

Chapter 1

Number Set

Several sets of numbers are used in digital imaging. Natural numbers are what we use for everyday counting, as one, two, three, Integer numbers are all the whole numbers, including the natural numbers and zero and negative numbers. Rational numbers also include fractions, where one integer is divided by another and the result terminates exactly, or repeats without end. Irrational numbers also include fractions which do not terminate or repeat, such as the square-root of 2. Real numbers include all of the above.

1.1 Set

A set is a collection of objects A, B, C, \dots , (the members of the set), with an equivalence relation $=$ between them. If A and B are alike in some way, we write $A = B$, other wise $A \neq B$. Equivalence is:

- Reflexive, that is, $A = A, B = B, C = C, \dots$ for every member of the set,
- Symmetric, that is, if $A = B$ then $B = A$,
- Transitive, that is, if $A = B$ and $B = C$, then $A = C$.

1.2 Natural Number

The set of natural numbers \mathbb{N} is constructed from five axioms¹ [1–3]:

¹Landau's formulation, following Dedekind and Peano. Edmund Georg Landau (1877–1938) was Professor at Göttingen and colleague of Klein and Hilbert. Richard Dedekind (1831–1916) was Gauss's last pupil, and Lecturer at Braunschweig. Guiseppe Peano (1856–1932) was Professor at Turin: he devised the logical symbols used by Whitehead and Russell in *Principia Mathematica*.

- I: 1 is a natural number. That is, our set is not empty; it contains an object called 1 (read “one”).
- II: For each A there exists exactly one natural number, called the *successor* of A, which will be denoted by A' .
- III: We always have $A' <> 1$. That is, there exists no number whose successor is 1.
- IV: If $A' = B'$ then $A = B$. That is, for any given number there exists either no number or exactly one number whose successor is the given number.
- V: (Axiom of Induction): Let there be given a set \mathbf{N} of natural numbers such that,
- 1 belongs to \mathbf{N} ,
 - If A belongs to \mathbf{N} then so does A' .

It follows that \mathbf{N} contains all the natural numbers.

The set of natural numbers is *ordered*, since for any two natural numbers A and B, one of the following holds:

A is greater than B: $A > B$,
 or A is equal to B: $A = B$,
 or A is less than B: $A < B$.

We define an *addition* operation $+$ on natural numbers, where

- $A + 1 = A'$. That is, adding 1 to any number gives the successor of that number.
- $A + B' = (A + B)'$. That is, adding the successor of any number to any number gives the successor of the sum.

By repeated induction on B we get $A + B =$ the Bth successor of A. We also define a *multiplication* operation \times on natural numbers, where

- $A \times 1 = A$. That is, multiplying any number by 1 gives the same number.
- $A \times B' = (A \times B) + A$. That is, multiplying any number by the successor of any number gives the same as A added to the product of the two numbers.

By repeated induction on B we get $A \times B = A$ added to itself B times.

The set of natural numbers \mathbf{N} is *closed* with respect to addition and multiplication; that is, every sum and product is also in \mathbf{N} .

1.3 Integer Number

The set of *integer numbers* \mathbf{Z} (for German Zahlen) is constructed from ordered pairs of natural numbers (A,B), and an equivalence relation \equiv between such pairs, where

$$(A,B) \equiv (C,D) \text{ if } A + D = B + C .$$

This is another way of saying that B subtracted from A is the same as D subtracted from C.

We define an *equivalence class* $[(A,B)]$ as that class which has (A,B) as a member; that is, all the pairs which are equinumerous with (A,B) . An *integer number* is an equivalence class $[(A,B)]$, and we write:

- Z if $A > B$ (a positive integer),
- 0 if $A = B$ (zero, neither positive nor negative),
- $-Z$ if $A < B$ (a negative integer).

For example:

$$\begin{aligned}
 & \dots \\
 2 &= [(3, 1)] = [(4, 2)] = \dots = [(N + 2, N)] \\
 1 &= [(2, 1)] = [(3, 2)] = \dots = [(N + 1, N)] \\
 0 &= [(1, 1)] = [(2, 2)] = \dots = [(N, N)] \\
 -1 &= [(1, 2)] = [(2, 3)] = \dots = [(N, N + 1)] \\
 -2 &= [(1, 3)] = [(2, 4)] = \dots = [(N, N + 2)] \\
 & \dots
 \end{aligned}$$

Addition, multiplication and subtraction between equivalence classes is defined as

$$\begin{aligned}
 [(A,B)] + [(C,D)] &= [(A + C, B + D)] \\
 [(A,B)] \times [(C,D)] &= [(AC + BD, AD + BC)] \\
 [(A,B)] - [(C,D)] &= [(A + D, B + C)] .
 \end{aligned}$$

Subtraction is the inverse of addition. The set of integer numbers \mathbf{Z} is closed with respect to addition, multiplication and subtraction; that is, every sum, product and difference of integers is also in \mathbf{Z} .

The natural numbers are a subset of the integers.

1.4 Rational Number

The set of *rational numbers* \mathbf{Q} (for Quotient) is constructed from (ordinary) division of one integer by another integer. Given integer A and integer B , we get the rational number A/B . This can also be represented as a *fraction* with an *integer part* (possibly 0) and a *fractional part* which terminates or repeats (possibly 0). For example:

$$\begin{aligned}
 1/1 &= 1.0 \\
 1/2 &= 0.5 \\
 1/3 &= 0.333\dots \\
 &\dots \\
 2/1 &= 2.0 \\
 2/2 &= 1.0 \\
 2/3 &= 0.666\dots \\
 &\dots
 \end{aligned}$$

Between any two rational numbers we can insert further rational numbers without limit. For example, between $1/1 = 1.0$ and $1/2 = 0.5$ we have $1/3 = 0.333\dots$, $1/4 = 0.25$, $1/5 = 0.2$,

Division is the inverse of multiplication. The set of rational numbers \mathbf{Q} is closed with respect to addition, multiplication, subtraction, and division; that is, every sum, product, difference, and quotient of rationals is also in \mathbf{Q} .

The integers are a subset of the rationals; for example

$$1 = 1/1 = 2/2 = 3/3\dots; 2 = 2/1 = 4/2 = 6/3\dots$$

An *arithmetic progression* is a succession of numbers with a constant difference between neighbours, such as the integers:

$$-\infty, \dots, -3, -2, -1, 0, 1, 2, 3, \dots, \infty.$$

A *harmonic progression* is a succession of reciprocals of numbers in an arithmetic progression:

$$-0, \dots, -1/3, -1/2, -1, \infty, 1, 1/2, 1/3, \dots, 0.$$

1.5 Irrational Number

An *irrational number* has a fractional part which does not terminate or repeat; for example:

$$\begin{aligned}
 \sqrt{2} &= 1.4142\dots \\
 \pi &= 3.1416\dots
 \end{aligned}$$

Between any two rational numbers we can insert further irrational numbers without limit.

1.6 Real Number

The integers, rationals, and irrationals together constitute the *continuum of real numbers* \mathbf{R} .

1.7 Number Variables

We shall generally use italic capitals I, J, K, \dots for variables taking integer values, and italic lower-case r, s, t, \dots for variables taking real values.

References

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