Chapter 2
Slice Regular Functions: Algebra

2.1 Definition and Main Results

In this chapter, we will present some basic material on slice regular functions, a generalization of holomorphic functions to the quaternions.

The skew field of quaternions $\mathbb{H}$ is defined as

$$\mathbb{H} = \{q = x_0 + ix_1 + jx_2 + kx_3 \ : \ x_0, \dots, x_3 \in \mathbb{R}\},$$

where the imaginary units i, j, k satisfy

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j.$$

It is a noncommutative field and since $\mathbb{C}$ can be identified (in a nonunique way) with a subfield of $\mathbb{H}$, it extends the class of complex numbers. On $\mathbb{H}$, we define the Euclidean norm

$$|q| = \sqrt{x_0^2 + x_1^2 + x_2^3 + x_3^2}.$$

The symbol $\mathbb{S}$ denotes the unit sphere of purely imaginary quaternion, i.e.,

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3, \text{ such that } x_1^2 + x_2^2 + x_3^3 = 1\}.$$

Note that if $I \in \mathbb{S}$, then $I^2 = -1$. For this reason, the elements of $\mathbb{S}$ are also called imaginary units. For any fixed $I \in \mathbb{S}$ we define

$$\mathbb{C}_I := \{x + Iy \ : \ x, y \in \mathbb{R}\}.$$

It is easy to verify that $\mathbb{C}_I$ can be identified with a complex plane, moreover $\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I$. The real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and thus a real quaternion can be associated with any imaginary unit I. Any nonreal quaternion $q = x_0 + ix_1 + jx_2 + kx_3$ is uniquely associated to the element $I_q \in \mathbb{S}$ defined by

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$$I_q := \frac{ix_1 + jx_2 + kx_3}{|ix_1 + jx_2 + kx_3|}.$$

It is obvious that q belongs to the complex plane $\mathbb{C}_{I_q}$.

Definition 2.1 Let U be an open set in $\mathbb{H}$ and let $f : U \to \mathbb{H}$ be real differentiable. The function f is said to be (left) slice regular or (left) slice hyperholomorphic if for every $I \in \mathbb{S}$, its restriction f_I to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ passing through origin and containing I and 1 satisfies

$$\overline{\partial}_I f(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0,$$

on $U \cap \mathbb{C}_I$. The class of (left) slice regular functions on U will be denoted by $\mathscr{R}(U)$.

Analogously, a function is said to be right slice regular in U if

$$(f_I\overline{\partial}_I)(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I\right) = 0,$$

on $U \cap \mathbb{C}_I$.

It is immediate to verify that:

Proposition 2.1 *Let U be an open set in $\mathbb{H}$. Then $\mathscr{R}(U)$ is a right linear space on $\mathbb{H}$.*

Let $f \in \mathscr{R}(U)$. The so-called left I-derivative of f at a point $q = x + Iy$ is defined by

$$\partial_I f_I(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) - I\frac{\partial}{\partial y} f_I(x + Iy)\right),$$

and the right I-derivative of f at $q = x + Iy$ is defined by

$$\partial_I f_I(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) - \frac{\partial}{\partial y} f_I(x + Iy)I\right).$$

Let us now introduce another suitable notion of derivative:

Definition 2.2 Let U be an open set in $\mathbb{H}$, and let $f : U \to \mathbb{H}$ be a slice regular function. The slice derivative $\partial_s f$ of f, is defined by:

$$\partial_s(f)(q) = \begin{cases} \partial_I(f)(q) & \text{if } q = x + Iy,\ y \neq 0, \\ \dfrac{\partial f}{\partial x}(x) & \text{if } q = x \in \mathbb{R}. \end{cases}$$

The definition of slice derivative is well posed because it is applied only to slice regular functions, thus

$$\frac{\partial}{\partial x} f(x + Iy) = -I \frac{\partial}{\partial y} f(x + Iy), \qquad \forall I \in \mathbb{S}.$$

Similarly to what happens in the complex case, we have

$$\partial_s(f)(x + Iy) = \partial_I(f)(x + Iy) = \partial_x(f)(x + Iy).$$

We will often write $f'(q)$ instead of $\partial_s f(q)$.
It is important to note that if $f(q)$ is a slice regular function then also $f'(q)$ is a slice regular function.
Let $I, J \in \mathbb{S}$ be such that I and J are orthogonal, so that $I, J, IJ = K$ is a basis of $\mathbb{H}$ and write the restriction $f_I(x + Iy) = f(x + Iy)$ of f to the complex plane $\mathbb{C}_I$ as $f = f_0 + If_1 + Jf_2 + Kf_3$, where $f_0, \ldots, f_3$ are $\mathbb{R}$-valued. In alternative, it can also be written as $f = F + GJ$ where $f_0 + If_1 = F$, and $f_2 + If_3 = G$ are $\mathbb{C}_I$-valued. This observation immediately gives the following result:

Lemma 2.1 (Splitting Lemma) *If f is a slice regular function on U, then for every $I \in \mathbb{S}$, and every $J \in \mathbb{S}$, perpendicular to I, there are two holomorphic functions $F, G : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that for any $z = x + Iy$*

$$f_I(z) = F(z) + G(z)J.$$

Proof Since f is slice regular, we know that

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

Therefore by decomposing the values of f_I into its complex components

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) F(x+Iy) + \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) G(x+Iy)J,$$

the statement immediately follows. □

We now consider slice regular functions on open balls $B(0; r)$ centered at the origin with radius $r > 0$. We have:

Theorem 2.1 *A slice regular function $f : B(0; r) \to \mathbb{H}$ has a series representation of the form*

$$f(q) = \sum_{n=0}^{\infty} q^n \frac{1}{n!} \cdot \frac{\partial^n f}{\partial x^n}(0) = \sum_{n=0}^{\infty} q^n a_n, \tag{2.1}$$

uniformly convergent on $B(0; r)$. Moreover, $f \in \mathscr{C}^\infty(B(0; R))$.

Proof Let us use the Splitting Lemma to write $f_I(z) = F(z) + G(z)J$, where $z = x + Iy$ and F, G are holomorphic. Then

$$f_I^{(n)}(z) = \partial_I^{(n)} f(z) = \frac{\partial^n}{\partial z^n} F(z) + \frac{\partial^n}{\partial z^n} G(z) J.$$

Now we use the fact that F and G admit power series expansions which converge uniformly and absolutely on any compact set in $B(0; r) \cap \mathbb{C}_I$:

$$\begin{aligned}
f_I(z) &= \sum_{n\geq 0} z^n \frac{1}{n!} \frac{\partial^n F}{\partial z^n}(0) + \sum_{n\geq 0} z^n \frac{1}{n!} \frac{\partial^n G}{\partial z^n}(0) J \\
&= \sum_{n\geq 0} z^n \frac{1}{n!} \left(\frac{\partial^n (F + GJ)}{\partial z^n}(0) \right) \\
&= \sum_{n\geq 0} z^n \frac{1}{n!} \left(\frac{\partial^n f}{\partial z^n}(0) \right) \\
&= \sum_{n\geq 0} z^n \frac{1}{n!} \left(\frac{1}{2} \left(\frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) \right)^n f(0) \\
&= \sum_{n\geq 0} z^n \frac{1}{n!} f^{(n)}(0).
\end{aligned}$$

The function f is infinitely differentiable in $B(0; r)$ by the uniform convergence on compact subsets. □

The proof of the following result is the same as the proof in the complex case.

Theorem 2.2 *Let $\{a_n\}$, $n \in \mathbb{N}$ be a sequence of quaternions and let*

$$r = \frac{1}{\overline{\lim}_{n\to\infty} |a_n|^{1/n}}.$$

If $r > 0$ then the power series $\sum_{N=0}^{\infty} q^n a_n$ converges absolutely and uniformly on compact subsets of $B(0; r)$. Its sum defines a slice regular function on $B(0; r)$.

Definition 2.3 A function slice regular on $\mathbb{H}$ will be called entire slice regular or entire slice hyperholomorphic.

Every entire regular function admits power series expansion of the form (2.1) which converges everywhere in $\mathbb{H}$ and uniformly on the compact subsets of $\mathbb{H}$.
A simple computation shows that since the radius of convergence is infinite we have

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0,$$

or, equivalently:

$$\lim_{n\to\infty} \frac{\log |a_n|}{n} = -\infty.$$

2.2 The Representation Formula

Slice regular functions possess good properties on specific open sets that are called axially symmetric slice domains. On these domains, slice regular functions satisfy the so-called Representation Formula which allows to reconstruct the values of the function once that we know its values on a complex plane $\mathbb{C}_I$. As we shall see, this will allow also to define a suitable notion of multiplication between two slice regular functions.

Definition 2.4 Let $U \subseteq \mathbb{H}$. We say that U is axially symmetric if, for every $x+Iy \in U$, all the elements $x + Jy$ for $J \in \mathbb{S}$ are contained in U. We say that U is a *slice domain* if it is a connected set whose intersection with every complex plane $\mathbb{C}_I$ is connected.

Definition 2.5 Given a quaternion $q = x + Iy$, the set of all the elements of the form $x + Jy$ where J varies in the sphere $\mathbb{S}$ is a two-dimensional sphere denoted by $[x + Iy]$ or by $x + \mathbb{S}y$.

The Splitting Lemma allows us to prove:

Theorem 2.3 (Identity Principle). *Let $f : U \to \mathbb{H}$ be a slice regular function on a slice domain U. Denote by $Z_f = \{q \in U : f(q) = 0\}$ the zero set of f. If there exists $I \in \mathbb{S}$ such that $\mathbb{C}_I \cap Z_f$ has an accumulation point, then $f \equiv 0$ on U.*

Proof The restriction $f_I = F + GJ$ of f to $U \cap \mathbb{C}_I$ is such that $F, G : U \cap \mathbb{C}_I \to \mathbb{C}_I$ are holomorphic functions. Under the hypotheses, F and G vanish on a set $\mathbb{C}_I \cap Z_f$ which has an accumulation point so F and G are both identically zero. So f_I vanishes on $U \cap \mathbb{R}$ and, in particular, f vanishes on $U \cap \mathbb{R}$. Thus the restriction of f to any other complex plane $\mathbb{C}_L$ vanishes on a set with an accumulation point and so $f_L \equiv 0$. Since

$$U = \bigcup_{I \in \mathbb{C}_I} U \cap \mathbb{C}_I$$

we have that f vanishes on U. □

The following result shows that the values of a slice regular function defined on an axially symmetric slice domain can be computed from the values of its restriction to a complex plane:

Theorem 2.4 (Representation Formula). *Let f be a slice regular function defined on an axially symmetric slice domain $U \subseteq \mathbb{H}$. Let $J \in \mathbb{S}$ and let $x \pm Jy \in U \cap \mathbb{C}_J$. Then the following equality holds for all $q = x + Iy \in U$:*

$$\begin{aligned} f(x+Iy) &= \frac{1}{2}\Big[f(x+Jy)+f(x-Jy)\Big] + I\frac{1}{2}\Big[J[f(x-Jy)-f(x+Jy)]\Big] \\ &= \frac{1}{2}(1-IJ)f(x+Jy) + \frac{1}{2}(1+IJ)f(x-Jy). \end{aligned} \tag{2.2}$$

Moreover, for all $x + Ky \subseteq U$, $K \in \mathbb{S}$, there exist two functions α, β, independent of I, such for any $K \in \mathbb{S}$ we have

$$\frac{1}{2}\Big[f(x+Ky)+f(x-Ky)\Big] = \alpha(x,y), \qquad \frac{1}{2}\Big[K[f(x-Ky)-f(x+Ky)]\Big] = \beta(x,y). \tag{2.3}$$

Proof If $\mathrm{Im}(q) = 0$, then q is real and the proof is immediate. Otherwise let us define the function $\psi : U \to \mathbb{H}$ by

$$\psi(q) = \frac{1}{2}\Big[f(\mathrm{Re}(q) + J|\mathrm{Im}(q)|) + f(\mathrm{Re}(q) - J|\mathrm{Im}(q)|)$$

$$+\frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|}J[f(\mathrm{Re}(q) - J|\mathrm{Im}(q)|) - f(\mathrm{Re}(q) + J|\mathrm{Im}(q)|)]\Big].$$

Using the fact that $q = x + Iy$, $x, y \in \mathbb{R}$, $y \geq 0$ and $I = \dfrac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|}$ we obtain

$$\psi(x+Iy) = \frac{1}{2}\Big[f(x+Jy)+f(x-Jy)+IJ[f(x-Jy)-f(x+Jy)]\Big].$$

Observe that for $I = J$ we have

$$\psi_J(q) = \psi(x+Jy) = f(x+Jy) = f_J(q).$$

Thus if we prove that ψ is regular on U, the first part of the assertion follows from the Identity Principle. Since f is regular on U, for any $I \in \mathbb{S}$ we have, on $U \cap \mathbb{C}_I$,

$$\begin{aligned} \frac{\partial}{\partial x}2\psi(x+yI) &= \frac{\partial}{\partial x}\Big[f(x+Jy)+f(x-Jy)+IJ[f(x-Jy)-f(x+Jy)]\Big] \\ &= \frac{\partial}{\partial x}f(x+Jy) + \frac{\partial}{\partial x}f(x-Jy) + IJ[\frac{\partial}{\partial x}f(x-Jy) - \frac{\partial}{\partial x}f(x+Jy)] \\ &= -J\frac{\partial}{\partial y}f(x+Jy) + J\frac{\partial}{\partial y}f(x-Jy) + IJ[J\frac{\partial}{\partial y}f(x-Jy) + J\frac{\partial}{\partial y}f(x+Jy)] \end{aligned}$$

$$
\begin{aligned}
&= -J\frac{\partial}{\partial y}f(x+Jy) + J\frac{\partial}{\partial y}f(x-Jy) - I[\frac{\partial}{\partial y}f(x-Jy) + \frac{\partial}{\partial y}f(x+Jy)] \\
&= -I\frac{\partial}{\partial y}\Big[f(x+Jy) + f(x-Jy) + IJ[f(x-Jy) - f(x+Jy)]\Big] \\
&= -I\frac{\partial}{\partial y}2\psi(x+yI)
\end{aligned}
$$

i.e.

$$
\frac{1}{2}(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y})\psi(x+yI) = 0. \tag{2.4}
$$

To prove (2.3) we take any $K \in \mathbb{S}$ and use equation (2.2) to show that

$$
\begin{aligned}
&\frac{1}{2}\Big[f(x+Ky) + f(x-Ky)\Big] \\
&= \frac{1}{2}\Big\{\frac{1}{2}\Big[f(x+Jy) + f(x-Jy)\Big] + K\frac{1}{2}\Big[J[f(x-Jy) - f(x+Jy)]\Big] \\
&+ \frac{1}{2}\Big[f(x+Jy) + f(x-Jy)\Big] - K\frac{1}{2}\Big[J[f(x-Jy) - f(x+Jy)]\Big]\Big\} \\
&= \frac{1}{2}\Big[f(x+Jy) + f(x-Jy)\Big] = \alpha(x,y)
\end{aligned}
$$

and that

$$
\begin{aligned}
&\frac{1}{2}\Big[K[f(x-Ky) - f(x+Ky)]\Big] \\
&= \frac{1}{2}K\Big\{\frac{1}{2}\Big[f(x+Jy) + f(x-Jy)\Big] - K\frac{1}{2}\Big[J[f(x-Jy) - f(x+Jy)]\Big] \\
&- \frac{1}{2}\Big[f(x+Jy) + f(x-Jy)\Big] - K\frac{1}{2}\Big[J[f(x-Jy) - f(x+Jy)]\Big]\Big\} \\
&= \frac{1}{2}K\Big[-K\Big[J[f(x-Jy) - f(x+Jy)]\Big] \\
&= \frac{1}{2}\Big[J[f(x-Jy) - f(x+Jy)]\Big] = \beta(x,y),
\end{aligned}
$$

and the proof is complete. □

Some immediate consequences are the following corollaries:

Corollary 2.1 *A slice regular function $f : U \to \mathbb{H}$ on an axially symmetric slice domain is infinitely differentiable on U. It is also real analytic on U.*

Corollary 2.2 *Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f : U \to \mathbb{H}$ be a slice regular function. For all $x_0, y_0 \in \mathbb{R}$ such that $x_0 + Iy_0 \in U$ there exist $a, b \in \mathbb{H}$ such that*

$$
f(x_0 + Iy_0) = a + Ib \tag{2.5}
$$

for all $I \in \mathbb{S}$. In particular, f is affine in $I \in \mathbb{S}$ on each 2-sphere $[x_0 + Iy_0]$ and the image of the $2-$sphere $[x_0 + Iy_0]$ is the set $[a + Ib]$.

Corollary 2.3 *Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f : U \to \mathbb{H}$ be a slice regular function. If $f(x + Jy) = f(x + Ky)$ for $J \neq K$ in $\mathbb{S}$, then f is constant on $[x + Iy]$. In particular, if $f(x + Jy) = f(x + Ky) = 0$ for $J \neq K$ in $\mathbb{S}$, then f vanishes on the whole $2-$sphere $[x + Iy]$.*

Corollary 2.4 *Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain, let $D \subseteq \mathbb{R}^2$ be such that $x + Iy \in U$ whenever $(x, y) \in D$ and let $f : U \to \mathbb{H}$. The function f is slice regular if and only if there exist two differentiable functions $\alpha, \beta : D \subseteq \mathbb{R}^2 \to \mathbb{H}$ satisfying $\alpha(x, y) = \alpha(x, -y)$, $\beta(x, y) = -\beta(x, -y)$ and the Cauchy–Riemann system*

$$\begin{cases} \partial_x \alpha - \partial_y \beta = 0 \\ \partial_x \beta + \partial_y \alpha = 0 \end{cases} \tag{2.6}$$

such that

$$f(x + Iy) = \alpha(x, y) + I\beta(x, y). \tag{2.7}$$

Proof A function of the form (2.7) where α and β satisfy the hypothesis in the statement is clearly slice regular. Conversely, a slice regular function on an axially symmetric slice domain satisfies the Representation formula and thus it is of the form (2.7), where α and β satisfy the Cauchy–Riemann system. The conditions $\alpha(x, y) = \alpha(x, -y), \beta(x, y) = -\beta(x, -y)$ can be easily verified from the definition of α and β given in the Representation Formula. □

Remark 2.1 Since α, β are quaternion-valued functions, for any given $I \in \mathbb{S}$ we select $J \in \mathbb{S}$ orthogonal to I and we write both α and β into their real components (omitting, for simplicity, the arguments of the functions):

$$\begin{aligned} f_I = \alpha + I\beta &= (a_0 + Ia_1 + Ja_2 + IJa_3) + I(b_0 + Ib_1 + Jb_2 + IJb_3) \\ &= (a_0 - b_1) + I(a_1 + b_0) + J(a_2 - b_3) + IJ(a_3 + b_2) \\ &= c_0 + Ic_1 + Jc_2 + IJc_3 = (c_0 + Ic_1) + (c_2 + Ic_3)J, \end{aligned}$$

where the functions a_ℓ, b_ℓ, c_ℓ, $\ell = 0, \ldots, 3$ are real-valued.
Imposing $(\partial_x + I\partial_y) f_I = 0$ we obtain the equations

$$\begin{cases} \partial_x c_0 - \partial_y c_1 = 0 \\ \partial_y c_0 + \partial_x c_1 = 0 \\ \partial_x c_2 - \partial_y c_3 = 0 \\ \partial_y c_2 + \partial_x c_3 = 0. \end{cases} \tag{2.8}$$

Requiring that the pair α, β satisfies the Cauchy–Riemann system, we obtain eight real equations:

$$\partial_x a_i - \partial_y b_i = 0 \qquad \partial_x b_i + \partial_y a_i = 0 \qquad i = 0, \ldots, 3. \tag{2.9}$$

System (2.8) is equivalent to the request that the functions $F = c_0 + Ic_1$ and $G = c_2 + Ic_3$ prescribed by the Splitting Lemma are holomorphic. As it can be easily verified, by setting $c_0 = a_0 - b_1, c_1 = a_1 + b_0, c_2 = a_2 - b_3, c_3 = a_3 + b_2$ the solutions to system (2.9) give solutions to system (2.8). Note that since the functions α and β satisfy the equation $(\partial_x + I\partial_y)(\alpha(x, y) + I\beta(x, y)) = 0$, they are harmonic in x, y.

The previous remark implies a stronger version of the Splitting Lemma for slice regular functions. To prove the result, we need to recall that complex functions defined on open sets $G \subset \mathbb{C}$ symmetric with respect to the real axis and such that $\overline{f(\bar{z})} = f(z)$ are called in the literature intrinsic, see [143].

Proposition 2.2 (Refined Splitting Lemma). *Let U be an axially symmetric slice domain in $\mathbb{H}$ and let $f \in \mathscr{R}(U)$. For any $I, J \in \mathbb{S}$ with J orthogonal to I, there exist four holomorphic intrinsic functions $h_\ell : U \cap \mathbb{C}_I \to \mathbb{C}_I$, $\ell = 0, \ldots, 3$ such that*

$$f_I(x + Iy) = h_0(x + Iy) + h_1(x + Iy)I + h_2(x + Iy)J + h_3(x + Iy)K,$$

where $K = IJ$.

Proof Corollary 2.4 shows that on an axially symmetric slice domain a function $f \in \mathscr{R}(U)$ can be written as $f(x + Iy) = \alpha(x, y) + I\beta(x, y)$. We can rewrite it (omitting the argument $x + Iy$ of the various functions) as:

$$\begin{aligned} f_I &= a_0 + Ia_1 + Ja_2 + Ka_3 + I(b_0 + Ib_1 + Jb_2 + Kb_3) \\ &= (a_0 + Ib_0) + (a_1 + Ib_1)I + (a_2 + Ib_2)J + (a_3 + Ib_3)K \\ &= h_0 + h_1 I + h_2 J + h_3 K, \end{aligned}$$

where a_ℓ, b_ℓ, $\ell = 0, \ldots, 3$ are real-valued. It follows from system (2.9) that h_ℓ, $\ell = 0, \ldots, 3$ satisfy the Cauchy–Riemann system and so they are holomorphic. The fact that the functions $\alpha(x, y)$ and $\beta(x, y)$ are even and odd, respectively, in the variable y implies that $a_\ell(x, y)$ and $b_\ell(x, y)$ are even and odd, respectively, in the variable y. Thus

$$h_\ell(x - Iy) = a_\ell(x, -y) + Ib_\ell(x, -y) = a_\ell(x, y) - Ib_\ell(x, y) = \overline{h_\ell(x + Iy)}$$

and so the functions h_ℓ, $\ell = 0, 1, 2, 3$ are holomorphic intrinsic. □

Another important consequence of the Representation Formula is the following result, which allows to obtain the slice regular extension of holomorphic maps:

Corollary 2.5 *Let U_J be a domain in $\mathbb{C}_J$ symmetric with respect to the real axis and such that $U_J \cap \mathbb{R} \neq \emptyset$. Let U be the axially symmetric slice domain defined by*

$$U = \bigcup_{x+Jy \in U_J,\ I \in \mathbb{S}} \{x + Iy\}.$$

If $f : U_J \to \mathbb{H}$ satisfies $\overline{\partial}_J f = 0$ then the function

$$\text{ext}(f)(x+Iy) := \frac{1}{2}\Big[f(x+Jy)+f(x-Jy)\Big]+I\frac{1}{2}\Big[J[f(x-Jy)-f(x+Jy)]\Big] \tag{2.10}$$

is the unique slice regular extension of f to U.

Definition 2.6 Let U_J be any open set in $\mathbb{C}_J$ and let

$$U = \bigcup_{x+Jy\in U_J,\ I\in\mathbb{S}} \{x+Iy\}. \tag{2.11}$$

We say that U is the axially symmetric completion of U_J in $\mathbb{H}$.

Corollary 2.4 implies that slice regular functions are a subclass of the following set of functions:

Definition 2.7 Let $U \subseteq \mathbb{H}$ be an axially symmetric open set. Functions of the form $f(q) = f(x+Iy) = \alpha(x,y) + I\beta(x,y)$, where α β are $\mathbb{H}$-valued functions such that $\alpha(x,y) = \alpha(x,-y)$, $\beta(x,y) = -\beta(x,-y)$ for all $x+Iy \in U$ are called *slice functions*. If α and β are continuous then f is said to be a slice continuous function.

Theorem 2.5 (General Representation Formula). *Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f \in \mathscr{R}(U)$. The following equality holds for all $q = x+Iy \in U$, $J, K \in \mathbb{S}$:*

$$f(x+Iy) = (J-K)^{-1}[Jf(x+Jy)-Kf(x+Ky)]+I(J-K)^{-1}[f(x+Jy)-f(x-Ky)]. \tag{2.12}$$

Proof If q is real the proof is immediate. Otherwise, for all $q = x+yI$, we define the function

$$\begin{aligned}&\phi(x+yI) = \\ &(J-K)^{-1}\Big[Jf(x+Jy)-Kf(x+Ky)\Big]+I(J-K)^{-1}\Big[f(x+Jy)-f(x+Ky)\Big] \\ &= [(J-K)^{-1}J+I(J-K)^{-1}]f(x+Jy)-[(J-K)^{-1}K+I(J-K)^{-1}]f(x+Ky).\end{aligned} \tag{2.13}$$

It is clear that for all $q = x \in U \cap \mathbb{R}$ we have

$$\phi(x) = f(x).$$

Therefore if we prove that ϕ is slice regular on U, the first part of the assertion will follow from the Identity Principle for slice regular functions. Indeed, since f is slice regular on U, for any $L \in \mathbb{S}$ we have $\frac{\partial}{\partial x} f(x+Ly) = -L\frac{\partial}{\partial y} f(x+Ly)$ on $U \cap \mathbb{C}_L$; hence

$$\frac{\partial \phi}{\partial x}(x+yI) = -[(J-K)^{-1}J + I(J-K)^{-1}]J\frac{\partial f}{\partial y}(x+Jy)$$
$$+ [(J-K)^{-1}K + I(J-K)^{-1}]K\frac{\partial f}{\partial y}(x+Ky)$$

and also

$$I\frac{\partial \phi}{\partial y}(x+yI) = I[(J-K)^{-1}J + I(J-K)^{-1}]\frac{\partial f}{\partial y}(x+Jy)$$
$$- I[(J-K)^{-1}K + I(J-K)^{-1}]\frac{\partial f}{\partial y}(x+Ky).$$

At this point, it is immediate that

$$\overline{\partial}_I \phi(x+yI) = 0$$

and equality (2.12) is proved. □

2.3 Multiplication of Slice Regular Functions

Because of the noncommutativity of quaternions, the pointwise product of two slice regular functions is not, in general, slice regular. Consider for example $f(q) = q - i$ (i being an imaginary unit) and the pointwise product

$$f(q) \cdot f(q) = (q-i)^2 = q^2 - qi - iq - 1.$$

The product is not slice regular because of the term iq, which is not slice regular.
In the case of slice regular functions defined on axially symmetric slice domains, we can define a suitable product, called $\star$-product, which preserves slice regularity. This product extends the very well known product for polynomials and series with coefficients in a ring, see e.g., [92] and [133].
Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f, g : U \to \mathbb{H}$ be slice regular functions. For any $I, J \in \mathbb{S}$, with $I \perp J$, the Splitting Lemma guarantees the existence of four holomorphic functions $F, G, H, K : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that for all $z = x + Iy \in U \cap \mathbb{C}_I$

$$f_I(z) = F(z) + G(z)J, \qquad g_I(z) = H(z) + K(z)J. \tag{2.14}$$

We define the function $f_I \star g_I : U \cap \mathbb{C}_I \to \mathbb{H}$ as

$$(f_I \star g_I)(z) = [F(z)H(z) - G(z)\overline{K(\bar{z})}] + [F(z)K(z) + G(z)\overline{H(\bar{z})}]J. \tag{2.15}$$

Then $(f_I \star g_I)(z)$ is obviously a holomorphic map and hence its unique slice regular extension to U defined, according to the extension formula (2.10) by

$$\text{ext}(f_I \star g_I)(q),$$

is slice regular on U.

Definition 2.8 Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f, g : U \to \mathbb{H}$ be slice regular. The function

$$(f \star g)(q) := \text{ext}(f_I \star g_I)(q)$$

defined as the extension of (2.15) is called the slice regular product of f and g. This product is also called $\star$-product.

Remark 2.2 It is immediate to verify that the $\star$-product is associative, distributive but, in general, not commutative.

Remark 2.3 Let $H(z)$ be a $\mathbb{C}_I$-valued holomorphic function in the variable $z \in \mathbb{C}_I$ and let $J \in \mathbb{S}$ be orthogonal to I. Then by the definition of $\star$-product we obtain $J \star H(z) = \overline{H(\bar{z})}J$.

To define the inverse of a function with respect to the $\star$-product we need to introduce the so-called conjugate and symmetrization of a slice regular function. These two notions, as we shall see, will be important also for other purposes.

Using the notations above we have:

Definition 2.9 Let $f \in \mathscr{R}(U)$ and let $f_I(z) = F(z) + G(z)J$. We define the function $f_I^c : U \cap \mathbb{C}_I \to \mathbb{H}$ as

$$f_I^c(z) := \overline{F(\bar{z})} - G(z)J. \tag{2.16}$$

Then $f_I^c(z)$ is a holomorphic map and we define the so-called conjugate function f^c of f as

$$f^c(q) = \text{ext}(f_I^c)(q).$$

Remark 2.4 Note that, by construction, $f^c \in \mathscr{R}(U)$. Note that if we write the function f in the form $f(x + Iy) = \alpha(x, y) + I\beta(x, y)$, then it is possible to show that

$$f^c(x + Iy) = \overline{\alpha(x, y)} + I\overline{\beta(x, y)}.$$

Remark 2.5 Some lengthy but easy computations show that if p is a fixed quaternion and $I \in \mathbb{S}$ there exists $J \in \mathbb{S}$ such that $Ip = pJ$. Thus we have

$$\begin{aligned} |f^c(x + Iy)| &= |\overline{\alpha(x, y)} + I\overline{\beta(x, y)}| = |\overline{\overline{\alpha(x, y)} + I\overline{\beta(x, y)}}| \\ &= |\alpha(x, y) - \beta(x, y)I| = |\alpha(x, y) + J\beta(x, y)| \\ &= |f(x + Jy)| \end{aligned} \tag{2.17}$$

for a suitable $J \in \mathbb{S}$.

Using the notion of $\star$-multiplication of slice regular functions, it is possible to associate to any slice regular function f its "symmetrization" also called "normal form", denoted by f^s. We will show that all the zeros of f^s are spheres of type $[x+Iy]$ (real points, in particular) and that, if $x+Iy$ is a zero of f (isolated or not) then $[x+Iy]$ is a zero of f^s.

Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f : U \to \mathbb{H}$ be a slice regular function. Using the notation in (2.16), we consider the function $f_I^s : U \cap \mathbb{C}_I \to \mathbb{C}_I$ defined by

$$\begin{aligned} f_I^s(z) = (f_I \star f_I^c)(z) &= (F(z)+G(z)J) \star (\overline{F(\bar{z})} - G(z)J) \\ &= [F(z)\overline{F(\bar{z})} + G(z)\overline{G(\bar{z})}] + [-F(z)G(z) + G(z)F(z)]J \\ &= F(z)\overline{F(\bar{z})} + G(z)\overline{G(\bar{z})} \\ &= (f_I^c \star f_I)(z). \end{aligned} \tag{2.18}$$

Then f_I^s is holomorphic. We give the following definition:

Definition 2.10 Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f : U \to \mathbb{H}$ be slice regular. The function

$$f^s(q) := \mathrm{ext}(f_I^s)(q)$$

defined by the extension of (2.18) is called the symmetrization (or normal form) of f.

Remark 2.6 Note that formula (2.18) yields that, for all $I \in \mathbb{S}$, $f^s(U \cap \mathbb{C}_I) \subseteq \mathbb{C}_I$.

We now show how the conjugation and the symmetrization of a slice regular function behave with respect to the $\star$-product:

Proposition 2.3 *Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f, g : U \to \mathbb{H}$ be slice regular functions. Then*

$$(f \star g)^c = g^c \star f^c$$

and

$$(f \star g)^s = f^s g^s = g^s f^s. \tag{2.19}$$

Proof It is sufficient to show that $(f \star g)^c = g^c \star f^c$. As customary, we can use the Splitting Lemma to write on $U \cap \mathbb{C}_I$ that $f_I(z) = F(z) + G(z)J$ and $g_I(z) = H(z) + K(z)J$. We have

$$(f_I \star g_I)(z) = [F(z)H(z) - G(z)\overline{K(\bar{z})}] + [F(z)K(z) + G(z)\overline{H(\bar{z})}]J$$

and hence

$$(f_I \star g_I)^c(z) = [\overline{F(\bar{z})}\,\overline{H(\bar{z})} - \overline{G(\bar{z})}K(z)] - [F(z)K(z) + G(z)\overline{H(\bar{z})}]J.$$

We now compute

$$g_I^c(z) \star f_I^c(z) = (\overline{H(\bar{z})} - K(z)J) \star (\overline{F(\bar{z})} - G(z)J)$$

$$= \overline{H(\bar{z})} \star \overline{F(\bar{z})} - \overline{H(\bar{z})} \star G(z)J - K(z)J \star \overline{F(\bar{z})} + K(z)J \star G(z)J$$

and conclude by Remark 2.3. □

Proposition 2.4 *Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f : U \to \mathbb{H}$ be a slice regular function. The function $(f^s)^{-1}$ is slice regular on $U \setminus \{q \in \mathbb{H} : f^s(q) = 0\}$.*

Proof The function f^s is such that $f^s(U \cap \mathbb{C}_I) \subseteq \mathbb{C}_I$ for all $I \in \mathbb{S}$ by Remark 2.6. Thus, for any given $I \in \mathbb{S}$, the Splitting Lemma implies the existence of a holomorphic function $F : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I^s(z) = F(z)$ for all $z \in U \cap \mathbb{C}_I$. The inverse of the function F is holomorphic on $U \cap \mathbb{C}_I$ outside the zero set of F. The conclusion follows by the equality $(f_I^s)^{-1} = F^{-1}$. □

The $\star$-product can be related to the pointwise product as described in the following result:

Theorem 2.6 *Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain, and let $f, g : U \to \mathbb{H}$ be slice regular functions. Then*

$$(f \star g)(q) = f(q)g(f(q)^{-1}qf(q)), \tag{2.20}$$

for all $q \in U$, $f(q) \neq 0$, while $(f \star g)(q) = 0$ when $q \in U$, $f(q) = 0$.

Proof Let $I \in \mathbb{S}$ and let $q = x + Iy$. If $f(x + Iy) \neq 0$, it is immediate that

$$f(x + Iy)^{-1}(x + Iy)f(x + Iy) = x + yf(x + Iy)^{-1}If(x + Iy)$$

with $f(x + Iy)^{-1}If(x + Iy) \in \mathbb{S}$. Applying the Representation Formula to the function g, we get

$$g(f(q)^{-1}qf(q)) = g(x + yf(x + Iy)^{-1}If(x + Iy))$$
$$= \frac{1}{2}\{g(x + Iy) + g(x - Iy) - f(x + Iy)^{-1}If(x + Iy)[Ig(x + Iy) - Ig(x - Iy)]\}.$$

Let us set

$$\psi(q) := f(q)g(f(q)^{-1}qf(q))$$
$$= \frac{1}{2}\{f(x + Iy)[g(x + Iy) + g(x - Iy)] - If(x + Iy)[Ig(x + Iy) - Ig(x - Iy)]\}.$$

If we prove that the function $\psi(q)$ is regular, then the statement follows by the Identity Principle, since formula (2.20) holds for power series, namely on a small open ball of Ω centered at a real point. In fact, let us consider the case of slice regular functions on an open ball centered at the origin. We have, if $g(q) = \sum_{n=0}^{\infty} q^n a_n$:

$$\begin{aligned}(f \star g)(q) &= f(q) \star \sum_{n=0}^{\infty} q^n a_n = \sum_{n=0}^{\infty} q^n f(q) a_n \\ &= f(q) f(q)^{-1} \sum_{n=0}^{\infty} q^n f(q) a_n = f(q) \sum_{n=0}^{\infty} (f(q)^{-1} q f(q))^n a_n \\ &= f(q) g(f(q)^{-1} q f(q)).\end{aligned}$$

Let us compute (with obvious notation for the derivatives):

$$\begin{aligned}&\frac{\partial}{\partial x}\psi(x+Iy) \\ &= \frac{1}{2}\{f_x(x+Iy)[g(x+Iy)+g(x-Iy)] - If_x(x+Iy)[Ig(x+Iy)-Ig(x-Iy)]\} \\ &+ \frac{1}{2}\{f(x+Iy)[g_x(x+Iy)+g_x(x-Iy)] - If(x+Iy)[Ig_x(x+Iy)-Ig_x(x-Iy)]\}\end{aligned}$$

and

$$\begin{aligned}&I\frac{\partial}{\partial y}\psi(x+Iy) \\ &= \frac{1}{2}\{If_y(x+Iy)[g(x+Iy)+g(x-Iy)] + f_y(x+Iy)[Ig(x+Iy)-Ig(x-Iy)]\} \\ &+ \frac{1}{2}\{If(x+Iy)[g_y(x+Iy)+g_y(x-Iy)] + f(x+Iy)[Ig_y(x+Iy)-Ig_y(x-Iy)]\}.\end{aligned}$$

By using the three relations

$$\begin{aligned}f_x(x+Iy) + If_y(x+Iy) &= g_x(x+Iy) + Ig_y(x+Iy) \\ &= g_x(x-Iy) - Ig_y(x-Iy) = 0,\end{aligned}$$

we obtain that $(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y})\psi(x+Iy) = 0$. The fact that I is arbitrary proves the assertion.

An immediate consequence is the following:

Corollary 2.6 *If $(f \star g)(q) = 0$ then we have that either $f(q) = 0$ or $f(q) \neq 0$ and $g(f(q)^{-1} q f(q)) = 0$.*

2.4 Quaternionic Intrinsic Functions

An important subclass of the class of slice regular functions on an open set U, denoted by $\mathcal{N}(U)$, is defined as follows:

$$\mathcal{N}(U) = \{f \text{ slice regular in } U \ : \ f(U \cap \mathbb{C}_I) \subseteq \mathbb{C}_I, \ \ \forall I \in \mathbb{S}\}.$$

Remark 2.7 If U is axially symmetric and if we denote by $\overline{q}$ the conjugate of a quaternion q, it can be shown that a function f belongs to $\mathcal{N}(U)$ if and only if it satisfies $f(q) = \overline{f(\bar{q})}$. In analogy with the complex case, we say that the functions such that $f(q) = \overline{f(\bar{q})}$ are *quaternionic intrinsic*.

If one considers a ball $B(0; R)$ with center at the origin, it is immediate that a function slice regular on the ball belongs to $\mathcal{N}(B(0; R))$ if and only if its power series expansion has real coefficients. Such functions are also said to be *real*. More in general, if U is an axially symmetric slice domain, then $f \in \mathcal{N}(U)$ if and only if $f(q) = f(x + Iy) = \alpha(x, y) + I\beta(x, y)$ with α, β real valued, in fact we have:

Proposition 2.5 *Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and consider the slice function $f(x + Iy) = \alpha(x, y) + I\beta(x, y)$. Then $f \in \mathcal{N}(U)$ if and only if*

$$f(x + Iy) = \alpha(x, y) + I\beta(x, y) \tag{2.21}$$

with α, β real valued and satisfying the Cauchy–Riemann system.

Proof If α, β are real valued and satisfy the Cauchy–Riemann system, then trivially f defined by (2.21) belongs to $\mathcal{N}(U)$. Conversely, assume that the slice function f belongs to $\mathcal{N}(U)$. Then $f_I(z) = F(z) = \alpha(x, y) + I\beta(x, y)$ satisfies the Cauchy–Riemann equation and so the pair α, β satisfies the Cauchy–Riemann system. By hypothesis, for any fixed $I \in \mathbb{S}$ the restriction f_I takes $\mathbb{C}_I$ to itself, and so the functions α, β are $\mathbb{C}_I$-valued. By the arbitrariness of I it follows that α and β are real valued. □

Remark 2.8 The class $\mathcal{N}(\mathbb{H})$ includes all elementary transcendental functions, in particular

$$\exp(q) = e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!},$$

$$\sin(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n+1}}{(2n+1)!},$$

$$\cos(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n}}{(2n)!}.$$

Note that these functions coincide with the analogous complex functions on any complex plane $\mathbb{C}_I$.

Let us now recall the definition of quaternionic logarithm, see [109], [127], which is another example of a quaternionic intrinsic function. It is the inverse function of the exponential function $\exp(q)$ in $\mathbb{H}$.

Definition 2.11 Let $U \subseteq \mathbb{H}$ be a connected open set. We define a branch of the quaternionic logarithm (or simply a *logarithm*) on U a function $f : U \to \mathbb{H}$ such that for every $q \in U$

$$e^{f(q)} = q.$$

Since $\exp(q)$ never vanishes, we suppose that $0 \notin U$. Recalling that

$$I_q = \begin{cases} \mathrm{Im}(q)/|\mathrm{Im}(q)| & \text{if } q \in \mathbb{H} \setminus \mathbb{R} \\ \text{any element of } \mathbb{S} \text{ otherwise} \end{cases}$$

we have that for every $q \in \mathbb{H} \setminus \{0\}$ there exists a unique $\theta \in [0, \pi]$ such that $q = |q|e^{\theta I_q}$. Moreover we have $\theta = \arccos(\mathrm{Re}(q)/|q|)$.

Definition 2.12 The function $\arccos(\mathrm{Re}(q)/|q|)$ will be called the principal quaternionic argument of q and it will be denoted by $\arg_{\mathbb{H}}(q)$ for every $q \in \mathbb{H} \setminus \{0\}$.

Below we define the principal quaternionic logarithm.

Definition 2.13 Let $\log(x)$ be the natural real logarithm of $x \in \mathbb{R}^+$. For every $q \in \mathbb{H} \setminus (-\infty, 0]$, we define the principal quaternionic logarithm (or, in short, principal logarithm) of q as

$$\mathrm{Log}(q) = \log|q| + \arccos\left(\frac{\mathrm{Re}(q)}{|q|}\right) I_q.$$

Remark 2.9 It is easy to directly verify that the principal quaternionic logarithm coincides with the principal complex logarithm on the complex plane $\mathbb{C}_I$, for any $I \in \mathbb{S}$.

We now go back to the properties of intrinsic slice regular functions. The following result has an elementary proof which is left to the reader.

Proposition 2.6 (Algebraic properties). *Let U, U' be two open sets in $\mathbb{H}$.*

(1) Let f and $g \in \mathcal{N}(U)$, then $f + g \in \mathcal{N}(U)$ and $fg = gf \in \mathcal{N}(U)$.
(2) Let f and $g \in \mathcal{N}(U)$ such that $g(q) \neq 0$ for all $q \in U$, then $g^{-1}f = fg^{-1} \in \mathcal{N}(U)$.
(3) Let $f \in \mathcal{R}(U')$, $g \in \mathcal{N}(U)$ with $g(U) \subseteq U'$. Then $f(g(q))$ is slice regular for $q \in U$.

Proposition 2.7 *Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $\{1, I, J, IJ\}$ be a basis of $\mathbb{H}$. Then*

$$\mathscr{R}(U) = \mathscr{N}(U) \oplus \mathscr{N}(U)I \oplus \mathscr{N}(U)J \oplus \mathscr{N}(U)IJ.$$

Proof By the Refined Splitting Lemma, there exist four functions h_0, h_1, h_2, h_3 holomorphic intrinsic on $U \cap \mathbb{C}_I$ such that

$$f_I(z) = h_0(z) + h_1(z)I + h_2(z)J + h_3(z)IJ,$$

and $f = \text{ext}(f_I) = \text{ext}(h_0) + \text{ext}(h_1)I + \text{ext}(h_2)J + \text{ext}(h_3)IJ$.
The extension of the complex intrinsic function h_ℓ which, for the sake of simplicity will be still denoted by h_ℓ, is a quaternionic intrinsic function, in fact, for $q = x + Ly$ we have

$$\begin{aligned} h_\ell(\bar{q}) &= \frac{1+LI}{2}h_\ell(z) + \frac{1-LI}{2}h_\ell(\bar{z}) = \frac{1+LI}{2}h_\ell(z) + \frac{1-LI}{2}\overline{h_\ell(z)} \\ &= \text{Re}h_\ell(z) + LI^2\text{Im}h_\ell(z) = \overline{h_\ell(q)}, \end{aligned}$$

so we obtain

$$\mathscr{R}(U) = \mathscr{N}(U) + \mathscr{N}(U)I + \mathscr{N}(U)J + \mathscr{N}(U)IJ.$$

To show that the sum is a direct sum, suppose that $f \in \mathscr{N}(U) \cap \mathscr{N}(U)I$. Then there exists $g \in \mathscr{N}(U)$, such that $f = gI$. So there exist k_1, k_2 complex intrinsic and holomorphic such that $f = \text{ext}(k_1)$, and $g = \text{ext}(k_2)$. Then $k_1 = k_2 I$, and for any $q \in U_I$ one has

$$k_2(q)I = k_1(q) = \overline{k_1(\bar{q})} = \overline{k_2(\bar{q})I} = -k_2(q)I,$$

then $k_2 = k_1 = 0$, and $\mathscr{N}(U) \cap \mathscr{N}(U)I = \{0\}$.

Similarly one can see that all the other intersections between $\mathscr{N}(U)$, $\mathscr{N}(U)I$, $\mathscr{N}(U)J$, $\mathscr{N}(U)IJ$, are reduced to the zero vector and the statement follows. □

2.5 Composition of Power Series

In this section, we introduce and study a notion of composition of slice regular functions, see also [93]. As it is well known, the composition $f \circ g$ of two slice regular functions is not, in general, slice regular, unless g belongs to the subclass of quaternionic intrinsic functions (see Proposition 2.6). Our choice of the notion of composition is based on the fact that slice regularity is not preserved by the pointwise product, but is preserved by the $\star$-product. Thus the power of a function is slice regular only if it is computed with respect to the $\star$-product and we will write $(w(q))^{\star n}$ to

denote that we are taking the n-th power with respect to this product. To introduce the notion of composition, we first treat the case of formal power series.

Definition 2.14 Let $g(q) = \sum_{n=0}^{\infty} q^n a_n$ and $w(q) = \sum_{n=1}^{\infty} q^n b_n$. We define

$$(g \bullet w)(q) = \sum_{n=0}^{\infty} (w(q))^{\star n} a_n.$$

Remark 2.10 When $w \in \mathscr{N}(B(0;1))$, then $g \bullet w = g \circ w$ where $\circ$ represents the standard composition of two functions. Moreover, when $w \in \mathscr{N}(B(0;1))$ then $(w(q))^{\star n} = (w(q))^n$ so, in particular, $q^{\star n} = q^n$.

Remark 2.11 The *order* of a series $f(q) = \sum_{n=0}^{\infty} q^n a_n$ can be defined as in [37] and we denote it by $\omega(f)$. It is the lowest integer n such that $a_n \neq 0$ (with the convention that the order of the series identically equal to zero is $+\infty$). Assume to have a family $\{f_i\}_{i \in \mathscr{I}}$ of power series where $\mathscr{I}$ is a set of indices. This family is said to be *summable* if for any $k \in \mathbb{N}$, $\omega(f_i) \geq k$ for all except a finite number of indices i. By definition, the sum of $\{f_i\}$ where $f_i(q) = \sum_{n=0}^{\infty} q^n a_{i,n}$ is

$$f(q) = \sum_{n=0}^{\infty} q^n a_n,$$

where $a_n = \sum_{i \in \mathscr{I}} a_{i,n}$. The definition of a_n is well posed since our hypothesis guarantees that for any n just a finite number of $a_{i,n}$ are nonzero.

Remark 2.12 In Definition 2.14, we require the hypothesis $b_0 = 0$. This is necessary in order to guarantee that the minimum power of q in the term $(w(q))^{\star n}$ is at least q^n or, in other words, that $\omega(w(q)^{\star n}) \geq n$ for all indices. With this hypothesis, the series $\sum_{n=0}^{\infty} (w(q))^{\star n} a_n$ is summable according to Remark 2.11, and we can regroup the powers of q.

Let us consider the following example: let $f(q) = q^2 c$, $g(q) = qa$ and $w(q) = q^2 b$. We have

$$((f \bullet g) \bullet w)(q) = q^4 b^2 a^2 c$$

while

$$(f \bullet (g \bullet w))(q) = q^4 babac$$

so

$$(f \bullet g) \bullet w \neq f \bullet (g \bullet w).$$

Thus we conclude that:

Proposition 2.8 *The composition $\bullet$ is, in general, not associative.*

However, we will prove that the composition is associative in some cases and to this end we need a preliminary lemma.

Lemma 2.2 *Let* $f_1(q) = \sum_{n=0}^{\infty} q^n a_n$, $f_2(q) = \sum_{n=0}^{\infty} q^n b_n$, *and* $g(q) = \sum_{n=1}^{\infty} q^n c_n$. *Then:*

(1) $(f_1 + f_2) \bullet g = f_1 \bullet g + f_2 \bullet g$;
(2) if g *has real coefficients* $(f_1 \star f_2) \bullet g = (f_1 \bullet g) \star (f_2 \bullet g)$;
(3) if g *has real coefficients* $f^{\star n} \bullet g = (f \bullet g)^{\star n}$.
(4) if $\{f_i\}_{i \in \mathscr{I}}$ *is a summable family of power series then* $\{f_i \bullet g\}_{i \in \mathscr{I}}$ *is summable and* $(\sum_{i \in \mathscr{I}} f_i) \bullet g = \sum_{i \in \mathscr{I}} (f_i \bullet g)$.

Proof Point (1) follows from

$$\begin{aligned}(f_1 + f_2) \bullet g &= \sum_{n=0}^{\infty} g^{\star n}(a_n + b_n) \\ &= \sum_{n=0}^{\infty} g^{\star n} a_n + \sum_{n=0}^{\infty} g^{\star n} b_n \\ &= f_1 \bullet g + f_2 \bullet g.\end{aligned}$$

To prove (2), recall that $(f_1 \star f_2)(q) = \sum_{n=0}^{\infty} q^n (\sum_{r=0}^{n} a_r b_{n-r})$ and taking into account that the coefficients of g are real we have:

$$((f_1 \star f_2) \bullet g)(q) = \sum_{n=0}^{\infty} (g(q))^n (\sum_{r=0}^{n} a_r b_{n-r}).$$

The statement follows since

$$\begin{aligned}(f_1 \bullet g)(q) \star (f_2 \bullet g)(q) &= \left(\sum_{n=0}^{\infty} g(q)^n a_n\right) \star \left(\sum_{m=0}^{\infty} g(q)^m b_m\right) \\ &= \sum_{n=0}^{\infty} g(q)^n (\sum_{r=0}^{n} a_r b_{n-r}).\end{aligned}$$

To prove point (3) we use induction. The statement is true for $n = 2$ since it follows from (2). Assume that the assertion is true for the n-th power. Let us show that it holds for $(n+1)$-th power. Let us compute

$$(f^{\star(n+1)} \bullet g)(q) = ((f^{\star(n)} \star f) \bullet g)(q) \overset{(2)}{=} (f \bullet g)^{\star n} \star (f \bullet g) = (f \bullet g)^{\star(n+1)},$$

and the statement follows.
To show (4) we follow [37, p. 13]. Let $f_i(q) = \sum_{n=0}^{\infty} q^n a_{i,n}$ so that, by definition

$$\sum_{i \in \mathscr{I}} f_i(q) = \sum_{n=0}^{\infty} q^n (\sum_{i \in \mathscr{I}} a_{i,n}).$$

Thus we obtain

$$(\sum_{i\in\mathscr{I}} f_i(q)) \bullet g = \sum_{n=0}^{\infty} g(q)^{\star n} (\sum_{i\in\mathscr{I}} a_{i,n}) \tag{2.22}$$

and

$$\sum_{i\in\mathscr{I}} (f_i \bullet g)(q) = \sum_{i\in\mathscr{I}} (\sum_{n=0}^{\infty} g(q)^{\star n} a_{i,n}). \tag{2.23}$$

By hypothesis on the summability of $\{f_i\}$, each power of q involves just a finite number of the coefficients $a_{i,n}$ so we can apply the associativity of the addition in $\mathbb{H}$ and so (2.22) and (2.23) are equal. □

Proposition 2.9 *If* $f(q) = \sum_{n=0}^{\infty} q^n a_n$, $g(q) = \sum_{n=1}^{\infty} q^n b_n$, $w(q) = \sum_{n=1}^{\infty} q^n c_n$ *and* w *has real coefficients, then* $(f \bullet g) \bullet w = f \bullet (g \bullet w)$.

Proof We follow the proof of Proposition 4.1 in [37]. We first prove the assertion in the special case in which $f(q) = q^n a_n$. We have:

$$((f \bullet g) \bullet w)(q) = (g^{\star n} \bullet w) a_n$$

and

$$(f \bullet (g \bullet w))(q) = (g \bullet w)^{\star n}(q) a_n.$$

Lemma 2.2, point (3) shows that $g^{\star n} \bullet w = (g \bullet w)^{\star n}$ and the equality follows. The general case follows by considering f as the sum of the summable family $\{q^n a_n\}$ and using the first part of the proof:

$$(f \bullet g) \bullet w = \sum_{n=0}^{\infty} (g^{\star n} \bullet w) a_n = \sum_{n=0}^{\infty} (g \bullet w)^{\star n} a_n = f \bullet (g \bullet w).$$

□

The discussion, so far, was on formal power series without specifying the set of convergence. We now consider this aspect, by proving the following result which is classical for power series with coefficients in a commutative ring, see [37].

Proposition 2.10 *Let* $f(q) = \sum_{n=0}^{\infty} q^n a_n$ *and* $g(q) = \sum_{n=1}^{\infty} q^n b_n$ *be convergent in the balls of nonzero radius* R *and* ρ, *respectively, and let* $h(q) = (f \bullet g)(q)$. *Then the radius of convergence of* h *is nonzero and it is equal to* $\sup\{r > 0 : \sum_{n=1}^{\infty} r^n |b_n| < R\}$.

Proof Let us compute

$$\begin{aligned}\left|\left(\sum_{m=1}^{\infty} q^m b_m\right)\star\left(\sum_{m=1}^{\infty} q^m c_m\right)\right| &= |\sum_{m=1}^{\infty} q^m(\sum_{r=0}^{m} b_r c_{m-r})| \\ &\leq \sum_{m=1}^{\infty} |q|^m(\sum_{r=0}^{m} |b_r||c_{m-r}|) \\ &= (\sum_{m=1}^{\infty} |q|^m|b_m|)(\sum_{m=1}^{\infty} |q|^m|c_m|),\end{aligned} \tag{2.24}$$

and thus the formula

$$\left|\left(\sum_{m=1}^{\infty} q^m b_m\right)^{\star n}\right| \leq \left(\sum_{m=1}^{\infty} |q|^m|b_m|\right)^n$$

is true for $n = 2$. The general formula follows recursively by using (2.24). So we have

$$\begin{aligned}\left|\sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} q^m b_m\right)^{\star n} a_n\right| &\leq \sum_{n=0}^{\infty}\left|\left(\sum_{m=1}^{\infty} q^m b_m\right)^{\star n}\right| |a_n| \\ &\leq \sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} |q|^m|b_m|\right)^n |a_n|.\end{aligned} \tag{2.25}$$

Since the series expressing g is converging on a ball of finite radius, there exists a positive number r, sufficiently small, such that $\sum_{n=1}^{\infty} r^n|b_n|$ is finite. Moreover, $\sum_{n=1}^{\infty} r^n|b_n| = r\sum_{n=1}^{\infty} r^{n-1}|b_n| \to 0$ for $r \to 0$ and so there exists r such that $\sum_{n=1}^{\infty} r^n|b_n| < R$. Thus, from (2.25), we deduce

$$\sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} r^m|b_m|\right)^n |a_n|,$$

and this is a convergent series of the form $\sum_{t=1}^{\infty} r^t\gamma_t$. Thus we have that $(g \bullet f)(q) = \sum_{m=0}^{\infty} q^m d_m$ and $|d_m| \leq \gamma_m$ and the radius of convergence of $g \bullet f$ is at least equal to r. □

Using this notion of composition, it is also possible to define, under suitable conditions, left and right inverses of a power series:

Proposition 2.11 *Let $g : B(0; R) \to \mathbb{H}$, $R > 0$, be a function slice regular of the form $g(q) = \sum_{n=0}^{\infty} q^n a_n$.*

(1) There exists a power series $g_r^{-\bullet}(q) = \sum_{n=0}^{\infty} q^n b_n$ convergent in a disc with positive radius, such that $(g \bullet g_r^{-\bullet})(q) = q$ and $g_r^{-\bullet}(0) = 0$ if and only if $g(0) = 0$ and $g'(0) \neq 0$.

(2) *There exists a power series* $g_l^{-\bullet}(q) = \sum_{n=0}^{\infty} q^n b_n$ *convergent in a disc with positive radius, such that* $(g_l^{-\bullet} \bullet g)(q) = q$ *and* $g_l^{-\bullet}(0) = 0$ *if and only if* $g(0) = 0$ *and* $g'(0) \neq 0$.

Proof To prove (1), assume that $g_r^{-\bullet}$ exists. Then $\sum_{n=0}^{\infty} \left(\sum_{m=1}^{\infty} q^m b_m\right)^{\star n} a_n = q$. By explicitly writing the terms of the equality we see that we have

$$a_0 + \left(\sum_{m=1}^{\infty} q^m b_m\right) a_1 + \left(\sum_{m=1}^{\infty} q^m b_m\right)^{\star 2} a_2 + \ldots + \left(\sum_{m=1}^{\infty} q^m b_m\right)^{\star n} a_n + \ldots = q \quad (2.26)$$

and so to have equality it is necessary that $a_0 = 0$, i.e., $g(0) = 0$, and $b_1 a_1 = 1$ and so $a_1 \neq 0$, i. e. $g'(0) \neq 0$. To prove that the condition is sufficient, we observe that for $n \geq 2$, the coefficient of q^n is zero on the right-hand side of (2.26) while on the left-hand side it is given by

$$b_n a_1 + P_n(b_1, \ldots, b_{n-1}, a_2, \ldots, a_n), \quad (2.27)$$

thus we have

$$b_n a_1 + P_n(b_1, \ldots, b_{n-1}, a_2, \ldots, a_n) = 0, \quad (2.28)$$

where the polynomials P_n are linear in the a_i's and they contain all the possible monomials $b_{j_1} \ldots b_{j_r}$ with $j_1 + \ldots + j_r = n$ and thus also b_1^n. In particular we have: $b_1 a_1 = 1$ and so $b_1 = a_1^{-1}$, then

$$b_2 a_1 + b_1^2 a_2 = 0$$

and so

$$b_2 = -a_1^{-2} a_2 a_1^{-1}.$$

Using induction, we can compute b_n if we have computed $b_1, \ldots, b_{n-1}$ by using (2.28) and the fact that a_1 is invertible. This concludes the proof since the function $g_r^{-\bullet}$ is a right inverse of g by construction.
We now show that $g_r^{-\bullet}$ converges in a disc with positive radius following the proof of [37, Proposition 9.1]. Construct a power series with real coefficients A_n which is a majorant of g as follows: set

$$\tilde{g}(q) = q A_1 - \sum_{n=2}^{\infty} q^n A_n$$

with $A_1 = |a_1|$ and $A_n \geq |a_n|$, for all $n \geq 2$. It is possible to compute the inverse of $\tilde{g}$ with respect to the (standard) composition to get the series $\tilde{g}^{-1}(q) = \sum_{n=1}^{\infty} q^n B_n$. The coefficients B_n can be computed with the formula

$$B_n A_1 + P_n(B_1, \ldots, B_{n-1}, A_2, \ldots, A_n) = 0,$$

analog of (2.27). Then we have

$$B_1 = A_1^{-1} = |a_1|^{-1},$$
$$B_2 = A_1^{-2}(-A_2)A_1^{-1} \geq |a_1|^{-2}|a_2| \cdot |a_1|^{-1} = |b_2|$$

and, inductively

$$B_n = Q_n(A_1, \ldots, A_n) \geq Q_n(|a_1|, \ldots, |a_n|) = |b_n|.$$

We conclude that the radius of convergence of $g_r^{-\bullet}$ is greater than or equal to the radius of convergence of $\tilde{g}^{-1}$ which is positive, see [37, p. 27].

Point (2) can be proved with similar computations and the function $g_l^{-\bullet}$ so obtained is a left inverse of g. □

Remark 2.13 As we have seen in Remark 2.8, the transcendental functions cosine, sine, exponential are entire slice regular functions. Let $f(q)$ be another entire function, for example a polynomial. It is then possible to define the composite functions

$$\exp_\star(f(q)) = e_\star^{f(q)} = \sum_{n=0}^{\infty} \frac{(f(q))^{\star n}}{n!},$$

$$\sin_\star(f(q)) = \sum_{n=0}^{\infty} (-1)^n \frac{(f(q))^{\star 2n+1}}{(2n+1)!},$$

$$\cos_\star(f(q)) = \sum_{n=0}^{\infty} (-1)^n \frac{(f(q))^{\star 2n}}{(2n)!}.$$

Comments to Chapter 2. The material in this chapter comes from several papers. The theory of slice regular functions started in [110] and [111] for functions defined on balls centered at the origin and then evolved in a series of papers, among which we mention [46], [47], [58], into a theory on axially symmetric domains. The theory was developed also for functions with values in a Clifford algebra, see [72], [73], and for functions with values in a real alternative algebra, see [118]. The composition of slice regular functions which appears in this chapter is taken from [93].

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