

2 Chapter 2: Theoretical Basis

This chapter provides a theoretical basis of the study. It has two sections. I first review the relevant literature and then I provide a theoretical framework for the study.

2.1 Review of Relevant Literature

In this literature review, the first section describes the van Hiele model and then summarizes studies guided by the model in investigating the learning of geometry. In addition, this section also summarizes research that addresses the knowledge of mathematics for teaching geometry. The second section describes Sfard's discursive framework and her communicational approach to the issue of learning and nature of mathematical thinking. Included in this section are summaries of discourse studies in the mathematics classrooms and a translation of geometric discourse that aligns the van Hiele model through the discursive lens. The final section raises general research questions in discursive terms.

For the teaching and learning of geometry, the van Hieles developed this influential model to distinguish levels of geometric thinking. In discussing the profound impact of the van Hiele model in mathematics education, Clements (2003) concluded that van Hiele "gave educators and researchers a model that promoted the understanding of important, conceptual based level of thinking... It is also a model of synergistic connections among theory, research, the practice of teaching, and students' thinking and learning" (p.151). To better describe the van Hiele model and how it has been used in the field of mathematics education, the following section provides the historical background and a general description of the theory.

2.1.1 *The van Hiele Model*

The root of the van Hiele model emanated from the task of improving the teaching of geometry. A Dutch husband and wife, Pierre Marie van Hiele and Dina van Hiele-Geldof, developed “the van Hiele model” in their doctoral dissertations at the University of Utrecht, Netherlands, in 1957. Dina died shortly after completing her dissertation, and Pierre continued to develop and disseminate the theory (van Hiele 1959/1985, 1986).

When Pierre and Dina worked at Montessori secondary schools as mathematics teachers, they were very disappointed with “students’ low-level knowledge of geometry” (van Hiele, 1959/1985, p.60). On the other hand, they also realized that teachers and students often fail to communicate with each other because they “speak a very different language” (p.61). For example, one of Pierre and Dina’s initial observations was that they seemed to speak about geometry in a different way than their students. When Pierre and Dina spoke about a square as a type of rectangle, students were confused because to them a square and a rectangle were quite different. This led Pierre and Dina to consider the existence of various levels of geometric thinking and the possibility that those students and teachers at different levels of thinking may have difficulty communicating with one another. Although Pierre and Dina developed the theory together, their views were quite different. As a result, Pierre’s dissertation focused on identifying students’ levels of thinking in learning geometry, while Dina’s dissertation was more about a teaching experiment designed to investigate how students move from level to level.

The van Hiele model includes five distinct levels that describe students’ thought levels in geometry. However, Pierre van Hiele suggested that mathematics educators should focus on the first four van Hiele levels, because those are what teachers have to deal with in school most of the time (van Hiele, 1986). As Pierre van Hiele (1959/1985) described in “the Children’s thought and geometry,” the five van Hiele levels are as follows (p.62-63): Base Level, figures are judged by their appearance; First Level, figures are bearers of their properties, and they are recognized by their properties but not yet ordered; Second Level, properties are ordered and they are deduced one from another; at

this level, definitions of figure come into play but students do not understand the meaning of deduction; Third Level, thinking is concerned with the meaning of deduction, with the converse of a theorem, with axioms, with necessary and sufficient conditions; Fourth Level, thinking is concerned with a variety of axiomatic systems that are non-Euclidean. Geometry is seen in the abstract.

As described in the levels, students' levels of thinking attached to the learning of a particular geometric topic are inductive in nature. At level n , the objects studied are now the statements that were explicitly made at the previous level (level $n-1$) as well as explicit statements that were only implicit at level $n-1$. At level $n-1$, certain geometric objects are studied. Students are able to state some of the relationships explicitly about the objects. Therefore, the objects at level n consist of extensions of the objects at level $n-1$. One major purpose of distinguishing the levels is to recognize obstacles that are presented to students. For example, when a student who is thinking at level $n-1$ confronts a problem that requires vocabulary, concepts or thinking at level n , the student is unable to make progress on the problem, with expected consequences such as frustration, anxiety and even anger.

The van Hiele levels have several important properties. First, the levels are discrete and sequential. *Discrete* indicates that the levels are qualitatively different from one another. *Sequential* implies that students pass through the levels in the same order, although varying at different rates, and it is not possible to skip levels. Second, that which was intrinsic at one level becomes extrinsic at the next level. For example, students operating at Level 1 are able to name geometric figures only by their appearance as a "whole"—the properties of a figure remain intrinsic. However, at Level 2, these properties become extrinsic and in fact are the new objects of study. Third, each level has its own language and symbols. The van Hieles believed that "in general, the teacher and the student speak a very different language" (van Hiele, 1986, p. 62). Therefore, teachers and students often have difficulty communicating with one another about geometric concepts. This linguistic challenge can also extend to communicational difficulties between students in a classroom when they are functioning at different thought levels. Finally,

instructional methods have a greater influence than either age or grade on a student's progress through the van Hiele levels. That is, a teacher's instructional activities can either foster or impede movement through the levels.

When assigning students to different van Hiele levels, Pierre van Hiele cautioned that it is possible to misjudge a student's level of thinking without careful analysis, because often students memorize or learn patterns in order to accomplish tasks, but do not really understand the underlying concepts. An example is when students recognize corresponding angles by finding the 'F' that is formed by parallel lines and the transversal (see Figure 2.1)

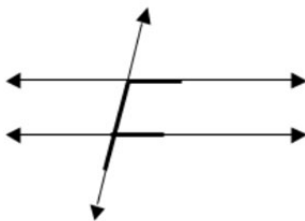


Figure 2.1. Corresponding angles of parallel lines intersected by a transversal.

This technique simplifies the relation between angles and lines. Pierre van Hiele claimed that it could be harmful to students if they only seek a quick result and avoid the “crisis of thinking.” In saying “crisis of thinking,” he meant the difficulties that students have to transition from one level to a higher level. It is possible for students to derive the answer without recognizing the relationships between the angles in the figure (e.g., supplementary angles, angles at a point, interior angles). Pierre van Hiele warned that these types of “tricks” might actually prevent students from moving to the subsequent level of reasoning (van Hiele, 1986, p.42).

The van Hiele model recognizes the importance of language, which plays a significant role in communicating geometric thinking. According to Pierre van Hiele, students' levels of thinking are important not in the sense of the way of their thinking, but in the results of thinking that are revealed in students' speaking and writing. For example, the meaning of

a statement like, “This figure is a rhombus” depends on how one argues about it. For a student who is at Basic level, her/his reasoning could be, “This figure has a shape that looks like what I learned to call ‘rhombus’.” In contrast, if another student has already obtained the first van Hiele level or higher, her/his reasoning could be quite different. The figure that the student refers to is a collection of properties and those properties he/she has learned to call “rhombus” (van Hiele, 1986, p.109). By making the same statement, “This figure is a rhombus,” two students could use different reasoning as a result two different levels of thinking. This example of students’ responses to a rhombus illustrates how the same statement provided by students can have different meanings because their geometric thinking varies among levels.

2.1.2 Research Guided by van Hiele Model

The van Hiele model has been influential and extensively studied. In this section, the review of the existing literatures focuses on how the model has been used in research in the years since it was developed. Following its incorporation into a new Soviet geometry curriculum, the model was introduced to the United States by the Russian mathematician Izaak Wirszup in a lecture titled “Some Breakthroughs in the Psychology of Learning and Teaching Geometry” at the Closing General Sessions of the National Council of Teachers of Mathematics in 1974 (Wirszup, 1976; van Hiele, 1959/1985; 1986). After the van Hiele levels were translated into English, they were widely used by many researchers in the United States. During the period of 1980-83, the National Science Foundation funded three major investigations using the van Hiele model in the United States: one directed by Burger and Shaughnessy at Oregon State University (1986), another by Fuys, Geddes, and Tischler at Brooklyn College (1988), and a third by Usiskin at the University of Chicago (1982). Burger and Shaughnessy set up a study using clinical interviews to determine the usefulness of van Hiele levels for describing geometric thinking in elementary, middle, and high school grades. Fuys and his colleagues (1988) focused their investigation on geometric thinking in adolescents using instructional models. Usiskin’s (1982) project used a large-scale survey to test

whether the van Hiele theory applied to the geometric reasoning of students enrolled in secondary geometry courses. These three intensive studies have been widely read, discussed, and cited. After these studies, dozens of other studies using the work of the van Hieles have been conducted in the United States. For instance, Mayberry (1983) used the model to investigate pre-service teachers' geometric thinking, and Senk (1983; 1989) used it to assess high school students' reasoning and proof in geometry.

There is also scarcity of research related to the van Hiele model of reasoning to assess students' geometric thinking. For example, Hoffer (1981) considers the model is integrated with "five skills in geometry" (Hoffer, 1981), whereas Micheal de Villiers (1987) identifies the model as "six geometric thought categories." Gutierrez and Jaime (1998) suggest different processes of reasoning including *recognition*, *definition*, *classification* and *proof* as characteristic of several van Hiele levels. Battista (2009) characterizes the van Hiele model as five levels of reasoning: visual-holistic reasoning (Level 1), descriptive-analytic reasoning (Level 2), relational-inferential reasoning (Level 3), formal deductive proof (Level 4), and rigor (Level 5) (pp. 92-94). These studies led to the conclusion that one cannot consider a level of reasoning as a singular process that is attained (or not) by students, but must be considered as a set of processes.

In the earlier writing of the van Hieles, the van Hiele levels of geometric thinking mainly refer to the classification of quadrilaterals (van Hiele, 1959/1985). At that time, levels were descriptors and they were not labeled by single words (e.g., "Visual" for Level 1, etc.). Almost thirty years later, however, van Hiele (1986) referred to the five levels of thinking as visual, descriptive, theoretical, formal logic, and rigor, and considered such classification to be suitable to a structure of mathematics (p.53).

Over the years, researchers not only used the levels to study students' levels of geometric thinking, but also expanded the area of research from classification of the quadrilaterals to the classification of similar figures, to reasoning and proof, to spatial geometry in three-dimension measurement, etc. In these studies, researchers have

proposed various descriptive labels for the van Hiele levels. As the first to name the van Hiele levels, Hoffer (1981) provided his descriptors, “levels of mental development in geometry” (p.13), which label Levels 1 through 5 as recognition, analysis, ordering, deduction, and rigor (p.13-14). Besides these five levels, Hoffer also suggested five basic skills in geometry that are expected at each level. These five skills are visual skills, verbal skills, drawing skills, logical skills, and applied skills. For instance, at Level 1 (recognition), the visual skills only focus on recognizing different figures from a picture, or on recognizing information labeled on a figure. At Level 2 (analysis), visual skills are developed to notice properties of a figure as well as to identify a figure as a part of a larger figure. At Level 3 (ordering), visual skills help to recognize interrelationships between different types of figures and common properties of different types of figures. At Level 4 (deduction), visual skills focus on using information about a figure to deduce more information. Finally, at Level 5 (rigor), visual skills are used to recognize unjustified assumptions made by using figures (p.15). Hoffer’s framework suggested that in developing geometric reasoning various geometric skills might be expected at different van Hiele levels of thinking.

Other “level indicators” suggested by Burger and Shaughnessy (1986), describe the five levels as visualization, analysis, informal deduction, formal deduction, and rigor, for Levels 0 through 4, respectively. Using Burger and Shaughnessy’s level indicators, Crowley (1987) provided additional examples of level-specific responses (except for Level 5) concerning how students would argue a given shape is a rectangle (p. 15).

- Level 1 “It looks like one.” Or “Because it looks like a door.”
- Level 2 “Four sides, closed, two long sides, two shorter sides, opposite sides parallel, four right angles ...”
- Level 3 “It is a parallelogram with right angles.”
- Level 4 “This can be proved if I know this figure is a parallelogram and that one angle is a right angle.”

Each response assigns to a level. The student at Level 1 thinking gives answers based on a visual model and is identifying the rectangle by its overall appearance. At Level 2 thinking, the student is aware that the rectangle has properties; however, redundancies (i.e., properties that can be derived from other properties) are not noticed. A student's thinking operating at Level 3 attempts to give a minimum number of properties (i.e., a definition), and finally, at Level 4 thinking, a student seeks to prove the fact deductively.

Micheal de Villiers (1987) identifies "six geometric thought categories" including recognition and represent of figure-types (Level 1), use and understanding of terminology (Level 2), verbal description of properties of figure-types (Level 2), hierarchical classification (Level 3), one step deduction (Level 3) and longer deduction (Level 4). Taking an intermediate position, Gutierrez and Jaime (1998) identify each process (recognition, definition, classification and proof) as a component of two or more van Hiele levels of reasoning. They argue that how a student considers and uses the processes is an indicator of the student's level of reasoning at each van Hiele level. For instance, at Level 1, recognition is limited to physical, global attributes of the figures. Students sometimes use geometric vocabulary, but these terms have a visual meaning more than a mathematical meaning. When students describe a rectangle, some of them use the term "perpendicular" for a side when they mean "vertical." In other cases, they are able to notice some mathematical properties of figures, but they are simple properties, such as the number of sides. However, Gutierrez and Jaime also note that the ability of recognition does not discriminate among students in the van Hiele levels 2, 3, or 4.

More recently, Battista (2009) elaborates and refines the van Hiele levels with regard to students' geometric reasoning. The descriptors of the levels he suggests are visual-holistic reasoning, descriptive-analytic reasoning, relational-inferential reasoning, formal deductive proof, and rigor (pp. 92-94), referring to Levels 1 through 5, respectively. For instance, at Level 1 (visual-holistic reasoning), students argue that a square is not a rectangle because a rectangle is "long," or claim that two figures have the "same shape" because they "look the same" (p. 92). At

this level, students' justifications of an argument are vague and holistic. At Level 2 (descriptive-analytic reasoning), students would assert that a square is a rectangle because "it has opposite sides equal and four right angles." At this level, students are able to explicitly specify shapes by their parts and spatial relationships among the parts; however, they describe parts and properties informally and imprecisely using strictly informal language learned from everyday life. At Level 3 (relational-inferential reasoning), students start with empirical inference to reason that if a quadrilateral has four right angles (and this is a rectangle), its opposite sides have to be equal because by drawing a rectangle with a sequence of perpendiculars, they cannot make the opposite sides unequal. They then use logical inference to recognize the classifications of shapes into a logical hierarchy (p.94).

These descriptors not only provide detailed information about how researchers identify students' levels of geometric thinking, but more importantly shed a light on ways thinking has been communicated through reasoning and the language skills that students need to develop at each van Hiele level. When conducting studies using the van Hiele model, some researchers use clinical interviews, while others prefer open-ended survey tests. Among all the van Hiele studies, Usiskin's Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project (Usiskin 1982; Senk 1983), and Burger and Shaughnessy's "Oregon Project" (1986) are two of the most frequently used and cited. In the following section, the method of inquiries used in the van Hiele studies are summarized, and some important findings are provided beginning with these two projects.

The Usiskin CDASSG project (Usiskin, 1982; Senk, 1983) used a pre-test and post-test involving four tests to assess 2699 students in full year geometry classes from 13 public high schools in five states. The four tests included Entering Geometry Test, van Hiele Geometry Test, Comprehensive Assessment Program Geometry Test, and Proof Test. The pre-test including Van Hiele Geometry Test and Entering Geometry Test was conducted during the first week of school, and the post-test including van Hiele Geometry Test, Proof Test, and Comprehensive

Assessment Program Geometry Test was conducted three to five weeks before the end of the school year.

The van Hiele Geometry Test was designed to predict students' van Hiele levels at the beginning and the end of the school year. This test consists of 25 multiple-choice items, with five foils per item and five items per level, and was designed to capture the key characteristic of each van Hiele level. In order to develop a rigorous test instrument that describes van Hiele levels in sufficient detail, researchers in the CDASSG project first reviewed nine original works of the van Hieles, including four originally written in English and five translated into English from Dutch, German or French. They compiled all the quotes from the van Hieles' writings that describe behaviors of students at a given level. As an example, the following is a selected list of Level 1 behaviors quoted from the van Hiele writings that Usiskin (1982) provided in the CDASSG project report:

Level 1 (also known as base level, level 0)

1. "Figures are judged according to their appearance."
2. "A child recognizes a rectangle by its form, shape"
3. "The rectangle seems different to him from a square."
4. "A child does not recognize a parallelogram in a rhombus."
5. "A student was able to produce these figures without error..."

The van Hiele Geometry Test instruments were based on the descriptions of students' behaviors at each given level. For example, Items 1-3 were derived from the description "figures are judged according to their appearances"; Item 4 was derived from the description "the rhombus is not a parallelogram. The rhombus appears as something quite different." Figure 2.2 presents one van Hiele Level 1 item. In Usiskin's study, students' responses to the van Hiele Geometry Test were graded using 3 out of 5 corrections and 4 out of 5 corrections. Usiskin and his colleagues compared the two grading methods criteria using the analyses of Type I and Type II error. The statistical analysis showed that depending on whether one wishes to reduce Type I or Type II error, the 3 of 5 correction method minimizes the chance of missing a student and yields an optimistic picture of students' van Hiele levels. The

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