2. NURBS Curves

This chapter provides an overview of the basic properties of NURBS curves and possible applications in the field of optimization in robotics. NURBS is an acronym that stands for *Non-Uniform Rational B-splines*, a class of parameterized geometric curves. For the sake of brevity, proofs are omitted in this chapter and the reader is referred to special literature such as [10].

2.1 Basics

A (in general multi-dimensional) NURBS curve $\mathbf{p}(t)$ of maximum degree n can be represented as a function of the parameter t,

$$\mathbf{p}(t) = \sum_{j=0}^{m-n-2} \mathbf{d}_j R_j^d(t), \quad t \in [a, b], \tag{2.1}$$

defined on m monotonically increasing knots t_k with

$$a = t_0 \leq t_1 \leq ... \leq t_{m-2} \leq t_{m-1} = b.$$

In general, see [7], the knots are chosen such that

$$a = t_0 = ... = t_d \leq t_{d+1} \leq ... \leq t_{m-d-2} \leq t_{m-d-1} = ... = t_{m-1} = b. \tag{2.2}$$

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The curve is called *uniform* if the interval sizes between knots (with the exception of any intervals with size zero at a or b; sometimes also referred to as *open-uniform*) are equal, otherwise it is called *non-uniform*. In (2.1) on the previous page $\mathbf{d}_j$ denotes the control points of the curve that form a polygon, the control polygon, in whose convex hull the curve lies, see the convex hull property in Section 2.1.1. $R_j^d(t)$ is a polynomial piecewise B-spline basis function of degree d that is defined by the rational expression

$$R_j^d(t) = \frac{w_j N_j^d(t)}{\sum\limits_{i=0}^{m-n-2} w_i N_i^d(t)}. \tag{2.3}$$

$R_j^d(t)$ allows to change the influence of each control point $\mathbf{d}_i$ using weight factors w_i and $N_i^d(t)$ are polynomial functions that are non-zero on the interval $[t_i, t_{i+d+1})$, see Section 2.1.1. For the computation of $N_j^d(t)$ see Section 2.1.2.

Rewriting the sum in (2.1) on the previous page, a matrix representation can be found which is advantageous especially for computational treatment, i.e.

$$\mathbf{p}(t) = \mathbf{D}\mathbf{R}^d(t)\,\mathbf{t} \tag{2.4}$$

where $\mathbf{D}$ is a matrix representation of the control points $\mathbf{d}_j$, i.e.

$$\mathbf{D} = \begin{pmatrix} \mathbf{d}_0 & \mathbf{d}_1 & \cdots & \mathbf{d}_{m-n-2} \end{pmatrix}.$$

A spline function $R_j^d(t)$ from (2.1) is a polynomial function of degree d that can be represented by its coefficient row vector $\left(\mathbf{r}_j^d(t)\right)^\top$ that is a piecewise constant function of the curve parameter t, and a vector $\mathbf{t}$ of powers of t, i.e.

$$R_j^d = \left(\mathbf{r}_j^d(t)\right)^\top \mathbf{t}, \tag{2.5}$$

where $\mathbf{t} = \begin{pmatrix} t^n & t^{n-1} & \cdots & t & 1 \end{pmatrix}^\top$. Using compact matrix notation in (2.4), the row vectors $\left(\mathbf{r}_j^d\right)^\top$ from (2.5) can be represented as a curve parameter-dependent matrix form, i.e.

$$\mathbf{R}^d(t) = \begin{pmatrix} \left(\mathbf{r}_0^d(t)\right)^\top \\ \left(\mathbf{r}_1^d(t)\right)^\top \\ \vdots \\ \left(\mathbf{r}_{m-n-3}^d(t)\right)^\top \\ \left(\mathbf{r}_{m-n-2}^d(t)\right)^\top \end{pmatrix}.$$

The matrix form of $\mathbf{p}(t)$ from (2.4) on the previous page additionally offers a simple way to compute the derivative of the curve with respect to the curve parameter, denoted as $\mathbf{p}'(t)$ since only the power vector $\mathbf{t}$ needs to be differentiated, i.e.

$$\mathbf{p}'(t) = \mathbf{D}\mathbf{R}^d(t)\,\mathbf{t}'$$

where

$$\mathbf{t}' = \operatorname{diag}(n, n-1, \dots, 2, 1, 0) \begin{pmatrix} t^{n-1} \\ t^{n-2} \\ \vdots \\ t \\ 1 \\ 0 \end{pmatrix}.$$

The computation of the time-dependent column vector in $\mathbf{t}'$ can be accomplished by shifting up $\mathbf{t}$ and filling the resulting gap with zero.

2.1.1 Properties of NURBS curves

In this section, a selection of properties of NURBS curves that are important for the application in this thesis are presented. The listing below is not exhaustive, a complete version is provided in [10].

- Partition of unity: $\sum_{j=0}^{m-n-2} R_j^d(t) = 1 \quad \forall\, t \in [a, b]$
- Local support: $R_j^d(t) = 0 \quad \text{for} \quad t \notin \left[t_j, t_{j+d+1}\right)$

- Knot multiplicity and continuity: At a knot t_l with multiplicity k_l, $R_j^d(t = t_l)$ is $d - k_l$ times continuously differentiable, i.e. $R_j^d(t = t_l) \in \mathcal{C}^{d-k_l}$.

- Convex hull: $\mathbf{p}(t)$ lies within the control polygon, i.e. the convex hull of the control points $\mathbf{d}_{j-d}, \dots, \mathbf{d}_j$ for $t \in [t_j, t_{j+1})$.

  In general, a convex set of two points $\mathbf{x}_i$ and $\mathbf{x}_j$ is their connecting line, i.e.

$$\operatorname{conv}(\mathbf{x}_i, \mathbf{x}_j) = \{\mathbf{x}_i + \lambda(\mathbf{x}_j - \mathbf{x}_i)\}, \quad 0 \le \lambda \le 1.$$

  In the two-dimensional example in Figure 2.1, the set $\mathcal{A}$ is a convex set of all points $\mathbf{x}_i$. In Figure 2.2 this is not the case, $\mathcal{B}$ is only the convex set of points $\mathbf{x}_1$ and $\mathbf{x}_3$, and $\mathbf{x}_2$ and $\mathbf{x}_5$ but not for any other combination of the depicted points. The convex hull of a set of points is the convex set of minimum size. Examples for the convex hull property can be found in Figure 2.5 and Figure 2.6 on page 13.

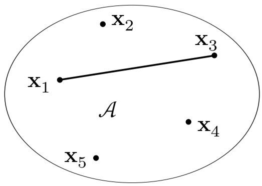

Figure 2.1: Example for a convex set

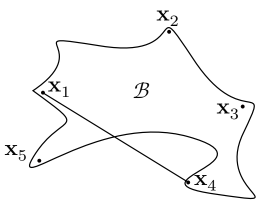

Figure 2.2: Example for a non-convex set

- Local approximation: If a control point $\mathbf{d}_j$ is moved, only the portion of $\mathbf{p}(t)$ on the interval $t \in [t_j, t_{j+d+1})$ is affected. An example for the local approximation property can be found in Figure 2.6 on page 13.

2.1.2 Recursion formula

A computation method for NURBS that is particularly interesting for computer implementation is the recursion formula by DE BOOR, COX and MANSFIELD.

The simple case of this recursion is called a base function, which in this case is a constant that is either one or zero, i.e.

$$N_j^0 = \begin{cases} 1 & t_j \leq t < t_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

where $j = 0, \ldots, m-2$. The general case that reduces the functions N_j^d towards the simple case $d = 0$ is

$$N_j^d(t) = \frac{t - t_j}{t_{j+d} - t_j} N_j^{d-1}(t) + \frac{t_{j+d+1} - t}{t_{j+d+1} - t_{j+1}} N_{j+1}^{d-1}(t)$$

where d is the local degree of the polynomial functions, n is the maximum degree of $N_j^d(t)$ and m is the number of knots. For knot multiplicities greater than zero, fractions $\frac{0}{0}$ may occur, which are defined to be zero, i.e. $\frac{0}{0} := 0$.

Following (2.3) on page 6, the control points $\mathbf{d}_j$ can be weighted using factors w_j. From the unity property from Section 2.1.1 follows that $R_j^d = N_j^d$ if all weights are equal, i.e. $w_i = w_j$, $\forall i, j$. If all weighting factors are one, the curve is called a B-spline curve.

Example

In this example, the quadratic ($n = 2$) basis functions N_j^d and the rational functions R_j^d are to be computed and the NURBS curve $\mathbf{p}(t)$ is to be visualized for the knots in the knot vector $\mathbf{T}$. This example is based on Exercise 4.1 from [10].

$$\begin{aligned} \mathbf{T} &= \begin{pmatrix} t_0 & t_1 & \ldots & t_6 & t_7 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 1 & 1 & 1 \end{pmatrix} \end{aligned}$$

$$\mathbf{d}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad w_0 = 1$$

$$\mathbf{d}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad w_1 = 4 \qquad \mathbf{d}_3 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad w_3 = 1$$

$$\mathbf{d}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad w_2 = 1 \qquad \mathbf{d}_4 = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \quad w_4 = 1$$

Following the de Boor, Cox and Mansfield algorithm presented in Section 2.1.2, the basis functions are found to be a scheme as depicted in Figure 2.3 on the next page where $\forall\, j \in \{0, 1, 2, 3, 4\}$

$$N_j^0(t) = \begin{cases} 1 & t_j \le t < t_{j+1} \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

$$N_{j+1}^0(t) = \begin{cases} 1 & t_{j+1} \le t < t_{j+2} \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

$$N_{j+2}^0(t) = \begin{cases} 1 & t_{j+2} \le t < t_{j+3} \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

$$N_j^1(t) = \frac{t - t_j}{t_{j+1} - t_j} N_j^0(t) + \frac{t_{j+2} - t}{t_{j+2} - t_{j+1}} N_{j+1}^0(t) \tag{2.9}$$

$$N_{j+1}^1(t) = \frac{t - t_{j+1}}{t_{j+2} - t_{j+1}} N_{j+1}^0(t) + \frac{t_{j+3} - t}{t_{j+3} - t_{j+2}} N_{j+2}^0(t) \tag{2.10}$$

$$N_j^2(t) = \frac{t - t_j}{t_{j+2} - t_j} N_j^1(t) + \frac{t_{j+3} - t}{t_{j+3} - t_{j+1}} N_{j+1}^1(t)\,. \tag{2.11}$$

Substituting (2.6) to (2.8) on the previous page in (2.9) and (2.10) yields

$$N_j^1(t) = \begin{cases} \frac{t-t_j}{t_{j+1}-t_j} & t_j \leq t < t_{j+1} \\ \frac{t_{j+2}-t}{t_{j+2}-t_{j+1}} & t_{j+1} \leq t < t_{j+2} \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

$$N_{j+1}^1(t) = \begin{cases} \frac{t-t_{j+1}}{t_{j+2}-t_{j+1}} & t_{j+1} \leq t < t_{j+2} \\ \frac{t_{j+3}-t}{t_{j+3}-t_{j+2}} & t_{j+2} \leq t < t_{j+3} \\ 0 & \text{otherwise} \end{cases} . \tag{2.13}$$

$$\begin{tikzcd}[column sep=small] & & N_j^2 \arrow[dl, dash] \arrow[dr, dash] & & \\ & N_j^1 \arrow[dl, dash] \arrow[dr, dash] & & N_{j+1}^1 \arrow[dl, dash] \arrow[dr, dash] & \\ N_j^0 & & N_{j+1}^0 & & N_{j+2}^0 \end{tikzcd}$$

Figure 2.3: NURBS example — Scheme of B-spline basis functions N_j^d

Substituting (2.12) and (2.13) in (2.11) yields

$$N_j^2(t) = \begin{cases} \frac{t-t_j}{t_{j+2}-t_j}\frac{t-t_j}{t_{j+1}-t_j} & t_j \leq t < t_{j+1} \\ \frac{t-t_j}{t_{j+2}-t_j}\frac{t_{j+2}-t}{t_{j+2}-t_{j+1}} + \frac{t_{j+3}-t}{t_{j+3}-t_{j+1}}\frac{t-t_{j+1}}{t_{j+2}-t_{j+1}} & t_{j+1} \leq t < t_{j+2} \\ \frac{t_{j+3}-t}{t_{j+3}-t_{j+1}}\frac{t_{j+3}-t}{t_{j+3}-t_{j+2}} & t_{j+2} \leq t < t_{j+3} \\ 0 & \text{otherwise} \end{cases} \quad \forall\, j = 0 \ldots 4.$$

Figure 2.4 on the next page shows the B-spline basis functions N_j^d for $j = 0 \ldots m-n-2 = 4$ and $d = 0 \ldots n = 2$.

The weighted functions R_j^d can be easily computed using (2.3) on page 6. Following (2.1), the NURBS function $\mathbf{p}(t)$ can now be obtained and visualized, see Figure 2.5.

To show the local approximation property from Section 2.1.1, one of the control points

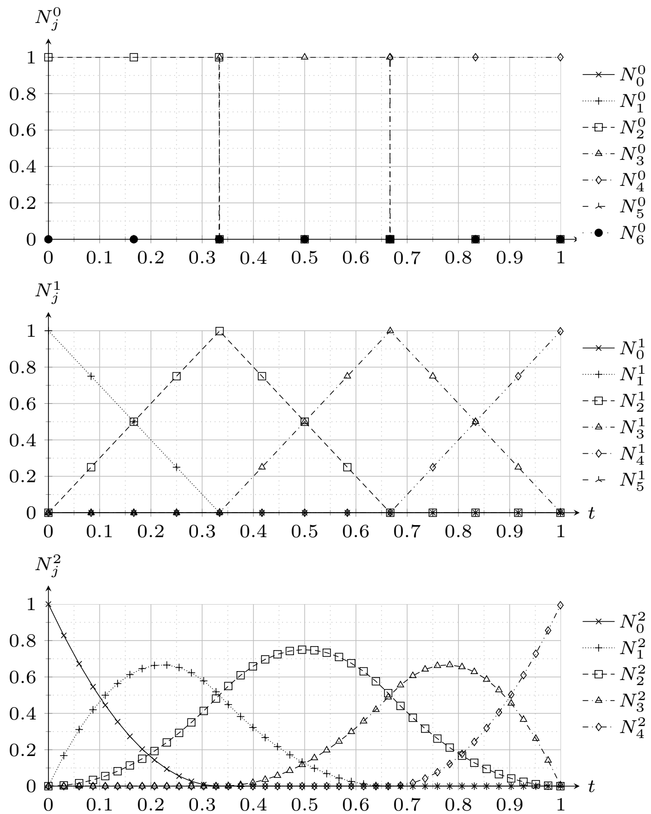

Figure 2.4: NURBS example — B-spline basis function N_j^d

from the example is modified, i.e.

$$\mathbf{d}_3 = \begin{pmatrix} 4.5 \\ 1.5 \end{pmatrix}.$$

The result is depicted in Figure 2.6 on the next page where the modified curve $\mathbf{p}_{\text{mod}}(t)$ only differs from the original curve $\mathbf{p}(t)$ in the area between the adjoining control points $\mathbf{d}_2$ and $\mathbf{d}_4$.

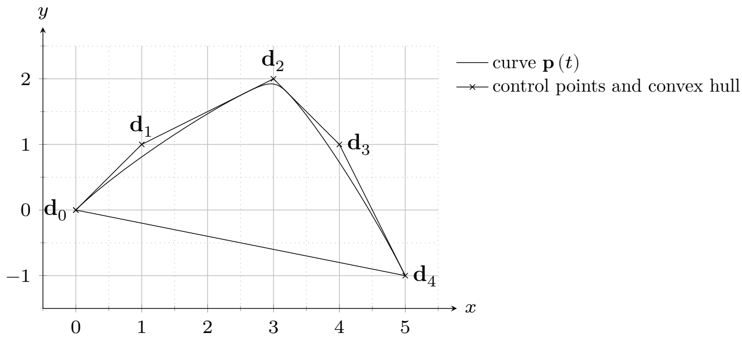

Figure 2.5: NURBS example — curve

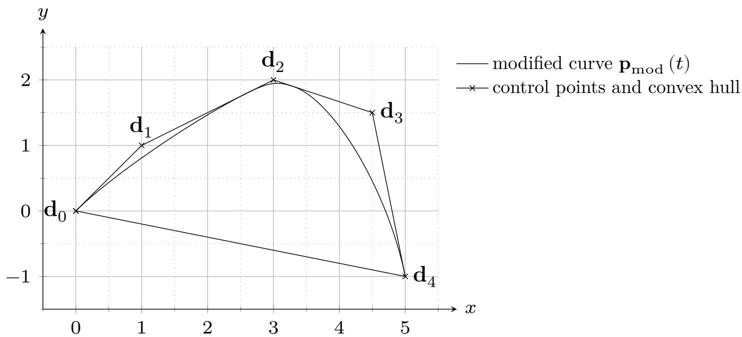

Figure 2.6: NURBS example — modified curve

2.1.3 Curve approximation

For the approximation of a series of $N \geq m - n - 1$ points $\mathbf{p}_i$ at parameters $\tau_i \in [a, b]$ with weights w_k where $k \in \{0, 1, \dots, m - n - 3, m - n - 2\}$ a NURBS curve $\tilde{\mathbf{p}}(t)$ of degree d with m knots t_k where $k \in \{0, 1, \dots, m - 2, m - 1\}$ is to be found such that $\tilde{\mathbf{p}}(t = \tau_i) \approx \mathbf{p}_i$, see [7].

The goal is to minimize the sum of error squares of the approximation $\tilde{\mathbf{p}}$ with respect to the given points $\mathbf{p}_i$ at the given parameters τ_i, i.e.

$$\sum_{i=0}^{N-1} |\mathbf{p}_i - \tilde{\mathbf{p}}(t = \tau_i)|^2 \quad \rightarrow \quad \min, \quad \tau_i \in [a, b], \quad i \in \{0, 1, \dots, N - 2, N - 1\}.$$

Substituting the sum representation from (2.1) on page 5 for $\tilde{\mathbf{p}}$ yields

$$\sum_{i=0}^{N-1} \left| \mathbf{p}_i - \sum_{j=0}^{m-n-2} \mathbf{d}_j R_j^d (\tau_i) \right|^2 \quad \rightarrow \quad \min, \quad \tau_i \in [a, b], \quad i \in \{0, 1, \dots, N-2, N-1\}.$$

Minimization yields a system of linear equations for the control points $\mathbf{d}_j$ of the approximated curve points $\tilde{\mathbf{p}}$,

$$\sum_{j=0}^{m-n-2} \sum_{i=0}^{N-1} R_l^d (\tau_i) R_j^d (\tau_i) \mathbf{d}_j = \sum_{i=0}^{N-1} R_l^d (\tau_i) \mathbf{p}_i$$

$$\begin{pmatrix} \sum_{i=0}^{N-1} R_0^d R_0^d & \sum_{i=0}^{N-1} R_0^d R_1^d & \cdots & \sum_{i=0}^{N-1} R_0^d R_{m-n-2}^d \\ \sum_{i=0}^{N-1} R_1^d R_j^d & \sum_{i=0}^{N-1} R_1^d R_j^d & \cdots & \sum_{i=0}^{N-1} R_1^d R_{m-n-2}^d \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{N-1} R_{m-n-2}^d R_0^d & \sum_{i=0}^{N-1} R_{m-n-2}^d R_1^d & \cdots & \sum_{i=0}^{N-1} R_{m-n-2}^d R_{m-n-2}^d \end{pmatrix} \begin{pmatrix} \mathbf{d}_0 \\ \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_{m-n-2} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{N-1} R_0^d \mathbf{p}_i \\ \sum_{i=0}^{N-1} R_1^d \mathbf{p}_i \\ \vdots \\ \sum_{i=0}^{N-1} R_{m-n-2}^d \mathbf{p}_i \end{pmatrix}$$

where $l \in \{0, 1, \dots, m-n-2\}$ and the argument τ_i of R was suppressed for the sake of compactness. The resulting band matrix can be efficiently solved for each of the components of $\mathbf{d}_j$. If any of the control points $\mathbf{d}_j$ are known, the order of the system of equation is reduced.

2.2 Application in optimization

A NURBS curve is defined by its maximum degree n, its range $t \in [a, b]$ with m knots t_k, its control points $\mathbf{d}_j$ and its weights w_j. The maximum degree and the range are mostly determined by the requirements of the application itself such as the continuity level. The number of knots and the knot positions, control points and the weights are possible parameters for optimization. The number and positions of knots can be adjusted in order to allow better local adaptivity while the control points and their weights influence the general shape of the curve. Due to the local approximation property from Section 2.1.1, local shape adjustments are possible. For some applications, such as the use as a time-dependent polynomial in the equations of motion of a manipulator, derivatives of the polynomial are necessary to compute. The matrix representation of a NURBS curve derived in (2.4) on page 6 allows to obtain those derivatives in a computationally cheap manner.

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