

Chapter 2

Basic Concepts in Ergodic Theory

2.1 Ergodicity, Freeness, and Poincaré recurrence

Ergodicity, freeness, and Poincaré recurrence are the three most basic properties in ergodic theory, and not coincidentally they can all be motivated in analogy with the simple picture that describes arbitrary group actions on ordinary sets.

The two phenomena that one observes in group actions $G \curvearrowright X$ on sets without extra structure are

- (i) the canonical decomposability into transitive pieces, each of which can be described as the orbit of any of its points and hence is indecomposable, and
- (ii) the possible lack of freeness, i.e., the fact that for some $x \in X$ the stabilizer group G_x , consisting of all $s \in G$ satisfying $sx = x$, might be nontrivial.

The orbit-stabilizer theorem combines these two aspects to give a structural description of the action as $G \curvearrowright \bigsqcup_{x \in R} G/G_x$ where R is a set of representatives for the transitive pieces and G acts componentwise according to the formula $s(tG_x) = stG_x$.

If we now consider a p.m.p. action $G \curvearrowright (X, \mu)$, then the orbit-stabilizer theorem still applies but is generally useless, as one runs into the problem of choosing representatives in measure-theoretically meaningful way. On the other hand, the properties of transitivity and freeness appearing in (i) and (ii) translate in the following fundamental ways. Recall that G -invariance for a measurable set $A \subseteq X$ means $\mu(sA \Delta A) = 0$ for all $s \in G$.

Definition 2.1 The action $G \curvearrowright (X, \mu)$ is said to be *ergodic* if $\mu(A) = 0$ or 1 for every G -invariant measurable set $A \subseteq X$.

Definition 2.2 The action $G \curvearrowright (X, \mu)$ is said to be (*essentially*) *free* if there is a G -invariant set $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that if $sx = x$ for some $x \in X_0$ and $s \in G$ then $s = e$.

As the brackets above indicate, we will drop the qualifier “essentially” when speaking of freeness for p.m.p. actions. There is no effective ambiguity in doing so,

since for most purposes in ergodic theory there is no harm in discarding a G -invariant null set.

Remark 2.3 When G is Abelian, for every subgroup $H \subseteq G$ the measurable set $\{x \in X : H \text{ is the stabilizer group of } x\}$ is G -invariant. Thus if G is a finitely generated Abelian group (in which case it has countably many subgroups) and the action $G \curvearrowright (X, \mu)$ is ergodic, then a.e. $x \in X$ has the same stabilizer group H , and by passing to the quotient we obtain an action $G/H \curvearrowright (X, \mu)$ which is free. In particular, if a p.m.p. action $\mathbb{Z} \curvearrowright (X, \mu)$ of the integers is ergodic then either (i) X is finite modulo a null set and the action is conjugate to the translation action on $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$, or (ii) X is atomless and the action is free.

Note the imbalance above between the formulation of ergodicity as a property that speaks of sets and freeness as a property that speaks of points. We can rectify this with the following characterization of freeness, and also similarly rephrase ergodicity in the language of set intersections. Here we see a basic consequence of our convention that G be countable and that (X, μ) be standard.

Proposition 2.4 *The action $G \curvearrowright (X, \mu)$ is free if and only if for every finite set $F \subseteq G$ and nonnull set $A \subseteq X$ there is a nonnull set $B \subseteq A$ such that $sB \cap tB = \emptyset$ for all distinct $s, t \in F$.*

Proof If the action is not free, then it follows from the countability of G that one of the sets $\{x \in X : sx = x\}$ for $s \in G \setminus \{e\}$, which are measurable by Proposition A.21, has positive measure. Then this set together with $\{e, s\}$ fails to have the property in the proposition statement.

In the converse direction, suppose that the action is free and let us establish the desired conclusion by induction on the number of elements in F . Let A be a nonnull subset of X . In the case that F is empty or a singleton, we get the conclusion by taking B to be A . Suppose now that F is an arbitrary nonempty finite subset of G and that there is a nonnull set $B_0 \subseteq A$ and $sB_0 \cap s'B_0 = \emptyset$ for all distinct $s, s' \in F$. Let $t \in G \setminus F$. Take an enumeration s_1, \dots, s_n of F and then apply Proposition A.22 recursively to obtain nonnull sets $B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots \supseteq B_n$ such that $tB_k \cap s_k B_k = \emptyset$ for $k = 1, \dots, n$. Then the set $B = B_n$ has the desired property with respect to $F \cup \{t\}$ and A . \square

Proposition 2.5 *For a p.m.p. action $G \curvearrowright (X, \mu)$ the following are equivalent:*

- (i) *the action is ergodic,*
- (ii) $\mu(A) = 0$ or 1 *for every measurable set $A \subseteq X$ satisfying $sA = A$ for all $s \in G$ (i.e., G -invariance in the strict sense),*
- (iii) *for all sets $A, B \subseteq X$ of positive measure there is an $s \in G$ such that $\mu(sA \cap B) > 0$.*

Proof (i) \Rightarrow (ii). Trivial.

(ii) \Rightarrow (iii). Let A and B be subsets of X with positive measure. Since the set $A' := \bigcup_{s \in G} sA$ satisfies $sA' = A'$ for all $s \in G$, it must have measure one by (ii),

and so its intersection with B has the same measure as B . Since G is countable it follows that at least one of the sets $sA \cap B$ for $s \in G$ has positive measure.

(iii) \Rightarrow (i). If A is a subset of X with $0 < \mu(A) < 1$, then by (iii) there exists an $s \in G$ such that $\mu(sA \cap (X \setminus A)) > 0$, so that A fails to be G -invariant. We thus obtain (i). \square

Ergodicity is a property that can be difficult to verify directly. One would like to be able to perform some kind of decomposition in order to simplify the problem of testing every measurable set for invariance. This can be done by passing to the function level and exploiting the Hilbert space structure of $L^2(X)$, which will enable us to easily establish ergodicity in many cases (see Section 2.3). In fact ergodicity is a property of the Koopman representation, and it is convenient to apply the terminology to general unitary representations.

Definition 2.6 A unitary representation $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ is *ergodic* if there are no nonzero G -invariant vectors in \mathcal{H} , i.e., $\pi(s)\xi = \xi$ for all $s \in G$ implies $\xi = 0$.

Proposition 2.7 *The action $G \curvearrowright (X, \mu)$ is ergodic if and only if the restriction of the Koopman representation to the orthogonal complement $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$ of the constant functions is ergodic.*

Proof If A is a G -invariant subset of X with $0 < \mu(A) < 1$ then the difference $\mathbf{1}_A - \mu(A)\mathbf{1}$ is a nonzero G -invariant vector in $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$.

Conversely, suppose that f is a nonzero G -invariant vector in $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$. Then we can find a measurable set $D \subseteq X$ such that $0 < \mu(f^{-1}(D)) < 1$. For every $s \in G$ the set $sf^{-1}(D) \Delta f^{-1}(D)$ has measure zero since it is contained in the union over $n \in \mathbb{N}$ of the sets $\{x \in X : |f(x) - sf(x)| > 1/n\}$, each of which has measure zero by virtue of the fact that $\|f - sf\|_2 = 0$. Thus the action fails to be ergodic. \square

Remark 2.8 If we were to relax our convention and not require G to be countable, then the above argument demonstrates that Proposition 2.7 is still valid using Definition 2.6 as our definition of ergodicity for unitary representations and Definition 2.1 as our definition of ergodicity for actions. This illustrates that, as opposed to condition (ii) in Proposition 2.5, Definition 2.1 is really the more natural formulation of ergodicity as a phenomenon that occurs at the level of the measure algebra and connects to behaviour in function spaces such as $L^2(X, \mu)$. We remark however that (i) and (iii) in Proposition 2.5 are still equivalent when G is uncountable, but the argument is more subtle.

Example 2.9 The one-dimensional unitary representations $\pi : \mathbb{Z} \rightarrow \mathcal{B}(\mathbb{C})$ of the integers are parametrized by elements z of the unit circle \mathbb{T} in \mathbb{C} according to the formula $\pi(n)\xi = z^n\xi$. Such a representation is ergodic precisely when $z \neq 1$. In this situation one observes that for any $\xi \in \mathbb{C}$ the averages

$$\frac{1}{n} \sum_{k=0}^{n-1} \pi(k)\xi = \frac{1}{n} \cdot \frac{1 - z^n}{1 - z} \xi$$

converge to zero as $n \rightarrow \infty$. This is an embryonic instance of an *ergodic theorem*, which exploits the approximate periodicity in an action or unitary representation in order to derive conclusions about the convergence of averages of orbits of functions or vectors. Here the distribution of the complex numbers z^n is either periodic or becomes more and more uniform.

By the spectral theorem (see Section 1.11), for every unitary representation $\pi : \mathbb{Z} \rightarrow \mathcal{B}(\mathcal{H})$ of the integers there are an index set I and finite Borel measures μ_i on \mathbb{T} such that we can express \mathcal{H} as $\bigoplus_{i \in I} L^2(\mathbb{T}, \mu_i)$ and $\pi(n)$ as summandwise multiplication by the function $z \mapsto z^n$. This can be thought of (and even formulated in a precise way) as an integral of one-dimensional representations. The invariant vectors are those whose support in each summand is either $\{1\}$ or empty, and so ergodicity occurs precisely when no μ_i has 1 as an atom. The approximate periodicity observed above for one-dimensional \mathcal{H} again takes hold: for each $i \in I$ the averages $(1/n) \sum_{k=0}^{n-1} \pi(k) \xi_i$ of the component ξ_i of ξ at i converge pointwise and hence, by the dominated convergence theorem, in the L^2 -norm to the orthogonal projection of ξ_i onto the subspace of invariant vectors, and in particular to zero when π is ergodic. This is the content of mean ergodic theorem of von Neumann, who established it using such a spectral argument. As we will see in Section 4.3, von Neumann's mean ergodic theorem applies more generally to unitary representations of amenable groups by averaging over a Følner sequence, which consists of asymptotically invariant finite subsets of the group such as the intervals $\{0, \dots, n-1\}$ in the prototypical case of \mathbb{Z} . Remarkably, for unitary representations of an arbitrary group G there is a canonical way of taking a G -invariant average over the orbit of a vector, which is then expressible in the more practical asymptotic form described above when specializing to amenable groups. One can then prove an abstract mean ergodic theorem which characterizes ergodicity by the vanishing of these averages (Theorem 2.21).

Although the measures in our p.m.p. framework are probability measures and hence finite, there was no reason to impose (and good reason not to impose) the analogous condition that the set X be finite in the second paragraph of the section in order to motivate the properties of ergodicity and freeness for p.m.p. actions. On the other hand, it is instructive to draw more specifically a conceptual comparison between actions on finite sets and general p.m.p. actions, in particular to appreciate how finiteness can be understood as an incompressibility property (*Dedekind finiteness*) that is meaningful in other structural contexts beyond the purely set-theoretic, as we will also see later in connection with amenability and soficity.

The distinctive feature of actions $G \curvearrowright X$ of an infinite group on a finite set is precisely their lack of freeness: if x is an element of X with trivial stabilizer group then $s \mapsto sx$ defines a bijection from G to Gx , contradicting the finiteness of X . This can be translated to p.m.p. actions $G \curvearrowright (X, \mu)$ of an infinite group by replacing points by sets of nonzero measure. Thus given a set $A \subseteq X$ of nonzero measure there exists a nontrivial $s \in G$ such that $\mu(sA \cap A) > 0$, for otherwise the sets sA for $s \in G$ are almost everywhere pairwise disjoint so that for any nonempty finite set $F \subseteq G$ we have $\mu(A) = (\sum_{s \in F} \mu(sA))/|F| \leq 1/|F|$, a contradiction. It is moreover interesting to observe, especially in anticipation of our discussions

on amenability and orbit equivalence, that one can reframe this more qualitatively as the preclusion of Hilbert's hotel by the incompressibility of the action: to reach a contradiction, take an enumeration $s_1 = e, s_2, s_3, \dots$ of the elements of G and set $f(s_n x) = s_{n+1}x$ for $x \in A$ and $n \in \mathbb{N}$ to almost everywhere define a measure-preserving map f from the set $A' := \bigcup_{n=1}^{\infty} s_n A$ to itself, and then observe that $\mu(A) = \mu(A' \setminus f(A')) = \mu(A') - \mu(f(A')) = 0$.

This kind of recurrence, which is a dynamical version of the pigeonhole principle, was first observed by Poincaré and predates the formalization of ergodicity, on which it does not depend. Poincaré showed in fact that, for single transformations, almost every point of a measurable subset returns to visit the set infinitely many times in the future. For general infinite G one can derive a similarly stronger conclusion, in analogy with the fact that for actions on a finite set each stabilizer group G_x is not only nontrivial but also infinite, having finite index $|Gx|$ in G .

Theorem 2.10 (Poincaré recurrence) *Let $G \curvearrowright (X, \mu)$ be a p.m.p. action of an infinite group and let A be a subset of X with $\mu(A) > 0$. Then for a.e. $x \in A$ the set of all $s \in G$ such that $sx \in A$ is infinite. In the case of a single p.m.p. transformation $T : X \rightarrow X$, for a.e. $x \in A$ there are infinitely many $n \in \mathbb{N}$ such that $T^n x \in A$.*

Proof Suppose the conclusion fails. Then there is a finite set $F \subseteq G$ for which the set $B := A \setminus \bigcup_{s \in G \setminus F} sA$ has nonzero measure. As G is infinite, by recursion we can construct an infinite set $I \subseteq G$ such that $s \notin tF$ for all distinct $s, t \in I$. Then the sets sB for $s \in I$ are pairwise disjoint and all have the same measure as B , a contradiction.

In the case of a single transformation T , we find instead an $n \geq 1$ such that the set $A \setminus \bigcup_{k=n+1}^{\infty} T^{-k}A$ has nonzero measure and proceed accordingly. \square

The analogue of the above for actions $G \curvearrowright X$ on ordinary sets would be that recurrence (however one might interpret it) is automatic when G is infinite and X is finite, in which case the action cannot be free. It is the probabilistic setting of Poincaré recurrence that allows it to coexist with freeness. This kind of tension between the infinite (freeness + infinite acting group) and the finite (automatic recurrence) is one of the hallmarks of ergodic theory and of functional analysis more generally. Structurally more profound manifestations of the infinite-versus-finite dialectic emerge in the study of weak mixing and compactness, which we introduce in the next two sections.

Having observed the basic phenomenon of Poincaré recurrence, we might now ask two different types of questions, one asymptotic and the other perturbative. These lie at the root of practically everything that will be encountered in the book henceforth.

- (i) How frequently and with what degree of overlap does recurrence occur asymptotically across orbits of sets?
- (ii) Is recurrence part of a more complete picture of the dynamics that one can obtain at a given scale of observation, say in terms of the approximate permutation of subsets of the space?

The theory of weak mixing and compactness provides a natural framework for addressing question (i) that leads to Furstenberg's celebrated multiple recurrence theorem. We can also view recurrence as the "order zero" case of combinatorial independence, for which the same kinds of asymptotic problems can be investigated in connection with both weak mixing and entropy, as will be done in Chapters 8 and 12.

Question (ii) speaks to the idea of approximation at a given scale, as opposed to asymptotic behaviour, and leads to the notion of an amenable group, which we treat in Chapter 4. It is precisely when G is amenable that every free p.m.p. action of G possesses the kind of approximate tileability alluded to in (ii), as captured by the Rokhlin-type quasitower theorem of Ornstein and Weiss (Section 4.6). Weak mixing and compactness again play a role here, but in a less overt way and with compactness appearing in a much weaker perturbative form, as described in the introduction to Chapter 4.

It is interesting to note that the phenomena typically associated with question (i), such as weak mixing and entropy, involve multiplicative structure in the form of probabilistic or combinatorial independence (compactness is an exception to this), while the phenomenon most immediately associated with question (ii), namely the Rokhlin lemma, is an expression of additive structure in which, roughly speaking, group elements permute disjoint subsets that cover the space (thus respecting a "direct sum" decomposition of the Boolean algebra of sets).

2.2 Mixing, Weak Mixing, and Compactness

In a probability space (X, μ) two measurable sets A and B need not satisfy the independence condition $\mu(A \cap B) = \mu(A)\mu(B)$, but for a p.m.p. action $G \curvearrowright (X, \mu)$ it can happen that for all such A and B the images sA for $s \in G$ become asymptotically independent with respect to B , as Bernoulli actions (Section 2.3) prototypically demonstrate.

Definition 2.11 The action $G \curvearrowright (X, \mu)$ is said to be *mixing* if G is infinite and for all measurable sets $A, B \subseteq X$ the function on G defined by

$$s \mapsto \mu(sA \cap B) - \mu(A)\mu(B)$$

vanishes at infinity (i.e., for every $\varepsilon > 0$ there is a finite subset of G off of which the function values are less than ε in absolute value).

The independence of two measurable sets A and B translates into the orthogonality of the vectors $\mathbf{1}_A - \mu(A)\mathbf{1}$ and $\mathbf{1}_B - \mu(B)\mathbf{1}$ in the Hilbert space $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$. Consequently, we can rephrase an asymptotic independence condition like mixing as an asymptotic orthogonality property for unitary representations, with the link between the two passing through the restriction of the Koopman representation to $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$.

Definition 2.12 A unitary representation $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ is *mixing* if G is infinite and for all $\xi, \zeta \in \mathcal{H}$ the function $s \mapsto \langle \pi(s)\xi, \zeta \rangle$ on G vanishes at infinity.

Example 2.13 For every infinite G , the left regular representation $\lambda : G \rightarrow \mathcal{B}(\ell_2(G))$, given by $(\lambda(s)f)(t) = f(s^{-1}t)$ for all $s, t \in G$ and $f \in \ell_2(G)$, is mixing. Indeed if ξ and ζ are vectors in $\ell_2(G)$ supported on finite sets E and F , respectively, then $\langle \lambda(s)\xi, \zeta \rangle = 0$ for all $s \in G$ lying outside the finite set FE^{-1} , and since finitely supported vectors are dense in $\ell_2(G)$ we obtain the vanishing condition in the above definition by a simple approximation.

Proposition 2.14 *The action $G \curvearrowright (X, \mu)$ is mixing if and only if the restriction of the Koopman representation π to the orthogonal complement $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$ of the constant functions is mixing.*

Proof Suppose first that the action is mixing. Let $\xi, \zeta \in L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$ and let us show that the function $s \mapsto \langle \pi(s)\xi, \zeta \rangle$ vanishes at infinity. We may assume by a straightforward approximation argument that there are finite partitions \mathcal{P} and \mathcal{Q} of X and scalars c_A and d_B such that ξ and ζ are of the form $\sum_{A \in \mathcal{P}} c_A \mathbf{1}_A$ and $\sum_{B \in \mathcal{Q}} d_B \mathbf{1}_B$, respectively. Then $\sum_{A \in \mathcal{P}} \sum_{B \in \mathcal{Q}} c_A \bar{d}_B \mu(A)\mu(B)$ is the product of the integrals of ξ and $\bar{\zeta}$ and hence is equal to zero, so that for all $s \in G$ we have by the triangle inequality

$$|\langle \pi(s)\xi, \zeta \rangle| \leq \sum_{A \in \mathcal{P}} \sum_{B \in \mathcal{Q}} |c_A \bar{d}_B| |\mu(sA \cap B) - \mu(A)\mu(B)|,$$

which yields the desired asymptotic vanishing.

For the converse direction, apply the mixing hypothesis to the vectors $\mathbf{1}_A - \mu(A)\mathbf{1}$ and $\mathbf{1}_B - \mu(B)\mathbf{1}$ in $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$. \square

Mixing is a powerful condition but it occurs relatively infrequently and does not fit naturally as a complement to other phenomena in a way that might lead to a deeper structure theory. These deficiencies are rectified, to remarkably fertile effect, by asking for the asymptotic independence to hold in a less strict sense referred to as *weak mixing*. As we will see in Chapter 8, one indication of the structural richness of weak mixing is the fact that it is a more natural condition from the viewpoint of combinatorial independence, which permits one to compare it with entropy in a simple way.

For a single measure-preserving transformation T of a probability space (X, μ) , weak mixing is customarily defined as the mean asymptotic independence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(T^{-k}A \cap B) - \mu(A)\mu(B)| = 0 \quad (2.1)$$

for all measurable sets $A, B \subseteq X$. A critical feature of this definition whose significance may not be initially evident is the asymptotic translation-invariance of the set of powers $\{0, \dots, n-1\}$ over which the averaging is carried out. This has the

effect that the averaging in (2.1) is asymptotically T -invariant, so that if we replace A by TA we get the same limiting value. The existence of a sequence of asymptotically invariant finite sets in a group is the Følner characterization of amenability (Theorem 4.4), and for amenable acting groups one can define weak mixing by averaging over a Følner sequence exactly as in (2.1).

Surprisingly, it turns out that for a p.m.p. action $G \curvearrowright (X, \mu)$ of any group, whether amenable or not, the functions

$$s \mapsto |\mu(sA \cap B) - \mu(A)\mu(B)| \quad (2.2)$$

for measurable $A, B \subseteq X$ can be averaged in a canonical G -invariant way. We can thus define weak mixing without recourse to amenability by asking for the vanishing of these averages, which we will do in Definition 2.15 once we have set up the terms more precisely. A function such as (2.2) is *weakly almost periodic*, which means that, viewing it as an element of $\ell_\infty(G)$ equipped with the G -action $(sf)(t) = f(s^{-1}t)$, the weak closure of its orbit is weakly compact. The weakly almost periodic functions in $\ell_\infty(G)$ form a G -invariant unital sub- C^* -algebra, which we denote by $\text{WAP}(G)$ (see Appendix D). In particular, if f is a weakly almost periodic function then so is $|f|$, and thus we see that (2.2) is weakly almost periodic by Proposition D.9. Now it is not always the case that $\ell_\infty(G)$ admits a G -invariant unital positive linear functional, which would give formal expression to the idea of G -invariant averaging. By definition such a functional exists precisely when G is amenable (Definition 4.1), and it is far from being unique when G is infinite. In contrast, there is always a unique G -invariant mean on $\text{WAP}(G)$ (Theorem D.13), which we denote by m . Thus m is a positive linear functional such that $m(\mathbf{1}) = 1$ and $m(sf) = m(f)$ for all $f \in \text{WAP}(G)$ and $s \in G$. This functional additionally satisfies the right invariance condition $m(fs) = m(f)$ where fs is the function $t \mapsto f(ts^{-1})$. When G is amenable m can be evaluated in a more concrete and practical form as a limit of averages over Følner sets as in (2.1), as recorded in Proposition D.17.

Definition 2.15 A p.m.p. action $G \curvearrowright (X, \mu)$ is *weakly mixing* if

$$m(s \mapsto |\mu(sA \cap B) - \mu(A)\mu(B)|) = 0$$

for all measurable $A, B \subseteq X$.

That this is equivalent to the classical definition (2.1) when $G = \mathbb{Z}$ is a consequence of Proposition D.17. We will discuss this equivalence for general amenable G in Section 4.3.

Proposition 2.16 *Every weakly mixing p.m.p. action is ergodic.*

Proof For a p.m.p. action $G \curvearrowright (X, \mu)$, the definition of weak mixing implies that for every G -invariant measurable $A \subseteq X$ we have $\mu(A) - \mu(A)^2 = 0$ and hence $\mu(A) = 0$ or 1 . \square

Like mixing, weak mixing depends only on the Koopman representation. The basic theory relies on Hilbert space techniques, and accordingly we will begin by developing it in the language of unitary representations, where probabilistic independence for pairs of measurable sets can be translated as orthogonality. So let G be a group and $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ a unitary representation. For $\xi, \zeta \in \mathcal{H}$ the function $f_{\xi, \zeta}$ on G given by $s \mapsto \langle \pi(s)\xi, \zeta \rangle$ is weakly almost periodic (Proposition D.8). Using the functional m on $\text{WAP}(G)$, we thus have two options for expressing orthogonality in mean: $m(f_{\xi, \zeta}) = 0$ or $m(|f_{\xi, \zeta}|) = 0$. The second provides the mechanism for defining weak mixing, which we now proceed to do in Definition 2.17, while the first gives a characterization of ergodicity.

In Definition 2.15 we could have first defined weak mixing locally as a property of a single measurable set $A \subseteq X$ by taking $B = A$, and then designated a p.m.p. action as being weakly mixing if all such A have this property. That this gives an equivalent formulation of weak mixing for p.m.p. actions can be gleaned from Lemma 2.19 using vectors of the form $\mathbf{1}_A - \mu(A)\mathbf{1}$ in $L^2(X, \mu)$. For representations we will take this local approach, as it will anticipate our treatment of the Hilbert module relativization in Chapter 3.

Definition 2.17 Let $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ be a unitary representation. A vector $\xi \in \mathcal{H}$ is *weakly mixing* if $m(|f_{\xi, \xi}|) = 0$. The representation π is *weakly mixing* if every vector in \mathcal{H} is weakly mixing.

Proposition 2.18 *Every weakly mixing unitary representation is ergodic.*

Proof It suffices to observe that if $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ is a unitary representation and ξ is a G -invariant vector in \mathcal{H} then the function $|f_{\xi, \xi}|$ takes the constant value $\|\xi\|^2$ and hence has this value as its mean. \square

Lemma 2.19 *Let $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ be a unitary representation. A vector $\xi \in \mathcal{H}$ is weakly mixing if and only if $m(|f_{\xi, \zeta}|) = 0$ for all $\zeta \in \mathcal{H}$.*

Proof For the nontrivial direction, suppose that ξ is a weakly mixing vector in \mathcal{H} and let $\zeta \in \mathcal{H}$. To show that the function $s \mapsto |\langle \pi(s)\xi, \zeta \rangle|$ has mean zero, we may assume that ζ has zero component in the orthogonal complement of the set $\{\pi(s)\xi : s \in G\}$, and by an approximation argument we may furthermore assume ζ to be of the form $\pi(t)\xi$ for some $t \in G$. Using the G -invariance of m , we then have

$$m(|f_{\xi, \zeta}|) = m(t|f_{\xi, \xi}|) = m(|f_{\xi, \xi}|) = 0.$$

\square

Lemma 2.19 immediately yields the following.

Proposition 2.20 *A unitary representation $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ is weakly mixing if and only if $m(|f_{\xi, \zeta}|) = 0$ for all $\xi, \zeta \in \mathcal{H}$.*

We will see as part of Theorem 2.25 that a p.m.p. action $G \curvearrowright (X, \mu)$ is weakly mixing if and only if the restriction of its Koopman representation to $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$ is weakly mixing.

The following abstract mean ergodic theorem will be useful in the proof of Theorem 2.23. It shows that a unitary representation $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ is ergodic if and only if $m(f_{\xi, \zeta}) = 0$ for all $\xi, \zeta \in \mathcal{H}$, and also that a p.m.p. action $G \curvearrowright (X, \mu)$ is ergodic if and only if

$$m(s \mapsto \mu(sA \cap B) - \mu(A)\mu(B)) = 0$$

for all measurable sets $A, B \subseteq X$, which we see by using Proposition 2.7 and the vectors $\mathbf{1}_A - \mu(A)\mathbf{1}$ and $\mathbf{1}_B - \mu(B)\mathbf{1}$ in $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$.

Theorem 2.21 *Let $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ be a unitary representation. Then $m(f_{\xi, \zeta}) = \langle P\xi, \zeta \rangle$ for all $\xi, \zeta \in \mathcal{H}$, where P is the orthogonal projection of \mathcal{H} onto the closed subspace of G -invariant vectors.*

Proof First note that the closed subspace of G -invariant vectors is the orthogonal complement of the set of vectors of the form $\pi(t)\eta - \eta$ for some $\eta \in \mathcal{H}$ and $t \in G$, for if ζ is a G -invariant vector then for every $\eta \in \mathcal{H}$ and $s \in G$ we have

$$\langle \pi(s)\eta - \eta, \zeta \rangle = \langle \eta, \pi(s^{-1})\zeta - \zeta \rangle = 0,$$

while if ζ is a vector which is orthogonal to every vector of the form $\pi(t)\eta - \eta$ then for every $s \in G$ and $\eta \in \mathcal{H}$ we have

$$\langle \pi(s)\zeta - \zeta, \eta \rangle = \langle \zeta, \pi(s^{-1})\eta - \eta \rangle = 0$$

so that $\pi(s)\zeta = \zeta$. Therefore to verify that $m(f_{\xi, \zeta}) = \langle P\xi, \zeta \rangle$ for prescribed $\xi, \zeta \in \mathcal{H}$ we may assume by an approximation argument that ζ is either G -invariant or of the form $\pi(s)\eta - \eta$. In the first case we have $f_{\xi, \zeta}(s) = \langle \xi, \pi(s^{-1})\zeta \rangle = \langle \xi, \zeta \rangle$ for all $s \in G$ and hence $m(f_{\xi, \zeta}) = \langle \xi, \zeta \rangle$, while in the second we have $m(f_{\xi, \zeta}) = m(sf_{\xi, \eta} - f_{\xi, \eta}) = 0$. \square

Definition 2.17 and Theorem 2.21 indicate that weak mixing is a positive version of ergodicity, an idea reinforced by condition (viii) in Theorem 2.23 below. The practical effect of this positivity is that, unlike for ergodicity, the vanishing mean condition has substantial algebraic ramifications, which connects it for example to amenability and property (T).

Definition 2.22 Let $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ be a unitary representation. A vector $\xi \in \mathcal{H}$ is said to be *compact* if $\overline{\pi(G)\xi}$ is compact. The representation π is said to be *compact* if every vector in \mathcal{H} is compact.

Theorem 2.23 *For a unitary representation $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$, the following are equivalent:*

- (i) π is weakly mixing,
- (ii) for every finite set $\Omega \subseteq \mathcal{H}$ and $\varepsilon > 0$ the set of all $s \in G$ such that $|\langle \pi(s)\xi, \zeta \rangle| < \varepsilon$ for all $\xi, \zeta \in \Omega$ is thickly syndetic (Definition D.15),
- (iii) for every finite set $\Omega \subseteq \mathcal{H}$ and $\varepsilon > 0$ there exists an $s \in G$ such that $|\langle \pi(s)\xi, \zeta \rangle| < \varepsilon$ for all $\xi, \zeta \in \Omega$,
- (iv) the only compact vector in \mathcal{H} is the zero vector,
- (v) π has no nonzero finite-dimensional subrepresentations,
- (vi) $\pi \otimes \rho$ is ergodic for every unitary representation ρ of G ,
- (vii) $\pi \otimes \rho$ is weakly mixing for every unitary representation ρ of G ,
- (viii) $\pi \otimes \bar{\pi}$ is ergodic.

Proof (i) \Rightarrow (ii). Apply Propositions 2.20 and D.16.

(ii) \Rightarrow (iii). Trivial.

(iii) \Rightarrow (iv). Let ξ be a nonzero vector in \mathcal{H} . Recursively applying (iii) we construct a sequence $\{s_n\}$ in G by setting $s_1 = e$ and choosing s_n for $n > 1$ so that for every $k = 1, \dots, n-1$ we have

$$|\langle \pi(s_n)\xi, \pi(s_k)\xi \rangle| < \frac{1}{2} \|\xi\|^2$$

and hence

$$\begin{aligned} \|\pi(s_n)\xi - \pi(s_k)\xi\|^2 &= \langle \pi(s_n)\xi - \pi(s_k)\xi, \pi(s_n)\xi - \pi(s_k)\xi \rangle \\ &= 2\|\xi\|^2 - 2\operatorname{re}\langle \pi(s_n)\xi, \pi(s_k)\xi \rangle \\ &\geq 2(\|\xi\|^2 - |\langle \pi(s_n)\xi, \pi(s_k)\xi \rangle|) \geq \|\xi\|^2. \end{aligned}$$

Then the set $\{\pi(s_n)\xi : n \in \mathbb{N}\}$ fails to be totally bounded, so that ξ is not compact. This yields (iv).

(iv) \Rightarrow (v). This follows from the fact that every closed bounded subset of a finite-dimensional Hilbert space is compact.

(v) \Rightarrow (vi). Let $\rho : G \rightarrow \mathcal{B}(\mathcal{H})$ be a unitary representation, and suppose that there is a nonzero vector $\xi \in \mathcal{H} \otimes \mathcal{H}$ such that $(\pi \otimes \rho)(s)\xi = \xi$ for all $s \in G$. Let T be the corresponding operator in $\operatorname{HS}(\overline{\mathcal{H}}, \mathcal{H})$ according to Section 1.8. Then for all $s \in G$ we have $\pi(s)T\bar{\rho}(s)^* = T$ and hence also $T^* = \bar{\rho}(s)T^*\pi(s)^*$ so that $TT^*\pi(s) = \pi(s)TT^*$. Since TT^* is a nonzero compact operator, it has a nonzero eigenvalue λ , and the associated eigenspace E_λ is finite-dimensional (Theorem 1.14). Then for $\zeta \in E_\lambda$ and $s \in G$ we have

$$TT^*\pi(s)\zeta = \pi(s)TT^*\zeta = \pi(s)\lambda\zeta = \lambda\pi(s)\zeta$$

so that $\pi(s)\zeta \in E_\lambda$. Thus E_λ is invariant and so π has a nonzero finite-dimensional subrepresentation, which contradicts (v). Hence $\pi \otimes \rho$ must be ergodic.

(vi) \Rightarrow (viii). Trivial.

(viii) \Rightarrow (i). Given $\xi \in \mathcal{H}$, by (viii) and Theorem 2.21 we have

$$\begin{aligned}
m(|f_{\xi,\xi}|^2) &= m(s \mapsto f_{\xi,\xi}(s) \overline{f_{\xi,\xi}(s)}) = m(s \mapsto \langle \pi(s)\xi, \xi \rangle \langle \bar{\pi}(s)\xi, \xi \rangle) \\
&= m(s \mapsto \langle (\pi \otimes \bar{\pi})(s)\xi \otimes \xi, \xi \otimes \xi \rangle) = 0,
\end{aligned}$$

in which case the Cauchy–Schwarz inequality gives

$$m(|f_{\xi,\xi}|)^2 = m(|f_{\xi,\xi}| \cdot \mathbf{1})^2 \leq m(|f_{\xi,\xi}|^2) m(\mathbf{1}) = 0,$$

yielding (i).

(vii) \Rightarrow (vi). Apply Proposition 2.18.

(vi) \Rightarrow (vii). Let ρ be a unitary representation of G . By (vi), for every unitary representation σ of G the representation $\pi \otimes (\rho \otimes \sigma)$ is ergodic. Expressing the latter as $(\pi \otimes \rho) \otimes \sigma$ we conclude that $\pi \otimes \rho$ is weakly mixing using the implication (vi) \Rightarrow (i), which we have established above. \square

Theorem 2.24 *Every unitary representation of G decomposes uniquely into a direct sum of weakly mixing and compact subrepresentations. Moreover, a unitary representation is compact if and only if it decomposes into a direct sum of finite-dimensional subrepresentations, which we can take to be irreducible.*

Proof Let $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ be a unitary representation. It is clear from Definition 2.22 that the compact vectors form a closed G -invariant subspace $\mathcal{H}_{\text{cpt}} \subseteq \mathcal{H}$. By Definition 2.17 and Lemma 2.19, the weakly mixing vectors also form a closed G -invariant subspace $\mathcal{H}_{\text{wm}} \subseteq \mathcal{H}$. From (i) \Leftrightarrow (iv) of Theorem 2.23 we see that these subspaces are orthogonal, and also that if the orthogonal complement of $\mathcal{H}_{\text{wm}} \oplus \mathcal{H}_{\text{cpt}}$ is nonzero then it must contain a nonzero compact vector, a contradiction. Consequently $\mathcal{H} = \mathcal{H}_{\text{wm}} \oplus \mathcal{H}_{\text{cpt}}$. The uniqueness of the decomposition also follows easily from (i) \Leftrightarrow (iv) of Theorem 2.23, completing the first part of the theorem.

An easy approximation argument using the compactness of balls in finite dimensions shows that a direct sum of finite-dimensional unitary representations is compact. Conversely, suppose that we are given a compact unitary representation $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$. Let \mathcal{C} be the collection of sets of pairwise orthogonal finite-dimensional subrepresentations of π . Ordering \mathcal{C} by inclusion, we observe that each totally ordered subcollection is bounded above by its union. It follows by Zorn's lemma that \mathcal{C} has a maximal element Ω . Then the direct sum $\bigoplus_{\rho \in \Omega} \rho$ must equal π , for otherwise its orthogonal complement would contain a finite-dimensional subrepresentation by (v) \Rightarrow (iv) of Theorem 2.23, contradicting maximality. Each representation in Ω can be further decomposed into irreducibles by a simple recursive splitting procedure, which terminates because there are only finitely many dimensions at play. \square

Note that the decomposition of a compact unitary representation into irreducibles is not unique in general, as illustrated by the tensor product of an irreducible finite-dimensional unitary representation with the identity representation on a two-dimensional Hilbert space.

Now we translate Theorem 2.23 to p.m.p. actions via the restriction of the Koopman representation to $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$. One issue to be careful about is that the tensor

product of two such restrictions is not equal to the corresponding restriction for the product action. Thus in condition (viii) below we must assume the action $G \curvearrowright (Y, \nu)$ to be ergodic. Another point is that conditions (iv) and (v) involve measurable sets instead of L^2 functions, and to connect these to the other conditions we will need a spectral argument, which appears in the proof of (v) \Rightarrow (vi). Note furthermore that (ix) in Theorem 2.25, in contrast to (viii) in Theorem 2.23, does not explicitly involve a conjugate, as the Koopman representation is isomorphic to its conjugate via the unitary operator $f \mapsto \bar{f}$. This reflects the natural presence of positivity in the structure of an action. Such positivity is also inherent in a representation of the form $\pi \otimes \bar{\pi}$ via its isomorphism with conjugation on Hilbert–Schmidt operators, but is absent in general for unitary representations.

For brevity we write $\mu(f)$ for the integral $\int_X f d\mu$.

Theorem 2.25 *For a p.m.p. action $G \curvearrowright (X, \mu)$, the following are equivalent:*

- (i) *the action is weakly mixing,*
- (ii) *the restriction of the Koopman representation to $L^2(X) \ominus \mathbb{C}\mathbf{1}$ is weakly mixing,*
- (iii) *for all $f, g \in L^2(X)$ one has $\lim_{s \rightarrow \infty} |\mu((sf)g) - \mu(f)\mu(g)| = 0$,*
- (iv) *for every finite collection Ω of measurable subsets of X and every $\varepsilon > 0$ the set of all $s \in G$ such that $|\mu(sA \cap B) - \mu(A)\mu(B)| < \varepsilon$ for all $A, B \in \Omega$ is thickly syndetic (Definition D.15),*
- (v) *for every finite collection Ω of measurable subsets of X and every $\varepsilon > 0$ there exists an $s \in G$ such that $|\mu(sA \cap B) - \mu(A)\mu(B)| < \varepsilon$ for all $A, B \in \Omega$,*
- (vi) *the only compact elements in $L^2(X)$ under the Koopman representation are the a.e. constant functions,*
- (vii) *the restriction of the Koopman representation to $L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$ has no nonzero finite-dimensional subrepresentations,*
- (viii) *for every ergodic p.m.p. action $G \curvearrowright (Y, \nu)$ the product action $G \curvearrowright (X \times Y, \mu \times \nu)$ is ergodic,*
- (ix) *the product action $G \curvearrowright (X \times X, \mu \times \mu)$ is ergodic.*

Proof (i) \Rightarrow (iv). Apply Proposition D.16.

(iv) \Rightarrow (v). Trivial.

(v) \Rightarrow (vi). Let f be a compact element in $L^2(X)$. To show that f is a.e. constant, it suffices to show, given a closed set $D \subseteq \mathbb{C}$, that the set $A := f^{-1}(D)$ has measure 0 or 1. Recursively applying (v) we construct a sequence $\{s_n\}$ in G such that

$$\lim_{n \rightarrow \infty} \mu(s_n A \cap s_m(X \setminus A)) = \mu(A)(1 - \mu(A))$$

for all $m \in \mathbb{N}$ by setting $s_1 = e$ and choosing s_n for $n > 1$ so that for each $k = 1, \dots, n-1$ we have $|\mu(s_n A \cap s_k(X \setminus A)) - \mu(A)\mu(X \setminus A)| < 1/n$. As f is compact, by passing to a subsequence we may assume that $\lim_{n, m \rightarrow \infty} \|s_n f - s_m f\|_2 = 0$. For $k \in \mathbb{N}$ write C_k for the measurable set of all $x \in X$ such that $f(x)$ does not lie within distance $1/k$ to some point in D . Since $X \setminus A$ is equal to the union of the increasing sequence of the sets C_k over k , given an $\varepsilon > 0$ we can find a particular k such that $\mu(X \setminus A) \leq \mu(C_k) + \varepsilon$. Then for all n and m we have

$$\mu(s_n A \cap s_m (X \setminus A)) \leq \mu(s_n A \cap s_m C_k) + \varepsilon.$$

Since

$$\|s_n f - s_m f\|_2^2 \geq \frac{1}{k^2} \mu(s_n A \cap s_m C_k)$$

for all n and m , we also have $\lim_{n,m \rightarrow \infty} \mu(s_n A \cap s_m C_k) = 0$. As ε may be taken arbitrarily small, it follows that $\lim_{n,m \rightarrow \infty} \mu(s_n A \cap s_m (X \setminus A)) = 0$. By our choice of the sequence $\{s_n\}$, we conclude that $\mu(A)$ is either 0 or 1, as desired.

(vi) \Rightarrow (vii). Apply (iv) \Rightarrow (v) from Theorem 2.23.

(vii) \Rightarrow (viii). Let $G \curvearrowright (Y, \nu)$ be an ergodic p.m.p. action. Then we can express $L^2(X \times Y) \ominus \mathbb{C}\mathbf{1}_{X \times Y}$ as the orthogonal direct sum of the G -invariant subspaces $(L^2(X) \ominus \mathbb{C}\mathbf{1}_X) \otimes L^2(Y)$ and $\mathbb{C}\mathbf{1}_X \otimes (L^2(Y) \ominus \mathbb{C}\mathbf{1}_Y)$. The action of G is ergodic on the first by (v) \Rightarrow (vi) in Theorem 2.23 and on the second by Proposition 2.7. Thus G acts ergodically on the direct sum, yielding (viii) in view of Proposition 2.7.

(viii) \Rightarrow (ix). Taking the action on Y in (viii) to be the trivial action on a one-point set shows that $G \curvearrowright (X, \mu)$ is ergodic. Now apply (viii) again to get (ix) by taking the action on Y to be $G \curvearrowright (X, \mu)$.

(ix) \Rightarrow (ii). Observe that the Koopman representation κ of $G \curvearrowright (X \times X, \mu \times \mu)$ is equivalent to the tensor product of the Koopman representation ρ of $G \curvearrowright (X, \mu)$ with itself via the canonical isomorphism $L^2(X \times X) \cong L^2(X) \otimes L^2(X)$. Using the fact that the unitary operator from $L^2(X)$ to the conjugate Hilbert space $\bar{L}^2(X)$ given by $f \mapsto \bar{f}$ intertwines ρ with its conjugate, we can thus identify κ with $\rho \otimes \bar{\rho}$ in such a way that the restriction of κ to $L^2(X \times X) \ominus \mathbb{C}\mathbf{1}_{X \times X}$ contains the tensor product of the restriction of ρ to $L^2(X) \ominus \mathbb{C}\mathbf{1}_X$ with its conjugate. This latter tensor product is ergodic by (ix) and Proposition 2.7, and so by (viii) \Rightarrow (i) from Theorem 2.23 the restriction of ρ to $L^2(X) \ominus \mathbb{C}\mathbf{1}_X$ is weakly mixing.

(ii) \Rightarrow (iii). Let $f, g \in L^2(X)$. Then $f - \mu(f)\mathbf{1}$ and $\bar{g} - \mu(\bar{g})\mathbf{1}$ lie in $L^2(X) \ominus \mathbb{C}\mathbf{1}$ and so $m(s \mapsto |\langle sf - \mu(f)\mathbf{1}, \bar{g} - \mu(\bar{g})\mathbf{1} \rangle|) = 0$, which can be reexpressed as the condition in (iii) by expanding the inner product and simplifying.

(iii) \Rightarrow (i). Apply (iii) to the indicator functions $\mathbf{1}_A$ and $\mathbf{1}_B$ for measurable sets $A, B \subseteq X$. \square

In reference to Definition 2.22 we make the following definition.

Definition 2.26 A p.m.p. action is said to be *compact* if its Koopman representation is compact.

In view of the discussion on von Neumann algebras and G -factor maps in Section 1.12, the following theorem shows that every p.m.p. action $G \curvearrowright (X, \mu)$ has a largest compact factor, i.e., there are a compact p.m.p. action $G \curvearrowright (Y, \nu)$ and a G -factor map $X \rightarrow Y$ such that if $X \rightarrow Z$ is a G -factor map onto another compact p.m.p. G -action then there is a G -factor map $Y \rightarrow Z$ for which $X \rightarrow Z$ is equal off a null set to the composition $X \rightarrow Y \rightarrow Z$.

Theorem 2.27 Let $G \curvearrowright (X, \mu)$ be a p.m.p. action. Then the set N of functions in $L^\infty(X)$ which are compact as elements of $L^2(X)$ under the Koopman representation

is a G -invariant von Neumann subalgebra, and the L^2 -closure of N in $L^2(X)$ is the subspace of compact vectors.

Proof It is clear from Definition 2.22 that N is a G -invariant linear subspace of $L^\infty(X)$, and that it is closed in $L^\infty(X)$ under the restriction of the L^2 -norm topology. Now let $f, g \in N$ and let us argue that $fg \in N$. We may assume that f and g have L^∞ -norm at most one. By the compactness of f and g , given an $\varepsilon > 0$ we can find a finite subset Ω of the union of the orbits of f and g which is an $(\varepsilon/2)$ -net for this union with respect to the L^2 -norm. Then for every $s \in G$ we can find $f', g' \in \Omega$ such that $\|sf - f'\|_2 < \varepsilon/2$ and $\|sg - g'\|_2 < \varepsilon/2$, in which case

$$\begin{aligned} \|s(fg) - f'g'\|_2 &\leq \|(sf - f')sg\|_2 + \|f'(sg - g')\|_2 \\ &\leq \|sf - f'\|_2 \|sg\|_\infty + \|f'\|_\infty \|sg - g'\|_2 < \varepsilon. \end{aligned}$$

Thus the set $\{hk : h, k \in \Omega\}$ is a finite ε -net for the orbit of fg , which shows that this orbit is totally bounded. As total boundedness is equivalent to precompactness in $L^2(X)$, we deduce that $fg \in N$. Since N is obviously closed under taking complex conjugates, we conclude that it is a G -invariant von Neumann subalgebra of $L^\infty(X)$.

We now turn to the second assertion. Let f be a compact element of $L^2(X)$. Let D be a closed bounded subset of \mathbb{C} and set $M = \sup\{|z| : z \in D\} + 1$ and $p = \mathbf{1}_{f^{-1}(D)}$. It is enough to show that the element pf of $L^\infty(X)$ is compact, since by the dominated convergence theorem we can choose it to be as close as we wish to f in L^2 -norm by taking D to be a sufficiently large disk centred at zero. Let $\varepsilon > 0$, and let us argue that there exists a $\delta > 0$ such that if s and t are any elements of G for which $\|sf - tf\|_2 < \delta$ then $\|s(pf) - t(pf)\|_2 < \varepsilon$. This is sufficient to conclude that pf is compact in view of the fact that total boundedness and precompactness are equivalent for subsets of $L^2(X)$. We may assume that $t = e$ by multiplying by t^{-1} . Take a $0 < \delta < 1$ such that the set

$$B := \{x \in X : f(x) \notin D \text{ and } \text{dist}(f(x), D) \leq \sqrt{\delta}\}$$

has measure less than $(\varepsilon/(6M))^2$. By shrinking δ we may assume that it is smaller than both $\varepsilon^2/(9M^2)$ and $\varepsilon/3$. Let $s \in G$ and suppose that $\|sf - f\|_2 < \delta$. Set $A = f^{-1}(D)$. Since $s(pf)$ and pf are zero on $A \setminus sA$ and $sA \setminus A$, respectively, and the sets A and B are disjoint, we have

$$\begin{aligned} \|\mathbf{1}_{(A \Delta sA) \cap (B \cup sB)}(s(pf) - pf)\|_2 &\leq \|\mathbf{1}_{(sA \setminus A) \cap B} \cdot s(pf)\|_2 + \|\mathbf{1}_{(A \setminus sA) \cap sB} \cdot pf\|_2 \\ &< M \cdot \frac{\varepsilon}{6M} + M \cdot \frac{\varepsilon}{6M} = \frac{\varepsilon}{3}. \end{aligned}$$

Since the set of all $x \in X$ such that $|f(x) - f(s^{-1}x)| \geq \sqrt{\delta}$ has measure at most δ and contains the set $(A \Delta sA) \setminus (B \cup sB)$, we also have

$$\|\mathbf{1}_{(A \Delta sA) \setminus (B \cup sB)}(s(pf) - pf)\|_2 < \sqrt{\delta}M \leq \frac{\varepsilon}{3}.$$

Thus, since $s(pf)$ and pf are both zero on the complement of $A \cup sA$,

$$\begin{aligned} \|s(pf) - pf\|_2 &\leq \|\mathbf{1}_{A \cap sA}(s(pf) - pf)\|_2 + \|\mathbf{1}_{(A \Delta sA) \cap (B \cup sB)}(s(pf) - pf)\|_2 \\ &\quad + \|\mathbf{1}_{(A \Delta sA) \setminus (B \cup sB)}(s(pf) - pf)\|_2 \\ &< \|sf - f\|_2 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon, \end{aligned}$$

as desired. \square

The orthogonal decomposition in Theorem 2.24 does not have a direct counterpart for actions as the dichotomy between weak mixing and compactness loses its symmetric nature in the dynamical setting due to the multiplicative structure associated to a probability space (X, μ) via its measure algebra (with its Boolean operations) or the von Neumann algebra $L^\infty(X)$. The natural replacement for an orthogonal direct summand in a Hilbert space is the dynamical notion of factor (or extension), which admits no notion of complement within the algebraic framework of $L^\infty(X)$. The basic asymmetry in the dichotomy is expressed by the following theorem, in which “weakly mixing” and “compact” cannot be interchanged. This result suggests that we aim for a structure theorem for actions in the spirit of Theorem 2.24 by relativizing the notion of weak mixing to extensions (which can be done using Hilbert modules) and then asking whether every action is a weakly mixing extension of a compact action. This is false in general, but by additionally relativizing compactness and allowing for a tower of extensions one can attain the desired structure theorem, which is the content of Section 3.2.

Theorem 2.28 *A p.m.p. action $G \curvearrowright (X, \mu)$ either is weakly mixing or admits a nontrivial compact factor.*

Proof Suppose that the action is not weakly mixing. By (vi) \Rightarrow (i) of Theorem 2.25 there is a compact function $f \in L^2(X)$ which is not a.e. constant. It follows by Theorem 2.27 and the discussion in the second last paragraph of Section 1.12 that the action has a nontrivial compact factor. \square

Remark 2.29 There exist groups G which admit no compact unitary representations except for direct sums of the trivial representation. This property clearly implies that every ergodic p.m.p. action of G is weakly mixing. Such groups are called *minimally almost periodic*. Examples can be found in [249, 69].

2.3 Examples

2.3.1 Bernoulli Actions

Let Y be a Polish space (e.g., a compact metrizable space, or even a finite set). We form the product topological space Y^G , which is again Polish, for Y^G is separable by

the countability of G and if d is a compatible complete metric on Y then taking an enumeration s_1, s_2, \dots of G we can define a compatible complete metric $d'(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min(d(x_{s_n}, y_{s_n}), 1)$ on Y^G . By definition the product topology on Y^G is generated by the cylinder sets $\prod_{s \in G} A_s$ where each A_s is open and $A_s = Y$ for all s outside of a finite subset of G . Every open set in Y^G is a countable union of such cylinder sets, which consequently generate the Borel σ -algebra.

Let ν be a Borel probability measure on Y . One can show that there is a unique Borel probability measure ν^G on Y^G which behaves as an ordinary product measure on Borel cylinder sets, i.e., if F is a finite subset of G , $\{A_s\}_{s \in F}$ is a collection of Borel subsets of Y , and $\pi_F : Y^G \rightarrow Y^F$ is the coordinate restriction map then

$$\nu^G \left(\pi_F^{-1} \left(\prod_{s \in F} A_s \right) \right) = \prod_{s \in F} \nu(A_s). \quad (2.3)$$

This is a special case of Kolmogorov's extension theorem. To verify it in the simplest case when Y is finite, we note that the algebra \mathcal{A} generated by the cylinder sets in Y^G (i.e., those sets of the form $\pi_F^{-1}(\prod_{s \in F} A_s)$ as above) consists entirely of clopen sets, which enables one to easily check that (2.3) defines a premeasure on \mathcal{A} by reducing countable additivity to finite additivity. Carathéodory's extension theorem then clinches the existence and uniqueness of ν^G .

Now we define the action $G \curvearrowright Y^G$ by $(sx)_t = x_{s^{-1}t}$ for all $s, t \in G$ and $x \in Y^G$. This action preserves the measure ν^G by the uniqueness property of the latter, and it is called a *Bernoulli action*. We refer to (Y, ν) or Y itself as the *base*. We say that a Bernoulli action is *trivial* if ν has a single atom of full measure, in which case ν^G also has a single atom of full measure.

To show that the Bernoulli action $G \curvearrowright (Y^G, \nu^G)$ is mixing when G is infinite, we use the following basic fact.

Proposition 2.30 *Let (X, μ) be a probability space with σ -algebra \mathcal{S} and let \mathcal{A} be an algebra of measurable subsets of X that generates \mathcal{S} as a σ -algebra modulo null sets. Then for every measurable set $B \subseteq X$ and $\varepsilon > 0$ there is an $A \in \mathcal{A}$ satisfying $\mu(A \Delta B) < \varepsilon$.*

Proof By hypothesis \mathcal{S} agrees modulo null sets with the smallest σ -algebra containing \mathcal{A} , and so it suffices to show that the collection \mathcal{B} of all measurable sets $B \subseteq X$ such that for every $\varepsilon > 0$ there is an $A \in \mathcal{A}$ satisfying $\mu(A \Delta B) < \varepsilon$ is a σ -algebra. Since \emptyset belongs to \mathcal{A} it also belongs to \mathcal{B} , and \mathcal{B} is closed under complementation since $(X \setminus A) \Delta (X \setminus B) = A \Delta B$ for all sets $A, B \subseteq X$. To verify closure under countable unions, let B_1, B_2, \dots be a sequence in \mathcal{B} and let $\varepsilon > 0$. Take an $n \in \mathbb{N}$ such that $\mu(\bigcup_{k=1}^{\infty} B_k \setminus \bigcup_{k=1}^n B_k) < \varepsilon/2$. For each $k \in \mathbb{N}$ find an $A_k \in \mathcal{A}$ such that $\mu(A_k \Delta B_k) < \varepsilon/(2n)$. Then

$$\mu \left(\left(\bigcup_{k=1}^n A_k \right) \Delta \left(\bigcup_{k=1}^{\infty} B_k \right) \right) \leq \sum_{k=1}^n \mu(A_k \Delta B_k) + \mu \left(\bigcup_{k=1}^{\infty} B_k \setminus \bigcup_{k=1}^n B_k \right) < \varepsilon,$$

showing that $\bigcup_{k=1}^{\infty} B_k$ lies in \mathcal{B} . \square

Now if $A = \prod_{s \in G} A_s$ and $B = \prod_{s \in G} B_s$ are Borel cylinder sets in Y^G such that $A_s = Y$ for all s outside the finite set E and $B_s = Y$ for all s outside the finite set F , then $\mu(sA \cap B) = \mu(A)\mu(B)$ for all $s \in G \setminus FE^{-1}$. It follows that if A and B are finite unions of cylinder sets then $\mu(sA \cap B) = \mu(A)\mu(B)$ for all s outside some finite subset of G . Since such sets form an algebra which generates the Borel σ -algebra, a simple approximation argument using Proposition 2.30 then shows that if G is infinite then $\lim_{s \rightarrow \infty} \mu(sA \cap B) = \mu(A)\mu(B)$ for all Borel $A, B \subseteq X$. Thus Bernoulli actions of infinite G are mixing, and in particular weakly mixing and ergodic.

The Koopman representation κ of the Bernoulli action $G \curvearrowright (Y^G, \nu^G)$ can be explicitly described as follows. Form the infinite tensor product Hilbert space $L^2(Y)^{\otimes G}$, which is the completion of the inner product space defined as the direct limit of the tensor products $L^2(Y)^{\otimes F}$ over the net of finite sets $F \subseteq G$. The embeddings $L^2(Y)^{\otimes E} \hookrightarrow L^2(Y)^{\otimes F}$ which are used to build the direct limit are determined on elementary tensors by sending $\bigotimes_{s \in E} \xi_s$ to $\bigotimes_{s \in F} \tilde{\xi}_s$ where $\tilde{\xi}_s = \xi_s$ if $s \in E$ and $\tilde{\xi}_s = \mathbf{1}$ if $s \in F \setminus E$. One can also describe such an embedding as composition with the coordinate restriction map $Y^F \rightarrow Y^E$. A perturbation argument then shows that $L^2(Y)^{\otimes G}$ can be identified with $L^2(Y^G)$ so that for every finite set $F \subseteq G$ the embedding $L^2(Y)^{\otimes F} \hookrightarrow L^2(Y)^{\otimes G}$ arising from the direct limit construction can be described as composition with the coordinate restriction map $Y^G \rightarrow Y^F$.

Now fix an orthonormal basis Ω for $L^2(Y)$ containing $\mathbf{1}$ and let \mathcal{C} be the collection of all maps $\xi : G \rightarrow \Omega$ such that $\xi_s = \mathbf{1}$ for all but finitely many $s \in G$. Then for such a ξ we can write $\bigotimes_{s \in G} \xi_s$ for the vector in $L^2(Y)^{\otimes G}$ which appears as the elementary tensor $\bigotimes_{s \in F} \xi_s$ in $L^2(Y)^{\otimes F}$ whenever F is a finite subset of G such that $\xi_s = \mathbf{1}$ for all $s \in G \setminus F$. The set $Z = \{\bigotimes_{t \in G} \xi_t : \xi \in \mathcal{C}\}$ is then easily checked to be an orthonormal basis for $L^2(Y)^{\otimes G}$, and this set is invariant under κ , yielding an action $G \curvearrowright Z$ given by $s(\bigotimes_{t \in G} \xi_t) = \bigotimes_{t \in G} \xi_{s^{-1}t}$. As in the discussion at the beginning of the chapter, we can express this action as $G \curvearrowright \bigsqcup_{\zeta \in R} G/G_\zeta$ where R is a choice of representatives for the transitive subsets of Z , G_ζ is the stabilizer subgroup $\{s \in G : s\zeta = \zeta\}$, and the componentwise action $G \curvearrowright G/G_\zeta$ is given by $s(tG_\zeta) = stG_\zeta$. Accordingly, κ decomposes as the direct sum $\bigoplus_{\zeta \in R} \lambda_{G/G_\zeta}$ where λ_{G/G_ζ} is the *left quasiregular representation* on $\ell_2(G/G_\zeta)$ given by $(\lambda_{G/G_\zeta}(s)f)(tG_\zeta) = f(s^{-1}tG_\zeta)$. When $\zeta = \bigotimes_{s \in G} \mathbf{1}$ we have $G_\zeta = G$, accounting for the trivial representation canonically contained in the Koopman representation. On the other hand, for every other $\zeta \in Z$ the stabilizer subgroup G_ζ is finite, since ζ appears in $L^2(Y)^{\otimes F}$ for some finite set $F \subseteq G$, in which case $G_\zeta \subseteq FF^{-1}$.

If G has no nontrivial finite subgroups (as in the case of \mathbb{Z}) and the Bernoulli action is nontrivial, then it is immediate from the above that κ is equivalent to $1_G \oplus \lambda_G^{\oplus \mathbb{N}}$ where 1_G is the trivial representation and λ_G is the left regular representation. For general G the representation κ is equivalent to a subrepresentation of $1_G \oplus \lambda_G^{\oplus I}$ for some countable index set I , since the quasiregular representation $\lambda_{G/H}$ for a finite subgroup $H \subseteq G$ is contained in the left regular representation λ_G via the isometric embedding of $\ell_2(G/H)$ into $\ell_2(G)$ given by $f \mapsto |H|^{-1/2}(f \circ q)$ where

$q : G \rightarrow G/H$ is the quotient map. Using this observation we can show as follows that κ is in fact equivalent to $1_G \oplus \lambda_G^{\oplus \mathbb{N}}$ whenever G is infinite and the Bernoulli action is nontrivial.

Assuming now that G is infinite and the Bernoulli action is nontrivial, we first demonstrate that κ contains a subrepresentation equivalent to $1_G \oplus \lambda_G^{\oplus \mathbb{N}}$. Recursively define finite subsets $F_1 \subseteq F_2 \subseteq \dots$ of G by setting $t_1 = e$ and $F_1 = \{t_1\}$ and then for every $n > 1$ choosing a $t_n \in G \setminus F_{n-1}F_{n-1}^{-1}F_{n-1}$ and setting $F_n = F_{n-1} \cup \{t_n\}$. Then given an $n \geq 3$ and an $s \in G \setminus \{e\}$ we must have $sF_n \neq F_n$, for otherwise there would be $1 \leq i, j < n$ with $st_i = t_j$ and $1 \leq k < n$ with $st_n = t_k$, in which case $t_n = s^{-1}t_k = t_it_j^{-1}t_k \in F_{n-1}F_{n-1}^{-1}F_{n-1}$, contradicting our choice of t_n . Pick an $\eta \in \Omega \setminus \{1\}$ and consider for each $n \geq 3$ the vector $\bigotimes_{s \in G} \xi_{n,s} \in L^2(Y)^{\otimes G}$ where $\xi_{n,s} = \eta$ if $s \in F_n$ and $\xi_{n,s} = 1$ otherwise. Each of these vectors generates a copy of the left regular representation by the property of the sets F_n that we verified above, and these copies are orthogonal since the sets F_n have different cardinalities. This means that κ contains a subrepresentation equivalent to $1_G \oplus \lambda_G^{\oplus \mathbb{N}}$, as desired. To conclude that κ is itself equivalent to $1_G \oplus \lambda_G^{\oplus \mathbb{N}}$, it remains to apply Proposition 2.32, which relies on the Cantor–Bernstein property of Lemma 2.31.

A set of operators on a Hilbert space is said to be *self-adjoint* if T^* belongs to the set whenever T does. For an operator T on a Hilbert space the following are equivalent: (i) T^*T is a projection, (ii) TT^* is a projection, (iii) T is a partial isometry (i.e., T is isometric on $\ker(T)^\perp$), (iii) T^* is a partial isometry (Theorem 2.3.3 of [190]). A partial isometry T maps $\ker(T)^\perp$ to $\ker(T^*)^\perp$ with T^* acting as the inverse of T between these two subspaces. The *commutant* of a set Ω of operators on a Hilbert space is the set of all operators which commute with every operator in Ω .

Lemma 2.31 *Let \mathcal{H} be a Hilbert space and let P and Q be projections in $\mathcal{B}(\mathcal{H})$ such that there exist $V, W \in \mathcal{B}(\mathcal{H})$ satisfying*

- (i) $V^*V = P$ and $VV^* \leq Q$,
- (ii) $W^*W = Q$ and $WW^* \leq P$.

*Then there exists a $Z \in \mathcal{B}(\mathcal{H})$ such that $Z^*Z = P$ and $ZZ^* = Q$. Moreover, if P, Q, V , and W all belong to the commutant of a given self-adjoint set of operators then Z may be chosen to belong to this commutant as well.*

Proof Set $P_0 = P$ and $Q_0 = Q$, and recursively define projections $P_0 \geq P_1 \geq \dots$ and $Q_0 \geq Q_1 \geq \dots$ by putting $P_{n+1} = WQ_nW^*$ and $Q_{n+1} = VP_nV^*$ for $n \geq 0$. Write P_∞ for the projection whose range is the intersection of the ranges of the P_n , and Q_∞ for the projection whose range is the intersection of the ranges of the Q_n . Observe that the Hilbert space \mathcal{H} decomposes into the orthogonal direct sum of the ranges of the projections P^\perp , P_∞ , and $P_n - P_{n+1}$ for $n \geq 0$, and also as the orthogonal direct sum of the ranges of the projections Q^\perp , Q_∞ , and $Q_n - Q_{n+1}$ for $n \geq 0$. The operator V is zero on $P^\perp\mathcal{H}$, maps $P_\infty\mathcal{H}$ isometrically onto $Q_\infty\mathcal{H}$, and maps $(P_n - P_{n+1})\mathcal{H}$ isometrically onto $(Q_{n+1} - Q_{n+2})\mathcal{H}$ for $n \geq 0$, while W is zero on $Q^\perp\mathcal{H}$, maps $Q_\infty\mathcal{H}$ isometrically onto $P_\infty\mathcal{H}$, and maps $(Q_n - Q_{n+1})\mathcal{H}$ isometrically onto $(P_{n+1} - P_{n+2})\mathcal{H}$ for $n \geq 0$. Let Z be the operator in $\mathcal{B}(\mathcal{H})$ which

- (i) is zero on $P^\perp \mathcal{H}$,
- (ii) agrees with V on $P_\infty \mathcal{H}$,
- (iii) for each $n \geq 0$ agrees with V on $(P_{2n} - P_{2n+1})\mathcal{H}$ and with W^* on $(P_{2n+1} - P_{2n+2})\mathcal{H}$ (which thus gets mapped to $(Q_{2n} - Q_{2n+1})\mathcal{H}$).

Then $Z^*Z = P$ and $ZZ^* = Q$.

For the last sentence of the lemma, note that Z can be expressed as the strong operator limit as $n \rightarrow \infty$ of the operators

$$VP_\infty + \sum_{k=0}^n V(P_{2k} - P_{2k+1}) + \sum_{k=0}^n W^*(P_{2k+1} - P_{2k+2})$$

and that the commutant of a given self-adjoint set of operators is self-adjoint and closed in the strong operator topology, as is easily verified. \square

Proposition 2.32 *Let $\pi_1 : G \rightarrow \mathcal{B}(\mathcal{H}_1)$ and $\pi_2 : G \rightarrow \mathcal{B}(\mathcal{H}_2)$ be unitary representations such that each is equivalent to a subrepresentation of the other. Then π_1 and π_2 are equivalent.*

Proof Form the direct sum representation $\pi := \pi_1 \oplus \pi_2$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$ and for $i = 1, 2$ write P_i for the orthogonal projection $\mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_i$. In the commutant $\pi(G)'$ we define the operator V as the composition of P_1 with an isometry $\mathcal{H}_1 \rightarrow 0 \oplus \mathcal{H}_2$ implementing an equivalence between π_1 and a subrepresentation of π_2 , and the operator W as the composition of P_2 with an isometry $\mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus 0$ implementing an equivalence between π_2 and a subrepresentation of π_1 . Then $V^*V = P_1$ and $VV^* \leq P_2$, while $W^*W = P_2$ and $WW^* \leq P_1$. It follows by Lemma 2.31 that there exists a $Z \in \pi(G)'$ such that $Z^*Z = P_1$ and $ZZ^* = P_2$. Then the restriction of Z to \mathcal{H}_1 is an isometry onto \mathcal{H}_2 which implements an equivalence between π_1 and π_2 . \square

2.3.2 Rotations of the Circle

Irrational rotations of the circle are basic examples of compact ergodic actions. Let μ be the normalized Haar measure on \mathbb{T} , which identifies with Lebesgue measure on $[0, 1)$ via the map $t \mapsto e^{2\pi it}$. Let $\theta \in [0, 1)$ and define a μ -preserving transformation T of \mathbb{T} by $Tz = e^{2\pi i\theta}z$, which we view as the generator for a p.m.p. \mathbb{Z} -action. For $k \in \mathbb{Z}$ write ξ_k for the function $z \mapsto z^k$ on \mathbb{T} , and observe that the Koopman representation $\pi : \mathbb{Z} \rightarrow \mathcal{B}(L^2(\mathbb{T}))$ satisfies $\pi(n)\xi_k = e^{-2\pi i kn\theta}\xi_k$. As the vectors ξ_k form an orthonormal basis for $L^2(\mathbb{T})$, this gives a decomposition of π into a direct sum of one-dimensional representations. Therefore the action is compact, and in particular fails to be weakly mixing. The restriction of π to the orthogonal complement of the vector ξ_0 will be ergodic if and only if each of its invariant one-dimensional summands is nontrivial, i.e., if for every $k \neq 0$ there is an n such that $e^{-2\pi i kn\theta} \neq 1$, which occurs precisely when θ is irrational.

2.3.3 Skew Transformations of the Torus

Let $\theta \in \mathbb{R}$ and define the transformation T of $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ by $T(x, y) = (x + \theta, x + y)$ modulo \mathbb{Z}^2 . This is a homeomorphism with inverse $(x, y) \mapsto (x - \theta, y - x + \theta)$. It preserves the normalized Haar measure μ on \mathbb{T}^2 , as can be seen by applying Fubini's theorem to characteristic functions of Borel sets and using the translation-invariance of Haar measure on each coordinate.

Proposition 2.33 *Suppose that θ is irrational. Then T is ergodic.*

Proof Let f be a vector in $L^2(\mathbb{T}^2)$ which is fixed by the unitary operator $g \mapsto g \circ T$. For every $n \in \mathbb{Z}$ define on $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ the function $f_n(x) = \int f(x, y) e^{-2\pi i n y} dy$. Then using the rotation-invariance of Haar measure on \mathbb{T} we have

$$\begin{aligned} f_n(x + \theta) &= \int f(x + \theta, x + y) e^{-2\pi i n(x+y)} dy \\ &= \int f(x, y) e^{-2\pi i n(x+y)} dy = e^{-2\pi i n x} f_n(x). \end{aligned} \quad (2.4)$$

Thus the function $|f_n|$ is invariant under rotation by θ , and hence is a.e. equal to a constant c_n since this rotation is ergodic by the irrationality of θ . From (2.4) we similarly see that f_0 itself is a constant. Now let n be a nonzero integer. For every $k \in \mathbb{Z}$, a computation similar to (2.4) using the formula

$$T^k(x, y) = (x + k\theta, kx + \tfrac{1}{2}k(k-1)\theta + y) \quad (2.5)$$

yields $f_n(x + k\theta) = e^{-\pi i n k(k-1)\theta} e^{-2\pi i n k x} f_n(x)$ and hence $\int f_n(x + k\theta) \overline{f_n(x)} dx = 0$ for $k \neq 0$. Since we can find a sequence $\{k_j\}$ of nonzero integers such that $k_j\theta$ converges to zero modulo \mathbb{Z} , we deduce using Egorov's theorem that $\int f_n(x) \overline{f_n(x)} dx = 0$, which implies that $f_n = 0$. Since we may express f as $(x, y) \mapsto \sum_{n \in \mathbb{Z}} f_n(x) e^{2\pi i n y}$ by Fourier analysis, it follows that f is constant. We conclude that T is ergodic. \square

Proposition 2.34 *The transformation T is neither weakly mixing nor compact.*

Proof Write U for the unitary operator $g \mapsto g \circ T$ on $L^2(\mathbb{T}^2)$. For each $n \in \mathbb{Z}$ the function $(x, y) \mapsto e^{2\pi i n x}$ on \mathbb{T}^2 is an eigenfunction with eigenvalue $e^{2\pi i n \theta}$ for U , and so the Koopman representation fails to be weakly mixing on $L^2(\mathbb{T}^2) \ominus \mathbb{C}\mathbf{1}$. On the other hand, using the formula (2.5) we see that the function $f(x, y) = e^{2\pi i y}$ satisfies $(U^n f)(x, y) = e^{\pi i n(n-1)\theta} e^{2\pi i n x} f(x, y)$. It follows that the functions $U^n f$ for $n \in \mathbb{Z}$ are pairwise orthogonal in $L^2(\mathbb{T}^2)$ and hence lie at distance $\sqrt{2}$ from each other, so that f is not compact for the Koopman representation. \square

In Definition 3.8 we relativize the notion of compactness to extensions, and in Example 3.10 we show that the projection map $\mathbb{T}^2 \rightarrow \mathbb{T}$ onto the first coordinate, which factors T onto rotation by the angle $2\pi\theta$, is a compact extension. Thus T , although not itself compact, is a compact extension of a compact action, which from

the viewpoint of the Furstenberg-Zimmer structure theorem (Theorem 3.15) is the closest step away from being compact.

2.3.4 Odometers

Besides an irrational rotation of the circle, the other basic example of a compact ergodic action is an odometer. As seen in the proof of Theorem 4.84, odometers play an important role in orbit equivalence theory owing to the fact that periodic approximation is built into their construction in a basic combinatorial way. Let $\{n_k\}$ be a sequence of positive integers and consider the product topological space $X = \prod_{k=1}^{\infty} \{0, \dots, n_k - 1\}$, which is compact. As in the discussion of Bernoulli actions above, Carathéodory's extension theorem yields the existence of a unique Borel probability measure μ on X which on a cylinder set

$$A_1 \times \dots \times A_m \times \{0, \dots, n_{m+1} - 1\} \times \{0, \dots, n_{m+2} - 1\} \times \dots$$

takes the value $\nu_1(A_1) \dots \nu_m(A_m)$, where ν_k is the uniform probability measure on $\{0, \dots, n_k - 1\}$. By an *odometer* (also called an *adding machine*) we mean the transformation T of such a product space X which is defined by addition by $(1, 0, 0, 0, \dots)$ with carry over to the right. That is, given an $(q_k)_k \in \prod_{k=1}^{\infty} \{0, \dots, n_k - 1\}$ we take the smallest k' for which $q_{k'} < n_{k'} - 1$ and set

$$T(q_k)_k = (0, \dots, 0, q_{k'} + 1, q_{k'+1}, q_{k'+2}, \dots)$$

where $q_{k'} + 1$ appears at the coordinate k' , unless it happens that $q_k = n_k - 1$ for all k , in which case we roll over to $(0, 0, 0, \dots)$. This map is clearly continuous and invertible and hence a homeomorphism. Since the measure $A \mapsto \mu(T^{-1}A)$ takes the same value as μ on each cylinder set, T is measure-preserving by the uniqueness of μ .

Again as in the discussion of Bernoulli actions, $L^2(X)$ can be written as the infinite tensor product $\bigotimes_{k=1}^{\infty} L^2(\{0, \dots, n_k - 1\}, \nu_k)$ which is constructed as a direct limit via the embeddings $\bigotimes_{j=1}^k L^2(\{0, \dots, n_j - 1\}, \nu_j) \hookrightarrow \bigotimes_{j=1}^l L^2(\{0, \dots, n_j - 1\}, \nu_j)$ for $k < l$ determined on elementary tensors by $\xi_1 \otimes \dots \otimes \xi_k \mapsto \xi_1 \otimes \dots \otimes \xi_k \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}$. Writing U for the unitary operator $f \mapsto f \circ T^{-1}$ on $L^2(X)$, for every $k \in \mathbb{N}$ the orbit $\{f, Uf, U^2f, \dots, U^{n_1 \dots n_k - 1}f\}$ of the elementary tensor $f = \bigotimes_{j=1}^k \sqrt{n_j} \mathbf{1}_{\{0\}}$ is an orthonormal basis for the finite-dimensional U -invariant subspace $\bigotimes_{j=1}^k L^2(\{0, \dots, n_j - 1\}, \nu_j)$, and via the finite-dimensional Fourier transform the restriction of U to this subspace has eigenvalues $1, \omega, \omega^2, \dots, \omega^{n_1 \dots n_k - 1}$ where ω is a primitive $n_1 \dots n_k$ th root of unity. The corresponding families of one-dimensional eigenspaces for $k = 1, 2, \dots$ are nested and by choosing a unit vector in each of these eigenspaces we obtain an orthonormal basis for $L^2(X)$, and none of

these vectors except the one corresponding to the eigenvalue 1 is invariant under U . Hence the odometer action is compact and ergodic.

The above odometer can also be described more abstractly as the inverse limit of the cyclic groups $\mathbb{Z}/(n_1 \cdots n_k \mathbb{Z})$ under the maps $\mathbb{Z}/(n_1 \cdots n_{k+1} \mathbb{Z}) \rightarrow \mathbb{Z}/(n_1 \cdots n_k \mathbb{Z})$ which reduce mod $n_1 \cdots n_k$. One can then obtain a simple picture of the Koopman representation using Pontrjagin duality.

2.3.5 Actions by Automorphisms of Compact Groups

Let X be a compact (Hausdorff topological) group. Let μ be the normalized Haar measure on X , which is the unique regular Borel probability measure on X satisfying the translation invariance $\mu(xA) = \mu(Ax) = \mu(A)$ for all $x \in X$ and Borel sets $A \subseteq X$ (for general locally compact groups one would need to specify left or right translation invariance and speak of left and right Haar measure, but for compact groups there is no distinction). Let $G \curvearrowright X$ be an action by (continuous) group automorphisms. By the uniqueness of the normalized Haar measure, this action is μ -preserving. A standard example is the \mathbb{Z} -action on the n -torus $\mathbb{R}^n/\mathbb{Z}^n$ generated by an $n \times n$ integer matrix with determinant ± 1 , which we will return to below.

When X is Abelian, like in the above toral example, these actions are referred to as *algebraic actions*. We will focus our discussion on this case, although a similar analysis can be carried out for general X using the Peter-Weyl theorem and the induced action on the dual \hat{X} , which is defined in this generality as the set of equivalence classes of irreducible unitary representations of X . So we assume henceforth that X is Abelian.

Now for any locally compact Abelian group A we define the Pontrjagin dual \hat{A} as the group of all continuous homomorphisms from A into \mathbb{T} (the *characters* of A) with the topology of uniform convergence on compact subsets. For example, $\hat{\mathbb{T}} \cong \mathbb{Z}$ and $\hat{\mathbb{Z}} \cong \mathbb{T}$. For each $a \in A$ the map $\varphi \mapsto \varphi(a)$ from \hat{A} to \mathbb{T} defines a continuous homomorphism, and Pontrjagin duality asserts that this map from A to $\hat{\hat{A}}$ is an isomorphism of topological groups. Consistent with its definition for more general locally compact groups, the dual \hat{A} can also be viewed as the set of (equivalence classes of) irreducible unitary representations of A , which are all one-dimensional as a consequence of Abelianness. If A is compact, as is the case for our group X , then \hat{A} appears in conjunction with $L^2(A)$ in two different guises which are compatible in the obvious way:

- (i) As functions into \mathbb{T} , the elements of \hat{A} form an orthonormal basis for $L^2(A)$, and each is a common eigenvector for A under the regular representation $\lambda : A \rightarrow \mathcal{B}(L^2(A))$, as we have $\lambda(a)\varphi = \varphi(a)^{-1}\varphi$ for every $\varphi \in \hat{A} \subseteq L^2(A)$ and $a \in A$.
- (ii) As irreducible representations, the elements of \hat{A} are subrepresentations of the regular representation $\lambda : A \rightarrow \mathcal{B}(L^2(A))$, and λ decomposes as the direct sum of these irreducibles.

For more about Pontrjagin duality and analysis on groups see [218, 125].

To an algebraic action $G \curvearrowright X$ we associate an action $G \curvearrowright \widehat{X}$ on the dual by automorphisms according to the formula $s\varphi(x) = \varphi(s^{-1}x)$ where $\varphi \in \widehat{X}$, $x \in X$, and $s \in G$. Since we can invert this relationship by Pontrjagin duality, we obtain a one-to-one correspondence between algebraic actions $G \curvearrowright X$ and actions of G on the discrete Abelian group \widehat{X} by automorphisms. An action of the latter kind endows \widehat{X} with the structure of a left module over the integral group ring $\mathbb{Z}G$ which uniquely determines it, thereby setting up a one-to-one correspondence between algebraic actions of G and left $\mathbb{Z}G$ -modules. This is explained in Chapter 13 and accounts for the adjective “algebraic”.

Given an algebraic action $G \curvearrowright X$, the dual action $G \curvearrowright \widehat{X}$ is replicated in the Koopman representation, with G acting on \widehat{X} in exactly the same way when the elements of \widehat{X} are viewed as orthonormal basis vectors in $L^2(X)$. The trivial subrepresentation of the regular representation of X , which corresponds to the identity in \widehat{X} , acts on the constant functions in $L^2(X)$. Now if the action $G \curvearrowright X$ fails to be mixing, then a simple approximation argument shows that there must be nonzero $\varphi, \rho \in \widehat{X}$ such that the set K of all $s \in G$ satisfying $s\varphi = \rho$ is infinite, in which case we have $s\varphi = \varphi$ for all s in the infinite set $K^{-1}K$. These observations yield the following.

Proposition 2.35 *The action $G \curvearrowright X$ is mixing if and only if the stabilizer subgroup $\{s \in G : s\varphi = \varphi\}$ is finite for every nontrivial $\varphi \in \widehat{X}$. In particular, if G is torsion-free then $G \curvearrowright X$ is mixing if and only if these stabilizer subgroups are all trivial.*

Given a $\varphi \in \widehat{X}$, if the G -orbit of φ is finite then the linear span of this orbit in $L^2(X)$ is a finite-dimensional invariant subspace for the Koopman representation of G . If on the other hand the G -orbit of φ is infinite, then every nonzero vector ξ in the closed linear span of this orbit in $L^2(X)$ fails to be compact, as is easy to see by tracking coefficients in orthonormal expansions with respect to the orbit of φ . Thus the decomposition of the Koopman representation into weakly mixing and compact parts (Theorem 2.24) is simple to describe: the weakly mixing subrepresentation acts on the closed linear span of the infinite G -orbits in $\widehat{X} \subseteq L^2(X)$, while the compact subrepresentation acts on the closed linear span of the finite G -orbits in $\widehat{X} \subseteq L^2(X)$. In particular, the action $G \curvearrowright X$ is compact if and only if the orbits of the dual action $G \curvearrowright \widehat{X}$ are all finite. Moreover, ergodicity and weak mixing are equivalent:

Proposition 2.36 *For an algebraic action $G \curvearrowright X$ the following are equivalent:*

- (i) *the action is ergodic,*
- (ii) *the action is weakly mixing,*
- (iii) *the orbit of every nontrivial element under the dual action $G \curvearrowright \widehat{X}$ is infinite.*

Proof (ii) \Rightarrow (i). This is true for general p.m.p. actions by Proposition 2.16.

(iii) \Rightarrow (ii). This follows from our observations above and (vi) \Rightarrow (i) of Theorem 2.25.

(i) \Rightarrow (iii). If φ is a nontrivial element of \widehat{X} with finite orbit $\{\varphi_1, \dots, \varphi_n\}$, then $\sum_{j=1}^n \varphi_j$ is a nonzero G -invariant vector in $L^2(X) \ominus \mathbb{C}\mathbf{1}$, contradicting ergodicity. \square

Since every nontrivial subgroup of \mathbb{Z} is infinite and has finite index, it follows from Propositions 2.35 and 2.36 that for algebraic actions of \mathbb{Z} the properties of ergodicity, weak mixing, and mixing are all equivalent.

We have thus seen that the issue of weak mixing and compactness for algebraic actions discretizes into the simple dichotomy between infiniteness and finiteness, since we have an orthonormal basis \widehat{X} which is permuted by the elements of G under the Koopman representation. At the representation level there is nothing special in Propositions 2.35 and 2.36 about \widehat{X} being a group, and one can restate these results so that they apply to the broader class of unitary representations of G for which there is a fixed orthonormal basis which is permuted by every group element.

Now let us return to the transformation T_A of the n -torus $\mathbb{R}^n/\mathbb{Z}^n$ defined by an $n \times n$ integer matrix A with determinant ± 1 , which we view as acting by $x \mapsto Ax$ on column vectors. To each $\varphi \in \widehat{\mathbb{R}^n/\mathbb{Z}^n}$ there is a tuple $(k_1, \dots, k_n) \in \mathbb{Z}^n$ such that $\varphi(x_1, \dots, x_n) = k_1 x_1 + \dots + k_n x_n$ modulo \mathbb{Z} for all $(x_1, \dots, x_n) \in \mathbb{R}^n$ modulo \mathbb{Z}^n , and this assignment establishes a group isomorphism between $\widehat{\mathbb{R}^n/\mathbb{Z}^n}$ and \mathbb{Z}^n . Making this identification, one can then check that the automorphism of \mathbb{Z}^n dual to T_A is given by $q \mapsto (A^t)^{-1}q$ where A^t is the transpose of A .

Proposition 2.37 *The transformation T_A is ergodic if and only if the matrix A has no eigenvalues which are roots of unity.*

Proof Since a square matrix and its transpose have the same eigenvalues, Proposition 2.36 reduces the problem to showing that A^t has a root of unity as an eigenvalue if and only if there exists a nonzero $q \in \mathbb{Z}^n$ and a $k \in \mathbb{N}$ such that $(A^t)^k q = q$. If the latter condition holds, then $(A^t)^k$ has 1 as an eigenvalue, which implies that A^t has an eigenvalue which is a k th root of unity. Conversely, suppose that A^t has an eigenvalue which is a k th root of unity for some $k \in \mathbb{N}$. Then there is a nonzero $y \in \mathbb{R}^n$ such that $((A^t)^k - I)y = 0$, and so by Gaussian elimination we can find a nonzero $q \in \mathbb{Z}^n$ such that $((A^t)^k - I)q = 0$, as desired. \square

2.3.6 Gaussian Actions

As described in Appendix E, for each unitary representation $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ on a separable Hilbert space there is an associated *Gaussian action* $G \curvearrowright (X, \mu)$ which has the property that its Koopman representation is equivalent to the representation $S(\pi \oplus \bar{\pi}) = \sum_{n=0}^{\infty} (\pi \oplus \bar{\pi})^{\odot n}$ on the symmetric Fock space $S(\mathcal{H} \oplus \mathcal{H}) = \sum_{n=0}^{\infty} (\mathcal{H} \oplus \mathcal{H})^{\odot n}$ (Definition E.16 and Theorem E.19). This action is constructed by first fixing an atomless standard probability space (X, μ) and identifying the realification $\mathcal{H}_{\mathbb{R}}$ with a closed subspace of $L^2_{\mathbb{R}}(X)$ consisting of centred Gaussian random variables (i.e., a Gaussian Hilbert space) which generates the σ -algebra, and then showing via an exponentiation procedure that the realification $\pi_{\mathbb{R}}$ canonically extends to an orthogonal representation of G on $L^2_{\mathbb{R}}(X)$ which is multiplicative on indicator functions and hence induces a p.m.p. action $G \curvearrowright (X, \mu)$. The main utility

of this construction is to show the existence of p.m.p. actions whose Koopman representation possesses certain prescribed properties. In particular, Gaussian actions are useful for converting representation-theoretic properties of groups into statements involving p.m.p. actions, as seen in the context of property (T) in Section 5.4.

As explained in Example E.17, the Gaussian action associated to the left regular representation $\lambda : G \rightarrow \mathcal{B}(\ell_2(G))$, or any countable multiple of it, is the Bernoulli action $G \curvearrowright (Y^G, \nu^G)$ where (Y, ν) is an atomless standard probability space. Beyond this simple case, it becomes difficult to give examples of ergodic Gaussian actions for which there is a structural description that goes beyond the representation-theoretic. One reason is that, while the compact actions of a group are the ones that tend to admit an explicit structural description (as illustrated by circle rotations and odometers), there do not exist any compact ergodic Gaussian actions, or even any ergodic Gaussian actions whose Furstenberg–Zimmer tower (Theorem 3.15) contains at least one compact extension. This is a consequence of Theorem 2.38, which we now aim to establish.

Theorem 2.38 *Let $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ be a unitary representation and $G \curvearrowright (X, \mu)$ the associated Gaussian action. Then the following are equivalent:*

- (i) $G \curvearrowright (X, \mu)$ is ergodic,
- (ii) $G \curvearrowright (X, \mu)$ is weakly mixing,
- (iii) π is weakly mixing.

Proof (iii) \Rightarrow (ii). Since weak mixing for unitary representations is preserved under taking tensor products (by Theorem 2.23), passing to subrepresentations, and taking conjugates (as is clear from the definitions), we see that the representation $\bigoplus_{n=1}^{\infty} (\pi \oplus \bar{\pi})^{\odot n}$ is weakly mixing. Theorem E.19 then yields (ii).

(ii) \Rightarrow (i). This is true for general p.m.p. actions by Proposition 2.16.

(i) \Rightarrow (iii). Let $\xi \in \mathcal{H}$. Then $\zeta := (\xi, 0) \otimes (0, \xi) + (0, \xi) \otimes (\xi, 0)$ is a vector in the symmetric tensor square $(\mathcal{H} \oplus \mathcal{H}) \odot (\mathcal{H} \oplus \mathcal{H})$. By Theorem E.19, $(\pi \oplus \bar{\pi}) \odot (\pi \oplus \bar{\pi})$ is a subrepresentation of the restriction of the Koopman representation to $L^2(X) \ominus \mathbb{C}\mathbf{1}$, and so by ergodicity we have $m(f_{\zeta, \zeta}) = 0$. But $f_{\zeta, \zeta} = 2|f_{\xi, \xi}|^2$ and therefore $m(|f_{\xi, \xi}|) = 0$ since by the Cauchy–Schwarz inequality we have $m(|f_{\xi, \xi}| \cdot \mathbf{1})^2 \leq m(|f_{\xi, \xi}|^2) m(\mathbf{1}^2)$. Thus π is weakly mixing. \square

To characterize mixing and compactness for Gaussian actions, we use the following two simple lemmas. These are verified by first observing that the local condition in the definition of mixing or compactness holds on elementary tensors and hence also on linear combinations of elementary tensors, which then implies that it holds on arbitrary vectors in $\mathcal{H} \otimes \mathcal{H}$ by a simple approximation argument using the fact that unitary operators are isometric.

Lemma 2.39 *Let $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ and $\sigma : G \rightarrow \mathcal{B}(\mathcal{K})$ be unitary representations, and suppose that π is mixing. Then $\pi \otimes \sigma$ is mixing.*

Lemma 2.40 *Let $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ and $\sigma : G \rightarrow \mathcal{B}(\mathcal{K})$ be compact unitary representations. Then $\pi \otimes \sigma$ is compact.*

Since mixing and compactness for unitary representations are both obviously preserved under passing to subrepresentations and taking conjugates, the following is now immediate from Theorem E.19 and Lemmas 2.39 and 2.40.

Proposition 2.41 *The Gaussian action associated to a unitary representation π is mixing if and only if π is mixing. It is compact if and only if π is compact.*

2.4 Notes and References

Standard references for the ergodic theory of single measure-preserving transformations are [122, 50, 251, 208, 219]. The more recent book [104] by Glasner presents the basic ergodic theory of p.m.p. actions of general countable groups and takes the representation-theoretic viewpoint that we have promoted here. Another recent introduction to ergodic theory is the book [72] by Einsiedler and Ward, which emphasizes applications to number theory.

For single measure-preserving transformations, Koopman and von Neumann developed the theory of weak mixing in [161] while Halmos and von Neumann gave a spectral classification in the ergodic compact case (“ergodic transformations with discrete spectrum”) and showed that such a transformation is conjugate to a rotation on a compact Abelian group [123]. The theory of weak mixing for unitary representations and p.m.p. actions of locally compact groups was worked out by Bergelson and Rosenblatt in [14].

In conjunction with Birkhoff’s pointwise ergodic theorem, which was obtained slightly later [17], the key event in the foundation of ergodic theory as a formal discipline was von Neumann’s mean ergodic theorem [248]. In its concrete form as a statement about averaging over partial orbits, this is a result that applies most generally to unitary representations of amenable groups [69] (see Section 4.3). Von Neumann was inspired by Koopman’s idea of studying Hamiltonian systems through their associated Hilbert space operators [160]. In [70] Eberlein proved an abstract mean ergodic theorem in the presence of an invariant mean, and in conjunction with Ryll-Nardzewski’s later fixed point theorem (see Appendix D) this yielded Theorem 2.21, which doesn’t require any amenability hypothesis.

The description of the Koopman representation for Bernoulli actions seems to be folklore. Proposition 2.32 and its utility in this context were pointed out to us by Ben Hayes. Lemma 2.31 is a special case of the Cantor–Bernstein property for Murray–von Neumann subequivalence in general von Neumann algebras (Proposition V.1.3 in [234]).

Skew-product transformations were studied by Anzai [5] and von Neumann. Via the seminal work of Furstenberg in [91] they motivated the development of the structure theory discussed in Sections 3.2 and 7.3. Along with circle rotations, automorphisms of the torus are important prototypes in dynamics from both the measurable and topological viewpoints. It is a remarkable fact that hyperbolic automorphisms

of the torus are conjugate to Bernoulli shifts [4, 140]. For an extensive source of information on algebraic \mathbb{Z}^d -actions, see the book [225] by Schmidt.

Theorem 2.38 appeared in the context of \mathbb{R} -actions in [177, 86, 116] and then later for general groups in [239]. An alternative C^* -algebraic approach to the construction of p.m.p. actions from representations via the canonical commutation relations can be found in Section 5.2.2.2 of [37].

Ergodic Theory

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