3.11 The Thom Isomorphism and Poincaré Duality

3.11.1 The Thom Isomorphism

Definition 3.11.1 Let $p: E \to B$ be a G-vector bundle and let γ be the associated representation of $\Pi_G B$. Let $T(p) = D(p)/_B S(p)$ be the *fiberwise Thom space* of p. A *Thom class* for p is an element $t \in \tilde{H}_G^{\gamma}(T(p); \overline{A}_{G/G})$ such that, for each G-map $b: G/K \to B$,

$$\begin{aligned} b^*(t) \in \tilde{H}_G^{\gamma(b)}(b^*T(p); \overline{A}_{G/G}) \\ &\cong \tilde{H}_G^{\gamma(b)}(G \times_K S^V; \overline{A}_{G/G}) \\ &\cong \tilde{H}_K^{V}(S^V; \overline{A}_{K/K}) \\ &\cong A(K) \end{aligned}$$

is a generator. Here, V is a representation of K such that $\gamma(b) \simeq G \times_K S^V$.

As in Remark 1.15.2, the Thom class must live in ordinary cohomology $\tilde{H}^*_{G,\delta}$ with $\delta = 0$.

Since a global Thom class is characterized by its local behavior, and locally a G-vector bundle is a V-bundle, the next proposition follows from the $RO(G)$-graded analogue, Proposition 1.15.3.

Proposition 3.11.2 *The following are equivalent for a cohomology class* $t \in \tilde{H}_G^{\gamma}(T(p); \overline{A}_{G/G})$.

(1) *t is a Thom class for p.*
(2) *For every subgroup $K \subset G$, $t|K \in \tilde{H}_K^{\gamma|K}(T(p); \overline{A}_{K/K})$ is a Thom class for p as a K-bundle.*
(3) *For every subgroup $K \subset G$, $t^K \in \tilde{H}_{WK}^{\gamma^K}(T(p)^K; \overline{A}_{WK/WK})$ is a Thom class for $p^K: E^K \to B^K$ as a WK-bundle.*
(4) *For every subgroup $K \subset G$, $t^K|e \in \tilde{H}^{|\gamma^K|}(T(p)^K; \mathbb{Z})$ is a Thom class for p^K as a nonequivariant bundle.*

□

Note that, in the last part of the proposition, $|\gamma^K|$ must be interpreted as a representation of the nonequivariant fundamental groupoid of B^K, so that the cohomology may be twisted.

The following generalizes Theorem C of [14].

Theorem 3.11.3 (Thom Isomorphism) *If $p: E \to B$ is any G-vector bundle and γ is the associated representation of $\Pi_G B$, then there exists a Thom class $t \in \tilde{H}_G^{\gamma}(T(p); \overline{A}_{G/G})$. For any Thom class t, the map*

$$t \cup -: H_{G,\delta}^{\alpha}(B; \overline{T}) \to \tilde{H}_{G,\delta}^{\alpha+\gamma}(T(p); \overline{T})$$

is an isomorphism for any familial δ.

Proof As in the proof of Theorem 1.15.5, the theorem is clear in the special case that p is a bundle over an orbit. The general case follows, as it does in the nonequivariant case for twisted coefficients, by a Mayer-Vietoris patching argument. The key point is that we can choose a compatible collection of local classes because the action of $\Pi_G B$ on γ is the same as the action on the fibers of p. □

3.11.2 Poincaré Duality

We are now in a position to describe Poincaré duality for arbitrary compact smooth G-manifolds. (Again, the noncompact case can be handled using cohomology with compact supports.)

Definition 3.11.4 Let M be a closed smooth G-manifold and let τ be the tangent representation of $\Pi_G M$, i.e., the representation associated with the tangent bundle. Think of M as a G-space over itself in the following. A *fundamental class* of M is a class $[M] \in \mathscr{H}_{\tau}^{G}(M; \overline{A}_{G/G})$ such that, for each point $m \in M$, thought of as the map $m: G/G_m \to M$ with image Gm, and tangent plane $\tau(m) = G \times_{G_m} V$, the image of $[M]$ in

$$\begin{aligned}
\mathscr{H}_{\tau}^{G}(M, M - Gm; \overline{A}_{G/G}) &\cong \tilde{\mathscr{H}}_{V}^{G}(G_+ \wedge_{G_m} S^{V-\mathscr{L}(G/G_m)}; \overline{A}_{G/G}) \\
&\cong \tilde{\mathscr{H}}_{V}^{G_m}(S^V; \overline{A}_{G/G}) \\
&\cong A(G_m)
\end{aligned}$$

is a generator.

The fundamental class $[M]$ is related to fundamental classes of the fixed submanifolds M^K as in Proposition 1.15.7.

Proposition 3.11.5 *The following are equivalent for a dual homology class $\mu \in \mathscr{H}_{\tau}^{G}(M; \overline{A}_{G/G})$.*

(1) *μ is a fundamental class for M as a G-manifold.*
(2) *For every subgroup $K \subset G$, $\mu|K \in \mathscr{H}_{\tau|K}^{K}(M; \overline{A}_{K/K})$ is a fundamental class for M as a K-manifold.*
(3) *For every subgroup $K \subset G$, $\mu^K \in \mathscr{H}_{\tau^K}^{WK}(M^K; \overline{A}_{WK/WK})$ is a fundamental class for M^K as a WK-manifold.*

(4) *For every subgroup $K \subset G$, $\mu^K|e \in H_{|\tau^K|}(M^K;\mathbb{Z})$ is a fundamental class for M^K as a nonequivariant manifold.* □

Note that, in the last part of the proposition, $|\tau^K|$ must be interpreted as a representation of the nonequivariant fundamental groupoid of M^K, so that the homology may be twisted.

We now have sufficient machinery in place to prove Poincaré duality for arbitrary G-manifolds either geometrically, along the lines of [52] or [14], or homotopically as in Sect. 1.15. The homotopical approach uses, of course, the duality of Theorem 2.9.11.

Theorem 3.11.6 (Poincaré Duality) *Every closed smooth G-manifold M has a fundamental class $[M] \in \mathscr{H}_\tau^G(M;\overline{A}_{G/G})$, and*

$$-\cap [M]\colon H^\gamma_{G,\delta}(M;\overline{T}) \to H^{G,\mathscr{L}-\delta}_{\tau-\gamma}(M;\overline{T})$$

is an isomorphism for every familial δ such that M is an $\mathscr{F}(\delta)$-manifold (i.e., $\mathscr{F}(\delta)$ contains every isotropy subgroup of M). □

In particular, when $\delta = 0$ we get the isomorphism

$$-\cap [M]\colon H^\gamma_G(M;\overline{T}) \to \mathscr{H}^G_{\tau-\gamma}(M;\overline{T})$$

and when $\delta = \mathscr{L}$ we get the isomorphism

$$-\cap [M]\colon \mathscr{H}^\gamma_G(M;\overline{T}) \to H^G_{\tau-\gamma}(M;\overline{T}).$$

If M is a compact G-manifold with boundary, then we get relative, or Lefschetz, duality.

Definition 3.11.7 Let M be a compact G-manifold with boundary, with tangent representation τ. A *fundamental class* of M is a dual homology class $[M,\partial M] \in \mathscr{H}_\tau^G(M,\partial M;\overline{A}_{G/G})$ such that, for each point $m \in M - \partial M$, thought of as the map $m\colon G/G_m \to M$ with image Gm, and tangent plane $\tau(m) = G \times_{G_m} V$, the image of $[M,\partial M]$ in

$$\begin{aligned}\mathscr{H}_\tau^G(M, M-Gm;\overline{A}_{G/G}) &\cong \tilde{\mathscr{H}}_V^G(G_+ \wedge_{G_m} S^{V-\mathscr{L}(G/G_m)};\overline{A}_{G/G})\\ &\cong \tilde{\mathscr{H}}_V^{G_m}(S^V;\overline{A}_{G/G})\\ &\cong A(G_m)\end{aligned}$$

is a generator.

There is an obvious relative version of Proposition 3.11.5.

Theorem 3.11.8 (Lefschetz Duality) *Every compact smooth G-manifold M has a fundamental class* $[M, \partial M] \in \mathscr{H}_\tau^G(M, \partial M; \overline{A}_{G/G})$*, and the following are isomorphisms for every familial dimension function* δ *such that M is an* $\mathscr{F}(\delta)$*-manifold:*

$$- \cap [M, \partial M] \colon H^\gamma_{G,\delta}(M; \overline{T}) \to H^{G,\mathscr{L}-\delta}_{\tau-\gamma}(M, \partial M; \overline{T})$$

and

$$- \cap [M, \partial M] \colon H^\alpha_{G,\delta}(M, \partial M; \overline{T}) \to H^{G,\mathscr{L}-\delta}_{\tau-\alpha}(M; \overline{T}).$$

□

As in the nonparametrized version, we get the following special cases when $\delta = 0$ or $\delta = \mathscr{L}$:

$$\begin{aligned}
&- \cap [M, \partial M] \colon H^\gamma_G(M; \overline{T}) \to \mathscr{H}^G_{\tau-\gamma}(M, \partial M; \overline{T}),\\
&- \cap [M, \partial M] \colon H^\gamma_G(M, \partial M; \overline{T}) \to \mathscr{H}^G_{\tau-\gamma}(M; \overline{T}),\\
&- \cap [M, \partial M] \colon \mathscr{H}^\gamma_G(M; \overline{T}) \to H^G_{\tau-\gamma}(M, \partial M; \overline{T}), \qquad \text{and}\\
&- \cap [M, \partial M] \colon \mathscr{H}^\gamma_G(M, \partial M; \overline{T}) \to H^G_{\tau-\gamma}(M; \overline{T}).
\end{aligned}$$

3.12 A Calculation

In this section, all cohomology is assumed to have coefficients in $\overline{A}_{G/G}$, which we will drop from the notation for simplicity.

In [66], the second author defined equivariant Chern classes in $RO(G)$-graded cohomology, for complex vector bundles modeled on a single representation. We can generalize that definition now as follows. Let ω be a complex vector bundle over the G-space B, and also write ω for the associated (real) representation of ΠB. By the Thom isomorphism (3.11.3), there exists a Thom class $t(\omega) \in \tilde{H}^\omega_G(T(\omega))$, and we let $c_\omega(\omega) = e(\omega) \in H^\omega_G(B)$ be its Euler class, the restriction of the Thom class to the zero section. Now consider the Gysin sequence of ω, the long exact sequence induced by the cofibration $S(\omega)_+ \to D(\omega)_+ \to T(\omega)$ over B. Part of that sequence is the following:

$$0 = H^{-2}_G(B) \to H^{\omega-2}_G(B) \to H^{\omega-2}_G(S(\omega)) \to H^{-1}_G(B) = 0.$$

When we pull back the bundle ω along the projection $p \colon S(\omega) \to B$, we get $p^*\omega \cong \omega' \oplus \mathbb{C}$, so we get the Euler class $e(\omega') \in H^{\omega-2}_G(B)$, which we write as $c_{\omega-2}(\omega)$.

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