2 L_∞-algebras

Let $\mathbb{K}$ denote here a fixed ground field of characteristic 0. All vector spaces, linear maps and tensor products will be defined with respect to/taken over this field, unless we explicitly state otherwise. A good overview of the subject of L_∞-algebras is provided in the n-lab ([12]).

2.1 Generalizing differential graded Lie algebras

In this section we will define L_∞-algebras as the generalization of Lie algebras. For doing so let us first review the notion of a Lie algebra: A Lie algebra is a vector space L with a skew-symmetric bilinear map $[\cdot,\cdot] : L \times L \to L$ satisfying the Jacobi identity:

$$[[x_1, x_2], x_3] - [[x_1, x_3], x_2] + [[x_2, x_3], x_1] = 0 \tag{1}$$

This can be rewritten in the following way, where $P = \{\left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)\} \subset S_3$:

$$\sum_{\sigma\in P} sgn(\sigma)[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0.$$

The set P consists precisely of those permutations, which move one element to the last position, without distorting the inner order of the others. As we will deal with identities of multi-brackets soon, we will have to generalize this notion of "moving one element out of three to the end" to "moving q elements out of $p + q$ to the end". The permutations doing that are exactly the (p, q)-unshuffles.

Definition 2.1. A permutation $\sigma \in S_{p+q}$ is a (p, q)-*unshuffle* if and only if $\sigma(i) < \sigma(i + 1)$ for $i \neq p$. We denote the set of of (p, q)-unshuffles by $ush(p, q) \subset S_{p+q}$.

The condition in the above definition guarantees that the first p and the last q elements stay in the same internal order. These permutations are called unshuffles, because their inverses correspond to shuffling a deck of p cards into a deck of q cards.

Let us now turn to the graded context. First we introduce a grading on our vector space: We set $L = \bigoplus_{i\in\mathbb{Z}} L_i$, where L_i ist the vector subspace of elements of degree i. We will write $|x| = i$ if $x \in L_i$. A Lie structure $[\cdot,\cdot]$ on such a vector space L should satisfy the following three conditions:

- $[L_i, L_j] \subset L_{i+j}$ (the bracket respects the grading)
- $[x_1, x_2] = -(-1)^{|x_1||x_2|}[x_2, x_1]$ for all homogenous $x_1, x_2 \in L$ (the bracket is graded skew-symmetric)
- $(-1)^{|x_1||x_3|}[x_1, [x_2, x_3]] + (-1)^{|x_1||x_2|}[x_2, [x_3, x_1]] + (-1)^{|x_2||x_3|}[x_3, [x_1, x_2]] = 0$ for all homogenous elements $x_1, x_2, x_3 \in L$ (the graded Jacobi identity holds)

If we try to bring the graded Jacobi identity into form of equation (1), we get:

$$[[x_1, x_2], x_3] - (-1)^{|x_2||x_3|}[[x_1, x_3], x_2] + (-1)^{|x_1||x_2|+|x_1||x_3|}[[x_2, x_3], x_1] = 0.$$

So $[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}]$ gets an additional sign for every transposition of two odd elements. This leads us to the definition of the Koszul sign ϵ of a permutation σ acting on elements $x_1, ..., x_n \in L$.

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Definition 2.2. Let $\sigma \in S_n$ be a permutation acting on elements $v_1, ..., v_n$ of a $\mathbb{Z}$-graded vector space V. Let $(v_{i_1}, ..., v_{i_k})$ be the ordered sublist of $v_1, ..., v_n$ including exactly the odd elements. Then there is a (unique) permutation $\tilde{\sigma} \in S_k$ such that $(v_{i_{\tilde{\sigma}(1)}}, ..., v_{i_{\tilde{\sigma}(k)}})$ is the ordered sublist of $v_{\sigma(1)}, ..., v_{\sigma(n)}$ including exactly the odd elements. Then the *Koszul sign of σ acting on $v_1, ..., v_n$* is defined by

$$\epsilon(\sigma, v_1, ..., v_n) := sgn(\tilde{\sigma}).$$

Remark 2.3. One can check that ϵ is well-behaved in the sense that

$$\epsilon(\sigma' \circ \sigma, v_1, ..., v_k) = \epsilon(\sigma', v_{\sigma(1)}, ..., v_{\sigma(k)}) \cdot \epsilon(\sigma, v_1, ..., v_k),$$

and that for a transposition τ_i interchanging v_i and v_{i+1} it holds that $\epsilon(\tau_i, v_1, ..., v_n) = (-1)^{|v_i||v_{i+1}|}$.

Thus, the graded Jacobi identity can be written in the following way:

$$\sum_{\sigma \in ush(2,1)} sgn(\sigma)\epsilon(\sigma, x_1, x_2, x_3)[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0 \tag{2}$$

Next we consider a graded vector space with a differential $d : L \to L$ satisfying $d(L_i) \subset L_{i+1}$ and $d^2 = 0$. This turns L into a differential graded vector space or, in other words, into a chain complex[1]:

$$\ldots \longrightarrow L_{i-2} \xrightarrow{d} L_{i-1} \xrightarrow{d} L_i \xrightarrow{d} L_{i+1} \xrightarrow{d} \ldots$$

Adopting the standard language, used e.g. for de Rham cohomology, we call $x \in L$ *closed* if $dx = d(x) = 0$ and *exact* if $x = dy$ for some $y \in L$. In the latter case y is called a *potential* for x.

A differential graded vector space (L, d) together with a graded Lie bracket $[\cdot, \cdot]$ on L is called a *differential graded Lie algebra* if the differential derives the bracket i.e. satisfies the following graded Leibniz rule (for $x_1, x_2 \in L$):

$$d[x_1, x_2] = [d(x_1), x_2] - (-1)^{|x_1|}[x_1, d(x_2)].$$

This can be rewritten as:

$$d[x_1, x_2] = [d(x_1), x_2] - (-1)^{|x_1||x_2|}[d(x_2), x_1].$$

The latter equation can also be written in terms of signed sums of unshuffles. In fact, it is equivalent to:

$$\sum_{\sigma \in ush(2,0)} sgn(\sigma)\epsilon(\sigma, x_1, x_2)d[x_{\sigma(1)}, x_{\sigma(2)}] = \sum_{\sigma \in ush(1,1)} sgn(\sigma)\epsilon(\sigma, x_1, x_2)[d(x_{\sigma(1)}), x_{\sigma(2)}].$$

[1]From the perspective of homological algebra it would be a cochain complex, as the differential has positive degree. The closed elements would be called cocycles and the exact elements coboundaries.

If we furthermore interpret the differential as a unary bracket and write $x \mapsto [x]$ instead of $x \mapsto d(x)$ we get:

$$\sum_{\sigma \in ush(2,0)} sgn(\sigma)\epsilon(\sigma, x_1, x_2)[[x_{\sigma(1)}, x_{\sigma(2)}]] = \sum_{\sigma \in ush(1,1)} sgn(\sigma)\epsilon(\sigma, x_1, x_2)[[x_{\sigma(1)}], x_{\sigma(2)}] \qquad (3)$$

Even the condition $d^2 = 0$ can be written as $[[x]] = 0$, or bringing it into the form of the other equations :

$$\sum_{\sigma \in ush(1,0)} sgn(\sigma)\epsilon(\sigma, x_1)[[x_{\sigma(1)}]] = 0 \qquad (4)$$

We have now learned how to describe a differential graded Lie algebra as a graded vector space L with a unary bracket $[\cdot]$ of degree one and a binary bracket $[\cdot,\cdot]$ of degree 0, which satisfy the equations (2), (3) and (4). These three equations are special cases of the below equation (5). We will now generalize the notion of differential graded Lie algebra to obtain a definition of an L_∞-algebra.

Definition 2.4. An *L_∞-algebra* (or *Lie-∞-algebra*) is a graded vector space $L = \bigoplus_{i\in\mathbb{Z}} L_i$ together with a family of graded skew-symmetric multilinear maps $\{l_i : \bigtimes^i L \to L \mid i \in \mathbb{N}\}$ such that l_i has degree $2-i$ and the following identity holds (for all $n \in \mathbb{N}$):

$$\sum_{i+j=n+1} (-1)^{i(j+1)} \sum_{\sigma \in ush(i,n-i)} sgn(\sigma)\epsilon(\sigma, x_1, ..., x_n)\, l_j(l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)} ..., x_{\sigma(n)}) = 0 \qquad (5)$$

To keep the definition transparent despite the use of (possibly infinitely many) multi-brackets we write l_n for the n-ary bracket and give a brief overview of the multilinear algebra surpressed in the definition.

- $\bigtimes^i L$ is the cross product of i copies of L.
- A multi-linear map $l_i : \bigtimes^i L \to L$ is a map linear in every component. One could equivalently define l_i as a linear map from $\bigotimes^i L$ to L. Here $\bigotimes^i L$ is the i-fold tensor product of the graded vector space L.
- Demanding l_i to have degree $2-i$ means that l_i restricted to $L_{k_1} \times L_{k_2} \times ... \times L_{k_i}$ must map into $L_{k_1+k_2+...+k_i+2-i}$. Turning $\bigotimes^i L$ into a graded vector space by defining the degree of $x_1 \otimes ... \otimes x_i$ as $\sum_{k=1}^i |x_k|$, this translates to saying that $l_i : \bigotimes^i L \to L$ is a linear map of degree $2-i$.
- The maps l_i being (graded) skew-symmetric means that for all $\sigma \in S_i$ the identity $l_i(x_1, ..., x_i) = sgn(\sigma)\epsilon(\sigma, x_1, ..., x_i) l_i(x_{\sigma(1)}, ..., x_{\sigma(i)})$ holds. Using the language of multilinear algebra this is equvialent to $l_i : \bigotimes^i L \to L$ descending to a map $l_i : E^i(L) \to L$, where $E^i(L)$ is the i-th (graded) exterior power of L.

- The reason why we only sum over the unshuffles is to avoid repetition: If any two permutations $\sigma, \sigma' \in S_n$ differ by a permutation which only interchanges the first i and the last $n-i$ elements (i.e. $\sigma = \tau \circ \sigma'$, where $\tau = (\tau_1, \tau_2) \in S_i \times S_{n-i} \subset S_n$) then $l_j(l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}...x_{\sigma(n)}) = \pm l_j(l_i(x_{\sigma'(1)}, ..., x_{\sigma'(i)}), x_{\sigma'(i+1)}...x_{\sigma'(n)})$. Thus, up to sign, we would add up essentially same elements several times. To avoid that, we need a system of representatives for $S_n/(S_i \times S_{n-i})$. That system of representatives is provided by the Unshuffles. All inner permutations of the first i and last $n-i$ elements are prohibited, as there is only one way to arrange an i-element (resp. $(n-i)$-element) subset of $\{1, ..., n\}$ in a strictly ascending order.

Next, let us take a closer look at the signs involved:

- The first sign in (5) is $(-1)^{i(j+1)}$. It only gives us an additional sign if the number of elements consumed by brackets is odd for the inner one (l_i) and even for the outer one (l_j). If we look at (5) for $n = 2$ we get equation (3). If not for the $(-1)^{i(j+1)}$ in our definition this equation would have an additional sign and the graded Leibniz rule would not be satisfied.

- $sgn(\sigma)$ is the usual sign of the permutation, not depending on the grading. It contributes a minus sign for every transposition in the permutation.

- $\epsilon(\sigma, x_1, ..., x_i)$ is the Koszul sign, which is highly dependant on the grading of the elements permuted. It gives us a factor of -1 for every transposition of two odd elements in the permutation.

Finally, by direct calculation, we can see that this definition indeed provides a generalization of a (differential graded) Lie algebra:

Example 2.5. A differential graded Lie algebra is precisely an L_∞-algebra with $l_i = 0$ for $i > 2$. Furthermore, a Lie algebra is an L_∞-algebra where the L is concentrated in degree zero (i.e. $L = L_0$).

2.2 L_∞-algebras as coalgebras with differentials

Having defined L_∞-algebra objects, we would now like to investigate their structure-preserving maps. Intuitively one would regard (say degree-zero) linear maps f which conserve the brackets i.e. $l'_i(f(x_1), ..., f(x_i)) = f(l_i(x_1, ..., x_i))$. Unfortunately this nontion of morphism is not flexible enough, and from our definition it is not clear what the right notion of more "flexible" maps would be. In order to advance, we will reformulate our definition in the language of "differential-graded coalgebras". First of all we will get rid of the different degrees of the l_n. We define the shift sL to be L with the grading shifted by one i.e. $(sL)_i = L_{i+1}$. To keep notation consistent we denote the multi-brackets by sl_i instead of l_i if we work with the shifted grading. Then we get

$$(sl_i)((sL)_{k_1}, ..., (sL)_{k_i}) = l_i(L_{k_1+1}, ..., L_{k_i+1}) \subset L_{k_1+1+...+k_i+1+2-i} = L_{k_1+...k_i+2} = (sL)_{k_1+...+k_i+1}.$$

So $(sl_i) : \bigtimes^i(sL) \to (sL)$ have degree one independantly of i. Unfortunately the sl_i are not in general skew-symmetric with respect to the new grading. But we can turn the l_i into graded

commutative maps by altering the signs.
Recall that a multi-linear map $f : \bigtimes^n V \to W$, where V is a graded vector space and W is any vector space, is called *graded-commutative* (or *graded-symmetric*) if it satisfies

$$f(y_1, ..., y_n) = \epsilon(\sigma, y_1, ..., y_n) f(y_{\sigma(1)}, ..., y_{\sigma(n)}) \quad \forall \sigma \in S_n \tag{6}$$

Writing $sx \in (sL)_n$ for an element $x \in L_{n+1}$, we now define $\hat{l}_n : \bigtimes^n (sL) \to (sL)$ by

$$\hat{l}_n(sx_1, ..., sx_n) = (-1)^{\alpha(x_1,...,x_n)}(sl_n)(sx_1, ..., sx_n)$$

with α defined in the following way:

$$\alpha(x_1, ..., x_n) = \alpha_n(x_1, ..., x_n) = \sum_{\substack{i \text{ is odd} \\ i\in\{1,...,n\}}} |x_i| \quad \text{when n is even,}$$

$$\alpha(x_1, ..., x_n) = \alpha_n(x_1, ..., x_n) = 1 + \sum_{\substack{i \text{ is even} \\ i\in\{1,...,n\}}} |x_i| \quad \text{when n is odd.}$$

Lemma 2.6. The maps $\hat{l}_n : \bigtimes^n (sL) \to sL$ are graded-symmetric multi-linear maps of degree 1 for all $n \in \mathbb{N}$.

Proof. The $\hat{l}_n$ are multi-linear by construction and $deg(\hat{l}_n) = 1$ follows from the discussion above. The only property, which remains to be checked is the graded symmetry (6). However, it is enough to check this property for the transpositions $\{\tau_i \mid 1 \le i < n\} \subset S_n$, where τ_i interchanges y_i and y_{i+1}, as these transpositions generate the full symmetric group. So we calculate:

$$\begin{aligned}\hat{l}_n(sx_1, ..., sx_n) &= (-1)^{\alpha(x_1,...,x_n)} sl_n(sx_1, ..., sx_n) = (-1)^{\alpha(x_1,...,x_n)} l_n(x_1, ..., x_n) \\ &= (-1)^{\alpha(x_1,...,x_n)} sgn(\tau_i)\epsilon(\tau_i, x_1, ..., x_n) l_n(x_{\tau_i(1)}, ..., x_{\tau_i(n)}).\end{aligned}$$

As τ_i consists of only one transposition we have $sgn(\tau_i) = -1$ and $\epsilon(\tau_i, x_1, ..., x_n) = (-1)^{|x_i||x_{i+1}|}$, thus

$$\begin{aligned}\hat{l}_n(sx_1, ..., sx_n) &= (-1)^{\alpha(x_1,...,x_n)}(-1)(-1)^{|x_i||x_{i+1}|} l_n(x_{\tau_i(1)}, ..., x_{\tau_i(n)}) \\ &= (-1)^{\alpha(x_1,...,x_n)}(-1)(-1)^{|x_i||x_{i+1}|} sl_n(sx_{\tau_i(1)}, ..., sx_{\tau_i(n)}) \\ &= (-1)^{\alpha(x_1,...,x_n)}(-1)(-1)^{|x_i||x_{i+1}|}(-1)^{\alpha(x_{\tau_i(1)},...,x_{\tau_i(n)})} \hat{l}_n(sx_{\tau_i(1)}, ..., sx_{\tau_i(n)}).\end{aligned}$$

But the sum $\alpha(x_1, ..., x_n)$ and $\alpha(x_{\tau_i(1)}, ..., x_{\tau_i(n)})$ only differ by one summand. One of them has the summand $|x_i|$ and the other one $|x_{i+1}|$. All the other signs cancel out, so we get:

$$\begin{aligned}\hat{l}_n(sx_1, ..., sx_n) &= (-1)^{|x_i|+|x_j|}(-1)(-1)^{|x_i||x_{i+1}|} \hat{l}_n(sx_{\tau_i(1)}, ..., sx_{\tau_i(n)}) = (-1)^{(|x_i|+1)(|x_{i+1}|+1)} \hat{l}_n(sx_{\tau_i(1)}, ..., sx_{\tau_i(n)}) \\ &= \epsilon(\tau_i, sx_1, ..., sx_n) \hat{l}_n(sx_{\tau_i(1)}, ..., sx_{\tau_i(n)}).\end{aligned}$$

□

Remark 2.7. In this proof the constant summand in the definition of α for odd n is not needed. It will turn out useful later, when we discuss the multi-bracket equation (5).

We can now regard $\hat{l}_n$ as a graded-symmetric linear map from $\bigotimes^n(sL)$ to sL of degree one. And, analogously, we can encode the symmetry properties of this linear map by modifying its domain. This time we need the graded-symmetric powers of (sL), which are denoted by $S^n(sL)$. Analogusly to the preceeding reasoning we can describe $\hat{l}_n$ equivalently as linear maps $\hat{l}_n : S^n(sL) \to sL$, where $S^n(\cdot)$ is defined as follows:

Definition 2.8. Let V be a $\mathbb{Z}$-graded vector space. *The k-th (graded) symmetrization operator* $Sym_k : \bigotimes^k V \to \bigotimes^k V$ is defined by

$$Sym_k(v_1 \otimes ... \otimes v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma, v_1, ..., v_k) v_{\sigma(1)} \otimes ... \otimes v_{\sigma(k)}.$$

The Image of Sym_k is a vector subspace and called *the k-th (graded) symmetric power of V*. We will denote $Sym_k(v_1 \otimes ... \otimes v_k)$ by $v_1 \odot ... \odot v_k$.

The sum of the $S^n(sL)$ forms an algebra $S^\bullet(sL) = \bigoplus_{n\in\mathbb{N}_0} S^n(sL)$, the so-called *free graded-commutative algebra* on sL. Setting $\hat{l}_0 : S^0(sL) = \mathbb{K} \to (sL)$ to be the zero map, we can combine the $\hat{l}_i$ to a linear map $\hat{l} : S^\bullet(sL) \to sL$ of degree 1. This map encodes all information of our L_∞-algebra except equation (5).

To encode equation (5) we have to change perspectives. Instead of regarding $S^\bullet(sL)$ as an algebra we will use its coalgebra structure. The reason for this is that the universal property of an algebra gives us a way to extend the domain of maps $sL \to W$. What we need here is the dual property: We want to extend a map $W \to sL$ to some map $W \to S^\bullet(sL)$ (in our case $W = S^\bullet(sL)$ and the map to be extended is $\hat{l}$). The most direct way to understand a coalgebra is by dualizing the diagrams encoding an algebra structure. Let us first recall the definition of a graded algebra in terms of diagrams:

Definition 2.9. Let A be a graded vector space. A *unital associative graded commutative algebra-structure on A* consists of two linear maps of degree zero, a linear multiplication map $m : A \otimes A \to A$ and a unit morphism $u : \mathbb{K} \to A$ such that the following diagrams commute, where $\tau : A \otimes A \to A \otimes A$ is defined by $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$:

associativity:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

left unit law:

$$\begin{tikzcd} \mathbb{K} \otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\lambda\otimes a \mapsto \lambda a"'] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

right unit law:

$$\begin{tikzcd} A \otimes \mathbb{K} \arrow[r, "id\otimes u"] \arrow[dr, "a\otimes\lambda \mapsto \lambda a"'] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

graded commutativity:

$$\begin{tikzcd} A \otimes A \arrow[r, "\tau"] \arrow[dr, "m"] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

"Reversing all arrows" gives us the definition of a coalgebra:

Definition 2.10. Let C be a graded vector space. A *counital coassociative graded cocommutative coalgebra-structure on C* consists of two linear maps of degree zero, a comultiplication map

$\Delta : C \otimes C \to C$ (also called diagonal) and a counit morphism $\pi : C \to \mathbb{K}$ such that the following diagrams commute:

coassociativity:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\Delta\otimes id"] \\ C \otimes C \arrow[r, "id\otimes\Delta"] & C \otimes C \otimes C \end{tikzcd}$$

left counit law:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "c\mapsto 1\otimes c"'] & C \otimes C \arrow[d, "\pi\otimes id"] \\ & \mathbb{K} \otimes C \end{tikzcd}$$

right counit law:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "c\mapsto c\otimes 1"'] & C \otimes C \arrow[d, "id\otimes\pi"] \\ & \mathbb{K} \otimes C \end{tikzcd}$$

graded cocommutativity:

$$\begin{tikzcd} C \arrow[d, "\Delta"] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[r, "\tau"] & C \otimes C \end{tikzcd}$$

One can turn $S^\bullet(sL)$ into a counital coassociative graded cocommutative coalgebra with the counit $\pi = \pi_0$ given by the projection onto $S^0(sL) = \mathbb{K}$ and the following diagonal:

$$\Delta(sx_1 \odot ... \odot sx_n) := \sum_{i=0}^{n} \sum_{\sigma\in ush(i,n-i)} \epsilon(\sigma, sx_1, ..., sx_n) \left(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)}\right) \otimes \left(sx_{\sigma(i+1)} \odot ... \odot sx_{\sigma(n)}\right)$$

More generally, this construction can be carried out for any graded vector space V. The resulting coalgebra structure on the graded vector space $S^\bullet V$ is then called the *cofree graded cocommutative coalgebra* of V. For any counital coassociative algebra C a homomorphism $\delta : C \to C$ is called a *coderivation* if the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "\Delta"] & C \arrow[d, "\Delta"] \\ C \otimes C \arrow[r, "a\otimes b\mapsto\delta(a)\otimes b+(-1)^{|\delta||a|}a\otimes\delta(b)"] & C \otimes C \end{tikzcd}$$

Returning to our case, we have a map $\hat{l} : S^\bullet(sL) \to sL = S^1(sL)$. It can be extended to a degree 1 coderivation $D : S^\bullet(sL) \to S^\bullet(sL)$ in the following way:

Definition 2.11. Let L be a $\mathbb{Z}$-graded vector space and $\{l_i \mid i \in \mathbb{N}\}$ a family of graded skew-symmetric maps. Let $\hat{l} : S^\bullet(sL) \to sL$ be the above defined map. Then we define the degree-one derivation $D : S^\bullet(sL) \to S^\bullet(sL)$ by the following formula:

$$D(sx_1 \odot ... \odot sx_n) = \sum_{i=1}^{n} \sum_{\sigma\in ush(i,n-i)} \epsilon(\sigma, sx_1, ..., sx_n) \left(\hat{l}(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)})\right) \odot sx_{\sigma(i+1)} \odot ... \odot sx_{\sigma(n)}.$$

Remark 2.12. As we defined $\hat{l}$ summand-wise by the $\hat{l}_i$, we could instead write

$$D(sx_1 \odot ... \odot sx_n) = \sum_{i=1}^{n} \sum_{\sigma\in ush(i,n-i)} \epsilon(\sigma, sx_1, ..., sx_n) \left(\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)})\right) \odot sx_{\sigma(i+1)} \odot ... \odot sx_{\sigma(n)},$$

and check that D indeed is a coderivation. As a concequence of corollary A.2 D is the unique coderivation satisfying $\pi_1 D = \hat{l}$.

With this extension done, we can finally reformulate condition (5).

Lemma 2.13. In the setting of the last definition the equation

$$\sum_{i+j=n+1} (-1)^{i(j+1)} \sum_{\sigma \in ush(i,n-i)} sgn(\sigma)\epsilon(\sigma, x_1, ..., x_n)\, l_j(l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}..., x_{\sigma(n)}) = 0$$

implies $D^2 = 0$.

Proof.

$$\begin{aligned} D^2(sx_1 \odot ... \odot sx_n) &= D\left(\sum_{i=1}^{n} \sum_{\sigma \in ush(i,n-i)} \epsilon(\sigma, sx_1, ..., sx_n) \left(\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)})\right) \odot sx_{\sigma(i+1)} \odot ... \odot sx_{\sigma(n)}\right) \\ &= \sum_{i=1}^{n} \sum_{\sigma \in ush(i,n-i)} \epsilon(\sigma, sx_1, ..., sx_n) D\left(\left(\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)})\right) \odot sx_{\sigma(i+1)} \odot ... \odot sx_{\sigma(n)}\right) \end{aligned}$$

Next we apply D to a term of length $n-i+1$. For that we will define $y_1^\sigma := \hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)})$ and $y_{k+1}^\sigma := sx_{\sigma(i+k)}$ for $k \in \{1, ..., n-i\}$. So we have:

$$\begin{aligned} &D\left(\left(\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)})\right) \odot sx_{\sigma(i+1)} \odot ... \odot sx_{\sigma(n)}\right) = D(y_1^\sigma \odot ... \odot y_{n-i+1}^\sigma) \\ &= \sum_{j=1}^{n-i+1} \sum_{\tau \in ush(j,n-i+1-j)} \epsilon(\tau, y_1^\sigma, ..., y_{n-i+1}^\sigma) \hat{l}_j(y_{\tau(1)}^\sigma \odot ... \odot y_{\tau(j)}^\sigma) \odot y_{\tau(j+1)}^\sigma \odot ... \odot y_{\tau(n-i+1)}^\sigma. \end{aligned}$$

As y_1^σ is a structurally different term than the other y_k^σ we will distinguish between τ satisfying $\tau(1) = 1$ denoted by $u_1(j, n, i)$ and $\tau(j + 1) = 1$ denoted by $u_2(j, n, i)$. Every element in $ush(j, n-i+1-j)$ is exactly in one of those two subsets. Let us first analyse the case $\tau \in u_2(j, n, i)$:

$$\begin{aligned} &\hat{l}_j(y_{\tau(1)}^\sigma \odot ... \odot y_{\tau(j)}^\sigma) \odot y_{\tau(j+1)}^\sigma \odot ... \odot y_{\tau(n-i+1)}^\sigma \\ =&\hat{l}_j(y_{\tau(1)}^\sigma \odot ... \odot y_{\tau(j)}^\sigma) \odot \hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)}) \odot y_{\tau(j+2)}^\sigma \odot ... \odot y_{\tau(n-i+1)}^\sigma. \end{aligned}$$

The total sign of this element is $\epsilon(\sigma, sx_1, ..., sx_n) \cdot \epsilon(\tau, y_1^\sigma, ..., y_{n-i+1}^\sigma)$. Now we regard the summand coming from $\tilde{i} = j, \tilde{j} = i$ and $\tilde{\sigma} \in ush(\tilde{i}, n-\tilde{i})$, $\tilde{\tau} \in u_2(\tilde{j}, n, \tilde{i})$ defined as follows:

- $\tilde{\sigma}$ is the unshuffle that, given the strictly ascending list $(sx_1, ..., sx_n)$, moves the elements $y_{\tau(1)}^\sigma, ..., y_{\tau(j)}^\sigma$ to the front.
- $\tilde{\tau}$ is the unshuffle that, given a list starting with $\hat{l}_{\tilde{i}}(y_{\tau(1)}^\sigma \odot ... \odot y_{\tau(j)}^\sigma)$ and continuing with a strictly ascending list of $\left\{y_{\tau(j+2)}^\sigma, ..., y_{\tau(n-i+1)}^\sigma\right\} \cup \left\{sx_{\sigma(1)}, ..., sx_{\sigma(i)}\right\}$, moves the elements $(sx_{\sigma(1)}, ..., sx_{\sigma(i)})$ to the front.

By construction the summands belonging to (i, j, σ, τ) and $(\tilde{i}, \tilde{j}, \tilde{\sigma}, \tau)$ are equal up to sign. The signs come from the odd transpositions in $\sigma, \tau, \tilde{\sigma}, \tilde{\tau}$ and from interchanging $\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)})$ with $\hat{l}_j(y_{\tau(1)}^\sigma \odot ... \odot y_{\tau(j)}^\sigma)$.

First of all let us take a look at the transpositions nessecary to move the elements $y_{\tau(k)}^\sigma$ with

$k \in \{j+2, ..., n-i+1\}$ to the end. So we fix such a k. In the summand belonging to (σ, τ), there is some subset A of $\{sx_{\sigma(1)}, ...sx_{\sigma(i)}\}$ which has to be moved to the front past $y^{\sigma}_{\tau(k)}$, while applying σ. After that operation there is a subset B of $\{y^{\sigma}_{\tau(1)}...y^{\sigma}_{\tau(j)}\}$ which has to be moved to the front past $y^{\sigma}_{\tau(k)}$. In the summand belonging to $(\tilde{\sigma}, \tilde{\tau})$ by construction first the elements of B and then the elements of A have to be moved past $y^{\sigma}_{\tau(k)}$. So $y^{\sigma}_{\tau(k)}$ is part of the same transpositions in both summands.

Next let us consider the transpositions of $sx_{\sigma(k)}$ for $k \in \{1, ..., i\}$ while applying σ. For fixed k $x_{\sigma(k)}$ has to be moved past some elements $A_k \subset \{y^{\sigma}_{\tau(1)}, ..., y^{\sigma}_{\tau(j)}\}$. Then in the course of applying τ $\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)})$ has to be moved past all $\{y^{\sigma}_{\tau(1)}, ..., y^{\sigma}_{\tau(j)}\}$. Analogously for $l \in \{1, ..., j\}$ while applying $\tilde{\sigma}$ we have to move $y^{\sigma}_{\tau(l)}$ past $B_l \subset \{sx_{\sigma(1)}, ..., \sigma(i)\}$. Also we have to move $\hat{l}_j(y^{\sigma}_{\tau(1)} \odot ... \odot y^{\sigma}_{\tau(j)})$ past all $\{sx_{\sigma(1)}, ..., sx_{\sigma(i)}\}$ while applying $\tilde{\tau}$. Note that,by construction, it holds that

$$\bigcup_{k \in \{1,...,i\}} (\{k\} \times A_k) \cup \bigcup_{l \in \{1,...,j\}} (B_l \times \{l\}) = \{1, ..., i\} \times \{1, ..., j\} \tag{7}$$

Note that $(\{k\} \times A_k) \cap (B_l \times \{l\}) = \emptyset$ for all k and l such that all unions involved are disjoint. Thus the relative sign of the two summands is $(-1)^c$ for

$$\begin{aligned} c = &\sum_{k=1}^{i} \sum_{l \in A_k} |sx_{\sigma(k)}||y^{\sigma}_{\tau(l)}| + \left(1 + \sum_{k=1}^{i} |sx_{\sigma(k)}|\right) \sum_{l=1}^{j} |y^{\sigma}_{\tau(l)}| + \sum_{l=1}^{j} \sum_{k \in B_l} |sx_{\sigma(k)}||y^{\sigma}_{\tau(l)}| \\ &+ \sum_{k=1}^{i} |sx_{\sigma(k)}| \left(1 + \sum_{l=1}^{j} |y^{\sigma}_{\tau(l)}|\right) + \left(1 + \sum_{k=1}^{i} |sx_{\sigma(k)}|\right)\left(1 + \sum_{l=1}^{j} |y^{\sigma}_{\tau(l)}|\right) \end{aligned}$$

The first summand comes from permuting $sx_{\sigma(k)}$ with $y^{\sigma}_{\tau(l)}$ via σ, the second one from permuting $\hat{l}_i(...)$ with $y^{\sigma}_{\tau(l)}$ via τ, the third one from permuting $y^{\sigma}_{\tau(l)}$ with $sx_{\sigma(k)}$ via $\tilde{\sigma}$, the fourth one from permuting $\hat{l}_j(...)$ with $sx_{\sigma(k)}$ and the last summand comes from interchanging $\hat{l}_i(...)$ with $\hat{l}_j(...)$. As already discussed, all other transpositions (i.e. those with $y^{\sigma}_{\tau(l)}$ for $l \geq j+2$) happen in both summands and thus give no relative sign. Using (7) we observe that:

$$\sum_{k=1}^{i} \sum_{l \in A_k} |sx_{\sigma(k)}||y^{\sigma}_{\tau(l)}| + \sum_{l=1}^{j} \sum_{k \in B_l} |sx_{\sigma(k)}||y^{\sigma}_{\tau(l)}| = \sum_{k=1}^{i} \sum_{l=1}^{j} |sx_{\sigma(k)}||y^{\sigma}_{\tau(l)}| = \left(\sum_{k=1}^{i} |sx_{\sigma(k)}|\right)\left(\sum_{l=1}^{j} |y^{\sigma}_{\tau(l)}|\right)$$

Writing p for $(\sum_{k=1}^{i} |sx_{\sigma(k)}|$ and q for $\sum_{l=1}^{j} |y^{\sigma}_{\tau(l)}|$ we get:

$$c = pq + (1+p)q + p(1+q) + (1+p)(1+q) = (1+p)(1+q) - 1 + (1+p)(1+q) \equiv 1 \mod 2$$

Thus the relative sign is -1 and we have shown, that summands coming from $u_2(j, n, i)$ cancel out pairwise.

So (without using equation (5)) we have shown:

$$D^2(sx_1 \odot ... \odot sx_n) = \sum_{i=1}^{n} \sum_{\sigma \in ush(i,n-i)} \epsilon(\sigma, sx_1, ..., sx_n) \sum_{j=1}^{n-i+1} \sum_{\tau \in u_1(j,n,i)} \epsilon(\tau, y_1^\sigma, ..., y_{n-i+1}^\sigma) \times \tag{8}$$
$$\hat{l}_j(\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)}) \odot y^\sigma_{\tau(2)} \odot ... \odot y^\sigma_{\tau(j)}) \odot y^\sigma_{\tau(j+1)} \odot ... \odot y^\sigma_{\tau(n-i+1)}$$

$$= \sum_{\substack{i \in \{1,...,n\} \\ \sigma \in ush(i,n-i) \\ j \in \{1,...,n-i+1\} \\ \tau \in u_1(j,n,i)}} \epsilon(\sigma, sx_1, ..., sx_n)\epsilon(\tau, y_1^\sigma, ..., y^\sigma_{n-i+1}) \times$$
$$\hat{l}_j(\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)}) \odot y^\sigma_{\tau(2)} \odot ... \odot y^\sigma_{\tau(j)}) \odot y^\sigma_{\tau(j+1)} \odot ... \odot y^\sigma_{\tau(n-i+1)}.$$

As expressions of different length can not cancel out each other, we can group elements of some fixed length d and thus obtain:

$$D^2(sx_1 \odot ... \odot sx_n) = \sum_{\substack{d \in \{1,...,n\} \\ i+j=d+1 \\ \sigma \in ush(i,n-i) \\ \tau \in u_1(j,n,i)}} \epsilon(\sigma, sx_1, ..., sx_n)\epsilon(\tau, y_1^\sigma, ..., y^\sigma_{n-i+1}) \times$$
$$\hat{l}_j(\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)}) \odot y^\sigma_{\tau(2)} \odot ... \odot y^\sigma_{\tau(j)}) \odot y^\sigma_{\tau(j+1)} \odot ... \odot y^\sigma_{\tau(n-i+1)}.$$

The last sum is zero if and only if it is zero for any fixed value of d. So it suffices to analyse the following sums for $d \in \{1, ..., n\}$:

$$\sum_{\substack{i+j=d+1 \\ \sigma \in ush(i,n-i) \\ \tau \in u_1(j,n,i)}} \epsilon(\sigma, sx_1, ..., sx_n)\epsilon(\tau, y_1^\sigma, ..., y^\sigma_{n-i+1}) y^\sigma_{\tau(j)}) \times \tag{9}$$
$$\hat{l}_j(\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)}) \odot y^\sigma_{\tau(2)} \odot ... \odot \odot y^\sigma_{\tau(j+1)} \odot ... \odot y^\sigma_{\tau(n-i+1)}$$

Now we have to regroup the permutations. Instead of moving $sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)}$ to the front and then $y^\sigma_{\tau(j+1)} \odot ... \odot y^\sigma_{\tau(n-i+1)}$ to the back, we first move $y^\sigma_{\tau(j+1)} \odot ... \odot y^\sigma_{\tau(n-i+1)}$ to the back via $\tilde{\tau} \in ush(d, n-d)$. Then we rename the elements $sx_{\tilde{\tau}(k)}$ with $k \in \{1, ..., d\}$ to $z_k^{\tilde{\tau}}$ and move $sx_{\sigma(1)} \odot ..., \odot sx_{\sigma(i)}$ to the front via some $\tilde{\sigma} \in ush\{i, d-i\}$. As we perform the same transpositions as before we can write the sign in terms of $\tilde{\tau}$ and $\tilde{\sigma}$. So (for each fixed d) the above sum is equal to:

$$\sum_{\substack{i+j=d+1 \\ \tilde{\sigma} \in ush(i,d-i) \\ \tilde{\tau} \in ush(d,n-d)}} \epsilon(\tilde{\sigma}, z_1^{\tilde{\tau}}, ... z_d^{\tilde{\tau}})\epsilon(\tilde{\tau}, sx_1, ... sx_n)\hat{l}_j(\hat{l}_i(z^{\tilde{\tau}}_{\tilde{\sigma}(1)} \odot ... \odot z^{\tilde{\tau}}_{\tilde{\sigma}(i)}) \odot z^{\tilde{\tau}}_{\tilde{\sigma}(i+1)} \odot ... \odot z^{\tilde{\tau}}_{\tilde{\sigma}(d)}) \odot sx_{\tilde{\tau}(d+1)} \odot ... \odot sx_{\tilde{\tau}(n)}$$

$$= \sum_{\tilde{\tau} \in ush(d,n-d)} \left(\sum_{\substack{i+j=d+1 \\ \tilde{\sigma} \in ush(i,d-i)}} \epsilon(\tilde{\sigma}, z_1^\tau, ... z_d^\tau)\hat{l}_j(\hat{l}_i(z^{\tilde{\tau}}_{\tilde{\sigma}(1)} \odot ... \odot z^{\tilde{\tau}}_{\tilde{\sigma}(i)}) \odot z^{\tilde{\tau}}_{\tilde{\sigma}(i+1)} \odot ... \odot z^{\tilde{\tau}}_{\tilde{\sigma}(d)}) \right)$$
$$\odot \epsilon(\tilde{\tau}, sx_1, ... sx_n) sx_{\tilde{\tau}(d+1)} \odot ... \odot sx_{\tilde{\tau}(n)}.$$

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