Chapter 2
Waves in Viscous, Dispersive, Nonlinear, Preliminary Deformable Layer with a Free Surface

2.1 Introduction

There are many works [47, 95, 96, 116], where propagation of nonlinear wave beams in infinite media with various physical properties is studied. For research of some problems of seismology and instrument-making it is necessary to study features of propagation and reflection from a free boundary of nonlinear wave bunches, as well as to consider a layer of a medium in the case, when excitation is set on one end, and an other end is free from stresses. It is interesting to consider such mathematical models of media, which reflect real properties of materials as better as possible. Sometimes a medium is preliminary deformed that can influence on behavior of a nonlinear wave. The works considering last factors are not numerous.

In recent years it is fashionable to solve nonlinear equations by numerical methods [47, 96]. In spite of appeal of numerical methods, importance of analytical methods hasn't lost its fundamental value. The analytical methods will be developed in the present chapter. And, in spite of the fact that the equations describing a wave behavior are nonlinear already in the first order, however in the same order boundary conditions are linear.

The present chapter is written on the base of works [32, 42].

2.2 The General Basic Equations

Let us consider a preliminary deformed isotropic layer of a medium. The medium has dissipation and dispersion.

The equations of a motion of the medium and of relations between tensors of stresses, deformations, and components of a vector of displacements have the following form [32, 42, 203]:

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$$\rho_0 \frac{\partial^2 u_i^*}{\partial t^2} = \frac{\partial \sigma_{ik}}{\partial x_k}, \quad u_i^* = u_i^0 + u_i \quad (i = 1, 2, 3). \tag{2.1}$$

$$\begin{aligned} \sigma_{ik} = a_1 \delta_{ik} \dot{\sigma}_{ll} + a_2 \dot{\sigma}_{ki} + a_2 \dot{\sigma}_{ik} = \lambda \delta_{ik} \varepsilon_{ll} + 2\mu \varepsilon_{ik} \\ + \frac{1}{4} A[e_{li}(e_{kl} + e_{lk}) + e_{ki}(e_{li} + e_{il})] \\ + \frac{1}{2} B[e_{lm}(e_{im} + e_{mi})\delta_{ik} + 2e_{ll}(e_{ki} + e_{ik})] \\ + C e_{ll}^2 \delta_{ik} + b_1 \dot{e}_{ll} \delta_{ik} + 2b_2 \dot{e}_{ik} \\ + d_1 \delta_{ik} \ddot{\varepsilon}_{ll} + 2d_2 \ddot{\varepsilon}_{ik} + n_1 \delta_{ik} \dddot{\varepsilon}_{ll} + 2n_2 \dddot{\varepsilon}_{ik}, \end{aligned} \tag{2.2}$$

$$\varepsilon_{ik} = \frac{1}{2}(e_{ik} + e_{ki} + e_{li} e_{lk}), \quad e_{ik} = \frac{\partial u_i}{\partial x_k} + \frac{\partial u_i^0}{\partial x_k}. \tag{2.3}$$

x_k are Lagrangian coordinates, u_i^*, u_i^0, and u_i are components, accordingly, of complete, initial and excited vector of displacements, ε_{ik} and e_{ik} are the strain tensors, σ_{ik} is a Lagrangian antisymmetric stress tensor, a_1 and a_2 are relaxation times, b_1, b_2, d_1, d_2, n_1, and n_2 are parameters of internal oscillators. u_i^0 is assumed to be known. $\frac{\partial u_i^0}{\partial x_k}$ is a constant in space and in time, which satisfies Eq. (2.1) in the case without excitations.

The relationship (2.2) between σ_{ik} and ε_{ik}, means that the medium is viscous and nonlinear (a physical and a geometrical nonlinearities are taken into account). There are oscillating masses with dissipation in the medium and their presence gives dispersing properties to the medium. If to neglect physical nonlinearity and preliminary deformability, then an expression looking like (2.2) can be used as a ground model [212]. In some cases it can be used as a mathematical model for a composite and alloys.

The coordinate system is chosen by the following way: axes x_1 and x_2 are in the plane of a layer, which is free from stresses. The x_3-axis is directed deep into the medium. In the plane $x_3 = 0$ true stresses are assumed to be equal to zero. The medium is preliminary deformed, and it is supposed that initial deformations are small, therefore they satisfy to the linear equations without dissipation and dispersion

$$\frac{\partial u_i^0}{\partial x_j} = 0 \quad \text{for } i = j; \quad \frac{\partial u_i^0}{\partial x_j} \neq 0 \quad \text{for } i \neq j. \tag{2.4}$$

The relationship (2.4) means that in preliminary deformations only the stretching and compression take place.

The specified assumptions are according to the boundary data: at $x_3 = 0$ $\sigma_{31}^0 = \sigma_{32}^0 = \sigma_{33}^0 = 0$ are the preliminary stresses existing in the medium before propagation of perturbation.

From the last equalities for negligible nonlinearity, viscosity and dispersion it is possible to obtain

$$\frac{\partial u_1^0}{\partial x_1} + \frac{\partial u_2^0}{\partial x_2} + \frac{\lambda + 2\mu}{\lambda} \cdot \frac{\partial u_3^0}{\partial x_3} = 0. \tag{2.5}$$

Equation (2.5) means that derivatives $\frac{\partial u_1^0}{\partial x_1}, \frac{\partial u_2^0}{\partial x_2}$, and $\frac{\partial u_3^0}{\partial x_3}$ have uniform orders. The condition (2.4) corresponds to the boundary conditions.

It is assumed that on some depth l a perturbation with Gaussian profile is formed. In the plane, which is perpendicular to x_3-axis and where the perturbation is generated, $u_3 \neq 0$, and $u_1 = u_2 = 0$, i.e. a quasilongitudinal wave arises in the medium. This wave is supposed to travel along an x_3-axis towards a plane $x_3 = 0$ and to reflect from it. Our purpose is to study features of the formed field between the planes $x_3 = 0$ and $x_3 = l$.

In the medium described by Eq. (2.2) the equilibrium and frozen waves can propagate.

2.3 Equilibrium Waves

In a certain time after generation of the perturbation the wave field comes to an equilibrium dynamic state that is called equilibrium. In this state σ_{ik} and ε_{ik} are assumed to be main terms in Eq. (2.2), which are considered as a basis during simplifications. Using a known method in the theory of wave diffraction (see, for example, [17, 31, 32, 42]), the following orders are taken for perturbations: $u_3 \sim \delta^2$, $\frac{\partial}{\partial x_i} \sim \delta^{-1/2}$ (i = 1, 2), $a_1, a_2, b_1 \sim \delta^2$, $d_1, d_2, \sim \delta^3$, $n_1, n_2 \sim \delta^4$, $\frac{\partial u_3}{\partial x_3} \sim \delta$. The accepted orders for coefficients mean that viscosity, dispersion, and dissipation are considered to be small.

It is convenient to write down Eq. (2.1) in terms of displacements, therefore σ_{ik} and ε_{ik} are excluded using expressions (2.2) and (2.3). After that the equations are simplified using the chosen orders and generation of the quasilongitudinal wave in the medium is taken into account, i.e. the longitudinal wave is considered to be a basic quantity, whereas the transverse waves are small. Therefore the equations for u_1 and u_2 become simpler up to terms $\delta^{-1/2}$, and the equations for u_3—up to δ. As a result, they take on the following form:

$$\rho_0 \frac{\partial^2 u_j}{\partial t^2} = Q_j \frac{\partial^2 u_3}{\partial x_j \partial x_3} + \mu_j \frac{\partial^2 u_j}{\partial x_3^2} \quad (j = 1, 2), \tag{2.6}$$

$$\begin{aligned}\rho_0\frac{\partial^2 u_3}{\partial t^2}+2a_2\rho_0\frac{\partial^2 u_3}{\partial t^2}+F_1\frac{\partial^3 u_3}{\partial x_3^2\partial t}+N_1\frac{\partial^2 u_3}{\partial x_3^2}\\ +P'_1\frac{\partial^2 u_1}{\partial x_1\partial x_3}+P''_1\frac{\partial^2 u_2}{\partial x_2\partial x_3}+G'_1\frac{\partial^2 u_3}{\partial x_1^2}+G''_1\frac{\partial^2 u_3}{\partial x_2^2}\\ +M_1\frac{\partial^2 u_3}{\partial x_3^2}\frac{\partial u_3}{\partial x_3}-(d_1+d_2)\frac{\partial^4 u_3}{\partial x_3^2\partial t^2}-(n_1+n_2)\frac{\partial^5 u_3}{\partial x_3^2\partial t^3}=0,\end{aligned}\tag{2.7}$$

where

$$\begin{aligned}
F_1 &= a_1(2\mu+3\lambda)-b_1-2b_2, \quad M_1=-\lambda-2\mu-2A-6B-2C,\\
N_1 &= -\lambda-2\mu-(2A+6B+2C)\frac{\partial u_3^0}{\partial x_3}-2B\left(\frac{\partial u_2^0}{\partial x_2}+\frac{\partial u_1^0}{\partial x_1}\right),\\
P'_1 &= -\lambda-\mu-\left(3B+2C+\frac{3}{4}A\right)\frac{\partial u_1^0}{\partial x_1}-B\frac{\partial u_2^0}{\partial x_2}-\left(3B+\frac{3}{4}A\right)\frac{\partial u_3^0}{\partial x_3},\\
P''_1 &= -\lambda-\mu-\left(3B+2C-\frac{3}{4}A\right)\frac{\partial u_2^0}{\partial x_2}-B\frac{\partial u_1^0}{\partial x_1}-\left(3B+\frac{3}{4}A\right)\frac{\partial u_3^0}{\partial x_3},\\
G'_1 &= -\mu-\left(B+\frac{1}{4}A\right)\frac{\partial u_1^0}{\partial x_1}-B\frac{\partial u_2^0}{\partial x_2}-\left(B+\frac{1}{4}A\right)\frac{\partial u_3^0}{\partial x_3},\\
G''_1 &= -\mu-B\frac{\partial u_1^0}{\partial x_1}-\left(B+\frac{1}{4}A\right)\left(\frac{\partial u_2^0}{\partial x_2}+\frac{\partial u_3^0}{\partial x_3}\right),\\
Q_1 &= \lambda+\mu+\left(B+\frac{3}{4}A\right)\frac{\partial u_1^0}{\partial x_1}+B\frac{\partial u_2^0}{\partial x_2}+\left(B+\frac{3}{4}A\right)\frac{\partial u_3^0}{\partial x_3},\\
Q_2 &= \lambda+\mu+B\frac{\partial u_1^0}{\partial x_1}+\left(B+\frac{3}{4}A\right)\frac{\partial u_2^0}{\partial x_2}+\left(B+\frac{3}{4}A\right)\frac{\partial u_3^0}{\partial x_3},\\
M_1 &= \mu+\left(3B+\frac{1}{4}A\right)\frac{\partial u_1^0}{\partial x_1}+B\frac{\partial u_2^0}{\partial x_2}+\left(2B+2C+\frac{1}{4}A\right)\frac{\partial u_3^0}{\partial x_3},\\
M_2 &= \mu+B\frac{\partial u_1^0}{\partial x_1}+\left(3B+\frac{1}{4}A\right)\frac{\partial u_2^0}{\partial x_2}+\left(3B+2C+\frac{1}{4}A\right)\frac{\partial u_3^0}{\partial x_3}.
\end{aligned}$$

Due to great values of nonlinear coefficients A, B, C, initial deformations give the contribution to the coefficients of Eqs. (2.6) and (2.7), and because of smallness of initial deformations the additions into the factors of Eqs. (2.6) and (2.7), caused by geometrical nonlinearity, are small and consequently are absent. It should be noted that in Eqs. (2.6) and (2.7) the terms containing derivatives on coordinates x_1 and x_2 have the different coefficients. This fact means that the initial deformed state breaks isotropy of the medium, namely, an axial symmetry is broken.

From Eqs. (2.6) and (2.7) follows, that in an one-dimensional case, when dispersion and dissipation are absent, the longitudinal wave travels in the medium with the velocity c:

$$c = -\frac{N_1}{\rho_0}. \tag{2.8}$$

From expression (2.8) ensues that c can be imaginary (physically it means that the medium becomes an impenetrable for wave propagation), when inequality

$$\lambda + 2\mu < \left| (2A + 6B + 2C)\frac{\partial u_3^0}{\partial x_3} + 2B\left(\frac{\partial u_2^0}{\partial x_2} + \frac{\partial u_1^0}{\partial x_1}\right) \right|$$

is valid. Validity of this inequality can be provided by the appropriate choice of materials and initial deformations.

The wave, which is generated in a plane $x_3 = l$, propagates till the free surface $x_3 = 0$ and is reflected. Thus, there are two wave fields between the planes $x_3 = l$ and $x_3 = 0$: falling and reflected.

All the functions in the set (2.6), (2.7) will be presented in the form of sums of two quantities, where terms with one prime correspond to the falling wave, and with two primes—to the reflected one. It is easy to note that the linear equations will be reduced to two new independent equations for the falling and reflected waves. According to [28, 30, 32] and as it was mentioned above in 1.5, in the first approximation the nonlinear equations can also be split on two new nonlinear independent equations. Thus, two new independent set of equations of type (2.6), (2.7) are derived for the falling and reflected waves. It means that in the medium two independent nonlinear bunches have been formed, which are interrelated through the boundary conditions and travel towards to each other.

2.4 Derivation of Evolution Equations

Let us introduce the new coordinate $\tau_1 = \tau_1' - t - lc^{-1}, \tau_1' = -\frac{x_3}{c}$ into the set of equations for functions with one prime.

After exclusion of functions u_1 and u_2 from the set of equations, as mentioned in [32], and equating with zero the factor at $(\rho_0 - N_1 c^{-1})\frac{\partial^2 u_3'}{\partial \tau_1^2}$ for the falling longitudinal wave, one can obtain the following equation:

$$\begin{gathered}\frac{2N_1}{c}\frac{\partial^2 u_3'}{\partial x_3 \partial \tau_1'} - (2a_2\rho_0 + Fc^{-2})\frac{\partial^3 u_3'}{\partial \tau_1^3} + \frac{M}{c^3}\frac{\partial u_3'}{\partial \tau_1}\frac{\partial^2 u_3'}{\partial \tau_1^2} \\ + Q_n'\left(\frac{\partial^2 u_3'}{\partial x_1^2} + \frac{Q_n''}{Q_n'}\frac{\partial^2 u_3'}{\partial x_3^2}\right) - \frac{d_1 + d_2}{c^2}\frac{\partial^4 u_3'}{\partial \tau_1^4} + \frac{n_1 + n_2}{c^2}\frac{\partial^5 u_3'}{\partial \tau_1^5} = 0,\end{gathered} \tag{2.9}$$

$$Q_n' = G_1' + \frac{P_1' Q_1^*}{c^2 V_1}, \quad Q_n'' = G_1'' + \frac{P_1' Q_2^*}{c^2 V_2}, \quad V_1 = \rho_0 - \frac{\mu_1}{c^2}, \quad V_2 = \rho_0 - \frac{\mu_2}{c^2}.$$

The substitutions $\frac{\partial}{\partial X_3} = c^{-1}\frac{\partial}{\partial t}$, $x = x_1$, $x_2 = \left(\frac{Q'_n}{Q''_n}\right)^{1/2} x_2$ are executed in Eq. (2.9). After derivation over τ_1, the following equation yields for $\psi_1 = \frac{\partial u'_3}{\partial \tau_1}$:

$$\frac{\partial^2 \psi_1}{\partial t \partial \tau_1} - \frac{1}{2} L(\psi_1) = c^{-1}\frac{\partial}{\partial \tau_1}\left[\Gamma \psi_1 \frac{\partial \psi_1}{\partial \tau_1} + D\frac{\partial^2 \psi_1}{\partial \tau_1^2} + d\frac{\partial^3 \psi_1}{\partial \tau_1^3} + n\frac{\partial^4 \psi_1}{\partial \tau_1^4}\right], \quad (2.10)$$

$$L = -Q'_n c^2 N_1^{-1}\Delta_\perp \psi, \quad \Gamma = \frac{1}{2}M_1 N_1^{-1}, \quad D = -\frac{1}{2}(2a_2\rho + F_1 c^2)N_1^{-1}c^3,$$
$$d = -\frac{1}{2}cN_1^{-1}(d_1 + d_2), \quad n = \frac{1}{2}cN_1^{-1}(n_1 + n_2).$$

It is supposed that $Q''_n(Q'_n)^{-1} > 0$, but this inequality is valid for not too great initial deformations. Equation (2.10) is an evolution equation for a falling wave. In the absence of initial deformations $Q''_n = Q'_n > 0$, the medium admits an axial symmetry.

For the set of equations with two primes, i.e. for the reflected wave, we shall enter a variable $\tau_2 = \tau'_2 - t - lc^{-1}$, where $\tau'_2 = -\tau'_1$, which differs from the similar variable for the falling wave by a sign of the wave velocity. By analogous calculations it is easy to obtain equation of type (2.10) for the reflected wave, where ψ_1 should be replaced by $\psi_2 = -\frac{\partial u''_3}{\partial \tau_2}$, and τ_1—by τ_2.

In the case of (2.10), terms $D\frac{\partial^3 \psi_{1,2}}{\partial \tau_{1,2}^3}$ and $n\frac{\partial^5 \psi_{1,2}}{\partial \tau_{1,2}^5}$ are provided by absorption, and, as follows from the expressions for the factors, the first term is caused by viscosity, the second one—by oscillating masses, and the term $\frac{\partial^4 \psi_{1,2}}{\partial \tau_{1,2}^4}$ is provided by a dispersion. It is interesting to note a physical effect: the factor Q'_n can change a sign depending on the chosen material and on the corresponding preliminary deformation that leads to change of properties of a medium, for example, the focusing medium can become defocusing, and vice versa.

2.5 The Equation of Modulation and Its Solution for Narrow Bunches

Presence of dissipation and dispersion smoothes a sawtooth wave and a quasi-monochromatic wave appears in the medium, therefore it is possible to search for the solution of Eq. (2.10) in the following form:

$$\psi_{1,2} = \frac{1}{2}\{A_{1,2}(\tau_{1,2}, t)\exp[(-\nu + i\alpha)\tau_{1,2} - (\nu + i\omega)t] + B_{1,2}(\tau_{1,2}, t)\exp[2(-\nu + i\alpha)\tau_{1,2} - 2(\nu + i\omega)t] + c.c.\}, \quad (2.11)$$

where $A_{1,2}$ and $B_{1,2}$ are the slowly varying amplitudes, accordingly, for the first and second harmonics, and the subscript "1" corresponds to the falling wave, whereas the subscript "2" stands for the reflected wave, ν is an absorption factor, and ω is an increment to the basic wave frequency α.

After calculation of derivatives from (2.11), substitution into (2.10) and equating to zero the factors at the first and second harmonics, it is possible to receive differential equations for the amplitudes $A_{1,2}$ and $B_{1,2}$.

It is assumed that amplitudes $A_{1,2}$ have the basic order in (2.11), whereas $B_{1,2}$ are small quantities of higher order and they arise on account of nonlinearities. Equating to zero the nondifferentiable terms of highest orders in the equation for the first harmonics both for the falling wave and for the reflected one, it is possible to receive the identical equations for the linear dispersion and for attenuation:

$$\omega = -\frac{d\alpha^3}{c}, \quad \nu = \frac{\alpha^4 n}{c} - \frac{D\alpha^2}{c}. \tag{2.12}$$

Equating differentiable terms of the next orders in the equations for amplitudes, we shall receive a set of differential equations for $A_{1,2}$ and $B_{1,2}$. If inequalities $\omega\tau'_{1,2} \gg 1$ and $\omega \ll \alpha$ are valid, it is possible to neglect derivatives in the equations for the second harmonic. As a result, the equation will become an algebraic one. Excluding $B_{1,2}$ by means of the last equation, we shall obtain the nonlinear Schrödinger equation, or the nonlinear equation of modulation. This equation will be studied in the stationary case, i.e. when amplitudes are constant in time. In this case from coordinates t and $\tau_{1,2}$ we pass to slow coordinate $\tau'_{1,2}$. Taking into account (2.12), after some simplifications we shall receive the following equation:

$$\begin{aligned}
&\left(3i\omega + i\alpha + \nu + \frac{2n\alpha^4}{c}\right)\frac{\partial A_{1,2}}{\partial \tau'_{1,2}} - \frac{1}{2}L(A_{1,2}) \\
&= \left(4i\nu\alpha - 12\alpha\omega + \frac{24in\alpha^5}{c}\right)^{-1} \\
&\quad \times \frac{\alpha^4}{2c}(1 + 8i\alpha^{-1}\nu)\Gamma^2 \exp\left(\tau'_{1,2} + \frac{1}{c}\right)|A_{1,2}|^2 A_{1,2}.
\end{aligned} \tag{2.13}$$

In order to find a solution, hereinafter it is supposed that the third and the fourth terms in the coefficient of Eq. (2.13) at $\frac{\partial A_{1,2}}{\partial \tau'_{1,2}}$ are negligible. This fact is in good agreement with condition $\omega \ll \alpha$, nevertheless the terms with ω are retained in this coefficient.

According to the procedure described in Chap. 1, namely, entering real amplitude and eikonal, one can derive equations for them, and then for narrow bunches three equations will be obtained for $f_{1,2}$, $\sigma_{1,2}$ and $R_{1,2}$. Let us write them again, as they have new factors:

$$\frac{d\sigma_{1,2}}{d\,\tau'_{1,2}} = Gf_{1,2}^{-2}, \tag{2.14}$$

$$R_{1,2}^{-1} = \frac{\alpha}{2}(1-3\xi)L_1^{-1}f_{1,2}^{-1}\frac{df_{1,2}}{d\tau'_{1,2}} + \frac{\kappa_2}{2}b_{1,2}^2\,\alpha^{-1}f_{1,2}^{-2}, \tag{2.15}$$

$$\frac{d^2f_{1,2}}{d\,\tau'^2_{1,2}} = Mf_{1,2}^{-3} + \frac{2\nu b_{1,2}}{f_{1,2}}\kappa_2; \tag{2.16}$$

$$M = \alpha^2(1-3\xi)^{-2}(L_1^2r_{1,2}^4 + 4\kappa_1^2b_{1,2}^2L_1r_0^{-2} - \kappa_2b_{1,2}^4),$$

$$\zeta = -\frac{\omega}{\alpha}, \quad L_1 = Q'_nc^2N_1^{-1},$$

$$\kappa_1 = \zeta(3\alpha^4\xi + 8\alpha^2\nu^2 + 48n\alpha^5\nu c^{-1}), \kappa_2 = -\zeta(\alpha\nu + 6n\alpha^5c^{-1} + 24\alpha^3c\xi),$$

$$\zeta = \Gamma^2(8c^2)^{-1}\left[9\xi^2 + (\nu\alpha^{-1} + 6n\alpha^3c^{-1})^2\right]^{-1}\exp\left[-2\nu\left(\frac{1}{c} + \tau'_{1,2}\right)\right],$$

$$G = \left(-2L_1\alpha^{-1}r_{1,2}^{-2} - \kappa_1b_{1,2}\alpha^{-1}\right)(1-3\xi)^{-1}.$$

As a layer is considered, it is necessary to set two boundary conditions: one—in plane $x_3 = l$, and another—in plane $x_3 = 0$. It is supposed that in plane $x_3 = l$, perturbation with Gaussian profile is given and the boundary conditions are taken similarly to formula (1.38) from Chap. 1, only zero should be replaced by l, and F_1 looks like $F_1 = \frac{2L_1}{\alpha(1-3\xi)}\left[\frac{\kappa_2b_1^2}{2} - \frac{1}{R_1(0)}\right]$.

Equations (2.14)–(2.16) should be written with index "1" and be solved with boundary condition (1.38) with the specified corrections.

The second boundary condition set at $x_3 = 0$, consists in $\sigma_{31} = \sigma_{32} = \sigma_{33} = 0$. As we study a bunch of quasilongitudinal waves, we shall consider only the terms of the highest orders in these equations. The next approach requires taking into account else the transverse waves. Stresses σ_{31}, σ_{32}, σ_{33} consist of two components: constants provided by initial deformations and variables caused by a wave. When the variable terms are equal to zero, i.e. a wave process has not yet begun, the constant terms equal to zero too, as the stresses are required to be equal to zero in plane $x_3 = 0$. In the highest orders of the equations, $\sigma_{31} = \sigma_{32} = \sigma_{33} = 0$ are separated, the conditions concerning with the transverse and longitudinal waves are divided. Then, in the highest order, equation $\sigma_{33} = 0$ in displacements gives condition (1.41).

In the highest orders of the equations, conditions $\sigma_{31} = \sigma_{32} = 0$ are satisfied.

The set of Eqs. (2.14)–(2.16) contains factor $\exp\left[-2\nu\left(\frac{1}{c} + \tau'_{1,2}\right)\right]$, that makes the procedure of solving more complicated. For problem simplification it is assumed a smallness of dissipation on width of area $\frac{\nu l}{c} \ll 1$ that is usually valid. Then in these equations it is possible to replace exponent by one and to omit the second term in the right-hand side. On the other hand, when in the equation of the second harmonic the derivatives with respect to $\tau'_{1,2}$ were negligible, there was an assumption $\omega\tau'_{1,2} \gg 1$. Further, if in the nonlinear terms the dissipation and the dispersion are supposed to be of the same order, it is possible to receive $\frac{\nu l}{c} \gg 1$ that gives small values of $\exp\left[-2\nu\left(\frac{l}{c} + \tau'_{1,2}\right)\right]$. Hence, the linear theory is suitable in this case.

When $\frac{\nu l}{c} \ll 1$, it is supposed that though the dispersion and the dissipation are small, the dispersion is more than the dissipation. Equations (2.14)–(2.16) are valid for all values of ω and ν, which satisfy conditions: $\omega, \nu \lll \alpha$.

Solutions of Eqs. (2.14)–(2.16) with boundary conditions (1.38) have the form:

$$f_1^2 = \left[\tau'_1 + \frac{l}{c} + F(F^2+M)^{-1}\right]^2 (F^2+M) + M(F^2+M)^{-1}, \tag{2.17}$$

$$\begin{aligned}\sigma_1 = \frac{G}{M^{1/2}} \operatorname{arctg}\left\{\frac{F^2+M}{M^{1/2}}\left[\tau'_1 + \frac{l}{c} + F(F^2+M)^{-1}\right]\right\} \\ - \operatorname{arctg}\left(\frac{F}{M^{1/2}}\right).\end{aligned} \tag{2.18}$$

Solutions of Eqs. (2.14) and (2.16) with boundary conditions (1.41), where $\frac{d f_{1,2}(0)}{d \tau'_{1,2}} = 0$, take on the form:

$$f_2^2(0) = f_1^2(0) + M f_1^2(0)\tau_1'^2, \tag{2.19}$$

$$\sigma_2 = GM^{-1/2}\operatorname{arctg}\left[M^{1/2} f_1^2(0)\right]\tau'_2 + \sigma_1(0). \tag{2.20}$$

The condition $\frac{d f_{1,2}(0)}{d \tau'_{1,2}} = 0$ limits distance l

$$l = F(F^2+M)^{-1}c \tag{2.21}$$

One can show that relations $f_1(\tau'_1) = f_2(\tau'_2)$ and $\sigma_1(\tau'_1) = \sigma_2(\tau'_2)$ are valid, if condition (2.21) is true. This fact points to existence of symmetry for the falling and reflected bunches.

2.6 Bistability

Keeping only the first harmonics in the relation (2.11) and separating the real part, it is possible to write down

$$\psi_1 = |\psi_1|\cos\Phi_1, \quad \psi_2 = -|\psi_2|\cos\Phi_2,$$
$$\Phi_{1,2} = \mp\frac{\alpha}{c}x_3 - \alpha t - \omega t + \frac{\alpha l}{c} + \sigma_{1,2}.$$

Values on a bunch axis are taken in the phase. On border x_3 = l, like in article [184], balance relations

$$|\psi_2|^2 = R|\psi_1|^2, \quad \psi_1 = K_0(1-R)^{1/2} - \psi_2 R^{1/2} \tag{2.22}$$

are set. Here R is a square of the reflection factor, K_0 is a value of intensity of the wave falling on border x_3 = l, out of the layer $K_0 = |K_0| \cos \alpha\tau$, $\tau = (\alpha + \omega)t$.

Carrying capacity looks like [184]:

$$P = |\psi_1|^2(1-R)|K_0|^{1/2}. \tag{2.23}$$

Averaging expression (2.22) on t in an interval (0, $2\pi\alpha^{-1}$), substituting in (2.23), it is possible to receive expression for throughput. From expressions (2.18) and (2.20), demanding F = $M^{1/2}$ [184], one can obtain the following expression for throughput

$$\begin{gathered} P = \left[1 + \frac{4R}{(1-R)^2}\sin^2\frac{\Phi_1 - \Phi_2}{2}\right]^{-1}, \\ \frac{1}{2}(\Phi_1 - \Phi_2) = -\frac{\alpha l}{c} + \frac{\pi}{4}GM^{-1/2}. \end{gathered} \tag{2.24}$$

Neglecting dissipative nonlinearity and entering $x' = \kappa r_1^2|\psi_1|^2 L_1^{-1}$ from (2.24) and (2.23), accordingly, it is possible to receive the equation

$$P = \left\{1 + \frac{4R}{(1-R)^2}\sin^2\left[-\frac{\alpha l}{c} + \frac{(2+x')\pi}{8(1+x')^{1/2}}\right]\right\}^{-1}, \quad P = \frac{L_1(1-R)}{\kappa_1 r_1^2|K_0|^2}. \tag{2.25}$$

The graphic solution of the received equations for enough big $|K_0|$ gives the multiple-valued solution, i.e. starting with some values of the amplitude of the wave falling on the layer, the solution jumps on the top branch and P substantially grows that corresponds to the phenomenon of bistability. In particular, for $\frac{\alpha l}{c} \approx \frac{3}{8}\pi$ and $L_1\kappa_1 r_1^{-1}|K_0|^2(1-R) \approx 10^{-3}$, a triple crossing occurs and P increases approximately in 30 times.

2.7 The "Frozen" Waves

Equations (2.1) and (2.2) admit also dynamic processes, therefore in (2.2) it is necessary to consider $\dot{\sigma}_{ik}$ and $\dot{\varepsilon}_{ik}$ as the major terms. Then it is necessary to enter the following orders:

$$u_3 \sim \delta^2, \quad \frac{\partial}{\partial x_{1,2}} \sim \delta^{-1/2}, \frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial t} \sim \delta^{-1}, \quad d_1, d_2 \sim \delta^2, \; n_1, n_2 \sim \delta^3.$$

These orders for the factors mean that viscosity, dispersion, and dissipation are assumed to be small. For simplicity we assume that $\frac{\partial u_3^0}{\partial x_{1,2,3}} = 0$, i.e. initial deformations and stresses are absent.

Taking into account the above-stated orders and repeating the procedure described in Sect. 2.1, from Eqs. (2.1) and (2.2) one can derive the following set of equations

$$G\frac{\partial^2 u_3}{\partial x_i \partial x_3} + T\frac{\partial^2 u_i}{\partial x_3^2} + a_2\rho_0\frac{\partial^2 u_i}{\partial t^2} = 0 \quad (i = 1, 2) \tag{2.26}$$

$$\begin{aligned} &\rho_0\frac{\partial^2 u_3}{\partial t^2} + F\frac{\partial^2 u_3}{\partial x_3^2 \partial t} + N\frac{\partial^2}{\partial x_3 \partial t}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) \\ &+ P\frac{\partial}{\partial t}(\Delta_\perp u_3) + a_2\rho_0\frac{\partial^3 u_3}{\partial t^3} - (\lambda + 2\mu)\frac{\partial^2 u_3}{\partial x_3^2} - (d_1 + d_2)\frac{\partial^4 u_3}{\partial x_3^2 \partial t^2} \\ &- (n_1 + n_2)\frac{\partial^5 u_3}{\partial x_3^2 \partial t^3} + M_2\left(\frac{\partial^2 u_3}{\partial x_3^2}\frac{\partial^2 u_3}{\partial x_3 \partial t} + \frac{\partial u_3}{\partial x_3}\frac{\partial^3 u_3}{\partial x_3^2 \partial t}\right) = 0, \end{aligned} \tag{2.27}$$

where: $G = a_1 (2\mu + 3\lambda) + a_2 (\mu + \lambda) - b_1 - b_2$, $T = 2 a_2\mu - b_2$,
$F = a_2 (2\mu + \lambda) + a_1 (2\mu + 3\lambda) - b_{1-2}b_2$,
$N = a_2 (\mu + \lambda) + a_2 (2\mu + 3\lambda) + 2a_1 (2B + 3C) - b_1 - 2b_2$,
$P = a_2\mu - b_2$, $M_2 = 2a_1 (A + 5B + 3C) + 2a_2 (A + 2B + C) - b_1 - 2b_2$.

In a linear homogeneous case from Eqs. (2.26) and (2.27) follows that the longitudinal wave propagates in the medium with the velocity

$$c_1 = \frac{F}{a_2\rho}.$$

Similarly to the equilibrium case, Eqs. (2.26) and (2.27) split on the equations for the falling and reflected waves. These equations have the following form, accordingly, for the falling and reflected waves:

$$\frac{\partial^2\psi_{1,2}}{\partial t\partial\tau_{1,2}}-\frac{1}{2}L_2(\psi_{1,2})=-c_1^{-1}\frac{\partial}{\partial\tau_{1,2}}\left[H\psi_{1,2}+d\frac{\partial^2\psi_{1,2}}{\partial\tau_{1,2}^2}+n\frac{\partial^3\psi_{1,2}}{\partial\tau_{1,2}^3}+\Gamma_2\psi_{1,2}\frac{\partial\psi_{1,2}}{\partial\tau_{1,2}}\right],\tag{2.28}$$

where $L_2=c_1^2Q_pF^{-1}\Delta_\perp\psi_i\ (i=1,2)$, $H=-\frac{1}{2}N_2c_1^3F^{-1}$, $d=\frac{1}{2}(d_1+d_2)c_1F^{-1}$, $n=-\frac{1}{2}(n_1+n_2)c_1F^{-1}, Q_p=-P+\frac{NG}{c_1^2(a_1\rho_0+Tc_1^{-2})}$.

We shall search for a solution of Eq. (2.28) in the form (2.11). Executing such calculations as during a derivation of Eqs. (2.13), one can obtain the following equations for the linear dispersion, attenuation and amplitude of the first harmonic:

$$\omega=-\frac{n}{c_1}\alpha^3,\quad \nu=Hc_1^{-1}-dc_1^{-1}\alpha^2,$$

$$\left(i\alpha-\nu-3i\omega-\frac{2d\alpha^2}{c_1}\right)\frac{\partial A_{1,2}}{\partial\tau'_{1,2}}-\frac{1}{2}L_2(A_{1,2})=\frac{\Gamma_2^2\alpha^3\exp\left(\tau'_{1,2}+\frac{1}{c}\right)}{4c^2\left(i\nu+6\omega+\frac{3id\alpha^2}{c}\right)}\left|A_{1,2}\right|^2A_{1,2}.\tag{2.29}$$

According to the analogous procedure as for equilibrium wave in 2.4, it is possible to receive equations of type (2.14), (2.15), and (2.16), where the factors have the form:

$$\kappa_1=\frac{6\omega}{36\omega^2+\left(\nu+\frac{3d\alpha^2}{c}\right)^2},\quad \kappa_2=-\frac{\nu+3d\alpha^2c^{-1}}{36\omega^2+\left(\nu+\frac{3d\alpha^2}{c}\right)^2}.$$

Solutions (2.17)–(2.21) remain valid, as well as symmetry of the falling and reflected waves.

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