Chapter 2
Autonomous Dynamical Systems

The theory of autonomous dynamical systems is now well established after being studied intensively over the past years. In this chapter we will provide a brief review of autonomous dynamical systems, as the background and motivation to introduce nonautonomous and random dynamical systems which are the major topics of the book.

An ordinary differential equation (ODE) is said to be an *autonomous differential equation* if the right hand side does not depend on time explicitly, i.e., it can be formulated as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x), \tag{2.1}$$

where $g : \mathcal{O} \subset \mathbb{R}^d \to \mathbb{R}^d$ is a mapping from an open subset $\mathcal{O}$ of $\mathbb{R}^d$ to $\mathbb{R}^d$. This is a particular case of the system in (1.6) where $I = \mathbb{R}$ and the vector field g does not depend on the time t. Associating Eq. (2.1) with an initial datum $x(t_0) = x_0 \in \mathcal{O}$ gives the following IVP

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x), \quad x(t_0) = x_0. \tag{2.2}$$

To apply the general results presented in Chap. 1 to obtain the existence and uniqueness of a maximal solution to Eq. (2.2), we only need to impose a locally Lipschitz assumption on function g. Note that if a function is locally Lipschitz with respect to all its variables, then it is continuous. But in general, the continuity of g guarantees only the existence of solutions to (2.1) (see e.g., [25]), but not the uniqueness.

Remark 2.1 One effective way to check if a function satisfies a Lipschitz condition is to check if it is continuously differentiable. A continuously differentiable function is always locally Lipschitz (see, e.g., [66]), hence every IVP problem (2.2) with $g \in C^1(\mathcal{O})$ possesses a unique maximal solution. Moreover, if the domain $\mathcal{O}$ is convex, then a continuously differentiable function is globally Lipschitz if and only if the partial derivatives $\frac{\partial g_i}{\partial x_j}(x), i, j = 1, 2 \cdots, d$, are globally bounded.

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A general theorem on existence and uniqueness of a maximal solution to the IVP (2.2) is stated below.

Theorem 2.1 (Existence and uniqueness of a maximal solution) *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ and assume that g is continuously differentiable on $\mathcal{O}$. Then for any $t_0 \in \mathbb{R}$ and any $x_0 \in \mathcal{O}$ the initial value problem (2.2) has a unique maximal solution $\varphi(\cdot; t_0, x_0)$ defined in its maximal open interval $I_{max} = I_{max}(t_0, x_0)$.*

Remark 2.2 As our main interest is the long term behavior of solutions to (2.2), we will focus on the cases when I_{max} contains the interval $[t_0, +\infty)$. This means that the solution $\varphi(\cdot; t_0, x_0)$ is globally defined in all future times, i.e., is a *global solution*. However, note that the existence of a global solution is not free; it requires conditions such as in Theorem 1.1.

An important property of autonomous ODEs, which can be easily verified, is that the solution mapping depends only on the elapsed time $t - t_0$ but not separately on the initial time t_0 and current time t (see, e.g., [51]), i.e.,

$$\varphi(t - t_0; 0, x_0) = \varphi(t; t_0, x_0), \quad \text{and} \quad I_{max}(t_0, x_0) = t_0 + I_{max}(0, x_0),$$

for all $x_0 \in \mathcal{O}$, $t_0 \in \mathbb{R}$, and $t \in I_{max}(t_0, x_0)$. Therefore for autonomous ODEs we can always focus on $t_0 = 0$. With $t_0 = 0$ the solution can be written as $\varphi(t; x_0)$ and the existence interval of maximal solution can be written as $I_{max}(x_0)$. It is straightforward to check that this solution mapping $\varphi(\cdot; \cdot) : \mathbb{R} \times \mathcal{O} \to \mathbb{R}^d$ satisfies the initial value property (1.3) and the group property (1.4) (when $I_{max} = \mathbb{R}$), and hence defines a dynamical system, namely an *autonomous dynamical system*.

Next we will provide a survey on the long term behavior for autonomous dynamical systems. In particular, we will start from the stability theory of linear ODE systems, followed by the stability of nonlinear ODE systems by the first approximation method. Then we will introduce basic Lyapunov theory for stability and asymptotic stability. Some comments on globally attracting sets will be provided via the LaSalle theorem and the Poincaré–Bendixson theorem in dimension $d = 2$, and serve as a motivation for the analysis of global asymptotic behavior in higher dimensions. In the end we will introduce the general concept of attractors and their main properties for autonomous dynamical systems.

2.1 Basic Stability Theory

For ease of understanding, we start from the stability of equilibrium points of system (2.1). Recall that an *equilibrium point (or steady state)* x^* of system (2.1) is a constant solution to (2.1) satisfying $g(x^*) = 0$.

Definition 2.1 An equilibrium x^* is said to be

- *stable* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|x_0 - x^*| < \delta(\varepsilon)$ then $I_{max}(x_0) \supseteq [0, +\infty)$ and

$$|\varphi(t; x_0) - x^*| < \varepsilon, \quad \forall t \geq 0.$$

(solutions starting close to one equilibrium point remain close to it in the future.)
- *convergent or attractive* if there exists $\delta > 0$ such that if $|x_0 - x^*| < \delta$ then $I_{max}(x_0) \supseteq [0, +\infty)$ and
$$\lim_{t\to\infty} \varphi(t; x_0) = x^*.$$

(solutions starting close to one equilibrium point will converge to it when time goes to infinity.)
- *asymptotically stable* if it is both stable and convergent.
- *exponentially stable* if there exist $\delta > 0$ and $\alpha, \lambda > 0$ such that if $|x_0 - x^*| < \delta$ then $I_{max}(x_0) \supseteq [0, +\infty)$ and

$$|\varphi(t; x_0) - x^*| < \alpha|x_0 - x^*|e^{-\lambda t}, \quad \forall t \geq 0.$$

Remark 2.3 Definition 2.1 includes only the stability for an equilibrium point, i.e., x^* is constant, but can be easily generalized to any non-constant particular solution of Eq. (2.1). More precisely, a particular solution $x^*(t)$ to (2.1) is said to be stable if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|x_0 - x^*(0)| < \delta(\varepsilon)$ then $I_{max} \supseteq [0, +\infty)$ and

$$|\varphi(t; x_0) - x^*(t)| < \varepsilon, \quad \forall t \geq 0.$$

Following a similar manner all the other concepts in Definition 2.1 can be generalized to any particular solution of (2.1).

It is worth mentioning that exponential stability implies asymptotic stability, and asymptotic stability implies stability and convergence. However, stability and convergence are independent properties. Convergence implies stability for linear ODEs (see, e.g., [81]), but does not imply stability in general. For example, the autonomous ODE system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{x^2(y-x)+y^5}{(x^2+y^2)(1+(x^2+y^2)^2)}; \qquad \frac{\mathrm{d}x(t)}{\mathrm{d}t} = 0 \text{ for } x=0,\ y=0, \tag{2.3}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2)^2)}; \qquad \frac{\mathrm{d}y(t)}{\mathrm{d}t} = 0 \text{ for } x=0,\ y=0. \tag{2.4}$$

has an isolated equilibrium at $(0, 0)$ that is convergent but unstable (see Fig. 2.1).

Remark 2.4 In the autonomous framework, all the stability concepts in Definition 2.1 are uniform in time, i.e., the choice of δ does not depend on time. But this does not hold true for nonautonomous ODEs, which requires new definitions to distinguish uniform and non-uniform type of stability (see, e.g., [71]).

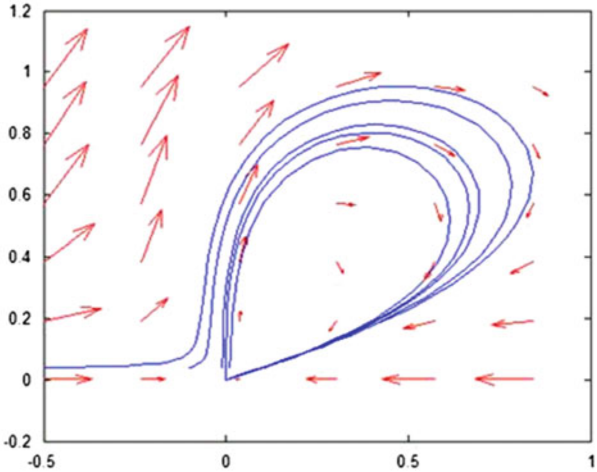

Fig. 2.1 Attractivity and convergence of example (2.3)–(2.4).

Notice that a very simple change of variable can reduce the problem of analyzing the stability of any solution to (2.1), to the problem of analyzing the stability of the zero (or trivial) solution $\varphi_0(t) \equiv 0$ (for all $t \in \mathbb{R}$) of a corresponding system of ODEs. More precisely, assume that $x^*(t)$ is a non-zero particular solution of system (2.1), and consider the change of variables $y(t) = x(t) - x^*(t)$, then it is easy to check that

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \tilde{g}(t, y(t)), \quad \text{where } \tilde{g}(t, y) = g(y + x^*(t)) - g(x^*(t)),$$

in which the right hand side satisfies $\tilde{g}(t, 0) = 0$ for all $t \in \mathbb{R}$. In the special case that $x^*(t) = x^*$ is a constant for all $t \in \mathbb{R}$, $\tilde{g}(t, y) = \tilde{g}(y) = g(y + x^*) - g(x^*)$. This observation allows us to focus the stability analysis to the zero solution by assuming that function g in (2.1) satisfies $g(0) = 0$.

Stability for Linear ODEs

One important fact of linear ODE systems is that all solutions possess the same type of stability. Moreover, the type of stability can be characterized by the asymptotic behavior of its fundamental matrix. For the autonomous system (2.1), the fundamental matrix can be determined by the eigenvalues of the matrix on the right hand side. More specifically, consider the following linear system of ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + b(t), \tag{2.5}$$

where $A = (a_{ij})_{i,j=1,\ldots,d}$ is a $d \times d$ matrix with real coefficients $a_{ij} \in \mathbb{R}$. Clearly if $x^*(t)$ is a solution to (2.5), then $y(t) = x(t) - x^*(t)$ is solution to the ODE system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Ay. \tag{2.6}$$

The same holds true if $x^*(t)$ is a solution to the homogeneous counterpart of Eq. (2.5), $x' = Ax$. Consequently, the stability of any solution of the linear ODE system (2.5) is equivalent to the stability of the zero solution to (2.6), regardless whether or not $b(t)$ is zero. The next theorem presents the stability of linear ODEs.

Theorem 2.2 *Let* $\{\lambda_j\}_{1 \le j \le d} \subset \mathbb{C}$ *be the set of eigenvalues for the matrix A. Then,*

(i) any solution to (2.6) *is exponentially stable if and only if the real parts of all the eigenvalues are negative, i.e.,* $\Re(\lambda_j) < 0$ *for all* $1 \le j \le d$.
(ii) any solution to (2.6) *is (uniformly) stable if and only if* $\Re(\lambda_j) \le 0$ *for all* $1 \le j \le d$ *and, if for those eigenvalues* λ_j $(1 \le j \le d)$ *such that* $\Re(\lambda_j) = 0$, *the dimension of the Jordan boxes associated to them in their canonical forms is* 1 *(in other words, their algebraic and geometric multiplicity coincide).*

Stability for Nonlinear ODEs

Theorem 2.2 can characterize completely the stability of linear ODE systems with constant coefficients. It can also be used to analyze the stability of nonlinear differential equations, by the so called *first approximation method* which can be briefly described as follows. Assume that the function g in (2.1) is continuously differentiable and satisfies $g(0) = 0$. Then according to the Taylor formula, g can be written as

$$g(x) = \mathrm{J}x + T_1(x), \qquad \mathrm{J} = \left(\frac{\partial g_i}{\partial x_j}(0)\right)_{i,j=1,\ldots,d} \tag{2.7}$$

where the higher order term $T_1(\cdot)$ is sufficiently small for small values of x in the sense that

$$\lim_{x \to 0} \frac{|T_1(x)|}{|x|} = 0.$$

We now state the following result on stability and instability of solutions to system (2.1).

Theorem 2.3 (Stability in first approximation) *Assume that* $g \in C^1(O)$ *and* $g(0) = 0$. *Let J be the Jacobian matrix defined in* (2.7). *Then,*

(i) if all the eigenvalues of matrix J have negative real parts, the trivial solution of (2.1) *is (locally) exponentially stable.*
(ii) if one of the eigenvalues of matrix J has positive real part, the trivial solution of (2.1) *is (locally) unstable.*

Stability by the Lyapunov Theory

Theorem 2.3 is based on a spectrum analysis of the linearization of system (2.1), and provides only local stability. For nonautonomous systems, such linearization requires additional justification (see, e.g., [55]). We next introduce the Lyapunov theory, that allows us to determine the stability of a system without either explicitly solving the differential Eq. (2.1) or approximating it by its first approximation linear system. Practically, the theory is a generalization of the idea that if there exists some "measure of energy" in one dynamical system, then we can study the rate of change of the energy to ascertain stability. In fact this "measure of energy" can be characterized by the so-called *Lyapunov function*. Roughly, if there exists a function $V : X \to \mathbb{R}$ satisfying certain conditions on V and $\dot{V}$ (the derivative along solution trajectories) that proves the Lyapunov stability of a system, we call it a Lyapunov function.

Definition 2.2 A continuous function $V : \mathcal{O} \in \mathbb{R}^d \to \mathbb{R}$ is said to be

- *positive definite* around $x = 0$ if $V(0) = 0$ and $V(x) > 0$ for all $x \in \mathcal{O}\backslash\{0\}$.
- *positive semi-definite* around $x = 0$ if $V(0) = 0$ and $V(x) \geq 0$ for all $x \in \mathcal{O}\backslash\{0\}$.
- *negative definite* or *negative semi-definite* if $-V$ is positive definite or positive semi-definite, respectively.

The following basic theorems provide sufficient conditions for the stability and instability of the origin of an autonomous dynamical system. First we recall that given a continuously differentiable function $V : \mathcal{O} \to \mathbb{R}$, the derivative of V along the trajectories of system (2.1), $\dot{V} : X \to \mathbb{R}$ is defined by

$$\dot{V}(x) = \sum_{i=1}^{d} \frac{\partial V}{\partial x_i}(x) g_i(x), \qquad x \in \mathcal{O}.$$

It is straightforward to check that if $x(\cdot)$ is a solution to (2.1), then

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \dot{V}(x(t)), \qquad \forall t \in \mathbb{R}.$$

Next we present the Lyapunov theorem on stability and the Tchetaev theorem on instability based on the Laypunov functions.

Theorem 2.4 (Lyapunov's stability theorem) *Let* $V : \mathcal{O} \to \mathbb{R}$ *be a continuously differentiable function with derivative* $\dot{V}$ *along the trajectories of the system* (2.1).

1. *If* V *is positive definite and* $\dot{V}$ *is negative semi-definite, then the zero solution is stable.*
2. *If* V *is positive definite and* $\dot{V}$ *is negative definite, then the zero solution is asymptotically stable.*
3. *If there exist some positive constants* a_1, a_2, a_3 *and* k *such that*

$$a_1|x|^k \leq V(x) \leq a_2|x|^k \quad \textit{and} \quad \dot{V}(x) \leq -a_3|x|^k, \quad \forall x \in \mathcal{O},$$

then the zero solution is exponentially stable.

Denote by $\mathbb{B}(x_0; r)$ the ball centered at x_0 with radius r.

Theorem 2.5 (Tchetaev's instability theorem) *Assume that there exists* $\rho > 0$ *and* $V \in C^1(\overline{\mathbb{B}}(0; \rho))$ *such that* $\mathbb{B}(0; \rho) \subseteq \mathcal{O}$ *and*

(i) $V(0) = 0$,
(ii) $\dot{V}$ *is positive definite in* $\mathbb{B}(0;\rho)$,
(iii) *for any* $\sigma \in (0,\rho)$ *there exists* $y_\sigma \in \mathbb{B}(0;\sigma)$ *such that* $V(y_\sigma) > 0$.

Then the zero solution is unstable.

Sometimes an equilibrium point can be asymptotically stable even if $\dot{V}$ is not negative definite. In fact, if we can find a Lyapunov function whose derivative along the trajectories of the system is only negative semi-definite, but we can further establish that no trajectory can stay identically at points where $\dot{V}$ vanishes, then the equilibrium must be asymptotically stable. This is the idea of LaSalle's invariance principle [54]. Before stating the principle we first introduce the definition of ω-limit set and invariant set, which is needed to state the LaSalle theorem.

Let $\varphi(t;x_0)$ be the autonomous dynamical system generated by the solutions of IVP (2.2).

Definition 2.3 A set $S \subset \mathbb{R}^d$ is said to be the *ω-limit set* of $\varphi(t;x_0)$ if for every $x \in S$, there exists a strictly increasing sequence of times t_n such that

$$\varphi(t_n;x_0) \to x \quad \text{as} \quad t_n \to \infty.$$

It is usual to write $S = \omega(x_0)$. In a similar way, it is defined the omega limit of a set $A \subset O$, and it is denoted as $\omega(A)$, as the set of points $x \in O$ such that there exist two sequences $\{x_n\} \subset A$, $t_n \to +\infty$ such that

$$\varphi(t_n;x_n) \to x, \quad \text{as } n \to +\infty.$$

Definition 2.4 A set $M \subset \mathbb{R}^d$ is said to be *(positively) invariant* if for all $x \in M$ we have

$$\varphi(t;x) \in M, \quad \forall t \geq 0.$$

Remark 2.5 The positive invariance means that as long as a solution passes a point inside M it will remain inside M forever, although the solution may have been outside of M for some previous instants of time.

Theorem 2.6 (LaSalle's Invariance Principle) *Let* $K \subset X$ *be a compact and positively invariant set,* $V : K \subset \mathbb{R}^d \to \mathbb{R}$ *be continuously differentiable with* $\dot{V} \leq 0$ *on* K*, and let* M *be the largest invariant set in* $E = \{x \in K : \dot{V} = 0\}$. *Then* $\varphi(t;x_0)$ *approaches* M *as* $t \to \infty$ *for every* $x_0 \in K$.

Remark 2.6 LaSalle's Invariance principle requires V to be continuously differentiable but not necessarily positive definite. It is applicable to any equilibrium set, rather than just an isolated equilibrium point. But when M is just a single point, it provides additional information about the type of stability of the equilibrium point. Indeed, when M is just a single point, and we are able to find a Lyapunov function which is only negative semi-definite, we can then ensure that this equilibrium is stable (thanks to Theorem 2.4) and also convergent as a consequence of LaSalle's principle, and hence is asymptotically stable.

To illustrate how the LaSalle invariance principle works, consider the following second order differential equation describing the movement of a pendulum with friction

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + k\sin x = 0,$$

where k and β are positive constants. First we transform the equation into an equivalent first order system by setting $x = y_1$, $\frac{dx}{dt} = y_2$ to obtain

$$\begin{aligned}\frac{dy_1}{dt} &= y_2,\\ \frac{dy_2}{dt} &= -\beta y_2 - k\sin y_1.\end{aligned}$$

Consider the function

$$V(y_1, y_2) = \frac{1}{2}y_2^2 + k(1 - \cos y_1).$$

It is easy to prove that V is positive definite on $\mathbb{B}(0; \pi)$ and satisfies

$$\dot{V}(y) = (k\sin y_1)\, y_2 + y_2\,(-\beta y_2 - k\sin y_1) = -\beta y_2^2 \le 0, \quad \forall (y_1, y_2) \in \mathbb{B}(0; \pi).$$

Therefore, $\dot{V}$ is negative semi-definite and, consequently, the trivial solution is stable. To prove that the trivial solution is also attractive, we will use LaSalle's invariance principle. Denote by

$$E := \{y \in \mathbb{B}(0; \pi) : \dot{V}(y) = 0\} \equiv \{(y_1, 0) : y_1 \in (-\pi, \pi)\}.$$

If we prove that $\{(0, 0)\}$ is the only positively invariant subset of E, then it will be also attractive, and thus the trivial solution will be uniformly asymptotically stable. To this end, pick $y_0 = (y_{01}, 0) \in E$, with $y_{01} \in (-\pi, \pi) \setminus \{0\}$, then the solution $\varphi(t; 0, y_0) = (\varphi_1(t), \varphi_2(t))$ satisfies the differential system as well as the initial condition $(\varphi_1(0), \varphi_2(0)) = y_0 = (y_{01}, 0)$. Notice that

$$\begin{aligned}&\varphi'(0) = \varphi_2(0) = 0,\\ &\varphi_2'(0) = -k\varphi_2(0) - k\sin\varphi_1(0) = -k\sin y_{01} \neq 0 \quad (y_{01} \in (-\pi, \pi) \setminus \{0\}).\end{aligned}$$

Therefore, the function φ_2 is strictly monotone in a neighborhood of $t = 0$, and since $\varphi_2(0) = 0$, there exists $\tilde{t} \in I_{max}(y_0)$ such that $\varphi_2(\tilde{t}) \neq 0$. Thus, solutions starting from points of $E \setminus \{(0, 0)\}$ leave this set and, therefore, the unique invariant subset is $(0, 0)$.

Remark 2.7 The LaSalle invariance principle is applicable to autonomous or periodic systems and can be extended to some specific nonautonomous systems (see, e.g., [65]), but not to general nonautonomous systems.

Remark 2.8 The Lasalle invariance principle provides a natural connection between the Lyapunov stability and the concept of attractors, to be introduced in the next section.

The largest invariant set M in Theorem 2.6 is the union of all invariant sets in the compact set K. It contains critical information on the asymptotic behavior of the system, as any solution has to approach this set as time goes on. In fact, the asymptotic dynamics of an autonomous dynamical system can be fully characterized by its invariant sets [51]. An invariant set possesses an independent dynamics inside itself and can also determine if any other trajectory outside the invariant set is approaching it (attractor) or being repelled from it (repeller). Next we will introduce in more details the concept of attractor, which is a compact invariant set that attracts all trajectories of a dynamical system starting either in a neighborhood (local attractor) or in the entire state space (global attractor).

2.2 Attractors

First we generalize the definition of invariance established in the previous section.

Definition 2.5 Let $\varphi : \mathbb{R} \times \mathcal{O} \to \mathcal{O}$ be a dynamical system on $\mathcal{O}$. A subset M of $\mathcal{O}$ is said to be

- invariant under φ (or φ-invariant), if

$$\varphi(t, M) = M \quad \text{for all } t \in \mathbb{R}.$$

- positively invariant under φ if

$$\varphi(t, M) \subset M \quad \text{for all } t \in \mathbb{R}.$$

- negatively invariant under φ if

$$\varphi(t, M) \supset M \quad \text{for all } t \in \mathbb{R}.$$

For any $x \in \mathcal{O}$, the function $\varphi(\cdot, x) : \mathbb{R} \to \mathcal{O}$ defines a solution curve, trajectory, or orbit of (2.2) passing through the point x_0 in $\mathcal{O}$. Graphically, the function $\varphi(\cdot, x)$ can be thought as an object moving along the curve

$$\gamma(x_0) := \{x \in \mathcal{O} \mid x = \varphi(t; x_0),\ t \in \mathbb{R}\}$$

defined by (2.2), as well as a possible parametrization of the orbit $\gamma(x_0)$ passing through the point x_0. A local attractor of a dynamical system is a compact invariant set that attracts all trajectories starting in some neighborhood of the attractor as $t \to \infty$, and a global attractor is such a compact invariant set that attracts not only trajectories in a neighborhood but trajectories in the entire state space. In this book we will focus on global attractor whose precise definition is given below.

Definition 2.6 A nonempty compact subset $\mathcal{A}$ is called a *global attractor* for a dynamical system φ on $\mathcal{O}$ if

(i) $\mathcal{A}$ is φ-invariant;
(ii) $\mathcal{A}$ attracts all bounded sets of $\mathcal{O}$, i.e.,

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t, B), \mathcal{A}) = 0 \quad \text{for any bounded subset } B \subset \mathcal{O},$$

where dist denotes the Hausdorff semi-distance given by

$$\operatorname{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} |a - b|.$$

The existence of attractors is mostly due to some dissipativity property (a loss of energy) of the dynamical system. The mathematical formulation for this concept is given in the following definition.

Definition 2.7 A nonempty bounded subset K of $\mathcal{O}$ is called an *absorbing set* of a dynamical system φ on $\mathcal{O}$ if for every bounded subset $B \subset \mathcal{O}$, there exists a time $T_B = T(B) \in \mathbb{R}$ such that $\varphi(t, B) \subset K$ for all $t > T_B$.

The following theorem stated in [51] provides the existence of a unique global attractor.

Theorem 2.7 *Assume that a dynamical system φ on $\mathcal{O}$ possesses a compact absorbing set K. Then, there exists a unique global attractor $\mathcal{A} \subset K$ which is given by*

$$\mathcal{A} = \omega(K).$$

If, in addition, K is positively invariant, then the unique global attractor is given by

$$\mathcal{A} = \bigcap_{t\geq 0} \varphi(t, K).$$

Global attractors are crucial for the analysis of dynamical systems, as they can characterize their asymptotic behavior. On the one hand, any trajectory starting from inside the global attractor is not allowed to leave it (because its invariance), and the original dynamical system restricted to the global attractor forms another dynamical system. On the other hand, any trajectory starting from a point outside the global attractor has to approach it, but can never touch it. Due to the complexity of the trajectories, the global attractor may eventually exhibit a strange and chaotic structure. Therefore, in addition to the general existence and continuity properties of the global attractors, geometrical structures of the global attractors can provide more detailed information about the long term dynamics of a dynamical system. We will not elaborate this point in this section, but will include below a few more comments including one example which can help the reader to understand the importance of the global attractor.

There are some particular dynamical systems, such as gradient dynamical systems, for which the internal geometrical structure of the global attractor is well known. These are the dynamical systems for which a "Lyapunov function" exists, but in this context is a continuous function V that (i) is nonincreasing along the trajectories of the dynamical system, i.e., the mapping $t \mapsto V(\varphi(t, x_0))$ is nonincreasing for any $x_0 \in \mathcal{O}$, and (ii) if it is constant along one trajectory, then this trajectory must be an equilibrium point. In fact, if $V(\varphi(t, x_0)) = V(x_0)$ for some $t > 0$, then x_0 is an equilibrium point. This concept is slightly different for the concept of Lyapunov function used in the stability analysis, but we will still adopt the same terminology while the context is clear.

One simple example of a gradient ordinary differential equation is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\nabla h(x), \tag{2.8}$$

where $h : \mathbb{R}^d \to \mathbb{R}$ is at least continuously differentiable. It is straightforward to verify that $V(x) := h(x)$ is a Lyapunov function for the dynamical system generated by (2.8). Let $\varphi(t, x_0)$ be the solution to (2.8) with initial condition $x(0) = x_0$, and let $\omega(x_0)$ be the $\omega-$limit set of the orbit through x_0. Then $\omega(x_0)$ is a compact invariant set in $\mathbb{R}^d$. According to the LaSalle invariance principle, for any $x \in \omega(x_0)$, the solution $\varphi(t, x)$ belongs to $\omega(x_0)$ and $V(\varphi(t, x)) = V(x)$ for all $t \in \mathbb{R}$. As a consequence, $\dot{V}(\varphi(t, x)) = 0$, which implies that $\nabla h(\varphi(t, x)) = 0$ for all $t \in \mathbb{R}$, i.e., $\omega(x_0)$ belongs to the set of equilibria of (2.8). More details on gradient dynamical systems can be found in [38].

The most interesting property of gradient dynamical systems is that their attractors are formed by the union of the unstable manifold of the equilibrium points (see, e.g., [53]). Briefly, given an equilibrium point x^* for the dynamical system φ, its unstable manifold is defined by

$$\mathcal{W}^u(x^*) = \{x \in \mathcal{O} : \varphi(t, x) \text{ is defined for } t \in \mathbb{R}, \text{ and } \varphi(t, x) \to x^*, \text{ as } t \to -\infty\}.$$

Then, if φ is a gradient dynamical system, it holds that the global attractor $\mathcal{A}$ is given by

$$\mathcal{A} = \bigcup_{x^* \in \mathcal{E}} \mathcal{W}^u(x^*),$$

where $\mathcal{E}$ denotes the set of all the equilibrium points.

Another interesting aspect of global attractors is related to *how* the global attractor determines the asymptotic dynamics of the system. According to the definition of the global attractor, we can say that any trajectory outside the global attractor can be tracked by some trajectories (or pieces of trajectories) inside the attractor. In other words, any external trajectory has a "target" on the attractor, that is getting closer to the trajectory as time passes. This property known as the "tracking property" reads as follows. Given a trajectory $\varphi(t, x_0)$ with x_0 not necessary inside the global attractor $\mathcal{A}$, and given any $\varepsilon > 0$ and $T > 0$, there exist a time $\tau = \tau(\varepsilon, T)$ and a point $v_0 \in \mathcal{A}$ such that

$$|\varphi(t+\tau, x_0) - \varphi(t, v_0)| \leq \varepsilon, \text{ for all } 0 \leq t \leq T.$$

If we wish to follow a trajectory for a time longer than T, then we may need to use more than one trajectory of $\mathcal{A}$ (see e.g., [68]).

Remark 2.9 There are several other interesting and important properties of the global attractor which can also help in understanding the dynamics of a dynamical system. The reader is referred to the monograph [68] for more details.

2.3 Applications

In this section we will introduce three applications of autonomous systems arising from different areas of the applied sciences. In particular we will discuss the long term dynamics including stability and existence of attractors for (1) a chemostat ecological model, (2) an SIR epidemic model and (3) a Lorenz-84 climate model. Later we will study the nonautonomous and random counterparts of these systems in Chaps. 3 and 4, respectively. To simplify the content and to avoid unnecessary repeated calculations, we will assume in this section that given any positive initial condition each of these systems possesses a continuous positive global solution. The proofs will be provided for their corresponding nonautonomous versions in Sect. 2.3.

2.3.1 Application to Ecology: A Chemostat Model

A chemostat is associated with a laboratory device which consists of three interconnected vessels and is used to grow microorganisms in a cultured environment (see Fig. 2.2). In its basic form, the outlet of the first vessel is the inlet for the second vessel and the outlet of the second vessel is the inlet for the third. The first vessel is called a feed bottle, which contains all the nutrients required to grow the microorganisms. All nutrients are assumed to be abundantly supplied except one, which is called a *limiting nutrient*. The contents of the first vessel are pumped into the second vessel, which is called the culture vessel, at a constant rate. The microorganisms feed on

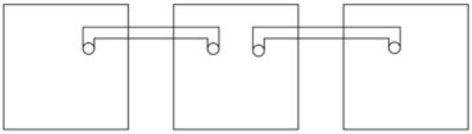

Fig. 2.2 A basic chemostat

nutrients from the feed bottle and grow in the culture vessel. The culture vessel is continuously stirred so that all the organisms have equal access to the nutrients. The contents of the culture vessel are then pumped into the third vessel, which is called a collection vessel. Naturally it contains nutrients, microorganisms and the products produced by the microorganisms [76].

As the best laboratory idealization of nature for population studies, the chemostat plays an important role in ecological studies [8, 13, 14, 32, 34, 41, 79, 82–84]. With some modifications it is also used as the model for waste-water treatment process [33, 52]. The chemostat model can be considered as the starting point for many variations that yield more realistic biological models, e.g., the recombinant problem in genetically altered organisms [56, 77] and the model of mammalian large intestine [36, 37]. More literature on the derivation and analysis of chemostat-like models can be found in [74, 75, 83] and the references therein.

Denote by x the growth-limiting nutrient and by y the microorganism feeding on the nutrient x. Assume that all other nutrients, except x, are abundantly available, i.e., we are interested only in the study of the effect of this essential limiting nutrient x on the species y. Under the standard assumptions of a chemostat, a list of basic parameters and functional relations in the system includes [76]:

- D, the rate at which the nutrient is supplied and also the rate at which the contents of the growth medium are removed.
- I, the input nutrient concentration which describes the quantity of nutrient available with the system at any time.
- a, the maximal consumption rate of the nutrient and also the maximum specific growth rate of microorganisms—a positive constant.
- U, the functional response of the microorganism describing how the nutrient is consumed by the species. It is known in literature as consumption function or uptake function. Basic assumptions on $U : \mathbb{R}^+ \to \mathbb{R}^+$ are given by

1. $U(0) = 0, \quad U(x) > 0$ for all $x > 0$.
2. $\lim_{x\to\infty} U(x) = L_1$, where $L_1 < \infty$.
3. U is continuously differentiable.
4. U is monotonically increasing.

Note that conditions 1 and 2 of the uptake function U ensure the existence of a positive constant $L > 0$ such that

$$U(x) \leq L \quad \text{for all} \quad x \in [0, \infty).$$

Throughout this book, when solid computations are needed, we assume that the consumption function follows the Michaelis-Menten or Holling type-II form:

$$U(x) = \frac{x}{\lambda + x}, \tag{2.9}$$

where $\lambda > 0$ is the half-saturation constant [76].

Denote by $x(t)$ and $y(t)$ the concentrations of the nutrient and the microorganism at any specific time, respectively. In the simplest model where I and D are both constants, the limited resource-consumer dynamics can be described by the following growth equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = D\,(I - x(t)) - a\frac{x(t)}{\lambda + x(t)}y(t), \tag{2.10}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -Dy(t) + a\frac{x(t)}{\lambda + x(t)}y(t). \tag{2.11}$$

Stability Analysis

We start from the behavior of equilibrium solutions. The equilibrium solutions to system (2.10)–(2.11) can be found by solving

$$D(I - x^*) - a\frac{x^*}{\lambda + x^*}y^* = 0,$$
$$-Dy^* + a\frac{x^*}{\lambda + x^*}y^* = 0,$$

which yields

$$(x^*, y^*) = (I, 0) \quad \text{or} \quad (x^*, y^*) = \left(\frac{D\lambda}{a - D}, I - \frac{D\lambda}{a - D}\right).$$

1. When $a < D$, system (2.10)–(2.11) has only one axial equilibrium $(I, 0)$, which is globally asymptotically stable. This means that the microorganisms y become extinct.
   Sketch of proof: First, it is straightforward to check that the positive quadrant is positively invariant. In fact, first notice that

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=0} = DI > 0.$$

   On the other hand, the set $\{(x, 0) : x > 0\}$ is formed by three orbits: (1) the equilibrium point $(I, 0)$, (2) the segment $\{(x, 0) : 0 < x < I\}$ which is parameterized by the solution $\{(I + (x_0 - I)e^{-Dt}, 0) : t \in (-D \log I(I - x_0)^{-1}, +\infty)\}$ for any fixed $x_0 \in (0, I)$, and (3) the unbounded segment $\{(x, 0) : x > I\}$ which is parameterized by the solution $\{(I + (x_0 - I)e^{-Dt}, 0) : t \in \mathbb{R}\}$. We now investigate the stability in first approximation by Theorem 2.3. The corresponding Jacobian of system (2.10)–(2.11) and its value at the point $(I, 0)$ are, respectively,

$$\mathrm{J}(x, y) = \begin{pmatrix} -D - \dfrac{a\lambda y}{(\lambda + x)^2} & -\dfrac{ax}{\lambda + x} \\ \dfrac{a\lambda y}{(\lambda + x)^2} & -D + \dfrac{ax}{\lambda + x} \end{pmatrix}, \qquad \mathrm{J}(I, 0) = \begin{pmatrix} -D & -\dfrac{aI}{\lambda + I} \\ 0 & -D + \dfrac{aI}{\lambda + I} \end{pmatrix}.$$

The eigenvalues of $\mathrm{J}(I, 0)$ are $-D$ and $-D + \dfrac{aI}{\lambda + I}$, which are both negative because $a < D$ is assumed. The equilibrium $(I, 0)$ is then asymptotically exponentially stable.

2. When $a > D$ and $\frac{\lambda D}{a-D} < I$, system (2.10)–(2.11) has two equilibria, among which the positive equilibrium $(x^*, y^*) = \left(\frac{D\lambda}{a-D}, I - \frac{\lambda D}{a-D}\right)$ is globally asymptotically stable. This means that the microorganisms y and the nutrient x co-exist.
Sketch of proof: In this case, the axial equilibrium becomes unstable because one of the eigenvalues of $\mathrm{J}(I, 0)$, $-D + \dfrac{aI}{\lambda + I}$, is positive due to the condition $\frac{\lambda D}{a-D} < I$. On the other hand, the positive equilibrium point (x^*, y^*) is now globally asymptotically (exponentially) stable, since the Jacobian evaluated at (x^*, y^*)

$$\mathrm{J}(x^*, y^*) = \begin{pmatrix} -D - \dfrac{a\lambda x^*}{(\lambda + x^*)^2} & -D \\ \dfrac{a\lambda y^*}{(\lambda + x^*)^2} & 0 \end{pmatrix},$$

has two negative eigenvalues $-D$ and $-\dfrac{a\lambda x^*}{(\lambda + x^*)^2}$.

Global Attractor

As each IVP associated to (2.10)–(2.11) corresponding to positive initial values has a positive global solution, this system generates an autonomous dynamical system $\varphi(t, x_0, y_0)$. Adding (2.10)–(2.11) we obtain immediately that

$$\frac{\mathrm{d}(x + y)}{\mathrm{d}t} = DI - D(x + y),$$

and given $x(0) = x_0$, $y(0) = y_0$ we have

$$x(t) + y(t) = I + (x_0 + y_0 - I)e^{-Dt}.$$

This implies that

$$K_\varepsilon := \left\{(x, y) \in \mathbb{R}^2_+ : x + y \leq I + \varepsilon\right\}$$

is a bounded absorbing set for the dynamical system φ generated by solutions of (2.10)–(2.11). Hence due to Theorem 2.7 φ possesses a global attractor $\mathscr{A}$ inside the nonnegative quadrant $\mathbb{R}^2_+$. Moreover, we obtain the geometric structure of the global attractor as follows.

1. When $a < D$, the attractor $\mathscr{A}$ has a single point $(I, 0)$.
2. When $a > D$ and $\frac{\lambda D}{a-D} < I$, the attractor $\mathscr{A}$ consists of two points, $(I, 0)$ and $\left(\frac{D\lambda}{a-D}, I - \frac{D\lambda}{a-D}\right)$, and heteroclinic solutions between them (solutions that converge in one time direction to a steady state and in the other direction to different steady state).

2.3.2 Application to Epidemiology: The SIR Model

The modeling of infectious diseases and their spread is crucial in the field of mathematical epidemiology, as it is an important and power tool for gauging the impact of different vaccination programs on the control or eradication of diseases. Here we will only introduce a simple autonomous model, which does not take into account age structure nor environmental fluctuation. More sophisticated models will be discussed in later chapters.

The classical work on epidemics is due to Kermack and McKendrick [42–44]. The Kermack-McKendrick model is essentially a compartmental model based on relatively simple assumptions on the rates of flows between different classes of members of the population. The population is divided into three classes labeled S, I and R, where $S(t)$ denotes the number of individuals who are not yet infected but susceptible to the disease, $I(t)$ denotes the number of infected individuals, assumed infectious and are able to spread the disease by contact with susceptibles, and $R(t)$ denotes the number of individuals who have been infected and then removed from the possibility of being infected again or of spreading infection. Removal is carried out through isolation from the rest of the population, through immunization against infection, recovery from the disease with full immunity against reinfection, or through death caused by the disease.

The terminology "SIR" is used to describe a disease that confers immunity against reinfection, to indicate that the passage of individuals is from the susceptible class S to the infective class I to the removed class R. Epidemics are usually diseases of this type. The terminology "SIS" is used to describe a disease with no immunity against re-infection, to indicate that the passage of individuals is from the susceptible class to the infective class and then back to the susceptible class. Usually, diseases caused by a virus are of SIR type, while diseases caused by bacteria are of SIS type.

In this book we will use SIR as an example and investigate dynamics of autonomous, nonautonomous and random SIR models. The simplest SIR model assumes that the total population size is held constant, i.e.,

$$S(t) + I(t) + R(t) = N,$$

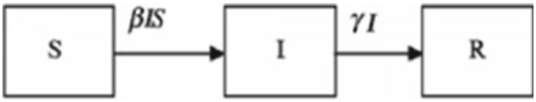

Fig. 2.3 Transition between SIR compartments

by making birth and death rates equal. Denote by ν the birth and death rate of all population, and assume that all newborns are susceptible to disease, we have the following basic SIR model.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \nu(S + I + R) - \frac{\beta}{N}SI - \nu S, \tag{2.12}$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta}{N}SI - \nu I - \gamma I, \tag{2.13}$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I - \nu R, \tag{2.14}$$

where γ is the fraction of infected individuals being removed per unit of time, and β is the contact rate of susceptible and infected individuals (see Fig. 2.3).

The vector field of system (2.12)–(2.14) is continuously differentiable, and hence ensures the maximal solution to the corresponding IVP exists and is unique. Furthermore the solution is defined globally in time (see Chap. 3 for more details). Notice that assumption of constant total population, $S(t)+I(t)+R(t) = N$, reduces system (2.12)–(2.14) to the following two dimensional system.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \nu N - \frac{\beta}{N}SI - \nu S, \tag{2.15}$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta}{N}SI - \nu I - \gamma I. \tag{2.16}$$

Note that such a reduction needs the positiveness of solutions which will be discussed later in Chap. 3.

Stability Analysis

The equilibrium solutions to system (2.15)–(2.16) can be found by solving

$$\nu N - \frac{\beta}{N}S^* I^* - \nu S^* = 0,$$
$$\frac{\beta}{N}S^* I^* - \nu I^* - \gamma I^* = 0,$$

which yields two equilibrium solutions

$$(S^*, I^*) = (N, 0), \quad (S^*, I^*) = \left(\frac{N(\nu+\gamma)}{\beta}, \nu N\left(\frac{1}{\nu+\gamma} - \frac{1}{\beta}\right)\right).$$

1. When $\nu + \gamma > \beta$, system (2.15)–(2.16) has only one axial equilibrium, $(N, 0)$, in the nonnegative quadrant, which is globally asymptotically stable. This means that all infected and removed individuals are cleared, the system is fully immunized.
Sketch of proof: The corresponding Jacobian of system (2.15)–(2.16) and its value at the equilibrium point $(N, 0)$ are, respectively,

$$\mathrm{J}(S, I) = \begin{pmatrix} -\frac{\beta}{N} I - \nu & \frac{\beta}{N} S \\ \frac{\beta}{N} I & \frac{\beta}{N} S - \nu - \gamma \end{pmatrix}, \qquad \mathrm{J}(N, 0) = \begin{pmatrix} -\nu & -\beta \\ 0 & \beta - \nu - \gamma \end{pmatrix}.$$

The eigenvalues of $\mathrm{J}(N, 0)$ are negative, and hence the axial equilibrium $(N, 0)$ is globally asymptotically stable.
2. When $\nu + \gamma < \beta$, system (2.15)–(2.16) has two equilibria, among which the positive equilibrium $(S^*, I^*) = \left(\frac{N(\nu+\gamma)}{\beta}, \nu N \left(\frac{1}{\nu+\gamma} - \frac{1}{\beta}\right)\right)$ is globally asymptotically stable and the axial equilibrium $(N, 0)$ is unstable. This means that the system is at endemic state, where infected, removed and susceptible individuals co-exist.

Sktech of proof: The instability of the axial equilibrium is obvious as $J(N, 0)$ now has a positive eigenvalue since $\nu + \gamma < \beta$. The Jacobian evaluated at the positive equilibrium (S^*, I^*) is

$$\mathrm{J}(S^*, I^*) = \begin{pmatrix} -\beta\nu\left(\frac{1}{\nu+\gamma} - \frac{1}{\beta}\right) - \nu & -(\nu + \gamma) \\ \beta\nu\left(\frac{1}{\nu+\gamma} - \frac{1}{\beta}\right) & 0 \end{pmatrix},$$

whose eigenvalues are the roots of the quadratic equation

$$r^2 + \left(\beta\nu\left(\frac{1}{\nu + \gamma} - \frac{1}{\beta}\right) + \nu\right) r + \beta(\nu + \gamma)\left(\frac{1}{\nu + \gamma} - \frac{1}{\beta}\right) = 0,$$

both of which have negative real part. Hence the positive equilibrium (S^*, I^*) is asymptotic exponential stable.

Attractors

Any IVP of (2.15)–(2.16) associated with positive initial values has positive global solutions (see Chap. 3 for detailed proof). Hence system (2.15)–(2.16) generates an autonomous dynamical system $\varphi(t, S_0, I_0)$. By adding (2.15)–(2.16) we obtain immediately that

$$\frac{\mathrm{d}(S + I)}{\mathrm{d}t} = \nu N - \nu(S + I) - \gamma I \leq \nu N - \nu(S + I),$$

and given $S(0) = S_0$, $I(0) = I_0$ we have

$$S(t) + I(t) \leq N + (S_0 + I_0 - N)e^{-\nu t}.$$

This implies that

$$K_\varepsilon := \left\{(S, I) \in \mathbb{R}_+^2 : S + I \leq N + \varepsilon\right\}$$

is a bounded absorbing set for the dynamical system φ generated by solutions of (2.15)–(2.16). Hence due to Theorem 2.7, φ possesses a global attractor $\mathcal{A}$ inside the nonnegative quadrant $\mathbb{R}_+^2$. Moreover, at light of the stability properties of the equilibrium points, we obtain the geometric structure of the global attractor as follows.

1. When $\nu + \gamma > \beta$, the attractor $\mathcal{A}$ has a single point $(N, 0)$.
2. When $\nu + \gamma < \beta$, the attractor $\mathcal{A}$ consists of two points, $(N, 0)$, and $\left(\frac{N(\nu+\gamma)}{\beta}, \nu N\left(\frac{1}{\nu+\gamma} - \frac{1}{\beta}\right)\right)$, and heteroclinic solutions between them.

2.3.3 Application to Climate Change: The Lorenz-84 Model

A model of atmospheric circulation was introduced by Lorenz in 1984, defined by a system of three nonlinear autonomous differential equations [58, 59, 80]. Let x represent the pole ward temperature gradient or the intensity of the westerly wind current, y and z be the strengths of cosine and sine phases of a chain of superposed waves transporting heat poleward, respectively. Then a modified Hadley circulation can be modeled by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax - y^2 - z^2 + aF, \tag{2.17}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y + xy - bxz + G, \tag{2.18}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -z + bxy + xz, \tag{2.19}$$

where the coefficient a, if less than 1, allows the westerly wind current to damp less rapidly than the waves, the terms in b represent the displacement of the waves due to interaction with the westerly wind, the terms in F and G are thermal forcings: F represents the symmetric cross-latitude heating contrast and G accounts for the asymmetric heating contrast between oceans and continents.

Despite the simplicity of the Lorenz-84 model, it addresses many key applications in climate studies such as how the coexistence of two possible climates combined with variations of the solar heating causes seasons with inter-annual variability [6, 58, 59, 67], how the climate is affected by the interactions between the atmosphere and oceans [7, 70], how the asymmetry between oceans and continents may result in complex behaviors of the system [62], etc. In addition to applications, the Lorenz-84 model has also attracted much attentions from mathematicians because of certain interesting and subtle mathematical aspects of its underlying differential equations such as multistability, intransitivity and bifurcation [35]. Here we will focus on the Lyapunov stability and existence of attractors for the Lorenz-84 model.

Notice that the vector field of system (2.17)–(2.19) is continuously differentiable, which ensures existence and uniqueness of maximal solutions to the corresponding IVP. Moreover, once we prove that the solutions are absorbed by a ball centered at zero, then it follows immediately from Theorem 1.5 that all solutions are defined globally in time (more details will be provided in Chap. 3).

Stability Analysis

The equilibrium points of system (2.17)–(2.19) can be calculated by solving

$$a(F - x^*)(1 - 2x^* + (1 + b^2)(x^*)^2) = G^2, \tag{2.20}$$

$$\frac{(1 - x^*)G}{1 - 2x^* + (1 + b^2)(x^*)^2} = y^*, \tag{2.21}$$

$$\frac{bx^*G}{1 - 2x^* + (1 + b^2)(x^*)^2} = z^*. \tag{2.22}$$

When $G = 0$, system (2.20)–(2.22) has only one solution, $(F, 0, 0)$, and hence (2.17)–(2.19) possesses only one equilibrium, $(F, 0, 0)$. To analyze its stability, we first shift the point $(F, 0, 0)$ to the origin $(0, 0, 0)$ by performing the change of variable $\tilde{x} = x - F$. The resulting system reads

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = -a\tilde{x} - y^2 - z^2, \tag{2.23}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y + (\tilde{x} + F)y - b(\tilde{x} + F)z, \tag{2.24}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -z + b(\tilde{x} + F)y + (\tilde{x} + F)z. \tag{2.25}$$

Though we can use again the first approximation method to analyze the stability of $(0, 0, 0)$, to present alternative methods we study the stability of the $(0, 0, 0)$ for this system, by the Lypunov theory and Tchetaev's theorem.

1. When $F < 1$, the equilibrium $(0, 0, 0)$ is globally asymptotically stable.
   Sketch of proof: Define the function

$$V(\tilde{x}, y, z) := \frac{1}{2}(\tilde{x}^2 + y^2 + z^2). \tag{2.26}$$

   Then $V(0, 0, 0) = 0$ and $V(\tilde{x}, y, z) > 0$ for all $(\tilde{x}, y, z) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, what implies that V is positive definite in any ball centered in $(0, 0, 0)$ (in particular, in the entire $\mathbb{R}^3$). Moreover, the derivative of V along trajectories of system (2.17)–(2.19) satisfies

$$\dot{V} = -a\tilde{x}^2 - (1-F)y^2 - (1-F)z^2,$$

which implies that V is negative definite when $F < 1$. The conclusion follows immediately from Theorem 2.4.

2. When $F > 1$, the equilibrium $(0, 0, 0)$ is unstable.
Sketch of proof: The previous Lyapunov function defined in (2.26) is no longer valid, but we can apply Tchetaev's theorem. To this end, let us consider the function

$$V(\tilde{x}, y, z) := \frac{1}{2}(-\tilde{x}^2 + y^2 + z^2). \tag{2.27}$$

Then, straightforward computations give

$$\dot{V} = a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2 + 2\tilde{x}(y^2 + z^2).$$

Observing that

$$\lim_{(\tilde{x},y,x)\to(0,0,0)} \frac{2\tilde{x}(y^2+z^2)}{a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2} = 0,$$

we can find a positive ρ such that

$$\left| \frac{2\tilde{x}(y^2+z^2)}{a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2} \right| \le \frac{1}{2}, \text{ for } (\tilde{x}, y, z) \in \mathbb{B}((0,0,0); \rho).$$

Hence

$$2\tilde{x}(y^2+z^2) \ge -\frac{1}{2}(a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2),$$

which implies that

$$\begin{aligned}\dot{V} &= a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2 + 2\tilde{x}(y^2+z^2)\\ &\ge \frac{1}{2}(a\tilde{x}^2 + (F-1)y^2 + (F-1)z^2)\end{aligned}$$

for all $(\tilde{x}, y, z) \in \mathbb{B}((0,0,0); \rho)$. The instability of $(0, 0, 0)$ then follows immediately from Tchetaev's theorem.

3. When $F = 1$, we cannot deduce any information from Theorem 2.3. However, since the Lyapunov function defined in (2.26) has a negative semi-definite $\dot{V}$, the equilibrium point $(0, 0, 0)$ is at least stable. Next we will apply the LaSalle invariance principle to prove that the equilibrium point is also attractive, and hence asymptotically stable. In fact, $\dot{V} = -a\tilde{x}^2 \le 0$ when $F = 1$. Denote by E the set

$$E := \{(\tilde{x}, y, z) \in \mathbb{R}^3 : \dot{V}(\tilde{x}, y, z) = 0\} = \{(0, y_0, z_0) : y_0, z_0 \in \mathbb{R}\}.$$

Let us prove that $(0, 0, 0)$ is the unique invariant subset of E. To this end, pick $(0, y_0, z_0) \in E \setminus \{(0, 0, 0)\}$, and denote by $\varphi(\cdot)$ the corresponding solution of the IVP. Then we only need to observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(0) = -y_0^2 - z_0^2 < 0.$$

Since $\varphi_1(0) = 0$, then there exists $\varepsilon > 0$ such that $\varphi_1(t) < 0$ for all $t \in (0, \varepsilon)$. This proves that $(0, 0, 0)$ is the only invariant subset of E, and therefore, the equilibrium point $(0, 0, 0)$ is asymptotically stable.

When $G \neq 0$, the dynamical behavior of system (2.17)–(2.19) becomes complicated and exhibits chaotic attractors. Note that from (2.20)–(2.22) it is difficult to obtain an explicit expression of the equilibrium points on parameters a, b, F and G. It is also difficult to determine how many equilibrium points there exist. Hence it is impossible to apply the Lyapunov method to obtain stability properties. However, we can still investigate the existence of global attractor.

Global Attractor

System (2.17)–(2.19) possesses a global solution that generates an autonomous dynamical system $\varphi(t, x_0, y_0, z_0)$ (see Chap. 3 for a detailed proof). To prove the existence of an attractor, we first prove the existence of a compact absorbing set. In fact, it is sufficient to prove the existence of a bounded absorbing set, as its closure will give us a compact absorbing set.

Let $D \subset \mathbb{R}^3$ be a bounded set. Then, there exists $k > 0$ such that $|u_0| \leq k$ for all $u_0 \in D$. Then, for any $u_0 = (x_0, y_0, z_0) \in D$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2 + z^2) &= 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} + 2z\frac{\mathrm{d}z}{\mathrm{d}t}\\ &= 2\left(-ax^2 - y^2 - z^2 + aFx + Gy\right)\\ &\leq -ax^2 - y^2 - 2z^2 + aF^2 + G^2\\ &\leq -\mu_1(x^2 + y^2 + z^2) + aF^2 + G^2,\end{aligned}$$

where $\mu_1 = \min\{a, 1\}$, and hence

$$\begin{aligned}x^2(t) + y^2(t) + z^2(t) &\leq (x_0^2 + y_0^2 + z_0^2)|u_0|^2 e^{-\mu_1 t} + \frac{aF^2 + G^2}{\mu_1}(1 - e^{-\mu_1 t})\\ &\leq k^2 e^{-\mu_1 t} + \frac{aF^2 + G^2}{\mu_1}.\end{aligned}$$

This implies that given any fixed positive ε, there exists $T = T(\varepsilon, D) >$ such that

$$x^2(t) + y^2(t) + z^2(t) \leq \frac{aF^2 + G^2}{\mu_1} + \varepsilon, \quad \text{for all } t \geq T. \tag{2.28}$$

In fact, simple calculations yield that we can take $T(\varepsilon, D) = -\frac{1}{\mu_1}\log\dfrac{\varepsilon}{k^2}$. Note that we can always take k such that $k^2 > \varepsilon$, so that $-\frac{1}{\mu_1}\log\dfrac{\varepsilon}{k^2} > 0$.

The inequality (2.28) implies that

$$K_\varepsilon := \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq \frac{aF^2 + G^2}{\mu_1} + \varepsilon\right\}$$

is a bounded absorbing set for φ. Hence by Theorem 2.7, we conclude immediately that the dynamical system generated by solutions to (2.17)–(2.18) possesses a global attractor $\mathscr{A}$. When $G = 0$, this attractor $\mathscr{A}$ consists of a single point, $(F, 0, 0)$. When $G \neq 0$, the geometric structure of $\mathscr{A}$ is difficult to obtain. In fact, numerical simulation shows that the attractor $\mathscr{A}$ exhibits chaotic behavior.

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