0,h}^{\infty}(\hat{u}) < \infty$. So Problem $(LQ)_{0,h}^{\infty}$ satisfies the FCC at h.

We next suppose that $h \in H$ satisfies (2.5) with $k = 0$. Since h satisfies (2.5), with $k = 0$, if and only if $h \in H_2$, we find that $J_{0,h}^{\infty}(0) < \infty$. So Problem $(LQ)_{0,h}^{\infty}$ satisfies the FCC at h. This ends the proof. □

From Theorem 1.4 and Lemma 2.1, we find that properties of subspaces $\big\{\mathscr{A}_k^Z, k \in \mathbb{N}\big\}$ play important roles in the studies of LPFS. Meanwhile, since h satisfies (2.5), with $k = 0$, if and only if $h \in H_2$, we find from Lemma 2.1 that when $h \in H_2$, Problem $(LQ)_{0,h}^{\infty}$ satisfies the FCC at h. Hence, the studies of the case when $h \in H_1$ is very important. We will see that it isindeed the key in the studies of LPFS.

Since H_1 is invariant with respect to $\mathscr{P}$ (see (*b*) of Proposition 1.4), we can define $\mathscr{P}_1 : H_1 \to H_1$ by setting

$$\mathscr{P}_1 \triangleq \mathscr{P}\big|_{H_1}. \tag{2.6}$$

Then, by (1.22), it follows that

$$\sigma(\mathscr{P}_1) \bigcap \mathbb{B} = \varnothing. \tag{2.7}$$

Lemma 2.2 *Let $\mathscr{P}_1$ and n_0 be given by (2.6) and (1.17), respectively. Suppose that $Z \subseteq U$ is a subspace with $\mathscr{A}_k^Z$ and $\hat{\mathscr{A}}_k^Z$ given by (2.2) and (2.4), respectively. Then for each $k \in \mathbb{N}$,*

$$\mathscr{A}_k^Z = \mathscr{A}_1^Z + \mathscr{P}\mathscr{A}_1^Z + \cdots + \mathscr{P}^{k-1}\mathscr{A}_1^Z; \ \hat{\mathscr{A}}_k^Z = \hat{\mathscr{A}}_1^Z + \mathscr{P}_1\hat{\mathscr{A}}_1^Z + \cdots + \mathscr{P}_1^{k-1}\hat{\mathscr{A}}_1^Z. \tag{2.8}$$

Furthermore, $\mathscr{P}_1$ is invertible and

$$\hat{\mathscr{A}}^Z = \hat{\mathscr{A}}_{n_0}^Z; \quad \mathscr{P}_1\hat{\mathscr{A}}^Z = \hat{\mathscr{A}}^Z = \mathscr{P}_1^{-1}\hat{\mathscr{A}}^Z, \tag{2.9}$$

where

$$\hat{\mathscr{A}}^Z \triangleq \bigcup_{k=1}^{\infty} \hat{\mathscr{A}}_k^Z. \tag{2.10}$$

Proof We begin with proving the first equality in (2.8) by the mathematical induction. Clearly, it stands when $k = 1$. Assume that it holds in the case when $k = k_0$ for some $k_0 \geq 1$, i.e.,

$$\mathscr{A}_{k_0}^Z = \mathscr{A}_1^Z + \mathscr{P}\mathscr{A}_1^Z + \cdots + \mathscr{P}^{k_0-1}\mathscr{A}_1^Z. \tag{2.11}$$

Because of (1.23) and (2.3), we have that $\Phi((k_0+1)T, T) = \Phi(T, 0)^{k_0} = \mathscr{P}^{k_0}$. This, along with (2.2), the T-periodicity of $B(\cdot)$ and (2.11), indicates that

$$\begin{aligned}
\mathscr{A}_{k_0+1}^Z &= \left\{ \int_0^{(k_0+1)T} \Phi((k_0+1)T, s)B(s)u(s)\mathrm{d}s \;\Big|\; u(\cdot) \in L^2(\mathbb{R}^+; Z) \right\} \\
&= \mathscr{P}^{k_0}\mathscr{A}_1^Z + \left\{ \int_0^{k_0 T} \Phi(k_0 T, s)B(s)u(s+T)\mathrm{d}s \;\Big|\; u(\cdot) \in L^2(\mathbb{R}^+; Z) \right\} \\
&= \mathscr{P}^{k_0}\mathscr{A}_1^Z + \mathscr{A}_{k_0}^Z = \mathscr{A}_1^Z + \mathscr{P}\mathscr{A}_1^Z + \cdots + \mathscr{P}^{k_0}\mathscr{A}_1^Z.
\end{aligned}$$

which leads to the first equality in (2.8).

We next show the second equality in (2.8). By (2.3) and (1.23), we have that $\mathscr{P}\mathbb{P} = \mathbb{P}\mathscr{P}$. Since $\mathbb{P}$ is a projection from H onto H_1 (see Proposition 1.4), the above, along with the first equality in (2.8) and (2.4), leads to the second equality in (2.8).

Then we show the first equality in (2.9). It follows from (2.10) and (2.8) that

$$\hat{\mathscr{A}}_{n_0}^Z \subseteq \hat{\mathscr{A}}^Z \ \text{ and } \ \hat{\mathscr{A}}_k^Z \subseteq \hat{\mathscr{A}}_{n_0}^Z, \ \text{ when } \ k \leq n_0. \tag{2.12}$$

Since $\dim H_1 = n_0$ (see (1.22)) and $\mathscr{P}_1 : H_1 \to H_1$ (see (2.6)), according to the Hamilton-Cayley theorem, each $\mathscr{P}_1^j$ with $j \geq n_0$ is a linear combination of $\{ I,\ \mathscr{P}_1^1,\ \mathscr{P}_1^2, \cdots ,\ \mathscr{P}_1^{(n_0-1)}\}$. This, along with the second equality in (2.8), indicates that

$$\hat{\mathscr{A}}_k^Z = \sum_{j=0}^{k-1} \mathscr{P}_1^j(\hat{\mathscr{A}}_1^Z) \subseteq \sum_{j=0}^{n_0-1} \mathscr{P}_1^j(\hat{\mathscr{A}}_1^Z) = \hat{\mathscr{A}}_{n_0}^Z, \quad \text{when } \ k \geq n_0. \tag{2.13}$$

Now the first equality in (2.9) follows from (2.12) and (2.13).

Finally, we show the non-singularity of $\mathscr{P}_1$ and the second equality in (2.9). By the first equality in (2.9) and the Hamilton-Cayley theorem, we see that

$$\mathscr{P}_1\hat{\mathscr{A}}^Z = \mathscr{P}_1\hat{\mathscr{A}}_{n_0}^Z = \mathscr{P}_1 \sum_{j=0}^{n_0-1} \mathscr{P}_1^j(\hat{\mathscr{A}}_1^Z) = \sum_{j=1}^{n_0} \mathscr{P}_1^j(\hat{\mathscr{A}}_1^Z) \subseteq \sum_{j=0}^{n_0-1} \mathscr{P}_1^j(\hat{\mathscr{A}}_1^Z) = \hat{\mathscr{A}}_{n_0}^Z,$$

from which, it follows that

$$\mathscr{P}_1\hat{\mathscr{A}}^Z \subseteq \hat{\mathscr{A}}^Z. \tag{2.14}$$

From (1.22) and (1.22), we find that $0 \notin \sigma(\mathscr{P}_1^C)$ and H_1^C (the domain of $\mathscr{P}_1^C$) is of finite dimension. Thus $\mathscr{P}_1^C$ is invertible. So is $\mathscr{P}_1$. This implies that $\dim(\mathscr{P}_1\hat{\mathscr{A}}^Z) = \dim\hat{\mathscr{A}}^Z$, which, together with (2.14), yields that $\mathscr{P}_1\hat{\mathscr{A}}^Z = \hat{\mathscr{A}}^Z$. This completes the proof. □

2.2 Three Criterions on Periodic Feedback Stabilization

The aim of this section is to prove the following theorem.

Theorem 2.1 *Let $\mathbb{P}$, $\mathscr{P}$ and H_j with $j = 1, 2$ be given by (2.3). Let n_0 be given by (1.17). Then, for each subspace $Z \subseteq U$, the following assertions are equivalent:*

(a) Equation (1.1) is LPFS with respect to Z, i.e., $Z \in \mathscr{U}^{FS}$.
(b) $\hat{\mathscr{A}}_{n_0}^Z = H_1$, where $\hat{\mathscr{A}}_{n_0}^Z$ is given by (2.4).
(c) If $\xi \in \mathbb{P}^* H_1$ and $\left(B(\cdot)\big|_Z\right)^* \Phi(n_0T, \cdot)^*\xi = 0$ over $(0, n_0T)$, then $\xi = 0$.
(d) If $\xi \in H^C$ satisfies that $\left(\mu I - \mathscr{P}^{*C}\right)\xi = 0$, with $\mu \notin \mathbb{B}$, and $\left(B(\cdot)\big|_Z\right)^{*C} \Phi(T, \cdot)^{*C}$ $\xi = 0$ over $(0, T)$, then $\xi = 0$.

2.2.1 Multi-periodic Feedback Stabilization

In this subsection, we will introduce three propositions. The first two propositions will be used in the proof of Theorem 2.1. The last one is independently interesting.

Defnition 2.2 (i) Equation (1.1) is said to be linear multi-periodic feedback stabilizable (LMPFS, for short) if there is a kT-periodic $K(\cdot) \in L^\infty\left(\mathbb{R}^+; \mathscr{L}(H, U)\right)$ for some $k \in \mathbb{N}$ so that Eq. (1.104) is exponentially stable. Any such a $K(\cdot)$ is called an LMPFS law for Eq. (1.1).
(ii) Equation (1.1) is said to be LMPFS with respect to a subspace Z of U if there is a kT-periodic $K(\cdot) \in L^\infty\left(\mathbb{R}^+; \mathscr{L}(H, Z)\right)$ for some $k \in \mathbb{N}$ so that Eq. (1.104) is exponentially stable. Any such a $K(\cdot)$ is called an LMPFS law for Eq. (1.1) with respect to Z.

Proposition 2.1 *Let n_0 and H_1 be given by (1.17) and (2.3) respectively. Suppose that $Z \subseteq U$ satisfies (b) of Theorem 2.1. Then, Eq. (1.1) is LMPFS with respect to Z.*

Proof Let $Z \subseteq U$ satisfy (b) of Theorem 2.1. We organize the proof by several steps as follows.

Step 1. To construct, for each $h_1 \in H_1$, $u^{h_1}(\cdot) \in L^2(\mathbb{R}^+; Z)$ so that $\mathbb{P}y(n_0T; 0, h_1, u^{h_1}) = 0$.

Because $\dim H_1 = n_0$ (see (1.22)), we can set $\{\eta_1, \cdots, \eta_{\hat{n}}\}$ to be an orthonormal basis of H_1. By (1.21), we see that $\mathbb{P}\mathscr{P}^{n_0}\eta_j \in H_1$ for each $j \in \{1, 2, \ldots, n_0\}$. Then it follows from (b) of Theorem 2.1 that $\mathbb{P}\mathscr{P}^{n_0}\eta_j \in \hat{\mathscr{A}}^Z_{n_0}$. Thus for each j, there is $u_j \in L^2(\mathbb{R}^+; Z)$ so that

$$\mathbb{P}\mathscr{P}^{n_0}\eta_j = \mathbb{P}\int_0^{n_0T} \Phi(n_0T, s)B(s)\hat{u}_j(s)\mathrm{d}s. \tag{2.15}$$

For any $h_1 \in H_1$, let

$$u^{h_1} = -\chi_{(0,n_0T)}\sum_{j=1}^{n_0}\langle h_1, \eta_j\rangle u_j. \tag{2.16}$$

Then we define an operator $\mathscr{L} : H_1 \to L^2(\mathbb{R}^+; Z)$ by setting

$$\mathscr{L}h_1(\cdot) = u^{h_1}(\cdot) \ \text{ for all } \ h_1 \in H_1. \tag{2.17}$$

Clearly, $\mathscr{L}$ is linear and bounded. Since $h_1 = \sum_{j=1}^{n_0}\langle h_1, \eta_j\rangle\eta_j$, it follows from (2.16) and (2.15) that

$$\begin{aligned}
&\mathbb{P}y(n_0T; 0, h_1, \mathscr{L}h_1) \\
&= \sum_{j=1}^{n_0}\langle h_1, \eta_j\rangle\mathbb{P}\mathscr{P}^{n_0}\eta_j - \sum_{j=1}^{n_0}\langle h_1, \eta_j\rangle\mathbb{P}\int_0^{n_0T}\Phi(n_0T, s)B(s)u_j(s)\mathrm{d}s \qquad (2.18)\\
&= \sum_{j=1}^{n_0}\langle h_1, \eta_j\rangle\left(\mathbb{P}\mathscr{P}^{n_0}\eta_j - \mathbb{P}\int_0^{n_0T}\Phi(n_0T, s)B(s)u_j(s)\mathrm{d}s\right) = 0.
\end{aligned}$$

Step 2. To show the existence of an $N_0 \in \mathbb{N}$ so that

$$\|y(NT;0,h,\mathscr{L}(\mathbb{P}h))\| \leq \delta_0\|h\| \quad \textit{for all} \quad h \in H \quad \textit{and} \quad N \geq N_0, \tag{2.19}$$

where

$$\delta_0 \triangleq (1+\hat{\delta})/2 < 1, \quad \textit{with} \ \ \hat{\delta} \ \ \textit{given by} \ \ (1.16) \tag{2.20}$$

Let $\rho_0 = -\ln \delta_0/T$. Then $0 < \rho_0 < -\ln \hat{\delta}/T \triangleq \hat{\rho}$. By (f) of Proposition 1.4, there is a constant $C_{\rho_0} > 0$ so that

$$\|y(kT;0,h_2,0)\| = \|\Phi(kT,0)h_2\| \leq C_{\rho_0}e^{-\rho_0 kT}\|h_2\| = C_{\rho_0}\delta_0^k\|h_2\|, \tag{2.21}$$

when $k \in \mathbb{N}$ and $h_2 \in H_2$. We claim that there is a constant $C > 0$ so that

$$\|y(NT;0,h_1,\mathscr{L}h_1)\| \leq CC_{\rho_0}\delta_0^{N-n_0}\|h_1\|, \quad \text{when} \ \ h_1 \in H_1, N \geq n_0, \tag{2.22}$$

where $\mathscr{L}$ is given by (2.17). In fact, because for each $h_1 \in H_1$,

$$\left\|y(n_0T;0,h_1,\mathscr{L}h_1)\right\| \leq \left\|\Phi(n_0T,0)h_1\right\| + \left\|\int_0^{n_0T}\Phi(n_0T,s)B(s)\mathscr{L}h_1(s)\mathrm{d}s\right\|,$$

we see that there is a constant $C > 0$ so that

$$\|y(n_0T;0,h_1,\mathscr{L}h_1)\| \leq C\|h_1\| \ \ \text{for all} \ \ h_1 \in H_1. \tag{2.23}$$

Here, we used facts that $B(\cdot) \in L^\infty(\mathbb{R}^+;\mathscr{L}(U;H))$ and $\mathscr{L}$ is linear and bounded. Meanwhile, it follows from (2.16) that $\mathscr{L}h_1(\cdot) = 0$ over $(n_0T,+\infty)$. This, together with (1.8) (where $u(\cdot) \equiv 0$), yields that for all $N \geq n_0$ and $h_1 \in H$,

$$\begin{aligned} &y(NT;0,h_1,\mathscr{L}h_1) = y(NT;n_0T,y(n_0T;0,h_1,\mathscr{L}h_1),\mathscr{L}h_1) \\ &= y(NT;n_0T,y(n_0T;0,h_1,\mathscr{L}h_1),0) = \Phi(NT,n_0T)y(n_0T;0,h_1,\mathscr{L}h_1) \\ &= \Phi((N-n_0)T,0)y(n_0T;0,h_1,\mathscr{L}h_1) = y((N-n_0)T;0,y(n_0T;0,h_1,\mathscr{L}h_1),0). \end{aligned} \tag{2.24}$$

Because of (2.18) and (1.20)–(1.21) with $t = 0$, we see that $y(n_0T;0,h_1,\mathscr{L}h_1) \in H_2$, when $h_1 \in H_1$. This along with (2.24), (2.21) and (2.23), leads to (2.22).

Let

$$N_0 = \max\left\{\frac{\ln C_{\rho_0} + \ln\left(C\delta_0^{-n_0}\|\mathbb{P}\| + \|I-\mathbb{P}\|\right)}{\ln(1/\delta_0)} + 2, \ n_0\right\}. \tag{2.25}$$

(Here, $[r]$ with $r \in \mathbb{R}$ denotes the integer so that $r - 1 < [r] \leq r$.) Then, it follows from (2.22), (2.21) and (2.25) that for all $N \geq N_0$, $h \in H$,

$$\|y(NT;0,h,\mathscr{L}(\mathbb{P}h))\| \leq C_{\rho_0}\delta_0^N(C\delta_0^{-n_0}\|\mathbb{P}\| + \|I-\mathbb{P}\|)\|h\| \leq \delta_0\|h\|.$$

Step 3. To study the value function associated with a family of LQ problems

Given $N \in \mathbb{N}$, $t \in [0, NT)$ and $h \in H$, write $y_N^Z(\cdot; t, h, u) \in C([t, NT]; H)$ for the solution to the equation:

$$\begin{cases} y'(s) = Ay(s) + D(s)y(s) + B(s)\big|_Z u(s) \ \text{ in } \ (t, NT), \\ y(t) = h, \end{cases} \tag{2.26}$$

where $B(s)\big|_Z$ is the restriction of $B(s)$ on the subspace Z. For each $\varepsilon > 0$, define the cost functional:

$$J_{t,h}^{NT,\varepsilon,Z}(u) = \int_t^{NT} \varepsilon \|u(s)\|_U^2 \mathrm{d}s + \|y_N^Z(NT; t, h, u)\|^2, \quad u \in L^2(t, NT; Z). \tag{2.27}$$

Then consider the following LQ problem

$$(LQ)_{t,h}^{NT,\varepsilon,Z}: \qquad W^{NT,\varepsilon,Z}(t, h) \triangleq \inf_{u \in L^2(t,NT;Z)} J_{t,h}^{NT,\varepsilon,Z}(u).$$

Let

$$\varepsilon_0 \overset{\Delta}{=} (\delta_0 - \delta_0^2)/(\|\mathscr{L}\|\|P\| + 1)^2, \quad \text{with} \ \ \delta_0 \ \ \text{and} \ \ \mathscr{L} \ \ \text{given by} \ \ (2.20) \ \ \text{and} \ \ (2.17). \tag{2.28}$$

We claim that when N_0 is given by (2.25),

$$W^{NT,\varepsilon,Z}(0, h) \le \delta_0 \|h\|^2 \ \text{ for all } \ h \in H, \quad \text{when} \ \ N \ge N_0 \ \ \text{and} \ \ \varepsilon \in (0, \varepsilon_0]. \tag{2.29}$$

In fact, it follows from (2.28) that for each $h \in H$ and $\varepsilon \in (0, \varepsilon_0]$,

$$\varepsilon \|\mathscr{L}(\mathbb{P}h)(\cdot)\|_{L^2(\mathbb{R}^+;Z)}^2 \le \varepsilon_0 \|\mathscr{L}\|^2 \|\mathbb{P}\|^2 \|h\|^2 \le (\delta_0 - \delta_0^2)\|h\|^2. \tag{2.30}$$

Since $\mathbb{P}$ is a projection from H to H_1 (see Proposition 1.4), it follows from (2.17) that

$$\mathscr{L}(\mathbb{P}h) \in L^2(\mathbb{R}^+; Z) \ \text{ for all } \ h \in H. \tag{2.31}$$

Since

$$y_N^Z\left(\cdot; t, h, u\big|_{[t,NT)}\right) = y(\cdot; t, h, u)\big|_{[t,NT]} \ \text{ for any } \ u \in L^2(t, +\infty; Z),$$

we see that

$$y_N^Z\left(NT; 0, h, \mathscr{L}(\mathbb{P}h)\big|_{(0,NT)}\right) = y(NT; 0, h, \mathscr{L}(\mathbb{P}h)).$$

This, together with (2.27), (2.31), (2.30) and (2.19), indicates that

$$W^{NT,\varepsilon,Z}(0,h) \le \varepsilon \int_0^{NT} \|\mathscr{L}(Ph)(s)\|^2 \mathrm{d}s + \|y_N^Z(NT;0,h,\mathscr{L}(\mathbb{P}h))\|^2 \le \delta_0\|h\|^2,$$

when $N \ge N_0$, $\varepsilon \in (0, \varepsilon_0]$, $h \in H$, i.e., (2.29) stands.

Step 4. To construct an NT-periodic $K_{\varepsilon,N}^Z(\cdot) \in L^\infty(\mathbb{R}^+; \mathscr{L}(H,Z))$.

Arbitrarily fix an $\varepsilon \in (0, \varepsilon_0]$ and an $N \ge N_0$, where N_0 and ε_0 are given by (2.25) and (2.28) respectively. By Theorem 1.2, we can derive that

$$W^{NT,\varepsilon,Z}(t,h) = \langle Q^{NT,\varepsilon,Z}(t)h, h\rangle \quad \text{for all} \quad h \in H,$$

where $Q^{NT,\varepsilon,Z}(\cdot)$ is the solution to the Riccati equation:

$$\begin{cases} \dot{\Upsilon}(t) + A(t)^*\Upsilon(t) + \Upsilon(t)A(t) - \dfrac{1}{\varepsilon}\Upsilon(t)B(t)\big|_Z\big(B(t)\big|_Z\big)^*\Upsilon(t) = 0, & t \in (0, NT), \\ \Upsilon(NT) = I. \end{cases} \tag{2.32}$$

Meanwhile, it follows form (2.27) that for each $h \in H$,

$$0 \le \big\langle h, Q^{NT,\varepsilon,Z}(t)h\big\rangle \le J_{t,h}^{NT,\varepsilon,Z}(0) \le \|\Phi(NT,t)\|^2\|h\|^2.$$

Define $K_{\varepsilon,N}^Z(\cdot) : [0, NT) \to \mathscr{L}(H; Z)$ by

$$K_{\varepsilon,N}^Z(t) = -\frac{1}{\varepsilon}\left(B(s)\big|_Z\right)^* Q^{NT,\varepsilon,Z}(t) \ \text{ for a.e. } \ t \in [0, NT). \tag{2.33}$$

One can easily check that $K_{\varepsilon,N}^Z(\cdot) \in L^\infty(0, NT; \mathscr{L}(H; Z))$. From this and $(\mathscr{H}_1)$-$(\mathscr{H}_2)$, the following feedback equation has a unique mild solution $y_{\varepsilon,N}^Z(\cdot; 0, h) \in C([0, NT]; H)$:

$$\begin{cases} y'(s) = Ay(s) + D(s)y(s) + B(s)\big|_Z K_{\varepsilon,N}^Z(s)y(s) \ \text{ in } \ (0, NT), \\ y(0) = h \in H. \end{cases} \tag{2.34}$$

Define the following function:

$$u_{\varepsilon,N,0,h}^Z(s) \triangleq K_{\varepsilon,N}^Z(s)y_{\varepsilon,N}^Z(s; 0, h) \qquad \text{for a.e. } \ s \in (0, NT). \tag{2.35}$$

By Theorem 1.2, we see that $u_{\varepsilon,N,0,h}^Z(\cdot)$ is the optimal control to $(LQ)_{0,h}^{NT,\varepsilon,Z}$. This yields that

$$W^{NT,\varepsilon,Z}(0,h) = J_{0,h}^{NT,\varepsilon,Z}\big(u_{\varepsilon,N,0,h}^Z\big), \quad \text{when} \quad h \in H. \tag{2.36}$$

By (2.26) with $t = 0$, (2.34) and (2.35), we see that

$$y_{\varepsilon,N}^Z(NT; 0, h) = y_N^Z(NT; 0, h, u_{\varepsilon,N,0,h}^Z).$$

From this, (2.27), (2.36) and (2.29), it follows that

$$\|y^Z_{\varepsilon,N}(NT;0,h)\|^2 \le J^{NT,\varepsilon,Z}_{0,h}(u^Z_{\varepsilon,N,0,h}) = W^{NT,\varepsilon,Z}(0,h) \le \delta_0\|h\|^2, \quad \text{when } \ h \in H. \tag{2.37}$$

Now, we extend NT-periodically $K^Z_{\varepsilon,N}(\cdot)$ over $\mathbb{R}^+$ by setting

$$K^Z_{\varepsilon,N}(t+kNT) = K^Z_{\varepsilon,N}(t) \ \text{ for all } \ t \in [0,NT), \ k \in \mathbb{N}. \tag{2.38}$$

Step 5. To prove that when $\varepsilon \in (0,\varepsilon_0]$ and $N \ge N_0$, $K^Z_{\varepsilon,N}(\cdot)$ *is an LMPFS law for Eq.* (1.1) *with respect to* Z.

Consider the feedback equation:

$$\begin{cases} y'(s) = Ay(s) + D(s)y(s) + B(s)\big|_Z K^Z_{\varepsilon,N}(s)y(s) & \text{in } \ \mathbb{R}^+, \\ y(0) = h \in H. \end{cases} \tag{2.39}$$

Since $K^Z_{\varepsilon,N}(\cdot) \in L^\infty(\mathbb{R}^+;\mathscr{L}(H;Z))$, we have that $D(\cdot) + B(\cdot)\big|_Z K^Z_{\varepsilon,N}(\cdot)$ belongs to $L^1_{loc}(\mathbb{R}^+;\mathscr{L}(H))$ (see $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$). Thus, for each $h \in H$, Eq. (2.39) has a unique mild solution $y^Z_\varepsilon(\cdot;0,h)$. Clearly, we have that $y^Z_\varepsilon(t;0,h) = y^Z_{\varepsilon,N}(t;0,h)$ for each $t \in [0,NT]$. Write $\Phi^Z_{\varepsilon,N}$ for the evolution generated by $A(\cdot) + B(\cdot)\big|_Z K^Z_{\varepsilon,N}(\cdot)$. By Proposition 1.2, we find that

$$y^Z_{\varepsilon,N}(NT;0,h) = \Phi^Z_{\varepsilon,N}(NT,0)h \quad \text{for all } \ h \in H.$$

This, along with (2.37) and (2.20), yields that

$$\|\Phi^Z_{\varepsilon,N}(NT,0)\| \le \sqrt{\delta_0} < 1. \tag{2.40}$$

Since $D(\cdot)$ and $B(\cdot)$ are T-periodic and $K^Z_{\varepsilon,N}(\cdot)$ is NT-periodic, it follows that

$$\Phi^Z_{\varepsilon,N}(s+NT,t+NT) = \Phi^Z_{\varepsilon,N}(s,t), \quad \text{when } \ 0 \le t \le s < +\infty. \tag{2.41}$$

By (2.41) and (2.40), we see that Eq. (2.39) is exponentially stable. Hence, $K^Z_{\varepsilon,N}(\cdot)$, with $\varepsilon \in (0,\varepsilon_0]$ and $N \ge N_0$, is an LMPFS law for Eq. (2.39). This ends the proof. □

Proposition 2.2 *Let Z be a subspace of U. Then, Eq. (1.1) is LPFS with respect to Z if and only if it is LMPFS with respect to Z.*

Proof Clearly, Eq. (1.1) is LPFS w.r.t. Z ⇒ Eq. (1.1) is LMPFS w.r.t. Z. To show the reverse, we suppose that Eq. (1.1) is LMPFS with respect to Z. Then there is an NT-periodic $\hat{K}^Z_N(\cdot) \in L^\infty(\mathbb{R}^+;\mathscr{L}(H;Z))$, with $N \in \mathbb{N}$, so that the following feedback equation is exponentially stable:

$$y'(s) = Ay(s) + D(s)y(s) + B(s)\big|_Z \hat{K}^Z_N(s)y(s), \quad s \ge 0. \tag{2.42}$$

Because $\hat{K}_N^Z(\cdot) \in L^\infty(\mathbb{R}^+; \mathscr{L}(H; Z))$ is NT-periodic, from the above, as well as assumptions $(\mathscr{H}_1)$-$(\mathscr{H}_2)$, one can verify that for each $t \geq 0$ and $h \in H$, the solution $\hat{y}_N^Z(\cdot; t, h)$ to the equation:

$$\begin{cases} y'(s) + Ay(s) + D(s)y(s) = B(s)\big|_Z \hat{K}_N^Z(s)y(s) \quad \text{in} \quad [t, \infty), \\ y(t) = h \end{cases} \tag{2.43}$$

satisfies that

$$\|\hat{y}_N^Z(s; t, h)\| \leq C_1 e^{-\delta_1(s-t)}\|h\|, \quad \text{when} \quad s \geq t \quad \text{and} \quad h \in H, \tag{2.44}$$

where C_1 and δ_1 are two positive constants independent of h, t and s. Given $h \in H$, let $u^h(\cdot) \triangleq \hat{K}_N^Z(\cdot)\hat{y}_N^Z(\cdot; 0, h)$. It follows from (2.44) that $u^h(\cdot) \in L^2(\mathbb{R}^+; Z)$. Moreover, it holds that $y(\cdot; 0, h, u^x) = \hat{y}_N^Z(\cdot; 0, h)$. Consider the LQ problem $(LQ)_{0,h}^\infty$, defined by (1.89) where $U = Z$, $Q = I_H$ and $R = I_U$. It follows from (2.44) that for each $h \in H$,

$$\begin{aligned} &\int_t^\infty \big[\|u^h(s)\|^2 + \|y(s; 0, h, u^h)\|^2\big]\mathrm{d}s \leq \int_t^\infty \big[(\|\hat{K}_N^Z(s)\|^2 + 1)\|y(s; 0, h, u^h)\|^2\big]\mathrm{d}s \\ &\leq \big(\|\hat{K}_N^Z\|^2 + 1\big)\int_t^\infty \|\hat{y}_N^Z(s; t, h)\|^2\mathrm{d}s \leq \frac{M_1^2}{2\delta_1}\big(\|\hat{K}_N^Z\|^2 + 1\big)\|h\|^2. \end{aligned}$$

Therefore, Problem $(LQ)_{0,h}^\infty$ satisfies the FCC for any $h \in H$. Then by Theorem 1.4, we see that Eq. (1.1) is LMPFS with respect to Z. This ends the proof. □

Proposition 2.3 *When both $D(\cdot)$ and $B(\cdot)$ are time invariant, i.e., $D(t) \equiv D$ and $B(t) \equiv B$ for all $t \geq 0$, the following statements are equivalent:*
(a) Equation (1.1) is linear $\hat{T}$-periodic feedback stabilizable for some $\hat{T} > 0$.
(b) Equation (1.1) is linear $\hat{T}$-periodic feedback stabilizable for any $\hat{T} > 0$.
(c) Equation (1.1) is linear time invariant feedback stabilizable.

Proof It suffices to show that $(a) \Rightarrow (c)$. Let $N \in \mathbb{N}$ with $N \geq 2$ and let $T = \hat{T}/N$. Since $D(\cdot)$ and $B(\cdot)$ are time invariant, Eq. (1.1) is T-periodic. Because of (a), there is an NT-periodic $\hat{K}_N^U(\cdot) \in L^\infty(\mathbb{R}^+; \mathscr{L}(H; U))$ so that the feedback Eq. (2.42), where $Z = U$, is exponentially stable. By Theorem 1.4, there is a pair (Q, R), with $Q \gg 0$ and $R \gg 0$, so that the Problem $(LQ)_{0,h}^\infty$ satisfies the FCC for each $h \in H$. Now, by the same way to show $(iii) \Rightarrow (i)$ in the proof of Theorem 1.4 (where $Z = U$), we see that $\bar{K}(\cdot)$, given by (1.109) with $Z = U$, is a LPFS law for Eq. (1.1). We claim that this $\bar{K}(\cdot)$ is time invariant in the case that $D(\cdot)$ and $B(\cdot)$ are time invariant. When this is done, $\bar{K}(\cdot) \equiv \bar{K} \in \mathscr{L}(H; U)$ is a feedback law for Eq. (1.1), which leads to (c).

The rest is to show that $\bar{K}(\cdot)$ is time invariant. Let $W^\infty(t, h)$ be the value function given by (1.89) with the aforementioned Q and R. By the time invariance of $B(\cdot)$, and by (1.108) and (1.109), it suffices to show $W^\infty(t, h)$ is time invariant. The later will be proved as follows. Since Eq. (1.1) is time invariant, it follows that when $t \in \mathbb{R}^+$, $h \in H$ and $u(\cdot) \in L^2(\mathbb{R}^+; U)$, we have that $y(s; t, h, u) = y(s - t; 0, h, v)$ for all

$s \geq t$, where $v(\cdot)$ is defined by $v(s) = u(s+t)$ for all $s \geq 0$. Hence, given $t \in \mathbb{R}^+$ and $h \in H$, we have that for each $u(\cdot) \in L^2(\mathbb{R}^+; U)$,

$$\begin{aligned} &\int_t^\infty \left(\|Q^{1/2}y(s;t,h,u(s))\|^2 + \|R^{1/2}u(s)\|_U^2\right) \mathrm{d}s \\ &= \int_0^\infty \left(\|Q^{1/2}y(r;0,h,u(r+t))\|^2 + \|R^{1/2}u(r+t)\|_U^2\right) \mathrm{d}r. \end{aligned}$$

Taking the infimum on the both sides of the above equation with respect to $u(\cdot) \in L^2(\mathbb{R}^+; U)$ leads to $W^\infty(t,h) = W^\infty(0,h)$. So the value function $W^\infty(t,h)$ is independent of t. This completes the proof. $\square$

Remark 2.1 By Proposition 2.3, we see that linear time-periodic functions $K(\cdot)$ will not aid in the linear stabilization of Eq. (1.1) when both $D(\cdot)$ and $B(\cdot)$ are time invariant. On the other hand, when Eq. (1.1) is T-periodically time varying, linear time-periodic functions $K(\cdot)$ do aid in the linear stabilization of Eq. (1.1). This can be seen from the following 2-periodic ODE: $y'(t) = \sum_{j=1}^\infty \left[\chi_{[2j,2j+1)}(t) - \chi_{[2j+1,2j+2)}(t)\right] u(t)$, $t \geq 0$. For each $k \in \mathbb{R}$, consider the equation: $y'(t) = \sum_{j=1}^\infty \left[\chi_{[2j,2j+1)}(t) - \chi_{[2j+1,2j+2)}(t)\right] ky(t)$, $t \geq 0$. Clearly, the corresponding periodic map $\mathscr{P}_k \equiv 1$. Thus any linear time invariant feedback equation is not exponentially stable. On the other hand, by a direct computation, one can easily check that the following 2-periodic map is an LPFS law:

$$k(t) = \sum_{j=1}^\infty \left[\chi_{[2j,2j+1)}(t) + 2\chi_{[2j+1,2j+2)}(t)\right], \ t \geq 0.$$

2.2.2 Proof of Theorem 2.1

We first show that $(a) \Leftrightarrow (b)$. To prove that $(b) \Rightarrow (a)$, suppose that $Z \subseteqq U$ satisfies (b) in Theorem 2.1. By Proposition 2.1, we see that Eq. (1.1) is LMPFS with respect to Z. This, along with Proposition 2.2, yields (a).

To verify that $(a) \Rightarrow (b)$, we suppose, by contradiction, that $Z \in \mathscr{U}^{FS}$, but (b) in Theorem 2.1 did not hold. Then $\hat{\mathscr{A}}_{n_0}^Z$ would be a proper subspace of H_1. This, along with (2.9), yields that $\hat{V}^Z$ is a proper subspace of H_1. One can directly check that

$$\left(H_1 \bigcap (\hat{\mathscr{A}}^Z)^\perp\right) \perp \hat{\mathscr{A}}^Z; \quad H_1 = \hat{\mathscr{A}}^Z \bigoplus \left(H_1 \bigcap (\hat{\mathscr{A}}^Z)^\perp\right). \tag{2.45}$$

Since $\hat{\mathscr{A}}^Z$ is a proper subspace of H_1 and $\dim H_1 = n_0$ (see (1.22)), we have that

$$n_0 \geq l \triangleq \dim \left(H_1 \bigcap (\hat{\mathscr{A}}^Z)^\perp\right) \geq 1. \tag{2.46}$$

By (2.45) and (2.46), we can let $\{\eta_1, \ldots, \eta_{n_0}\}$ be a basis of H_1 so that $\{\eta_1, \cdots, \eta_l\}$ and $\{\eta_{l+1}, \cdots, \eta_{n_0}\}$ are bases of $H_1 \bigcap (\hat{\mathscr{A}}^Z)^\perp$ and $\hat{\mathscr{A}}^Z$. By (2.9) in Lemma 2.2, $\hat{\mathscr{A}}^Z$ is an invariant subspace under $\mathscr{P}_1$. Thus there are matrices $A_1 \in \mathbb{R}^{l\times l}$, $A_2 \in \mathbb{R}^{(n_0-l)\times l}$, $A_3 \in \mathbb{R}^{(n_0-l)\times(n_0-l)}$ so that

$$\mathscr{P}_1\left(\eta_1, \cdots, \eta_{n_0}\right) = \left(\eta_1, \cdots, \eta_{n_0}\right)\begin{pmatrix} A_1 & 0_{l\times(n_0-l)} \\ A_2 & A_3 \end{pmatrix}. \tag{2.47}$$

Let P_{11} be the orthogonal projection from H_1 onto $H_1 \bigcap (\hat{\mathscr{A}}^Z)^\perp$. Define a linear bijection $\mathscr{J} : \mathbb{R}^l \to \left(H_1 \bigcap (\hat{\mathscr{A}}^Z)^\perp\right)$ by setting

$$\mathscr{J}(\alpha) \triangleq (\eta_1, \ldots, \eta_l)\alpha \quad \text{for each column vector } \alpha \in \mathbb{R}^l. \tag{2.48}$$

By (2.47) and (2.48), we see that for all $\alpha \in \mathbb{R}^l$ and $k \in \mathbb{N}$,

$$\begin{aligned} &P_{11}\mathscr{P}_1^k \mathscr{J}(\alpha) = P_{11}\mathscr{P}_1^k(\eta_1, \cdots, \eta_l)\alpha \\ &= P_{11}(\eta_1, \cdots, \eta_l, \mid \eta_{l+1}, \cdots, \eta_{n_0})\begin{pmatrix} A_1 & 0_{l\times(n_0-l)} \\ A_2 & A_3 \end{pmatrix}^k \begin{pmatrix} \alpha \\ 0_{(n_0-l)\times 1} \end{pmatrix} \\ &= (\eta_1, \cdots, \eta_l)A_1^k\alpha. \end{aligned} \tag{2.49}$$

On the other hand, since $Z \in \mathscr{U}^{FS}$, there is a T-periodic $K(\cdot) \in L^\infty(\mathbb{R}^+; \mathscr{L}(H; Z))$ so that Eq. (1.104) is exponentially stable, which implies that for all $h \in H$,

$$\lim_{t\to+\infty} y_K(t; 0, h) = 0, \tag{2.50}$$

where $y_K(\cdot; 0, h)$ denotes the solution of Eq. (1.104) with the initial condition that $y(0) = h$. Let $u_K^h(t) \triangleq K(t)y_K(t; 0, h)$ for a.e. $t \geq 0$. Then by (1.8), we have that

$$y_K(t; 0, h) = \Phi(t, 0)h + \int_0^t \Phi(t, s)B(s)u_K^h(s)\mathrm{d}s, \quad \text{when } t \in \mathbb{R}^+ \text{ and } h \in H. \tag{2.51}$$

From (2.51) and (2.2), it follows that

$$P_{11}Py_K(kT; 0, h) \in P_{11}\mathbb{P}(\mathscr{P}^k h + \mathscr{A}_k^Z) \quad \text{for all } h \in H \text{ and } k \in \mathbb{N}. \tag{2.52}$$

Since $\mathbb{P}\mathscr{A}_k^Z \triangleq \hat{\mathscr{A}}_k^Z \subseteq \hat{\mathscr{A}}^Z$ for all $k \in \mathbb{N}$ (see (2.4) and (2.10)) and P_{11} is the orthogonal projection from H_1 onto $H_1 \bigcap (\hat{\mathscr{A}}^Z)^\perp$, it follows from (2.45) that $P_{11}P\mathscr{A}_k^Z \subset P_{11}\hat{\mathscr{A}}^Z = \{0\}$. Because $\mathbb{P}\mathscr{P}^k = \mathscr{P}^k\mathbb{P}$ for all $k \in \mathbb{N}$ (see Parts (a) and (e) in Proposition 1.4), the above, along with (2.52), indicates that

$$P_{11}\mathbb{P}y_K(kT; 0, h) = P_{11}\mathbb{P}\mathscr{P}^k h = P_{11}\mathscr{P}^k\mathbb{P}h \quad \text{for all } h \in H \text{ and } k \in \mathbb{N}. \tag{2.53}$$

Since $\mathbb{P} : H \to H_1$ is a projection (see Proposition 1.4), from (2.50) and (2.53), we see that

$$\lim_{k\to+\infty} P_{11}\mathscr{P}^k h = 0, \quad \text{when} \;\; h \in H_1. \tag{2.54}$$

Now by (2.54), (2.49) and (2.6), we have that $\lim_{k\to\infty} A_1^k\alpha = 0$, when $\alpha \in \mathbb{R}^l$. Thus, we have that

$$\sigma(A_1) \in \mathbb{B} \;\; \text{(the open unit ball in } \;\mathbb{C}^1). \tag{2.55}$$

By (2.47), we find that $\sigma(A_1) \subset \sigma(\mathscr{P}_1)$. This, together with (2.55) and (2.7), leads to a contradiction. Hence, $(a) \Rightarrow (b)$. This completes the proof of $(a) \Leftrightarrow (b)$.

We next show that $(b) \Leftrightarrow (c)$. First of all, we mention that (b) means that Eq. (1.1) over $(0, n_0T)$ is null controllable under the projection $\mathbb{P}$ with respect to the initial data $h \in H_1$, while (c) says that the adjoint equation

$$\psi_t(t) + A^*\psi(t) + D(t)^*\psi(t) = 0 \;\; \text{for a.e. } t \in (0, n_0T), \quad \psi(n_0T) = \xi \tag{2.56}$$

with the initial data in $\mathbb{P}^* H_1$ has the unique continuation property. Such two properties are equivalent in finite dimensional spaces. The detailed proof is as follows. We introduce two complex adjoint equations as follows:

$$\psi'(t) + A^{*C}\psi(t) + D(t)^{*C}\psi(t) = 0 \;\; \text{in} \;\; (0, n_0T), \quad \psi(n_0T) \in H^C; \tag{2.57}$$

$$\psi'(t) + A^{*C}\psi(t) + D(t)^{*C}\psi(t) = 0 \;\; \text{in} \;\; (0, T), \quad \psi(T) \in H^C. \tag{2.58}$$

For each $\xi \in H^C$, Eq. (2.57) (or (2.58)) with the initial condition that $\psi^\xi_{n_0}(n_0T) = \xi$ (or $\psi^\xi(T) = \xi$) has a unique solution in $C([0, n_0T]; H^C)$ (or $C([0, T]; H^C)$). We denote this solution by $\psi^\xi_{n_0}(\cdot)$ (or $\psi^\xi(\cdot)$). Clearly, when $\xi \in H$, $\psi^\xi_{n_0}(\cdot) \in C([0, n_0T]; H)$ and $\psi^\xi(\cdot) \in C([0, T]; H)$ are accordingly the solutions of (2.57) and (2.58) where A^C and $D(t)^C$ are replaced by A and $D(t)$ respectively. One can easily check that

$$\psi^\xi(0) = \mathscr{P}^{*C}\xi \;\; \text{and} \;\; \psi^\xi_{n_0}(0) = \left(\mathscr{P}^{*C}\right)^{n_0}\xi \;\; \text{for all} \;\; \xi \in H^C. \tag{2.59}$$

By the T-periodicity of $D^*(\cdot)$, we see that for each $\xi \in H^C$,

$$\psi^\xi_{n_0}((k-1)T+t) = \psi^{\xi_k}(t), \;\; t \in [0, T], \; k \in \{1, \dots, n_0\}, \quad \text{where} \;\; \xi_k \triangleq \left(\mathscr{P}^{*C}\right)^{n_0-k}\xi. \tag{2.60}$$

Now we carry out the proof of $(b) \Leftrightarrow (c)$ by several steps as follows.

Step 1. To prove that (b) *is equivalent to the following property*:

$$\forall\, h \in H, \; \exists\, u^h(\cdot) \in L^2(\mathbb{R}^+; Z) \; s.t. \; \mathbb{P}y(n_0T; 0, h, u^h) = 0 \tag{2.61}$$

Suppose that (b) holds. Then by (2.2), we have that

$$\mathbb{P}\Big\{\int_0^{n_0T}\Phi(n_0T,t)B(t)u(t)\mathrm{d}t \;\Big|\; u(\cdot)\in L^2(\mathbb{R}^+;Z)\Big\}=H_1. \tag{2.62}$$

Given $h\in H$, it holds that $\mathbb{P}\Phi(n_0T,0)h\in H_1$ (see (1.21)). From this and (2.62), there is a $u^h(\cdot)\in L^2(\mathbb{R}^+;Z)$ so that

$$\mathbb{P}y(n_0T;0,h,u^h)=\mathbb{P}\Phi(n_0T,0)h+\mathbb{P}\int_0^{n_0T}\Phi(n_0T,t)B(t)u^h(t)\mathrm{d}t=0,$$

which leads to (2.61).

Conversely, assume that (2.61) holds. Then for any $h\in H$, there exists $u^h(\cdot)\in L^2(\mathbb{R}^+;Z)$ so that $\mathbb{P}y(n_0T;0,h,u^h)=0$. Thus, we find that

$$\begin{aligned}H_1\supseteq\hat{\mathscr{A}}^Z_{n_0}&\triangleq\mathbb{P}\Big\{\int_0^{n_0T}\Phi(n_0T,t)B(t)u(t)\mathrm{d}t \;\Big|\; u(\cdot)\in L^2(\mathbb{R}^+;Z)\Big\}\\ &\supseteq\mathbb{P}\Big\{\int_0^{n_0T}\Phi(n_0T,t)B(t)u^h(t)\mathrm{d}t \;\Big|\; h\in H\Big\}\\ &=-\mathbb{P}\Big\{\Phi(n_0T,0)h \;\Big|\; h\in H\Big\}=\mathbb{P}\mathscr{P}^{n_0}H.\end{aligned} \tag{2.63}$$

Since $\mathbb{P}\mathscr{P}=\mathscr{P}\mathbb{P}$ (see (1.23)), $\mathbb{P}H=H_1$ and $\mathscr{P}H_1=\mathscr{P}_1H_1=H_1$ (see (2.6) and Lemma 2.2), we see that $\mathbb{P}\mathscr{P}^{n_0}H=H_1$. This, together with (2.63), leads to (b).

Step 2. To show that $\xi\in\mathbb{P}^*H_1$ *and* $\psi^\xi_{n_0}(0)=0\Rightarrow\xi=0$

Recall Proposition 1.5. Because $\tilde{H}_1$ is invariant under $\mathscr{P}^*$, it follows from (2.59) that

$$\psi^\xi_{n_0}(0)=(\mathscr{P}^*)^{n_0}\xi=\big(\mathscr{P}^*\big|_{\tilde{H}_1}\big)^{n_0}\xi\in\tilde{H}_1,\ \text{ when }\xi\in\tilde{H}_1. \tag{2.64}$$

By Proposition 1.5, we find that $\sigma\big(\mathscr{P}^{*C}|_{\tilde{H}_1^C}\big)\bigcap\mathscr{B}=\varnothing$ and $\dim\tilde{H}_1=n_0<\infty$. Thus, the map $\big(\mathscr{P}^*\big|_{\tilde{H}_1}\big)^{n_0}$ is invertible from $\tilde{H}_1$ onto $\tilde{H}_1$. Then by (2.64), we see that $\xi=0$, when $\xi\in\tilde{H}_1$ and $\psi^\xi_{n_0}(0)=0$. This, together with (1.43), implies that $\xi=0$, when $\xi\in P^*H_1$ and $\psi^\xi_{n_0}(0)=0$.

Step 3. To show that (2.61) $\Rightarrow$ (c)

Clearly, when $\eta,\ h\in H$ and $u(\cdot)\in L^2(\mathbb{R}^+;Z)$, we have that

$$\big\langle\psi^\eta_{n_0}(0),\ h\big\rangle=\big\langle\eta,\ y(n_0T;0,h,u)\big\rangle-\int_0^{n_0T}\big\langle\big(B(t)\big|_Z\big)^*\psi^\eta_{n_0}(t),\ u(t)\big\rangle\mathrm{d}t. \tag{2.65}$$

Suppose that ξ satisfies conditions in (c). Then by (2.65) where $\eta=\xi$ and $\psi^\xi_{n_0}(t)=\Phi(n_0T,t)^*\xi$, we find that

$$\big\langle\psi^\xi_{n_0}(0),\ h\big\rangle=\big\langle\xi,\ y(n_0T;0,h,u)\big\rangle,\ \text{ when }h\in H\text{ and }u(\cdot)\in L^2(\mathbb{R}^+;Z). \tag{2.66}$$

By (2.61), given $h \in H$, there is a $u^h(\cdot) \in L^2(\mathbb{R}^+; Z)$ so that

$$\mathbb{P}y(n_0T; 0, h, u^h) = 0. \tag{2.67}$$

Since $\xi \in \mathbb{P}^* H_1$, there is $g \in H_1$ with $\xi = \mathbb{P}^* g$. This, along with (2.66) and (2.67), indicates that

$$\begin{aligned}\left\langle \psi_{n_0}^{\xi}(0),\ h\right\rangle &= \left\langle \xi,\ y(n_0T; 0, h, u^h)\right\rangle = \left\langle \mathbb{P}^* g,\ y(n_0T; 0, h, u^h)\right\rangle \\ &= \left\langle g,\ \mathbb{P}y(n_0T; 0, h, u^h)\right\rangle = 0 \ \text{ for all } \ h \in H.\end{aligned}$$

Hence, $\psi_{n_0}^{\xi}(0) = 0$. Then by the conclusion of Step 2, we have that $\xi = 0$. So (c) holds.

Step 4. To show that $(c) \Rightarrow$ (2.61)

Assume that (c) holds. Define two subspaces

$$\begin{aligned}\Gamma &\triangleq \left\{ \left(B(\cdot)\big|_Z\right)^* \psi_{n_0}^{\xi}(\cdot) \;\middle|\; \xi \in \mathbb{P}^* H_1 \right\} \subseteq L^2(0, n_0T; Z);\\ \Gamma_0 &\triangleq \left\{ \psi_{n_0}^{\xi}(0) \;\middle|\; \xi \in \mathbb{P}^* H_1 \right\} \subseteq H.\end{aligned} \tag{2.68}$$

By (c) and the conclusion of Step 2, we see that the following map $\mathscr{L}_1 : \Gamma \to \Gamma_0$ is well defined:

$$\mathscr{L}_1\left(\left(B(\cdot)\big|_Z\right)^* \psi_{n_0}^{\xi}(\cdot) \right) = \psi_{n_0}^{\xi}(0) \quad \text{for all} \quad \xi \in P^* H_1. \tag{2.69}$$

Clearly, it is linear. Given $h \in H$, define a linear functional $\mathscr{F}^h$ on Γ by

$$\mathscr{F}^h(\gamma) = \langle \mathscr{L}_1(\gamma), h\rangle \quad \text{for all} \quad \gamma \in \Gamma. \tag{2.70}$$

Since $\dim(\mathbb{P}^* H_1) = \dim \tilde{H}_1 = n_0 < \infty$, it holds that $\dim \Gamma < \infty$. Thus, $\mathscr{F}^h \in \mathscr{L}(\Gamma; \mathbb{R})$. By the Hahn-Banach theorem , there is a $\bar{\mathscr{F}}^h \in \mathscr{L}(L^2(0, n_0T; Z); \mathbb{R})$ so that

$$\bar{\mathscr{F}}^h(\gamma) = \mathscr{F}^h(\gamma) \ \text{ for all } \ \gamma \in \Gamma; \quad \text{and} \quad \|\bar{\mathscr{F}}^h\| = \|\mathscr{F}^h\|. \tag{2.71}$$

Then by the Riesz representation theorem (see p. 59 in [32]), there exists a function $u^h(\cdot)$ in $L^2(0, n_0T; Z)$ so that

$$\bar{\mathscr{F}}^h(\gamma) = -\int_0^{n_0T} \langle u^h(t), \gamma(t)\rangle_U \mathrm{d}t \ \text{ for all } \ \gamma \in L^2(0, n_0T; Z). \tag{2.72}$$

Since $\mathbb{P}^* H_1 = \mathbb{P}^* H$ (see (1.42)), it follows from (2.69) to (2.72) that

$$-\int_0^{n_0T} \left\langle \left(B(t)\big|_Z\right)^* \psi_{n_0}^{\mathbb{P}^*\hat{\eta}}(t),\ u^h(t)\right\rangle \mathrm{d}t = \left\langle \psi_{n_0}^{\mathbb{P}^*\hat{\eta}}(0),\ h\right\rangle \quad \text{for all} \quad \hat{\eta} \in H.$$

Meanwhile, it follows by (2.65) that for each $\hat{\eta} \in H$,

$$\big\langle \psi_{n_0}^{\mathbb{P}^*\hat{\eta}}(0),\ h\big\rangle = \big\langle \mathbb{P}^*\hat{\eta},\ y(n_0T;0,h,u^h)\big\rangle - \int_0^{n_0T} \big\langle \big(B(t)\big|_Z\big)^* \psi_{n_0}^{\mathbb{P}^*\hat{\eta}}(t),\ u^h(t)\big\rangle \mathrm{d}t.$$

The above two equalities imply that

$$\big\langle \hat{\eta},\ \mathbb{P}y(n_0T;0,h,u^h)\big\rangle = \big\langle \mathbb{P}^*\hat{\eta},\ y(n_0T;0,h,u^h)\big\rangle = 0 \quad \text{for all} \ \ \hat{\eta} \in H.$$

So $\mathbb{P}y(n_0T;0,h,u^h) = 0$, which leads to (2.61).

From Step 1–Step 4, we end the proof of $(b) \Leftrightarrow (c)$.

We then show $(c) \Leftrightarrow (d)$. To show that $(c) \Rightarrow (d)$, we suppose that Z satisfies (c). Let μ and ξ satisfy the conditions in (d) with the aforementioned Z. Then by (1.44), we find that $\xi \in \tilde{H}_1^C$. Hence, we can write $\xi \triangleq \xi_1 + i\xi_2$ with $\xi_1,\ \xi_2 \in \tilde{H}_1$. By (1.43), we have $\xi_1, \xi_2 \in \mathbb{P}^*H_1$. By the second condition in (d), we see that

$$\big(B(t)\big|_Z\big)^{*C}\ \psi^{\xi}(t) = 0 \ \ \text{for a.e.} \ \ t \in (0,T).$$

Then by (2.60) and the first condition in (d), we find that for all $t \in [0,T]$ and $k = 1,\ldots,n_0$,

$$\psi_{n_0}^{\xi}((k-1)T + t) = \psi^{\mu^{n_0-k}\xi}(t) = \mu^{n_0-k}\psi^{\xi}(t).$$

Since $\psi_{n_0}^{\xi}(\cdot) = \psi_{n_0}^{\xi_1}(\cdot) + i\psi_{n_0}^{\xi_2}(\cdot)$, the above two equations yield that

$$\big(B(\cdot)\big|_Z\big)^*\ \psi_{n_0}^{\xi_1}(\cdot) + i\ \big(B(\cdot)\big|_Z\big)^*\ \psi_{n_0}^{\xi_2}(\cdot) = \big(B(\cdot)\big|_Z\big)^*\ \psi_{n_0}^{\xi}(\cdot) = 0 \ \ \text{over} \ \ (0,n_0T).$$

Since $\xi_1, \xi_2 \in \mathbb{P}^*H_1$, the above-mentioned equation, along with (c), leads to $\xi_1 = \xi_2 = 0$, i.e., $\xi = 0$. Hence, Z satisfies (d). Thus, we have proved that $(c) \Rightarrow (d)$.

To show that $(d) \Rightarrow (c)$, we suppose that $Z \subseteq U$ satisfies (d). In order to show that Z satisfies (c), it suffices to prove that

$$\hat{\xi} \in (\mathbb{P}^*H_1)^C \ \ \text{and} \ \ \big(B(\cdot)\big|_Z\big)^{*C}\ \psi_{n_0}^{\hat{\xi}}(\cdot) = 0 \ \ \text{over} \ \ (0,n_0T) \Rightarrow \hat{\xi} = 0. \tag{2.73}$$

Notice that $(\mathbb{P}^*H_1)^C = \tilde{H}_1^C$ and $\dim \tilde{H}_1^C = n_0$ (see Proposition 1.5). Simply write

$$\mathscr{Q} \triangleq \mathscr{P}^{*C}\big|_{\tilde{H}_1^C} \in \mathscr{L}(\tilde{H}_1^C) \ \ \text{and} \ \ B_1(\cdot) \triangleq \Big(\big(B(\cdot)\big|_Z\big)^{*C}\Big)\Big|_{(0,T)} \in L^2(0,T;\mathscr{L}(H,Z)).$$

By Proposition 1.5 and (1.37), we have that $\sigma(\mathscr{Q}) = \{\bar{\lambda}_j\}_{j=1}^{\hat{n}}$; l_j is the algebraic multiplicity of $\bar{\lambda}_j$. Hence, $p(\lambda) \triangleq \prod_{j=1}^{\hat{n}} \big(\lambda - \bar{\lambda}_j\big)^{l_j}$ is the characteristic polynomial of $\mathscr{Q}$. Write $\hat{l}_j$ for the geometric multiplicity of $\bar{\lambda}_j$. Clearly, $\hat{l}_j \leq l_j$ for all j. Let

$\beta \triangleq \{\beta_1, \ldots, \beta_{n_0}\}$ be a basis of $(\mathbb{P}^* H)^C = \tilde{H}_1^C$ so that

$$\mathcal{Q}(\beta_1, \ldots, \beta_{n_0}) = J(\beta_1, \ldots, \beta_{n_0}). \tag{2.74}$$

Here J is the Jordan matrix: $\mathrm{diag}\{J_{11}, \ldots, J_{1\hat{l}_1}, J_{21}, \ \ldots, \ J_{2\hat{l}_2}, \ldots, J_{\hat{n}1}, \ \ldots, \ J_{\hat{n}\hat{l}_{\hat{n}}}\}$ with

$$J_{jk} = \begin{pmatrix} \bar{\lambda}_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \bar{\lambda}_j \end{pmatrix} \ \text{a} \ \ d_{jk} \times d_{jk} \ \ \text{matrix},$$

where $j = 1, \ldots, \hat{n}$, $k = 1, \ldots, \hat{l}_j$, and for each j, $\{d_{jk}\}_{k=1}^{\hat{l}_j}$ is decreasing. It is clear that $\sum_{k=1}^{\hat{l}_j} d_{jk} = l_j$ for each $j = 1, \ldots, \hat{n}$, and $\sum_{j=1}^{\hat{n}} \sum_{k=1}^{\hat{l}_j} d_{jk} = n_0$. We rewrite the basis β as

$$\beta \triangleq \left\{\xi_{111}, \ldots, \xi_{11d_{11}}, \xi_{1\hat{l}_1 1}, \ldots, \xi_{1\hat{l}_1 d_{1\hat{l}_1}}, \ldots, \xi_{\hat{n}11}, \ldots, \xi_{\hat{n}1d_{\hat{n}1}}, \xi_{\hat{n}\hat{l}_{\hat{n}}1}, \ldots, \xi_{\hat{n}\hat{l}_{\hat{n}} d_{\hat{n}\hat{l}_{\hat{n}}}}\right\}.$$

Then by (2.74), one can easily check that for each $j \in \{1, \ldots, \hat{n}\}$ and $k \in \{1, \ldots, \hat{l}_j\}$,

$$\left(\bar{\lambda}_j I - \mathcal{Q}\right)^q \xi_{jkr} = \begin{cases} \xi_{jk(r-q)} & \text{when } r > q, \\ 0 & \text{when } r \le q. \end{cases} \tag{2.75}$$

Now we assume $\hat{\xi}$ satisfies the conditions on the left side of (2.73). Since $\hat{\xi} \in (\mathbb{P}^* H_1)^C = \tilde{H}_1^C$, there is a vector

$$\left(C_{111}, \ldots, C_{11d_{11}}, C_{1\hat{l}_1 1}, \ldots, C_{1\hat{l}_1 d_{1\hat{l}_1}}, \ldots, C_{\hat{n}11}, \ldots, C_{\hat{n}1d_{\hat{n}1}}, C_{\hat{n}\hat{l}_{\hat{n}}1}, \ldots, C_{\hat{n}\hat{l}_{\hat{n}} d_{\hat{n}\hat{l}_{\hat{n}}}}\right)^* \in \mathbb{C}^{n_0},$$

so that

$$\hat{\xi} = \sum_{j=1}^{\hat{n}} \sum_{k=1}^{\hat{l}_j} \sum_{r=1}^{d_{jk}} C_{jkr} \xi_{jkr}. \tag{2.76}$$

From (2.59) and the second condition on the left side of (2.73), it follows that for each $m \in \{0, \ldots, n_0 - 1\}$, $B_1(\cdot)\psi_{n_0}^{\hat{\xi}}(\cdot)\big|_{((n_0-m-1)T, (n_0-m)T)} = 0$, that is,

$$\sum_{j=1}^{\hat{n}} \sum_{k=1}^{\hat{l}_j} \sum_{r=1}^{d_{jk}} C_{jkr} B_1(t) \psi^{\mathcal{Q}^m \xi_{jkr}}(t) = 0 \ \text{ for a.e. } \ t \in (0, T).$$

From this, we see that for any polynomial g with $\mathrm{degree}(g) \le n_0 - 1$,

$$\sum_{j=1}^{\hat{n}} \sum_{k=1}^{\hat{l}_j} \sum_{r=1}^{d_{jk}} C_{jkr} B_1(\cdot) \psi^{g(\mathscr{Q})\xi_{jkr}} = 0 \text{ over } (0, T). \tag{2.77}$$

Given $\widetilde{j} \in \{1, \ldots, \hat{n}\}$, let

$$p_{\widetilde{j}}(\lambda) = \prod_{j=1, j \neq \widetilde{j}}^{\hat{n}} \left(\lambda - \bar{\lambda}_j\right)^{l_j}.$$

By taking $g(\lambda) = \lambda^m p_{\widetilde{j}}(\lambda)$, with $m = 0, 1, \ldots, l_{\widetilde{j}} - 1$, in (2.77), we find that

$$\sum_{j=1}^{\hat{n}} \sum_{k=1}^{\hat{l}_j} \sum_{r=1}^{d_{jk}} C_{jkr} B_1(\cdot) \psi^{\mathscr{Q}^m p_{\widetilde{j}}(\mathscr{Q})\xi_{jkr}}(\cdot) = 0 \text{ over } (0, T), \quad \text{when } m \in \{0, 1, \ldots, l_{\widetilde{j}} - 1\}.$$

By (2.75), we see that $p_{\widetilde{j}}(\mathscr{Q})\xi_{jkr} = 0$, when $j \in \{1, \ldots, \hat{n}\}$, $j \neq \widetilde{j}$, $k \in \{1, \ldots, \hat{l}_j\}$ and $r \in \{1, \ldots, d_{jk}\}$. The above two equations imply that for each $m \in \{0, 1, \ldots, l_{\widetilde{j}} - 1\}$,

$$\sum_{k=1}^{\hat{l}_{\widetilde{j}}} \sum_{r=1}^{d_{\widetilde{j}k}} C_{\widetilde{j}kr} B_1(\cdot) \psi^{\mathscr{Q}^m p_{\widetilde{j}}(\mathscr{Q})\xi_{\widetilde{j}kr}}(\cdot) = 0 \text{ over } (0, T),$$

from which, it follows that for any polynomial f with degree$(f) \leq l_{\widetilde{j}} - 1$,

$$\sum_{k=1}^{\hat{l}_{\widetilde{j}}} \sum_{r=1}^{d_{\widetilde{j}k}} C_{\widetilde{j}kr} B_1(\cdot) \psi^{f(\mathscr{Q}) p_{\widetilde{j}}(\mathscr{Q})\xi_{\widetilde{j}kr}}(\cdot) = 0 \text{ over } (0, T). \tag{2.78}$$

Given $m \in \{0, 1, 2, \cdots, l_{\widetilde{j}} - 1\}$, since $p_{\widetilde{j}}(\lambda)$ and $(\lambda - \bar{\lambda}_{\widetilde{j}})^{m+1}$ are coprime, there are polynomials $g_m^1(\lambda)$ and $g_m^2(\lambda)$ with degree$(g_m^1) \leq m$ and degree$(g_m^2) \leq$ degree $(p_{\widetilde{j}}) - 1$, respectively, so that

$$g_m^1(\lambda) p_{\widetilde{j}}(\lambda) + g_m^2(\lambda)(\lambda - \bar{\lambda}_{\widetilde{j}})^{m+1} \equiv 1.$$

Thus, for all $m \in \{0, 1, \cdots, l_{\widetilde{j}} - 1\}$, $k \in \{1, 2, \cdots, \hat{l}_{\widetilde{j}}\}$, and $r \in \{1, 2, \cdots, d_{\widetilde{j}k}\}$,

$$(\mathscr{Q} - \bar{\lambda}_{\widetilde{j}} I)^{l_{\widetilde{j}} - m - 1} g_m^1(\mathscr{Q}) \mathbb{P}_{\widetilde{j}}(\mathscr{Q}) \xi_{\widetilde{j}kr} + g_m^2(\mathscr{Q})(\mathscr{Q} - \bar{\lambda}_{\widetilde{j}} I)^{l_{\widetilde{j}}} \xi_{\widetilde{j}kr} \equiv (\mathscr{Q} - \bar{\lambda}_{\widetilde{j}} I)^{l_{\widetilde{j}} - m - 1} \xi_{\widetilde{j}kr}. \tag{2.79}$$

By (2.75), we have that

$$(\hat{\mathscr{Q}} - \bar{\lambda}_{\widetilde{j}} I)^{l_{\widetilde{j}}} \xi_{\widetilde{j}kr} = 0 \text{ for all } k \in \{1, 2, \cdots, \hat{l}_{\widetilde{j}}\}, \ r \in \{1, 2, \cdots, d_{\widetilde{j}k}\}. \tag{2.80}$$

Taking $f(\lambda) = (\lambda - \bar{\lambda}_{\tilde{j}})^{l_{\tilde{j}}-m-1} g_m^1(\lambda)$, with $m = 0, \ldots, l_{\tilde{j}} - 1$, in (2.78), using (2.79) and (2.80), we find that for each $m \in \{0, 1, \ldots, l_{\tilde{j}} - 1\}$,

$$\sum_{k=1}^{\hat{l}_{\tilde{j}}} \sum_{r=1}^{d_{\tilde{j}k}} C_{\tilde{j}kr} B_1(\cdot) \psi^{(\mathcal{Q} - \bar{\lambda}_{\tilde{j}} I)^m \xi_{\tilde{j}kr}}(\cdot) = 0 \ \text{ over } \ (0, T). \tag{2.81}$$

Now we are on the position to show that

$$C_{\tilde{j}kr} = 0 \ \text{ for all } \ k \in \{1, 2, \cdots, \hat{l}_{\tilde{j}}\}, \ r \in \{1, \ldots, d_{\tilde{j}k}\}, \tag{2.82}$$

which leads to $\hat{\xi} = 0$ because of (2.76). For this purpose, we write

$$K_{\tilde{j}}^m = \left\{ k \in \{1, 2, \ldots, \hat{l}_{\tilde{j}}\} \mid d_{\tilde{j}k} > m \right\}, \quad m = 0, 1, \ldots, l_{\tilde{j}} - 1.$$

One can easily check that (2.82) is equivalent to

$$\mathscr{C}_{\hat{m}} \stackrel{\Delta}{=} \left\{ C_{\tilde{j}k\hat{m}}, \ \ k \in K_{\tilde{j}}^{\hat{m}-1} \right\} = \{0\} \ \text{ for all } \ \hat{m} \in \{1, \ldots, d_{\tilde{j}1}\}. \tag{2.83}$$

We will use the mathematical induction with respect to $\hat{m}$ to prove (2.83). (Notice that $d_{\tilde{j}k}$ is decreasing with respect to k.) First of all, we let

$$\mathbb{Q}_{\tilde{j}}^m(\lambda) = \left(\bar{\lambda}_{\tilde{j}} - \lambda\right)^m, \quad m = 0, 1, \ldots, l_{\tilde{j}} - 1, \tag{2.84}$$

In the case that $\hat{m} = d_{\tilde{j}1}$, it follows from (2.84) and (2.75) that

$$\mathbb{Q}_{\tilde{j}}^{\hat{m}-1}(\mathcal{Q}) \xi_{\tilde{j}k\hat{m}} = \left(\bar{\lambda}_{\tilde{j}} I - \mathcal{Q}\right)^{\hat{m}-1} \xi_{\tilde{j}k\hat{m}} = \xi_{\tilde{j}k1}, \quad \text{when } \ k \in K_{\tilde{j}}^{\hat{m}-1},$$

and

$$\mathbb{Q}_{\tilde{j}}^{\hat{m}-1}(\mathcal{Q}) \xi_{\tilde{j}kr} = 0, \qquad \text{when } k \in K_{\tilde{j}}^{\hat{m}-1}, \ r < \hat{m}; \ \text{ or } k \notin K_{\tilde{j}}^{\hat{m}-1}, \ r \in \{1, \ldots, d_{\tilde{j}k}\}.$$

These, alone with (2.81) (where $m = \hat{m} - 1$), imply that

$$\sum_{k \in K_{\tilde{j}}^{\hat{m}-1}} C_{\tilde{j}k\hat{m}} B_1(\cdot) \psi^{\xi_{\tilde{j}k1}}(\cdot) = 0 \quad \text{over} \ \ (0, T).$$

Let

$$\bar{\xi}_{\hat{m}} \stackrel{\Delta}{=} \sum_{k \in K_{\tilde{j}}^{\hat{m}-1}} C_{\tilde{j}k\hat{m}} \xi_{\tilde{j}k1} \quad \text{with } \ \hat{m} = 1, \ldots, d_{\tilde{j}1}.$$

Then, it holds that

$$B_1(\cdot)\psi^{\bar{\xi}_{\hat{m}}}(\cdot) = 0 \ \text{ over } \ (0, T). \tag{2.85}$$

Since for each $k \in \{1, \ldots, \hat{l}_{\tilde{j}}\}$, $\xi_{\tilde{j}k1}$ is an eigenfunction of $\mathscr{Q}$ with respect to the eigenvalue $\bar{\lambda}_{\tilde{j}}$, it follows from the definition of $\bar{\xi}_{\hat{m}}$ that

$$(\bar{\lambda}_{\tilde{j}} I - \mathscr{Q})\bar{\xi}_{\hat{m}} = 0.$$

This, along with (2.85) and (d), yields that $\bar{\xi}_{\hat{m}} = 0$, i.e., $\bar{\xi}_{d_{\tilde{j},1}} = 0$, which leads to $\mathscr{C}_{d_{\tilde{j}1}} = 0$ because of the linear independence of the group $\{\xi_{\tilde{j}k1},\ k \in K_{\tilde{j}}^{\hat{m}-1}\}$. Hence, (2.83) holds when $\hat{m} = d_{\tilde{j}1}$. Suppose inductively that (2.83) holds when $\tilde{m} + 1 \le \hat{m} \le d_{\tilde{j}1}$ for some $\tilde{m} \in \{1, \ldots, d_{\tilde{j}1} - 1\}$, i.e.,

$$\mathscr{C}_{\hat{m}} = \{0\}, \quad \text{when } \ \tilde{m} + 1 \le \hat{m} \le d_{\tilde{j}1}. \tag{2.86}$$

We will show that (2.83) holds when $\hat{m} = \tilde{m}$. In fact, it follows from (2.75) that

$$\mathbb{Q}_{\tilde{j}}^{\tilde{m}-1}(\mathscr{Q})\xi_{\tilde{j}kr} = \begin{cases} \xi_{\tilde{j}k(r-\tilde{m}+1)}, & \text{when } k \in K_{\tilde{j}}^{\tilde{m}-1},\ r \ge \tilde{m}, \\ 0, & \text{when } k \in K_{\tilde{j}}^{\tilde{m}-1},\ r < \tilde{m}, \\ 0, & \text{when } k \notin K_{\tilde{j}}^{\tilde{m}-1},\ r \in \{1, \ldots, d_{\tilde{j}k}\}. \end{cases}$$

This, alone with (2.81) (where $m = \tilde{m} - 1$), indicates that

$$\sum_{k=1}^{\hat{l}_{\tilde{j}}} \sum_{r=1}^{d_{\tilde{j}k}} C_{\tilde{j}kr} B_1(\cdot)\psi^{p_{\tilde{j}}^{\tilde{m}-1}(\mathscr{Q})\xi_{\tilde{j}kr}}(\cdot) = \sum_{k \in K_{\tilde{j}}^{\tilde{m}-1}} \sum_{r=\tilde{m}}^{d_{\tilde{j}k}} C_{\tilde{j}kr} B_1(\cdot)\psi^{\xi_{\tilde{j}k(r-\tilde{m}+1)}}(\cdot) = 0 \ \text{ over } \ (0, T).$$

Then, by (2.86), we have that

$$\sum_{k \in K_{\tilde{j}}^{\tilde{m}-1}} C_{\tilde{j}k\tilde{m}} B_1(\cdot)\psi^{\xi_{\tilde{j}k1}}(\cdot) = 0 \ \text{ over } \ (0, T). \tag{2.87}$$

Let

$$\bar{\xi}_{\tilde{m}} \triangleq \sum_{k \in K_{\tilde{j}}^{\tilde{m}-1}} C_{\tilde{j}k\tilde{m}} \xi_{\tilde{j}k1}.$$

Then, it follows from (2.87) that

$$B_1(\cdot)\psi^{\bar{\xi}_{\tilde{m}}}(\cdot) = 0 \ \text{ over } \ (0, T). \tag{2.88}$$

Since for each $k \in \{1, \ldots, \hat{l}_{\tilde{j}}\}$, $\xi_{\tilde{j},k,1}$ is an eigenfunction of $\mathscr{Q}$ with respect to the eigenvalue $\bar{\lambda}_{\tilde{j}}$, it holds that $(\bar{\lambda}_{\tilde{j}} I - \mathscr{Q})\bar{\xi}_{\tilde{m}} = 0$. This, along with (2.88) and (d), yields that $\bar{\xi}_{\tilde{m}} = 0$. Hence, $\mathscr{C}_{\tilde{m}} = \{0\}$ because of the linear independence of the group

$\{\xi_{\tilde{j}k1},\ k \in K_{\tilde{j}}^{\tilde{m}-1}\}$. In summary, we conclude that $(d) \Rightarrow (c)$. This completes the proof of Theorem 2.1. □

2.3 Applications

Some applications of Theorem 2.1 will be given in this section.

2.3.1 Feedback Realization in Finite Dimensional Subspaces

When Eq. (1.1) is LPFS, can we find a finite dimensional subspace Z of U so that $Z \in \mathscr{U}^{FS}$? The answer is positive. This might help us to design a feedback law numerically. To prove the above-mentioned positive answer, the following lemma is needed.

Lemma 2.3 *For each subspace $Z \subseteqq U$, there is a finite dimensional subspace $\hat{Z} \subseteqq Z$ so that*

$$\hat{\mathscr{A}}_{n_0}^{Z} = \hat{\mathscr{A}}_{n_0}^{\hat{Z}} \quad \text{and} \quad \dim \hat{Z} \le n_0 \tag{2.89}$$

where $\hat{\mathscr{A}}_{n_0}^{Z}$ and $\hat{\mathscr{A}}_{n_0}^{\hat{Z}}$ are defined by (2.4), and n_0 is given by (1.17).

Proof We carry out the proof by two steps.

Step 1. To show that there is a finite-dimensional subspace $\tilde{Z}$ of U *so that $\hat{\mathscr{A}}_{n_0}^{Z} = \hat{\mathscr{A}}_{n_0}^{\hat{Z}}$*
Let Z be a subspace of U. Since $\hat{\mathscr{A}}_{n_0}^{Z}$ is a subspace of H_1 and $\dim H_1 = n_0 < \infty$ (see (1.22)), we can assume that $\dim \hat{\mathscr{A}}_{n_0}^{Z} \triangleq m \le n_0$. Write $\{\xi_1, \dots, \xi_m\}$ for an orthonormal basis of $\hat{\mathscr{A}}_{n_0}^{Z}$. By (2.4) and (2.2), there are $u_j(\cdot) \in L^2(\mathbb{R}^+; Z)$, $j = 1, \dots, m$, so that

$$\mathscr{L}_1 u_j = \xi_j \quad \text{for all} \quad j = 1, \dots, m, \tag{2.90}$$

where $\mathscr{L}_1 : L^2(\mathbb{R}^+; Z) \to H_1$ is defined by

$$\mathscr{L}_1 u \triangleq \int_0^{n_0 T} \mathbb{P}\Phi(n_0 T, s) D(s) u(s) \mathrm{d}s,\ u \in L^2(\mathbb{R}^+; Z).$$

From the orthonormality, it follows that

$$\det \left(\langle \mathscr{L}_1 u_i,\ \mathscr{L}_1 u_j \rangle \right)_{ij} = 1 \ne 0. \tag{2.91}$$

By the definition of the Bochner integration (*see* [14]), for each $j \in \{1, \ldots, m\}$, there is a sequence of simple functions, denoted by $\{v_j^k\}_{k=1}^{\infty}$, so that

$$\lim_{k\to\infty} \int_0^{n_0 T} \|v_j^k(s) - u_j(s)\|_U \mathrm{d}s = 0.$$

This, along with (2.91), yields that there is a k_0 such that $\det\Big(\langle \mathscr{L}_1 v_i^{k_0},\ \mathscr{L}_1 v_j^{k_0}\rangle\Big)_{ij} \neq 0$. Let

$$\eta_j = \mathscr{L}_1 v_j^{k_0}, \quad j = 1, \ldots, m. \tag{2.92}$$

Then, $\{\eta_1, \ldots, \eta_m\}$ is a linearly independent group in the subspace $\hat{\mathscr{A}}_{n_0}^Z$. Hence, $\{\eta_1, \ldots, \eta_m\}$ is a basis of $\hat{\mathscr{A}}_{n_0}^Z$. Write

$$v_j^{k_0}(\cdot) = \sum_{l=1}^{k_j} \chi_{E_{jl}}(\cdot) z_{jl} \ \text{ over } \ (0, n_0 T), \quad j = 1, \ldots, m, \tag{2.93}$$

with $z_{jl} \in Z$, E_{jl} measurable sets in $(0, n_0 T)$ and $\chi_{E_{jl}}$ the characteristic function of E_{jl}. Let

$$\tilde{Z} = \text{span}\big\{z_{11}, \ldots, z_{1k_1}, z_{21}, \ldots, z_{2k_2}, \ldots, z_{m1}, \ldots, z_{mk_m}\big\}.$$

Clearly, $\tilde{Z}$ is a finite-dimensional subspace of Z and all $v_j^{k_0}(\cdot)$, $j = 1, \ldots, m$, (given by (2.93)) belong to $L^2(\mathbb{R}^+; \tilde{Z})$. The later, along with (2.92), yields that $\eta_j \in \hat{V}_{n_0}^{\tilde{Z}}$ for each $j = 1, \ldots, m$. Hence,

$$\hat{\mathscr{A}}_{n_0}^Z \supseteq \hat{\mathscr{A}}_{n_0}^{\tilde{Z}} \supseteq \text{span}\{\eta_1, \ldots, \eta_m\} = \hat{\mathscr{A}}_{n_0}^Z.$$

This leads to $\hat{\mathscr{A}}_{n_0}^Z = \hat{\mathscr{A}}_{n_0}^{\tilde{Z}}$.

Step 2. To show that there is a subspace $\hat{Z}$ of $\tilde{Z}$ (*which is constructed in Step 1) such that* $\hat{\mathscr{A}}_{n_0}^{\hat{Z}} = \hat{\mathscr{A}}_{n_0}^{\tilde{Z}}$ and $\dim \hat{Z} \leq n_0$

Write $\{\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_{l_0}\}$ for an orthonormal basis of $\tilde{Z}$. Then it holds that

$$\hat{\mathscr{A}}_{n_0}^{\tilde{Z}} = \sum_{j=1}^{l_0} \hat{\mathscr{A}}_{n_0}^{Z_j}, \quad \text{with} \quad Z_j = \text{span}\{\zeta_j\}, \quad j = 1, 2, \ldots, l_0. \tag{2.94}$$

For each $j \in \{1, \ldots, l_0\}$, let $\{\eta_{j1},\ \eta_{j2},\ \ldots,\ \eta_{jk_j}\}$ be a basis of $\hat{\mathscr{A}}_{n_0}^{Z_j}$. Denote by S a maximal independent group of the set:

$$\{\eta_{11}, \ldots, \eta_{1,k_1}, \eta_{21}, \ldots, \eta_{2,k_2},\ \ldots,\ \eta_{l_0,1}, \ldots, \eta_{l_0,k_{l_0}}\}.$$

Then

$$\operatorname{span}\left\{\eta \mid \eta \in S\right\} = \mathscr{A}_{n_0}^{\hat{\tilde{Z}}}, \tag{2.95}$$

and S contains $m \le n_0$ elements. Let

$$J = \left\{j = 1, \ldots, l_0 \mid \eta_{jk} \in S \ \text{ for some } \ k \in \{1, \ldots, k_j\}\right\}.$$

Then the number of elements contained in J equals to or is less than $m \le n_0$. Set $\hat{Z} = \sum_{j \in J} Z_j$. Then it holds that $\mathscr{A}_{n_0}^{\hat{\hat{Z}}} \subseteq \mathscr{A}_{n_0}^{\hat{\tilde{Z}}}$ and $\operatorname{span}\left\{\eta \mid \eta \in S\right\} \subseteq \sum_{j \in J} \mathscr{A}_{n_0}^{\hat{Z}_j} = \mathscr{A}_{n_0}^{\hat{\hat{Z}}}$. These, together with (2.95), yield that $\mathscr{A}_{n_0}^{\hat{\hat{Z}}} = \mathscr{A}_{n_0}^{\hat{\tilde{Z}}}$. Because of the number of elements contained in J equals to or is less than $m \le n_0$, it holds that $\dim \hat{Z} \le m \le n_0$. This completes the proof. $\square$

Now a main result of this subsection is presented by the following theorem.

Theorem 2.2 *Equation (1.1) is LPFS if and only if it is LPFS with respect to a subspace Z of U with $\dim Z \le n_0$.*

Proof Clearly, it suffices to show the *only if* part. Assume that Eq. (1.1) is LPFS. By the equivalence of (a) and (b) in Theorem 2.1, we find that $\mathscr{A}_{n_0}^{\hat{U}} = H_1$. Meanwhile, by Lemma 2.3, there is a subspace of $\hat{Z}$ of U so that $\mathscr{A}_{n_0}^{\hat{U}} = \mathscr{A}_{n_0}^{\hat{\hat{Z}}}$ with $\dim \hat{Z} \le n_0$. Thus, we find that $\mathscr{A}_{n_0}^{\hat{\hat{Z}}} = H_1$. This, along with the equivalence of (a) and (b) in Theorem 2.1, indicates that Eq. (1.1) is LPFS with respect to $\hat{Z}$. This ends the proof. $\square$

2.3.2 Applications to Heat Equations

In this subsection, we will present some applications of Theorem 2.1 to heat equations with time-periodic potentials.

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d \ge 1)$ with a C^2-smooth boundary $\partial\Omega$. Write $Q \triangleq \Omega \times \mathbb{R}^+$ and $\sum \triangleq \partial\Omega \times \mathbb{R}^+$. Let $\omega \subseteq \Omega$ be a non-empty open subset with its characteristic function χ_ω. Let $T > 0$ and $a \in C(\bar{Q})$ be T-periodic (with respect to the time variable t), i.e., for each $t \in \mathbb{R}^+$, $a(\cdot, t) = a(\cdot, t + T)$ over Ω. One can easily check that the function a can be treated as a T-periodic function in $L^1_{loc}(\mathbb{R}^+; \mathscr{L}(L^2(\Omega)))$. Consider the following controlled heat equation:

$$\begin{cases} \partial_t y(x,t) - \triangle y(x,t) + a(x,t)y(x,t) = \chi_\omega(x)u(x,t) & \text{in } \ Q, \\ y(x,t) = 0 & \text{on } \sum, \end{cases} \tag{2.96}$$

where $u \in L^2(\mathbb{R}^+; L^2(\Omega))$. Given $y_0 \in L^2(\Omega)$ and $u \in L^2(\mathbb{R}^+; L^2(\Omega))$, Eq. (2.96), with the initial condition that $y(x, 0) = y_0(x)$, has a unique solution $y(\cdot; 0, y_0, u)$ in the space $C(\mathbb{R}^+; L^2(\Omega))$. Let $H = U = L^2(\Omega)$ and $A = \triangle$ with $\mathscr{D}(A) = H_0^1(\Omega) \bigcap$

$H_2(\Omega)$. Define, for each $t \in \mathbb{R}^+$, $D(t)\colon H \to H$ by $D(t)z(x) = -a(x,t)z(x)$, $x \in \Omega$, and $B(t)\colon U \to H$ by $B(t)v(x) = \chi_\omega(x)v(x)$, $x \in \Omega$. Clearly, A generates a compact semigroup on $L^2(\Omega)$ and both $D(\cdot) \in L^1_{loc}(\mathbb{R}^+; \mathscr{L}(L^2(\Omega)))$ and $B(\cdot) \in L^\infty(\mathbb{R}^+; \mathscr{L}(U; H))$ are T-periodic. Thus, we can study Eq. (2.96) under the framework (1.1). Write Ψ_a for the evolution generated by $A + D(\cdot)$. We use notations n_0, $\mathbb{P}$, H_j (with $j = 1, 2$), $\mathscr{A}_k^Z$ and $\hat{\mathscr{A}}_k^Z$ (with $k \in \mathbb{N}$) to denote the same subjects as those introduced in the first section of this chapter.

Corollary 2.1 *Equation (2.96) is LPFS with respect to a subspace Z of $L^2(\Omega)$ with $dim Z \le n_0$.*

Proof We will provide two ways to show that Eq. (2.96) is LPFS. We first use the equivalence $(a) \Leftrightarrow (c)$ in Theorem 2.1. In fact, $\psi(\cdot) \triangleq \Psi_a(n_0T, \cdot)^*\xi$ with $\xi \in H$ is the solution to the equation:

$$\begin{cases} \partial_t \psi(x,t) + \triangle\psi(x,t) - a(x,t)\psi(x,t) = 0 & \text{in } \Omega \times (0, n_0T), \\ \psi(x,t) = 0 & \text{on } \partial\Omega \times (0, n_0T), \\ \psi(x, n_0T) = \xi(x) & \text{in } \Omega. \end{cases} \tag{2.97}$$

Moreover, we have that

$$B(t)\eta = \chi_\omega \eta \quad \text{for any} \quad \eta \in H \quad \text{and} \quad t \in [0, T]. \tag{2.98}$$

These, along with the unique continuation property of parabolic equations established in [60] (see also [74, 75]), lead to the condition (c) in Theorem 2.1 for the current case. Then, according to the equivalence $(a) \Leftrightarrow (c)$ in Theorem 2.1, Eq. (2.96) is LPFS.

We next use the equivalence $(a) \Leftrightarrow (b)$ in Theorem 2.1. Without loss of generality, we can assume that $n_0 \ge 1$, for otherwise Eq. (2.96), with the null control, is stable. When $n_0 \ge 1$, we have $H_1 \ne \{0\}$ and $||P|| > 0$. Write $\left\{\xi_1, \ldots, \xi_{n_0}\right\}$ for an orthonormal basis of H_1. By the approximate controllability of the heat equation (*see* [33]), $\mathscr{A}_1^U$ is dense in H. Thus there are η_j, $j = 1 \ldots, n_0$, in $\mathscr{A}_1^U$ so that

$$\|\eta_j - \xi_j\| \le \frac{1}{16n_0\|\mathbb{P}\|} \quad \text{for all} \quad j = 1, \ldots, n_0. \tag{2.99}$$

Since P is a projection from H onto H_1, we have that $\mathbb{P}\xi_j = \xi_j$ for all $j = 1, \ldots, n_0$. This, along with (2.99), yields that for each $j \in \{1, \ldots, n_0\}$,

$$\|\mathbb{P}\eta_j\| \le \|\xi_j\| + \|\mathbb{P}\|\|\eta_j - \xi_j\| \le 1 + \frac{1}{16n_0}; \tag{2.100}$$

and

$$\langle \mathbb{P}\eta_j, \xi_j \rangle \ge 1 - \frac{1}{16n_0}. \tag{2.101}$$

Since $\mathbb{P}\eta_j \in H_1$ and $\{\xi_k\}_{k=1}^{n_0}$ is an orthonormal basis of H_1, we find that

$$\|\mathbb{P}\eta_j\|^2 = \sum_{k=1}^{n_0} |\langle \mathbb{P}\eta_j, \xi_k\rangle|^2, \quad \text{when} \quad j = 1, \dots, n_0. \tag{2.102}$$

From (2.102), (2.100) and (2.101), it follows that for each $j \in \{1, \dots, n_0\}$,

$$\begin{aligned}\sum_{k\neq j} \big|\langle \mathbb{P}\eta_j, \xi_k\rangle\big| &\leq (n_0-1)^{1/2}\Big(\sum_{k\neq j}\big|\langle \mathbb{P}\eta_j, \xi_k\rangle\big|^2\Big)^{1/2} \\ &= (n_0-1)^{1/2}\big(\|\mathbb{P}\eta_j\|^2 - |\langle \mathbb{P}\eta_j, \xi_j\rangle|^2\big)^{1/2} \\ &\leq n_0^{1/2}\big((1+1/(16n_0))^2 - (1-1/(16n_0))^2\big)^{1/2} = 1/2.\end{aligned}$$

This, together with (2.101), indicates that

$$\langle \mathbb{P}\eta_j, \xi_j\rangle \geq 1 - 1/(16n_0) > 1/2 \geq \sum_{k\neq j}\big|\langle \mathbb{P}\eta_j, \xi_k\rangle\big|, \;\; j = 1, 2, \cdots, n_0. \tag{2.103}$$

We claim that $\{\mathbb{P}\eta_1, \dots, \mathbb{P}\eta_{n_0}\}$ is a linearly independent group. In fact, suppose that

$$\sum_{j=1}^{n} c_j \mathbb{P}\eta_j = 0 \;\; \text{for some} \;\; c_1, \dots, c_{n_0} \in \mathbb{R}. \tag{2.104}$$

Write

$$\hat{A} \triangleq (\langle \mathbb{P}\eta_j, \xi_k\rangle)_{j,k} \in \mathbb{R}^{n_0\times n_0} \quad \text{and} \quad \hat{c} \triangleq (c_1, \dots, c_{n_0})^* \in \mathbb{R}^{n_0}.$$

By (2.103), the matrix $\hat{A}$ is diagonally dominant, hence it is invertible. Then, from (2.104), it follows that $\hat{A}^*\hat{c} = 0$, which implies $\hat{c} = 0$. Hence, $\mathbb{P}\eta_1, \dots, \mathbb{P}\eta_{n_0}$ are linearly independent.

Since $\dim H_1 = n_0$, it follows that

$$span\{\mathbb{P}\eta_1, \dots, \mathbb{P}\eta_{n_0}\} = H_1.$$

Therefore, we have that

$$H_1 \supseteq \hat{V}_{n_0}^U \supseteq \hat{V}_1^U = \mathbb{P}V_1^U \supseteq \text{span}\{\mathbb{P}\eta_1, \cdots, \mathbb{P}\eta_{n_0}\} = H_1,$$

from which, it follows that $H_1 = \hat{V}_{n_0}^U$. This, along with the equivalence of (a) and (b) in Theorem 2.1, indicates that Eq. (2.96) is LPFS.

Finally, according to Theorem 2.2, there is a subspace Z of $L^2(\Omega)$ with $\dim Z \leq n_0$ so that Eq. (2.96) is LPFS with respect to Z. This completes the proof. $\square$

Corollary 2.2 *Equation (2.96) is LPFS with respect to the subspace* $\mathbb{P}^* H$.

Proof Let $Z = \mathbb{P}^* H$. By the equivalence between (a) and (d) in Theorem 2.1, it suffices to show that Z satisfies (d), i.e.,

$$\left.\begin{array}{r}\mu \notin \mathbb{B},\ \xi \in H^C,\ \big(\mu I - \mathscr{P}^{*C}\big)\xi = 0, \\ \big(B(\cdot)\big|_Z\big)^{*C} \Psi_a(T,\cdot)^{*C}\xi = 0 \ \text{ over } \ (0,T)\end{array}\right\} \Rightarrow \xi = 0. \tag{2.105}$$

Suppose that μ and ξ satisfy the conditions on the left side of (2.105). Write $\xi = \xi_1 + i\xi_2$ where $\xi_1, \xi_2 \in H$. Then, we have that

$$\big(B(\cdot)\big|_Z\big)^* \Psi_a(T,\cdot)^*\xi_j = 0 \ \text{ over } \ (0,T); \quad j = 1,2.$$

Since $\psi_j(\cdot) \triangleq \Psi_a(T,\cdot)^*\xi_j$ (with $j = 1,2$) is the solution to the Eq. (2.97) (where $n_0 T$ and ξ are replaced by T and ξ_j respectively), $\Psi_a(T,\cdot)^*\xi_j$ is continuous on $[0,T]$ and $B(t)$ is independent of t, the above yields that

$$\big(B(0)\big|_Z\big)^* \Psi_a(T,0)^*\xi_j = 0, \quad \text{with} \ \ j = 1,2.$$

Since $\mathbb{P}^*\eta \in \mathbb{P}^* H$ for each $\eta \in H$, the above yields that

$$\begin{aligned}&\big\langle \big(B(0)\big|_Z\big)^* \Psi_a(T,0)^*\xi_j,\ \mathbb{P}^*\eta\big\rangle = \big\langle \Psi_a(T,0)^*\xi_j,\ \big(B(0)\big|_Z\big)\mathbb{P}^*\eta\big\rangle \\ &= \big\langle \Psi_a(T,0)^*\xi_j,\ \chi_\omega \mathbb{P}^*\eta\big\rangle = \big\langle \mathbb{P}\chi_\omega \Psi_a(T,0)^*\xi_j,\ \eta\big\rangle,\ \ j = 1,2.\end{aligned}$$

Hence, we have that

$$\mathbb{P}\chi_\omega \Psi_a(T,0)^*\xi_j = 0,\ \ j = 1,2,$$

from which, it follows that

$$\langle \mathbb{P}^*\Psi_a(T,0)^*\xi_j,\ \chi_\omega \Psi_a(T,0)^*\xi_j\rangle = \langle \Psi_a(T,0)^*\xi_j,\ \mathbb{P}\chi_\omega \Psi_a(T,0)^*\xi_j\rangle = 0,\ \ j = 1,2. \tag{2.106}$$

Two facts are as follows. First, it follows from (1.23) that

$$\mathbb{P}^*\Psi_a(T,0)^*\xi_j = \Psi_a(T,0)^*\mathbb{P}^*\xi_j,\ \ j = 1,2. \tag{2.107}$$

Second, by (1.44), (1.43), and the first three conditions on the left side of (2.105), we have $\xi \in \tilde{H}_1^C$. Since $\mathbb{P}^* = \tilde{\mathbb{P}}$ and $\tilde{\mathbb{P}}$ is a projection from H to $\tilde{H}_1$ (see Proposition 1.5), we see that $\mathbb{P}^* : H \to \tilde{H}_1$ is a projection. Hence, $\mathbb{P}^{*C} : H^C \to \tilde{H}_1^C$ is a projection. These two facts yields that $\mathbb{P}^{*C}\xi = \xi$, from which, it follows that $\mathbb{P}^*\xi_j = \xi_j$, $j = 1,2$. This along with (2.106) and (2.107), indicates that $\big\|\chi_\omega \Psi_a(T,0)^{*C}\xi\big\| = 0$. By the unique continuation property of parabolic equations established in [60] (see also [74, 75]), we find that $\xi_j = 0$, $j = 1,2$, which leads to $\xi = 0$. This completes the proof. $\square$

We next introduce a controlled heat equation which is not LPFS. Write λ_1 and λ_2 for the first and the second eigenvalues of the operator $-\triangle$ with $\mathscr{D}(-\triangle) = H_0^1(\Omega) \cap H^2(\Omega)$, respectively. Let ξ_j, $j = 1, 2$, be an eigenfunction corresponding to λ_j. Consider the following heat equation:

$$\begin{cases} \partial_t y(x,t) - \triangle y(x,t) - \lambda_2 y(x,t) = \langle u(t), \xi_1 \rangle \xi_1(x) & \text{in } Q, \\ y(x,t) = 0 & \text{on } \sum. \end{cases} \tag{2.108}$$

where $u(\cdot) \in L^2(\mathbb{R}^+; L^2(\Omega))$. By a direct calculation, one has that $V_{n_0} = \text{span}\{\xi_1\}$ and $H_1 \supseteq \text{span}\{\xi_1, \xi_2\}$. These, along with $(a) \Leftrightarrow (b)$ in Theorem 2.1, indicates that (2.108) is not LPFS.

We end this subsection with the following note: It should be an interesting problem how to find a finitely dimensional subspace Z from $\mathscr{U}^{FS}$ so that it has the minimal dimension. (Here, $\mathscr{U}^{FS}$ is given by (2.1)) In general, we are not able to solve this problem. However, in some cases, it can be done. In Example 4.2, by applying Theorem 2.1, as well as Theorem 2.2 and Corollary 2.1, to a controlled heat equation, we solved this problem. From this point of view, Example 4.2 is also an application of Theorem 2.1 to controlled heat equations. The reasons that we put this example at the end of the last section of Chap. 4 are as follows: First, this problem can be understood as designs of a kind of simple control machines in infinitely dimensional cases. Second, the problems of designs of some simple control machines in finitely dimensional settings will be introduced in Chap. 4.

Miscellaneous Notes

There have been studies on equivalence conditions of periodic feedback stabilization for linear periodic evolution systems. In [31, 67], the authors established an equivalent condition on stabilizability for linear time-periodic parabolic equations with open-loop controls. Their equivalence (see Theorem 3.1 in [67] and Proposition 3.1 in [31]) can be stated, under the framework of Sect. 1.1, as follows: the condition (d) (in Theorem 2.1 where $Z = U$) is equivalent to the statement that for any $h \in H$, there is a control $u^h(\cdot) \in C(\mathbb{R}^+; U)$, with $\sup_{t \in \mathbb{R}^+} \|e^{\bar{\delta} t} u^h(t)\|_U$ bounded (where $\bar{\delta}$ is given by (1.16)), so that the solution $y(\cdot; 0, h, u^h)$ is stable. Meanwhile, it was pointed out in [67] that when open-looped stabilization controls exist, one can construct a periodic feedback stabilization law through using a method provided in [30]. From this point of view, the equivalence $(a) \Leftrightarrow (d)$ in Theorem 2.1 has been built up in [31, 67], through a different way. The method to construct the stabilization feedback law in this chapter is different from that in [30]. Besides, we would like to mention the paper [7] where the authors built up a feedback law for some nonlinear time-periodic evolution systems.

Proposition 2.3 (see also Remark 2.1) is a byproduct of the main study in this chapter. It shows that when both $D(\cdot)$ and $B(\cdot)$ are time invariant, linear time-period functions $K(\cdot)$ will not aid the linear stabilization of Eq. (1.1), i.e., Eq. (1.1) is linear $\hat{T}$-periodic feedback stabilizable for some $\hat{T} > 0$ if and only if Eq. (1.1) is linear time invariant feedback stabilizable. On the other hand, when Eq. (1.1) is periodic time varying, linear time-periodic $K(\cdot)$ do aid in the linear stabilization of this equation.

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