

2 Theoretical Background

This chapter will introduce theoretical tools as well as physical concepts that are applied in this work. It is divided into two sections. The first will discuss the current understanding of thermoacoustic instabilities and their interplay with noise. The second section is dedicated to dynamical systems theory: A mathematical approach to gain qualitative statements about the asymptotic evolution of systems described by ordinary differential equations (ODEs).

2.1 Thermoacoustic Instability

Thermoacoustic instability has been described in a scientific way first by Rijke (1859). Rijke observed it in a vertical pipe of roughly 1 m height and 0.1 m diameter with a gauze, located in the lower half of the duct. The gauze was heated by a flame beneath the lower end of the pipe. Due to thermal convection a mean air flow through the pipe established. When the gauze was heated sufficiently a loud tone in the frequency range of the first acoustic mode of the pipe appeared. Derivations of the so called Rijke tube are still used to study thermoacoustic instabilities due to its simple geometry and easy to control parameters (Matveev, 2003; Moeck et al., 2009; Juniper, 2010)

Rayleigh (1896) gave a first explanation to the effect: “If heat be given to the air at the moment of greatest condensation, or be taken from it at the moment of greatest rarefaction, the vibration is encouraged. On the other hand, if heat be given at the moment of greatest rarefaction, or abstracted at the moment of greatest condensation, the vibration is discouraged.” In other words, if the unsteady heat release rate $\dot{q}'$ of the heat source is in phase with the pressure fluctuations p' a self-sustaining feedback loop is established. In a mathematical

form, stating a necessary condition, it can be written as:

$$\int_V p' \dot{q}' dV > 0, \quad (2.1)$$

with V being the volume of the respective confinement. Since the heat transfer to the air is dominated by convection (in the case of gauze as a heat source), $\dot{q}'$ is proportional to the velocity fluctuations v' at the location of the heat source. However, due to the inertia of the transfer process a phase difference between $\dot{q}'$ and v' appears; this enables the pressure fluctuations p' , which are shifted against v' by π , to be partially in phase with the heat release fluctuations, satisfying Rayleigh's criterion (equation 2.1). If the energy which is transferred from heat to acoustic energy by this process exceeds dissipation due to acoustic damping a thermoacoustic instability appears (Huang and Yang, 2009).

As stated above, Rayleigh's criterion defines a necessary condition for the onset of an instability. It does not, however, capture e.g. dissipation or energy transfer between acoustic modes and thus does not serve to predict thermoacoustic instabilities.

2.2 Dynamical Systems Theory

Dynamical systems theory is a vast field of research. The intention of this section is to present the idea of the concept and introduce the terms used later on in this work. A clear and comprehensive elaboration of the subject can be found in Strogatz (2000).

Dynamical systems theory approaches the solving of the set of n ODEs (equation 2.2), depending on the n -dimensional vector $\vec{x}$, its time derivative and a parameter μ , in a graphical way.

$$\frac{d\vec{x}}{dt} := \dot{\vec{x}} = F(\vec{x}, \mu) \quad (2.2)$$

The solution $\vec{x}(t)$ to a given set of initial values $\vec{x}(t_0)$ is viewed as a curve, the trajectory, in the n -dimensional space with the coordinates $(x_1, x_2, \dots, x_n)$ called *phase space*. For example, the harmonic oscillator defined by equation 2.3 assigns the vector $(\dot{x}_1, \dot{x}_2) = (x_2, -\mu x_1)$ to each point (x_1, x_2) in the two dimensional

phase space ($n = 2$) of the oscillator.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mu x_1\end{aligned}\tag{2.3}$$

Drawing the *phase portrait*, starting at an arbitrary point $(\dot{x}_{1,0}, \dot{x}_{2,0})$, following the vectors defined by 2.3 one will see that the trajectories are circles around the origin. The only exception being the point $(\dot{x}_{1,0}, \dot{x}_{2,0}) = (0, 0)$. The trajectory starting here will remain at the origin of the phase space. By assigning the (angular) position θ to x_1 and the velocity v to $x_2 = \dot{x}_1$ the face portrait gives an intuitive picture of the harmonic oscillator without actually (analytically) solving the ODE: The system will either oscillate at an amplitude defined by the initial conditions (and a frequency defined by $\sqrt{\mu}$), or remain stationary at the origin if no initial displacement is imposed (2.1.a). These two states are considered *stable* in dynamical systems parlance: Without external forcing, the system will not leave the respective state. In particular the oscillatory state is referred to as a stable *limit cycle*, the origin as a stable *focus* or *fixed point*.

The introduced phase portrait of 2.3 will not change its general appearance when μ changes. However, for the undamped pendulum in a gravitational field, for example, a third stable state besides oscillating around the resting position and remaining still is possible: The continuous rotation around the pendulum's bearing. Figure 2.1.b shows the rotating state as wavelike trajectories with either all positive or all negative velocity v .

The transition from oscillation to rotation can be achieved by either choosing a sufficiently high initial velocity v_0 or by reducing the gravitational acceleration g . The latter is of importance, despite remaining a theoretical example, because the transition from one state to the other happened by changing the system parameter, in this case g . This could be done while the pendulum oscillates; such a change in a system's qualitative behaviour is known as *bifurcation*, the parameter g plays the role of the *bifurcation parameter*. A bifurcation will come along with qualitative changes in the system's phase portrait: New stable states may emerge and old ones disappear. In a *bifurcation diagram* the bifurcation parameter μ is plotted on the x-axis and a characteristic state variable x on the y-axis. The stable states of x as a function of μ are represented as lines in the diagram and by that the dynamics of x are visualised.

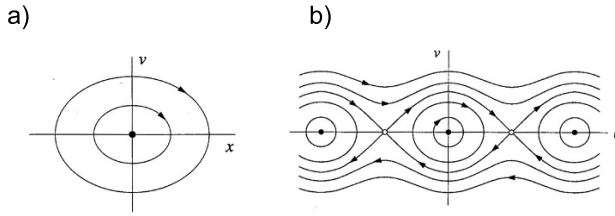


Figure 2.1: Phase portraits of the harmonic oscillator (a) and the undamped pendulum in a potential field (b) (Strogatz, 2000).

The definition of stability given in this section can be contradicting to the terminology used for thermoacoustic systems. When a thermoacoustic instability appears, the system is still in a stable state from the dynamical systems point of view, namely a stable limit cycle. What has become unstable during transition, however, is the fixed point at the origin of the phase plane; disturbances of any amplitude to the fixed point will lead to the limit cycle oscillation referred to as thermoacoustic instability.

Effect of Noise on a Model Thermoacoustic System at
its Stability Boundary

Steinert, R.

2016, XIV, 44 p. 23 illus., Softcover

ISBN: 978-3-658-13822-6