2 Why another Multipole Family?

This chapter serves as a phenomenological motivation for introducing toroidal moments. With this, it lays the ground on which we will unfold the mathematical details in the following chapters. We start with a short overview regarding the research efforts concerning toroidal moments in the last decades. This past research already provides a considerable understanding of the toroidal moments, but also of the concept of multipole expansion in general. A discussion of several prototypical charge and current distributions that possess specific electromagnetic multipole moments is done in the following. Hereafter, we consider symmetry arguments and fundamental physical concepts to support the idea that there exist three multipole families, i.e. electric, magnetic, and toroidal multipoles. In the last section we discuss the distinction between the toroidal moment and the anapole.

2.1 A Brief History of Toroidal Moments

Toroidal moments have not been examined for a long time in the theory of electromagnetic multipole expansion. In 1957, Zel'dovich discussed parity violation of elementary particles and postulated that spin-1/2 Dirac particles must have an "anapole" [14]. In the late 1960s and early 1970s, Dubovik *et al.* [15] connected the quantum description of the anapole to classical electrodynamics by introducing the "polar toroidal multipole moments". The term "toroidal" stems from current distributions in the shape of a circular coil that were first shown to have a toroidal moment. "Polar" indicates that the polar toroidal moment transforms under parity as a polar vector. Dubovik *et al.* showed that the toroidal moments form, like the electric and magnetic multipoles, a family of multipole moments and managed to express the toroidal moments in the language of the classical multipole expansion [12]. Already at that time it was pointed out that for symmetry reasons a fourth multipole family potentially could exist. They were called "axial toroidal moments", because the dipole moment, which belongs to this family, transforms under parity like an axial vector. We will outline this later, but it is worth emphasizing that a distinction is made here between axial and polar toroidal moments. However, we do not encounter the axial toroidal moments due to the non-existence of magnetic charges and currents [16]. In 1986 [17], Dubovik *et al.* discussed the existence of such axial toroidal moments in continuous media and related it to phenomenons like phase-transitions and non-linear optical effects.

Toroidal moments were not acknowledged outside the Soviet Union as being an important part of the multipole expansion until the 1990s and remained neglected from a big part of the research community even in the 1990s and early 2000s. In 1988, Bladel [18] discussed several current

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distributions that required the concept of toroidal moments for their understanding. He pointed out that another term, besides the well-known electric and magnetic multipole moments, is necessary to properly describe an arbitrary current distribution. He was probably not aware of toroidal moments, since he neither referred to the papers of Dubovik *et al.* nor mentioned that this term belongs to another multipole family. In their 2005 published book [10], the authors Raab and Lange discuss the difference between fields originating from different definitions of the multipole moment tensors, but do not mention a connection to the third multipole family.

In the 1990s, Afanasiev *et al.* published a series of papers where they analyzed in detail the fields of toroidal moments [19] and discussed specific current distributions which produce time-dependent toroidal moments [20]. In 2000, Dubovik *et al.* [21] discussed the properties of continuous media with toroidal moments, introducing the so-called “toroidization”, an analogon to polarization and magnetization, which characterizes the effect of a toroidal multipole field.

However, despite this increasing theoretical understanding of toroidal moments, experimental evidence was scarce. The toroidal moments only came into consciousness of the research community with the advent of metamaterials. They are composed out of basic unit cells, called meta-atoms. For meta-atoms with a strong electric dipolar response, strong dispersion in the permittivity is observed. A meta-atom with a strong magnetic dipolar response causes a strong dispersion in the effective permeability. Extrapolating from these insights, it should be possible to identify meta-atoms that have a strong toroidal moment.

Basically with the possibility to design meta-atoms with high toroidal dipole radiation, it became necessary to take this contribution into account. In 2002 [22], Radescu and Vaman published a detailed analysis of radiation properties of arbitrary charge-current distributions including the toroidal moments. In the same year, Vrejoiu [23] proposed an algorithm which makes it possible to calculate the toroidal moments from the usual Cartesian multipole expansion.

In the early 2000s, Zheludev, Fedotov *et al.* began working on various properties of toroidal metamaterials. They carried out lots of simulations and analytic calculations, which improved the understanding of the properties of toroidal moments. In 2007, they calculated the anapole moment of nanocrystals [24]. Turning to possible applications of toroidal moments, they showed in 2009 that a torus-shaped structure provides optical activity [25]. In 2013 and 2014, they analyzed non-radiating structures [26] and transmission and reflection properties of thin layers of metamaterials consisting of toroidal dipoles [27]. Most recently, toroidal moments have been found in dielectric nanoparticles [7] and nanowires [8]. In 2010, Kaelberer *et al.* carried out an experiment measuring toroidal dipoles in a metamaterial slab consisting of split-ring resonators [28].

In the following we provide a condensed summary from the insights of this past research to understand on the grounds of basic considerations regarding charge-current distributions the origin of toroidal multipole moments.

2.2 Characterization of General Charge and Current Distributions

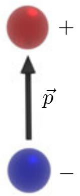
Every multipole moment is associated with a specific charge-current distribution. An arbitrary source distribution can be expressed through a sum of terms, called “multipole expansion”. Each term is a “multipole” and consists of a numeric weight factor, called “moment” or “multipole moment”, and a functional dependency of the coordinates. Moments are tensorial quantities and enumerated as exponentiation with the base two, thus, the n-th term in the sum is a 2^n-pole. [29]. Each of the multipoles has a different radiation pattern. It turns out that all radiation fields can be classified by using the symmetries of spatial inversion and time reversal. This will be analyzed in the next section.

To illustrate this concept of fundamental charge and current distributions, we discuss now the sources of the four dipoles, i.e. the electric, magnetic, polar toroidal and axial toroidal dipole. They are illustrated in Fig. 2.1. The simplest radiating configuration is given by two opposite and separated charges, Fig. 2.1a. This configuration generates the electric dipole moment $\vec{p}$. Inverting this configuration reverses the direction of the electric dipole moment. This means that the electric dipole moment is antisymmetric under parity transformation. Inverting the time does not change anything for the dipole, because by convention, the electric charge does not change sign under time inversion [9, p. 271].

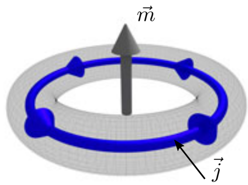
Another basic current configuration is the circular current, pictured in Fig. 2.1b. This current distribution is divergence-free, as all current lines are closed. Such a configuration causes a magnetic dipole moment $\vec{m}$ that is perpendicular to the plane in which the current flows. The direction of $\vec{m}$ depends on the direction in which the current flows. If the current direction (or equivalently the time) in Fig. 2.1b is reversed, the magnetic dipole moment would point downwards. This means that the magnetic dipole moment is antisymmetric under time inversion. Inverting the space changes nothing for the magnetic dipole moment, since the inversion of the spatial coordinates and of the current direction compensate each other.

The current configuration, which produces a polar toroidal dipole $\vec{t}$, is pictured in Fig. 2.1c. It consists of circular currents which are symmetrically arranged on the surface of a torus. Often, one finds also pictures which show a coil bent into a ring [6]. Like the circular current before, the divergence of this current distribution is zero. Inverting the directions of the current will invert the direction of the toroidal dipole. This can be achieved by inverting the time or the spatial coordinates. Thus, the polar toroidal dipole moment is antisymmetric under both space and time inversion.

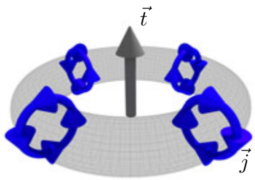
The charge configuration in Fig. 2.1d that shall illustrate the axial toroidal dipole moment $\vec{g}$ will be discussed in the next section.



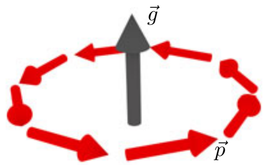
(a) Electric dipole moment $\vec{p}$ between a negative and a positive charge.



(b) Circular current $\vec{j}$ inducing a magnetic dipole moment $\vec{m}$.



(c) Circular currents $\vec{j}$ on a torus inducing polar toroidal dipole moment $\vec{t}$.



(d) Circular arranged electric dipoles $\vec{p}$ inducing axial toroidal dipole moment $\vec{g}$.

Figure 2.1: The four principle charge-current configurations generating dipoles. After [28], [27] and [16].

2.3 Necessity for Three Multipole Families

In this section, we want to motivate why there is a necessity for three multipole families. Historically, for decades only two families were considered, namely the electric moments and the magnetic moments. This seemed sufficient, as the electric multipole moments were attributed to static electric charges, whereas the magnetic multipole moments were explained through moving electric charges. If we want to discuss what is missing in this picture, it is most instructive to discuss the symmetry properties of the electric and magnetic multipole families first and then to generalize this.

We will now consider the two discrete symmetries of parity and time inversion on the base of equations. As pointed out in the last section, the electric dipole moment behaves under spatial inversion like a polar vector, thus it holds [9, p. 271]

$$\mathcal{P}\vec{p} = -\vec{p}\,.$$

Here we used the parity operator $\mathcal{P}$, which is defined as such that it inverts spatial coordinates,

$$\mathcal{P}\vec{r} := -\vec{r}\,. \tag{2.1}$$

The magnetic dipole moment behaves under spatial inversion like a pseudovector, meaning that it remains invariant,

$$\mathcal{P}\vec{m} = \vec{m}\,.$$

Now we turn to the time inversion. The time reversal operator $\mathcal{T}$ is defined through

$$\mathcal{T}t := -t\,, \tag{2.2}$$

meaning that it changes the sign of the time variables. The electric dipole is the result of separated charges, and it holds [9, p. 271]

$$\mathcal{T}\vec{p} = \vec{p}\,.$$

Electric currents, as time derivatives of charges, change sign under time inversion [9, p. 271]. Hence, for the magnetic dipole moment caused by currents, it holds

$$\mathcal{T}\vec{m} = -\vec{m}\,,$$

as it was made obvious in the last section.

Summarizing, $\vec{p}$ is even under time reversal, but odd under spatial inversion. $\vec{m}$ is odd under time reversal, but even under spatial inversion. The limitation to electric and magnetic moments thus misses a dipole moment which is odd under both time and spatial inversion, and another dipole moment, which is even under both transformations. We will illustrate this observation in the following table [16]:

Table 2.1: Behavior of the four dipoles under space and time inversion.

$\mathcal{T}$ \ $\mathcal{P}$	$+$	$-$
$+$	$\vec{g}$	$\vec{p}$
$-$	$\vec{m}$	$\vec{t}$

The "+" indicates even and the "−" odd behavior under the corresponding transformation. We introduced the dipole moments $\vec{g}$ and $\vec{t}$; we will call $\vec{g}$ the "axial toroidal dipole moment", and its multipole family "axial toroidal moments". The $\vec{t}$ is the "(polar) toroidal dipole moment", but in this thesis, we will from now on always refer to it only as "toroidal dipole moment". It is the lowest order of the (polar) toroidal multipole moment family. Toroidal moments interact only with time-dependent external electromagnetic fields [6], whereas electric and magnetic moments interact also with static external fields.

Following this table, one would expect that four multipole families are present in electrodynamic multipole expansions. An example of a charge distribution [16] generating an axial toroidal dipole moment is pictured in Fig. 2.1d. However, it has been argued [16, 17] that such a charge distribution is not stable in Maxwell-Lorentz electrodynamics. Therefore, axial toroidal multipole moments usually are considered as not possible in microscopic charge-current systems. For continuous media, the situation is different. There, axial toroidal moments can be generated e.g. by spin currents [30]. Also, such moments are suggested to appear in microscopic crystals with a specfic lattice structure [24]. Since we are discussing metamaterials rather than continuous media in this thesis, we will only consider (polar) toroidal moments in the following.

The previous reasoning was made for dipole moments only. One has to be careful when going to higher orders than the dipolar order. For example, the relation when applying the parity transformation to an arbitrary n-th order electric multipole moment tensor $\hat{P}^{(n)}$ is

$$\mathcal{P}\hat{P}^{(n)} = (-1)^n \hat{P}^{(n)} \,.$$

For arbitrary n-th poles, we have the following transformation rules [31, p. 227 and p. 257]:

Table 2.2: Behavior of the four general n-th pole moments under space and time inversion.

$\mathcal{T}$ \ $\mathcal{P}$	$(-1)^{n+1}$	$(-1)^n$
$+1$	$\hat{G}^{(n)}$	$\hat{P}^{(n)}$
-1	$\hat{M}^{(n)}$	$\hat{T}^{(n)}$

From the table it follows that n-th poles of the same order of electric and toroidal kind give the same fields that are not distinguishable with spatial manipulations of the experiment, since they share the same parity. But with respect to time it should be in principle possible to decide whether a charge-current distribution is characterized by an electric or toroidal n-th pole moment. However, no experiment has been carried out so far which provides such distinction.

2.4 Distinction between Toroidal Moment and Anapole

In literature, there is sometimes a confusion between the toroidal moment and the anapole [32]. The anapole is important in the theory of weak interaction where it was first postulated [14]. However, since we are interested in metamaterials rather than in particle physics, we focus here on the anapole description in classical electrodynamics. An anapole is in this context a charge-current distribution that neither radiates nor interacts with external fields [6].

An anapole can be realized in classical electrodynamics by a suitable combination of an electric and a toroidal dipole. Compared to the field of an electric dipole moment, the field of a toroidal dipole moment is scaled with the wave number k and has a phase-shift of $\pi/2$ relative to the field of the electric dipole [26]. Thus, by designing a charge-current configuration where the electric and toroidal dipoles point in the same direction and where the toroidal dipole exceeds the electric by a factor of k, the two fields annihilate each other exactly. In this configuration, however, the vector potential is non-zero, enabling Aharonov-Bohm like effects [19].

Because it does not radiate, the anapole is not a “moment” like the other multipole moments. This is the origin of the name “anapole” (from Greek ’ana’, ’without’, thus meaning ’without poles’). Nevertheless, the anapole is very often improperly denoted as “anapole moment” (e.g. in [32]). Because ideal non-radiating charge-current configurations are not possible [4], describing a structure as “non-radiating” refers only to the first orders of multipole moments. This is the reason why the anapole is usually only related with the toroidal dipole, not with higher toroidal moments.

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