

## Chapter 2

# Preliminaries on Dynamical Systems and Stability Theory

In this chapter, we will review some fundamental concepts on dynamical systems and stability theory, and give some evaluations and opinions on some problems, which will be helpful for understanding the main contents of this book.

### 2.1 Overview of Dynamical Systems

The concept of a dynamical system has its origins in Newtonian mechanics. There, as in other natural sciences and engineering disciplines, the evolution rule of dynamical systems is an implicit relation that gives the state of the system for only a short time into the future. (The relation is either a differential equation, difference equation, or other timescale.) To determine the state for future time requires iterating the relation many times, each advancing time a small step. The iteration procedure is referred to as solving the system or integrating the system. If the system can be solved, given an initial point it is possible to determine all its future positions, a collection of points known as a trajectory or orbit.

Before the advent of computers, finding an orbit required sophisticated mathematical techniques and could be accomplished only for a small class of dynamical systems. Numerical methods implemented on electronic computing machines have simplified the task of determining the orbits of a dynamical system.

For simple dynamical systems, knowing the trajectory is often sufficient, but most dynamical systems are too complicated to be understood in terms of individual trajectories. The difficulties are listed as follows:

(1) The systems studied may only be known approximately—the parameters of the system may not be known precisely or terms may be missing from the equations. The approximations used bring into question the validity or relevance of numerical solutions. To address these questions several notions of stability have been introduced in the study of dynamical systems, such as Lyapunov stability, Lagrange stability,

Hurwitz stability, and structural stability or connective stability. The stability of dynamical systems implies that there is a class of models or initial conditions for which the trajectories would be equivalent. The operation for comparing orbits to establish their equivalence changes with the different notions of stability.

(2) The type of trajectory may be more important than one particular trajectory. Some trajectories may be periodic, whereas others may wander through many different states of the system. Applications often require enumerating these classes or maintaining the system within one class. Classifying all possible trajectories has led to the qualitative study of dynamical systems, that is, properties that do not change under coordinate transformations. Linear dynamical systems and systems that have two numbers describing a state are examples of dynamical systems where the possible classes of orbits are understood.

(3) The behavior of trajectories as a function of a parameter may be what is needed for an application. As a parameter varies, the dynamical systems may have bifurcation points where the qualitative behavior of the dynamical system changes. For example, it may go from having only periodic motions to having apparently erratic behavior, as in the transition to turbulence of a fluid.

(4) The trajectories of the system may appear erratic randomly. In these cases it may be necessary to compute averages using one very long trajectory or many different trajectories. The averages are well defined for ergodic systems and a more detailed understanding has been worked out for hyperbolic systems. Understanding the probabilistic aspects of dynamical systems has helped to establish the foundations of statistical mechanics and of chaos.

Many people regard Henri Poincaré as the founder of dynamical systems. Poincaré published two classical monographs, “New Methods of Celestial Mechanics” (1892–1899) and “Lectures on Celestial Mechanics” (1905–1910). In the monographs, he successfully applied the results of his research to the problem of the motion of three bodies and studied in detail the behavior of solutions (frequency, stability, asymptotic property, and so on). These papers included the Poincaré recurrence theorem, which states that certain systems will, after a sufficiently long but finite time, return to a state very close to the initial state.

Aleksandr Lyapunov developed many important approximation methods. His methods, which was developed in 1899, make it possible to define the stability of sets of ordinary differential equations. He created the modern theory of the stability of a dynamic system. In 1913, George David Birkhoff [1] proved Poincaré’s “Last Geometric Theorem,” a special case of the three-body problem, a result that made him world famous. In 1927, he published his “Dynamical Systems.” Birkhoff’s most durable result was his 1931 discovery of what is now called the ergodic theorem. Combining insights from physics on the ergodic hypothesis with measure theory, this theorem solved, at least in principle, a fundamental problem of statistical mechanics. The ergodic theorem has also had repercussions for dynamics. Stephen Smale made significant advances as well. His first contribution is the Smale horseshoe that jump started significant research in dynamical systems. He also outlined a research program carried out by many others. Oleksandr Mykolaiovych Sharkovsky developed Sharkovsky’s theorem on the periods of discrete dynamical systems in 1964. One of

the implications of the theorem is that if a discrete dynamical system on the real line has a periodic point of period 3, then it must have periodic points of every other period.

Some theories related to dynamical systems are simply described as follows:

#### (1) Arithmetic dynamics

Arithmetic dynamics is a field emerged in the 1990s that amalgamates two areas of mathematics, dynamical systems and number theory. Classically, discrete dynamics refers to the study of the iteration of self-maps of the complex plane or real line. Arithmetic dynamics is the study of the number-theoretic properties of integer, rational,  $p$ -adic, and/or algebraic points under repeated application of a polynomial or rational function.

#### (2) Chaos theory

Chaos theory describes the behavior of certain dynamical systems, that is, systems whose state evolves with time, that may exhibit dynamics that are highly sensitive to initial conditions (popularly referred to as the butterfly effect). As a result of this sensitivity, which manifests itself as an exponential growth of perturbations in the initial conditions, the behavior of chaotic systems appears random. This happens even though these systems are deterministic, meaning that their future dynamics are fully defined by their initial conditions, with no random elements involved. This behavior is known as deterministic chaos or simply chaos.

#### (3) Complex systems

Complex systems are a scientific field, which study the common properties of systems considered complex in nature, society and science. It is also called complex systems theory, complexity science, study of complex systems and/or sciences of complexity. The key problems of such systems are difficulties with their formal modeling and simulation. From such perspective, in different research contexts complex systems are defined on the basis of their different attributes. The study of complex systems is bringing new vitality to many areas of science where a more typical reductionist strategy has fallen short. Complex systems are, therefore, often used as a broad term encompassing a research approach to problems in many diverse disciplines including neurosciences, social sciences, meteorology, chemistry, physics, computer science, psychology, artificial life, evolutionary computation, economics, earthquake prediction, molecular biology, and inquiries into the nature of living cells themselves.

#### (4) Control theory

Control theory is an interdisciplinary branch of engineering and mathematics, which deals with how to influence the behavior of dynamical systems subjectively.

#### (5) Ergodic theory

Ergodic theory is a branch of mathematics that studies dynamical systems with an invariant measure and related problems. Its initial proposal was motivated by problems of statistical physics.

### (6) Functional analysis

Functional analysis is the branch of mathematics, and specifically of analysis concerned with the study of vector spaces and operators acting upon them. It has its historical roots in the study of functional spaces, in particular transformations of functions, such as the Fourier transform, as well as in the study of differential and integral equations. This usage of the word “functional” goes back to the calculus of variations, implying a function whose argument is a function. Its use in general has been attributed to mathematician and physicist Vito Volterra and its founding is largely attributed to mathematician Stefan Banach.

### (7) Graph dynamical systems

The concept of graph dynamical systems (GDS) can be used to capture a wide range of processes taking place on graphs or networks. A major theme in the mathematical and computational analysis of GDS is to relate their structural properties (e.g., the network connectivity) and the global dynamics that result.

### (8) Projected dynamical systems

Projected dynamical systems are a mathematical theory investigating the behaviour of dynamical systems where solutions are restricted to a constraint set. The discipline shares connections to and applications with both the static world of optimization and equilibrium problems and the dynamical world of ordinary differential equations. A projected dynamical system is given by the flow to the projected differential equation.

### (9) Symbolic dynamics

Symbolic dynamics are the practice of modeling a topological or smooth dynamical system by a discrete space consisting of infinite sequences of abstract symbols, each of which corresponds to a state of the system, with the dynamics (evolution) given by the shift operator.

### (10) System dynamics

System dynamics are an approach to understanding the behavior of complex systems over time. It deals with internal feedback loops and time delays that affect the behavior of the entire system. What makes system dynamics different from other approaches to studying complex systems is the use of feedback loops and stocks and flows. These elements help describe how even seemingly simple systems display baffling nonlinearity.

### (11) Topological dynamics

Topological dynamics is a branch of the theory of dynamical systems in which qualitative, asymptotic properties of dynamical systems are studied from the viewpoint of general topology.

## 2.2 Definition of Dynamical System and Its Qualitative Analysis

A dynamical system is a concept in mathematics where a fixed rule describes how a point in a geometrical space depends on time. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number of fish each spring time in a lake. At any given time a dynamical system has a state given by a set of real numbers (a vector) that can be represented by a point in an appropriate state space (a geometrical manifold). Small changes in the state of the system create small changes in the numbers. The evolution rule of the dynamical system is a fixed rule that describes what future states follow from the current state. The rule is deterministic; in other words, for a given time interval only one future state follows from the current state.

In the following, several types of definitions of dynamical system are provided.

(1) A *dynamical system* is a four-tuple  $\{T, X, A, S\}$  where  $T$  denotes time set,  $X$  is the state space (a metric space with metric  $d$ ),  $A$  is the set of initial states, and  $S$  denotes a family of motions. When  $T = \mathbb{R}^+ = [0, \infty)$ , we speak of a continuous-time dynamical system; and when  $T = N = \{0, 1, 2, \dots\}$ , we speak of a discrete-time dynamical system. For any motion  $x(\cdot; x_0, t_0) \in S$ , we have  $x(t_0; x_0, t_0) = x_0 \in A \subset X$  and  $x(t, x_0, t_0) \in X$  for all  $t \in [t_0, t_1] \cap T$ ,  $t_1 > t_0$ , where  $t_1$  may be finite or infinite. The set of motions  $S$  is obtained by varying  $(t_0, x_0)$  over  $T \times A$ . A dynamical system is said to be *autonomous*, if every  $x(\cdot, x_0, t_0) \in S$  is defined on  $T \cap [t_0, \infty)$  and if for each  $x(\cdot, x_0, t_0) \in S$  and for each  $\tau$  such that  $t_0 + \tau \in T$ , there exists a motion  $x(\cdot, x_0, t_0 + \tau) \in S$  such that  $x(t + \tau; x_0, t_0 + \tau) = x(t; x_0, t_0)$  for all  $t$  and  $\tau$  satisfying  $t + \tau \in T$ .

A set  $M \subset A$  is said to be *invariant* with respect to the set of motions  $S$  if  $x_0 \in M$  implies that  $x(t, x_0, t_0) \in M$  for all  $t \geq t_0$ , for all  $t_0 \in T$ , and for all  $x(\cdot; x_0, t_0) \in S$ . A point  $p \in X$  is called an *equilibrium* for the dynamical system  $\{T, X, A, S\}$  if the singleton  $\{p\}$  is an invariant set with respect to the motion  $S$ . The term *stability* (more specially, *Lyapunov stability*) usually refers to the qualitative behavior of motions relative to an invariant set (resp. an equilibrium), whereas the term *boundedness* (more specially, *Lagrange stability*) refers to the (global) boundedness properties of the motions of a dynamical system. Of the many different types of *Lyapunov stability* that have been considered in the literature, perhaps the most important ones include stability, uniform stability, asymptotic stability, uniform asymptotic stability, exponential stability, asymptotic stability in the large, uniform asymptotic stability in the large, exponential stability in the large, instability, and complete instability. The most important *Lagrange stability* types include boundedness, uniform boundedness, and uniform ultimate boundedness of motions.

(2) A *dynamical system* (geometrical definition) is the tuple  $\langle M, f, T \rangle$ , with  $M$  a manifold (locally a Banach space or Euclidean space),  $T$  the domain for time (nonnegative reals, the integers) and  $f$  an evolution rule  $t \rightarrow f(t)$  (with  $t \in T$ ) such that  $f(t)$  is a diffeomorphism of the manifold to itself. So,  $f$  is a mapping of the time

domain into the space of diffeomorphisms of the manifold to itself. In other terms,  $f(t)$  is a diffeomorphism, for every time  $t$  in the domain  $T$ .

(3) A *dynamical system* (measure theoretical definition) may be defined formally, as a measure-preserving transformation of a sigma-algebra, the quadruplet  $(X, \Sigma, \mu, \tau)$ . Here,  $X$  is a set, and  $\Sigma$  is a sigma-algebra on  $X$ , so that the pair  $(X, \Sigma)$  is a measurable space.  $\mu$  is a finite measure on the sigma-algebra, so that the triplet  $(X, \Sigma, \mu)$  is a probability space. A map  $\tau : X \rightarrow X$  is said to be  $\Sigma$ -measurable if and only if, for every  $\sigma \in \Sigma$ , one has  $\tau^{-1}\sigma \in \Sigma$ . A map  $\tau$  is said to preserve the measure if and only if, for every  $\sigma \in \Sigma$ , one has  $\mu(\tau^{-1}\sigma) = \mu(\sigma)$ . Combining the above, a map  $\tau$  is said to be a measure-preserving transformation of  $X$ , if it is a map from  $X$  to itself, it is  $\Sigma$ -measurable, and is measure-preserving. The quadruple  $(X, \Sigma, \mu, \tau)$ , for such a  $\tau$ , is then defined to be a dynamical system.

The map  $\tau$  embodies the time evolution of the dynamical system. Thus, for discrete dynamical systems the iterations  $\tau^n = \tau \circ \tau \circ \cdots \circ \tau$  for integer  $n$  are studied. For continuous dynamical systems, the map  $\tau$  is understood to be a finite time evolution map and the construction is more complicated.

(4) A *dynamical system* is a manifold  $M$  called the phase (or state) space endowed with a family of smooth evolution functions  $\Phi(t)$  that for any element of  $t \in T$ , the time, map a point of the phase space back into the phase space. The notion of smoothness changes with applications and the type of manifold. There are several choices for the set  $T$ . When  $T$  is taken to be the reals, the dynamical system is called a flow, and if  $T$  is restricted to the nonnegative reals, then the dynamical system is a semi-flow. When  $T$  is taken to be the integers, it is a cascade or a map, and the restriction to the nonnegative integers is a semi-cascade.

For example, the evolution function  $\Phi(t)$  is often the solution of a differential equation of motion  $\dot{x}(t) = v(x(t))$  or  $\dot{x} = v(x)$  for brevity. The equation gives the time derivative, represented by the dot, of a trajectory  $x(t)$  on the phase space starting at some point  $x_0$ . The vector field  $v(x)$  is a smooth function that at every point of the phase space  $M$  provides the velocity vector of the dynamical system at that point. (These vectors are not vectors in the phase space  $M$ , but in the tangent space  $T_x M$  of the point  $x$ .) Given a smooth function  $\Phi(t)$ , an autonomous vector field can be derived from it. There is no need for higher order derivatives in the equation, nor for time dependence in  $v(x)$  because these can be eliminated by considering systems of higher dimensions. Other types of differential equations can be used to define the evolution rule:  $G(x, \dot{x}) = 0$  is an example of an equation that arises from the modeling of mechanical systems with complicated constraints.

The differential equations determining the evolution function  $\Phi(t)$  are often ordinary differential equations: in this case the phase space  $M$  is a finite dimensional manifold. Many of the concepts in dynamical systems can be extended to infinite-dimensional manifolds—those that are locally Banach spaces—in which case the differential equations are partial differential equations. In the late twentieth century the dynamical system perspective to partial differential equations started gaining popularity.

The *qualitative properties* of dynamical systems do not change under a smooth change of coordinates (this is sometimes taken as a definition of qualitative): a

singular point of the vector field (a point where  $v(x) = 0$ ) will remain a singular point under smooth transformations; a periodic orbit is a loop in phase space and smooth deformations of the phase space cannot alter it to be a loop. It is in the neighborhood of singular points and periodic orbits that the structure of a phase space of a dynamical system can be well understood. In the qualitative study of dynamical systems, the approach is to show that there is a change of coordinates (usually unspecified, but computable) that makes the dynamical system as simple as possible.

A similar concept to the qualitative properties of dynamical systems is the *rectification*. A flow in most small patches of the phase space can be made very simple. If  $y$  is a point where the vector field  $v(y) \neq 0$ , then there is a change of coordinates for a region around  $y$  where the vector field becomes a series of parallel vectors of the same magnitude. This is known as the rectification theorem. The rectification theorem says that away from singular points the dynamics of a point in a small patch is a straight line. The patch can sometimes be enlarged by stitching several patches together, and when this works out in the whole phase space  $M$  the dynamical system is integrable. In most cases the patch cannot be extended to the entire phase space. There may be singular points in the vector field (where  $v(x) = 0$ ) or the patches may become smaller and smaller as some point is approached. The more subtle reason is a global constraint, where the trajectory starts out in a patch, and after visiting a series of other patches comes back to the original one. If the next time the orbit loops around phase space in a different way, then it is impossible to rectify the vector field in the whole series of patches.

Dynamical systems theory is an area of mathematics used to describe the behavior of complex dynamical systems, usually by employing differential equations or difference equations. When differential equations are employed, the theory is called continuous dynamical systems. When difference equations are employed, the theory is called discrete dynamical systems. When the time variable runs over a set that is discrete over some intervals and continuous over other intervals or is any arbitrary time—set such as a cantor set—one gets dynamic equations on timescales. Some situations may also be modeled by mixed operators, such as differential–difference equations. This theory deals with the long-term qualitative behavior of dynamical systems, and studies the solutions of the equations of motion of systems that are primarily mechanical in nature, although this includes both planetary orbits as well as the behavior of electronic circuits and the solutions to partial differential equations that arise in biology. Much of modern research is focused on the study of chaotic systems. This field of study is also called just dynamical systems, mathematical dynamical systems theory, and mathematical theory of dynamical systems.

Dynamical systems theory and chaos theory deal with the long-term qualitative behavior of dynamical systems. Here, the focus is not on finding precise solutions to the equations defining the dynamical system (which is often hopeless), but rather to answer questions like, “Will the system settle down to a steady state in the long term, and if so, what are the possible steady states?” or “Does the long-term behavior of the system depend on its initial condition?” An important target is to describe the fixed points, or steady states of a given dynamical system; these are values of the variable



that do not change over time. Some of these fixed points are attractive, meaning that if the system starts out in a nearby state, it converges toward the fixed point.

Similarly, one is interested in periodic points, states of the system that repeat after several time steps. Periodic points can also be attractive. Sharkovskii's theorem is an interesting statement about the number of periodic points of a one-dimensional discrete dynamical system. Even simple nonlinear dynamical systems often exhibit seemingly random behavior that has been called chaos. The branch of dynamical systems that handles the clear definition and investigation of chaos is called chaos theory.

## 2.3 Lyapunov Stability of Dynamical Systems

Various types of stability may be discussed for the solutions of differential equations describing dynamical systems. The most important type is that concerning the stability of solutions near to a point of equilibrium. This may be discussed by the theory of Lyapunov. In simple terms, if all solutions of the dynamical system that start out near an equilibrium point  $x_e$  stay near  $x_e$  forever, then  $x_e$  is *Lyapunov stable*. More strongly, if  $x_e$  is Lyapunov stable and all solutions that start out near  $x_e$  converge to  $x_e$ , then  $x_e$  is asymptotically stable. The notion of exponential stability guarantees a minimal rate of decay, i.e., an estimate of how quickly the solutions converge. The idea of Lyapunov stability can be extended to infinite-dimensional manifolds, where it is known as structural stability, which concerns the behavior of different but “nearby” solutions to differential equations. Input-to-state stability (ISS) applies Lyapunov notions to systems with inputs.

Lyapunov stability is named after Aleksandr Lyapunov, a Russian mathematician who published his book “The General Problem of Stability of Motion” in 1892 [2]. Lyapunov was the first to consider the modifications necessary in nonlinear systems to the linear theory of stability based on linearizing near a point of equilibrium. His work, initially published in Russian and then translated to French, received little attention for many years. Interest in it started suddenly during the Cold War (1953–1962) period when the so-called “Second Method of Lyapunov” was found to be applicable to the stability of aerospace guidance systems which typically contain strong nonlinearities not treatable by other methods. A large number of publications appeared then and since in the control and systems literature [3–7]. More recently, the concept of the Lyapunov exponent (related to Lyapunov's First Method of discussing stability) has received wide interest in connection with chaos theory. Lyapunov stability methods have also been applied to finding equilibrium solutions in traffic assignment problems [8].

### (1) Definition of Lyapunov stability for continuous-time systems

Consider an autonomous nonlinear dynamical system

$$\dot{x}(t) = f(x(t)) \text{ with } x(0) = x_0, \quad (2.1)$$



where  $x(t) \in D \subseteq \mathbb{R}^n$  denotes the system state vector,  $D$  an open set containing the origin, and  $f : D \rightarrow \mathbb{R}^n$  continuous on  $D$ . Suppose  $f$  has an equilibrium at  $x_e$  so that  $f(x_e) = 0$ , then

- (1) This equilibrium is said to be *Lyapunov stable*, if, for every  $\epsilon > 0$ , there exists a  $\delta = \delta(\epsilon)$  such that, if  $\|x(0) - x_e\| < \delta$ , then for every  $t \geq 0$  we have  $\|x(t) - x_e\| < \epsilon$ .
- (2) The equilibrium of the above system is said to be *asymptotically stable* if it is Lyapunov stable and there exists  $\delta > 0$  such that if  $\|x(0) - x_e\| < \delta$ , then  $\lim_{t \rightarrow \infty} \|x(t) - x_e\| = 0$ .
- (3) The equilibrium of the above system is said to be *exponentially stable* if it is asymptotically stable and there exist  $\alpha > 0$ ,  $\beta > 0$ ,  $\delta > 0$  such that if  $\|x(0) - x_e\| < \delta$ , then  $\|x(t) - x_e\| \leq \alpha \|x(0) - x_e\| e^{-\beta t}$ , for  $t \geq 0$ .

Conceptually, the meanings of aforementioned terms are the following:

- (1) Lyapunov stability of an equilibrium means that solutions starting “close enough” to the equilibrium (within a distance  $\delta$  from it) remain “close enough” forever (within a distance  $\epsilon$  from it). Note that this must be true for any  $\epsilon$  that one may want to choose.
- (2) Asymptotic stability means that solutions that start close enough not only remain close enough but also eventually converge to the equilibrium.
- (3) Exponential stability means that solutions not only converge, but in fact converge faster than or at least as fast as a particular known rate  $\alpha \|x(0) - x_e\| e^{-\beta t}$ .

The trajectory  $x$  is (locally) *attractive* if  $\|y(t) - x(t)\| \rightarrow 0$  (where  $y(t)$  denotes the system output) for  $t \rightarrow \infty$  for all trajectories that start close enough, and globally attractive if this property holds for all trajectories. That is, if  $x$  belongs to the interior of its stable manifold, it is asymptotically stable if it is both attractive and stable. (There are counterexamples showing that attractivity does not imply asymptotic stability.)

#### (2) Lyapunov's second method for stability

Lyapunov, in his original 1892 work, proposed two methods for demonstrating stability. The first method developed the solution in a series which was then proved convergent within limits. The second method, which is almost universally used nowadays, makes use of a Lyapunov function  $V(x)$  which has an analogy to the potential function of classical dynamics. It is introduced as follows for a system having a point of equilibrium at  $x = 0$ .

Consider a function  $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

- (1)  $V(x) \geq 0$  with equality if and only if  $x = 0$  (positive definite).
- (2)  $\dot{V}(x(t)) = \frac{dV(x(t))}{dt} \leq 0$  with equality not constrained to only  $x = 0$  (negative semidefinite. Note: for asymptotic stability,  $\dot{V}(x(t))$  is required to be negative definite!).

Then  $V(x)$  is called a Lyapunov function candidate and the system is stable in the sense of Lyapunov. Furthermore, the system is asymptotically stable, in the sense of Lyapunov, if  $\dot{V}(x(t)) \leq 0$  with equality if and only if  $x = 0$ . Global asymptotic stability (GAS) follows similarly.

Note that, (1)  $V(0) = 0$  is required; otherwise for example  $V(x) = \frac{1}{1+|x|}$  would “prove” that  $\dot{x}(t) = x(t)$  is locally stable. (2) An additional condition called “properness” or “radial unboundedness” is required in order to conclude global stability.

It is easier to visualize this method of analysis by thinking of a physical system (e.g., vibrating spring and mass) and considering the energy of such a system. If the system loses energy over time and the energy is never restored, then eventually the system must grind to a stop and reach some final resting state. This final state is called the attractor. However, finding a function that gives the precise energy of a physical system can be difficult, and for abstract mathematical systems, economic systems or biological systems, the concept of energy may not be applicable. *Lyapunov's realization* is that stability can be proven without requiring knowledge of the true physical energy, provided that a Lyapunov function can be found to satisfy the above constraints.

Lyapunov stability method is mainly focused on the system (2.1), i.e., a system with zero input. In fact, many systems have external control inputs. If the control law is in the form of state feedback, then the closed-loop systems is equivalent to the system (2.1). In this case, Lyapunov stability theory can be directly applied. If the external input exists and is different from the system states, some variants of Lyapunov stability theory should be investigated. In the following, we will introduce some other stability analysis methods.

### (3) Stability for systems with inputs

A system with inputs (or controls) has the form

$$\dot{x}(t) = f(x(t), u(t)), \quad (2.2)$$

where the (generally time-dependent) input  $u(t)$  may be viewed as a control, external input, stimulus, disturbance, or forcing function. The study of such systems is the subject of control theory and applied in control engineering. For systems with inputs, one must quantify the effect of inputs on the stability of the system. The main two approaches to this analysis are BIBO stability (for linear systems) and input-to-state (ISS) stability (for nonlinear systems)

### (4) Barbalat's lemma and stability of time-varying systems

Assume that  $f(t)$  is function of time only.

- (1) Having  $\dot{f}(t) \rightarrow 0$  does not imply that  $f(t)$  has a limit as  $t \rightarrow \infty$ . For example,  $f(t) = \sin(\ln(t))$ ,  $t > 0$ .
- (2) Having  $f(t)$  approaching a limit as  $t \rightarrow \infty$  does not imply that  $\dot{f}(t) \rightarrow 0$ . For example,  $f(t) = \frac{\sin(t^2)}{t}$ .
- (3) Having  $f(t)$  lower bounded and decreasing ( $\dot{f}(t) \leq 0$ ) implies it converges to a limit. But it does not say whether or not  $\dot{f}(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

**Barbalat's Lemma says:** If  $f(t)$  has a finite limit as  $t \rightarrow \infty$  and if  $\dot{f}(t)$  is uniformly continuous (or  $\ddot{f}(t)$  is bounded), then  $\dot{f}(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

Usually, it is difficult to analyze the asymptotic stability of time-varying systems because it is very difficult to find Lyapunov functions with a negative definite derivative. We know that in case of autonomous (time-invariant) systems, if  $\dot{V}(x(t))$  is negative semidefinite, then also, it is possible to know the asymptotic behavior by invoking invariant set theorems. However, this flexibility is not available for time-varying systems. This is where "Barbalat's lemma" comes into picture. It says:

If  $V(x(t), t)$  satisfies following conditions: (1)  $V(x(t), t)$  is lower bounded. (2)  $\dot{V}(x(t), t)$  is negative semidefinite. (3)  $\dot{V}(x(t), t)$  is uniformly continuous in time (satisfied if  $\ddot{V}(x(t), t)$  is finite), then  $\dot{V}(x(t), t) \rightarrow 0$  as  $t \rightarrow \infty$ .

#### (5) Differential inequality methods

This kind of stability analysis methods generally do not need the so-called Lyapunov functions, and can also determine the stability properties of the concerned system. Such kinds of methods include, but not limited to, Halanay inequality, contraction principle, Gronwall's inequality, comparison theorems, and so on [9–13]. Even by constructing Lyapunov functions, except the Lyapunov stability theory, there are still many methods to derive the stability criteria for the concerned systems, e.g., Barbalat's lemma. However, a key point should be kept in mind, that is, the definition of stability property must be declared beforehand. In this way, there are many different senses of stability definition in the literature. In the research of neural network stability theory, more emphasis is placed on the stability of Lyapunov sense. Fortunately, other stability definitions in the Hopfield sense and Lagrange sense are also paid attention. Except Lyapunov function method, input–output method and differential inequality methods are also powerful to analyze the qualitative characteristics of dynamical systems.

## 2.4 Stability Theory

In mathematics, *stability theory* addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions. The heat equation, for example, is a stable partial differential equation because small perturbations of initial data lead to small variations in temperature at a later time as a result of the maximum principle. One must specify the metric used to measure the perturbations when claiming a system is stable. In partial differential equations one may measure the distances between functions using  $L_p$  norms or the *sup* norm, while in differential geometry one may measure the distance between spaces using the Gromov–Hausdorff distance.

In dynamical systems, an *orbit* is called Lyapunov stable if the forward orbit of any point is in a small enough neighborhood or it stays in a small (but perhaps, larger) neighborhood. Various criteria have been developed to prove stability or instability of an orbit. Under favorable circumstances, the question may be reduced to a well-studied problem involving eigenvalues of matrices. A more general method involves Lyapunov functions. In practice, any one of different stability criteria is applied.

Many parts of the qualitative theory of differential equations and dynamical systems deal with asymptotic properties of solutions and the trajectories—what happens with the system after a long period of time. The simplest kind of behavior is exhibited by equilibrium points or fixed points, and by periodic orbits. If a particular orbit is well understood, it is natural to ask next whether a small change in the initial condition will lead to similar behavior. *Stability theory* addresses the following questions: (1) will a nearby orbit indefinitely stay close to a given orbit? (2) will it converge to

the given orbit? (this is a stronger property). In the former case, the orbit is called stable and in the latter case, asymptotically stable, or attracting.

*Stability* means that the trajectories do not change too much under small perturbations. The opposite situation, where a nearby orbit is getting repelled from the given orbit, is also of interest. In general, perturbing the initial state in some directions results in the trajectory asymptotically approaching the given one and in other directions to the trajectory getting away from it. There may also be directions for which the behavior of the perturbed orbit is more complicated (neither converging nor escaping completely), and then stability theory does not give sufficient information about the dynamics.

One of the key ideas in stability theory is that the *qualitative behavior* of an orbit under perturbations can be analyzed using the linearization of the system near the orbit. In particular, at each equilibrium of a smooth dynamical system with an  $n$ -dimensional phase space, there is a certain  $n \times n$  matrix  $A$  whose eigenvalues characterize the behavior of the nearby points (Hartman–Grobman theorem). More precisely, if all eigenvalues are negative real numbers or complex numbers with negative real parts, then the point is a stable attracting fixed point, and the nearby points converge to it at an exponential rate. If none of the eigenvalues is purely imaginary (or zero) then the attracting and repelling directions are related to the eigenspaces of the matrix  $A$  with eigenvalues whose real part is negative and, respectively, positive. Analogous statements are known for perturbations of more complicated orbits.

### (1) *Stability of fixed points*

The simplest kind of an orbit is a fixed point or an equilibrium. If a mechanical system is in a stable equilibrium state, then a small push will result in a localized motion, for example, small oscillations as in the case of a pendulum. In a system with damping, a stable equilibrium state is moreover asymptotically stable. On the other hand, for an unstable equilibrium, such as a ball resting on a top of a hill, certain small pushes will result in a motion with a large amplitude that may or may not converge to the original state. There are useful tests of stability for the case of a linear system. Stability of a nonlinear system can often be inferred from the stability of its linearization.

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuously differentiable function with a *fixed point*  $a$ ,  $f(a) = a$ . Consider the dynamical system obtained by iterating the function  $f$ :

$$x_{n+1} = f(x_n), n = 0, 1, 2, 3, \dots, \quad (2.3)$$

The fixed point  $a$  is stable if the absolute value of the derivative of  $f$  at  $a$  is strictly less than 1, and unstable if it is strictly greater than 1. This is because near the point  $a$ , the function  $f$  has a *linear approximation* with slope  $\dot{f}(a)$ :

$$f(x) \approx f(a) + \dot{f}(a)(x - a). \quad (2.4)$$

Thus

$$\frac{x_{n+1} - a}{x_n - a} = \frac{f(x_n) - a}{x_n - a} = \dot{f}(a), \quad (2.5)$$

which means that the derivative measures the rate at which the successive iterates approach the fixed point  $a$  or diverge from it. If the derivative at  $a$  is exactly 1 or  $-1$ , then more information is needed in order to decide stability.

There is an analogous criterion for a continuously differentiable map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with a fixed point  $a$ , expressed in terms of its Jacobian matrix at  $a$ ,  $J = J_a(f)$ . If all eigenvalues of  $J$  are real or complex numbers with absolute value strictly less than 1 then  $a$  is a stable fixed point; if at least one of them has absolute value strictly greater than 1 then  $a$  is unstable. Just as for  $n = 1$ , the case of the largest absolute value being 1 needs to be investigated further—the Jacobian matrix test is inconclusive. The same criterion holds more generally for diffeomorphisms of a smooth manifold.

For the linear autonomous systems, the stability of fixed points of a system of constant coefficient linear differential equations of first order can be analyzed using the eigenvalues of the corresponding matrix. An autonomous system  $\dot{x} = Ax$ , where  $x(t) \in \mathbb{R}^n$  and  $A$  is an  $n \times n$  matrix with real entries, has a constant solution  $x(t) = 0$ . (In a different language, the origin  $0 \in \mathbb{R}^n$  is an equilibrium point of the corresponding dynamical system.) This solution is *asymptotically stable* as  $t \rightarrow \infty$  (“in the future”) if and only if for all eigenvalues  $\lambda$  of  $A$ ,  $\operatorname{Re}(\lambda) < 0$  (which means the real part of eigenvalue  $\lambda$  is negative). Similarly, it is asymptotically stable as  $t \rightarrow -\infty$  (“in the past”) if and only if for all eigenvalues  $\lambda$  of  $A$ ,  $\operatorname{Re}(\lambda) > 0$ . If there exists an eigenvalue  $\lambda$  of  $A$  with  $\operatorname{Re}(\lambda) > 0$  then the solution is unstable for  $t \rightarrow \infty$ . Application of this result in practice, in order to decide the stability of the origin for a linear system, is facilitated by the Routh–Hurwitz stability criterion. The eigenvalues of a matrix are the roots of its characteristic polynomial. A polynomial in one variable with real coefficients is called a Hurwitz polynomial if the real parts of all roots are strictly negative. The Routh–Hurwitz theorem implies a characterization of Hurwitz polynomials by means of an algorithm that avoids computing the roots.

For a nonlinear autonomous systems, asymptotic stability of fixed points can often be established using the *Hartman–Grobman theorem*. Suppose that  $v$  is a  $C^1$ -vector field in  $\mathbb{R}^n$  which vanishes at a point  $p$ ,  $v(p) = 0$ . Then the corresponding autonomous system  $\dot{x} = v(x)$  has a constant solution  $x(t) = p$ . Let  $J = J_p(v)$  be the  $n \times n$  Jacobian matrix of the vector field  $v$  at the point  $p$ . If all eigenvalues of  $J$  have strictly negative real part, then the solution is asymptotically stable. This condition can be tested using the Routh–Hurwitz criterion.

For the general dynamical systems, a general way to establish Lyapunov stability or asymptotic stability of a dynamical system is by means of Lyapunov functions, which has been introduced above.

## (2) Structural stability

Let  $G$  be an open domain in  $\mathbb{R}^n$  with compact closure and smooth  $(n - 1)$ -dimensional boundary. Consider the space  $X^1(G)$  consisting of restrictions to  $G$  of  $C^1$  vector fields on  $\mathbb{R}^n$  that are transversal to the boundary of  $G$  and are inward oriented. This space is endowed with the  $C^1$  metric in the usual fashion. A vector

field  $F \in X^1(G)$  is weakly structurally stable if for any sufficiently small perturbation  $F_1$ , the corresponding flows are topologically equivalent on  $G$ : there exists a homeomorphism  $h : G \rightarrow G$  which transforms the oriented trajectories of  $F$  into the oriented trajectories of  $F_1$ . If, moreover, for any  $\epsilon > 0$  the homeomorphism  $h$  may be chosen to be  $C_\epsilon^0$ -close to the identity map when  $F^1$  belongs to a suitable neighborhood of  $F$  depending on  $\epsilon$ , then  $F$  is called (strongly)structurally stable.

These definitions are extended in a straightforward way to the case of  $n$ -dimensional compact smooth manifolds with boundary. Andronov and Pontryagin originally considered the strong property. Analogous definitions can be given for diffeomorphisms in place of vector fields and flows: in this setting, the homeomorphism  $h$  must be a topological conjugacy.

In mathematics, structural stability is a fundamental property of a dynamical system which means that the qualitative behavior of the trajectories is unaffected by small perturbations (to be exact  $C^1$ -small perturbations). Examples of such qualitative properties are numbers of fixed points and periodic orbits (but not their periods). Unlike Lyapunov stability, which considers perturbations of initial conditions for a fixed system, structural stability deals with perturbations of the system itself. Variants of this notion apply to systems of ordinary differential equations, vector fields on smooth manifolds and flows generated by them, and diffeomorphisms.

Structurally stable systems were introduced by Andronov and Pontryagin [14] under the name “systemes grossiers” or rough systems [14, 15]. They announced a characterization of rough systems in the plane, the Andronov–Pontryagin criterion. In this case, structurally stable systems are typical, they form an open dense set in the space of all systems endowed with appropriate topology. In higher dimensions, this is no longer true, indicating that typical dynamics can be very complex (e.g., strange attractor). An important class of structurally stable systems in arbitrary dimensions is given by Anosov diffeomorphisms and flows.

*Structural stability* of the system provides a justification for applying the qualitative theory of dynamical systems to analysis of concrete physical systems. The idea of such qualitative analysis goes back to the work of Henri Poincaré on the three-body problem in celestial mechanics. Around the same time, Aleksandr Lyapunov rigorously investigated stability of small perturbations of an individual system. In practice, the evolution law of the system (i.e., the differential equations) is never known exactly, due to the presence of various small interactions. It is, therefore, crucial to know that basic features of the dynamics are the same for any small perturbation of the “model” system, whose evolution is governed by a certain known physical law. Qualitative analysis was further developed by George Birkhoff in the 1920s [16], but was first formalized with introduction of the concept of rough system by Andronov and Pontryagin in 1937 [14]. This was immediately applied to analysis of physical systems with oscillations by Andronov, Witt, and Khaikin. The term “structural stability” is due to Solomon Lefschetz, who oversaw translation of their monograph into English. Ideas of structural stability were taken up by Stephen Smale and his school in the 1960s in the context of hyperbolic dynamics [17]. Earlier, Marston Morse and Hassler Whitney initiated and Rene Thom developed a parallel theory of stability for differentiable maps, which forms a key part of singularity theory.

Thom envisaged applications of this theory to biological systems. Both Smale and Thom worked in direct contact with Mauricio Peixoto, who developed Peixoto's theorem in the late 1950s.

When Smale started to develop the theory of hyperbolic dynamical systems, he hoped that structurally stable systems would be "typical." This would have been consistent with the situation in low dimensions: dimension two for flows and dimension one for diffeomorphisms. However, he soon found examples of vector fields on higher dimensional manifolds that cannot be made structurally stable by an arbitrarily small perturbation. This means that in higher dimensions, structurally stable systems are not dense. In addition, a structurally stable system may have transversal homoclinic trajectories of hyperbolic saddle closed orbits and infinitely many periodic orbits, even though the phase space is compact. The closest higher dimensional analog of structurally stable systems considered by Andronov and Pontryagin is given by the Morse–Smale systems.

### (3) *Rough system* (structurally stable dynamical system)

A smooth dynamical system is called rough systems if the following properties hold. For any  $\epsilon > 0$ , there is a  $\delta > 0$  such that for any perturbation of the system by not more than  $\delta$  in the  $C^1$ -metric, there exists a homeomorphism of the phase space which displaces the points by not more than  $\epsilon$  and converts the trajectories of the unperturbed system into trajectories of the perturbed system.

Formally, this definition assumes that a certain Riemannian metric is given on the phase manifold. In fact, one speaks of a structurally stable system when the phase manifold is closed, or else if the trajectories form part of some compact domain  $G$  with a smooth boundary not tangent to the trajectories; here the perturbation and the homeomorphism are considered on  $G$  only. In view of the compactness, the selection of the metric is immaterial.

Thus, a small (in the sense of  $C^1$ ) perturbation of a structurally stable system yields a system equivalent to the initial one as regards all its topological properties (however, this definition comprises one additional requirement, i.e., this equivalence must be realized by a homeomorphism close to the identity). The terms "roughness" and "(structural) stability" are used in a broader sense, e.g., to mean merely the preservation of some property of the system under a small perturbation (in such a case it is preferable to speak of the structural stability of the property in question).

As said above, *structurally stable systems* were introduced by A.A. Andronov and L.S. Pontryagin. If the dimension of the phase manifold is small (one for discrete time and one or two for continuous time), structurally stable systems can be simply characterized in terms of the qualitative properties of behavior of trajectories (then they are the so-called Morse–Smale systems, cf. Morse–Smale system); in that case they form an open everywhere-dense set in the space of all dynamical systems, provided with the  $C^1$ -topology. Thus, systems whose trajectories display a behavior which is more complex and more sensitive to small perturbations are considered here as exceptional. If the dimensions are larger, none of these facts hold, as was established by S. Smale. He advanced the hypothesis according to which, irrespective of all these complications, it is possible in the general case to formulate the following necessary



and sufficient conditions for structural stability in terms of a qualitative picture of the behavior of the trajectories: (1) the non-wandering points (cf. Non-wandering point) should form a hyperbolic set  $\Omega$ , in which the periodic trajectories are everywhere dense (the so-called Smale's Axiom A); and (2) the stable and unstable manifolds of any two trajectories from  $\Omega$  should intersect transversally (the strong transversality condition). That these conditions are sufficient have now been proved in almost all cases; as regards their necessity, proof is only available if the definition of structural stability is somewhat changed.

#### (4) *Absolute stability*

Absolute stability practically means that a system is convergent for any choice of parameters and nonlinear functions, within specified and well-characterized sets.

In 1944, when studying the stability of an autopilot, Lur'e and Postnikov introduced the concept of absolute stability and the Lur'e problem [18]. Since then, the problem of absolute stability for Lur'e-type systems has received considerable attention and many fruitful results, such as Popov's criterion, circle criterion, and Kalman–Yakubovich–Popov (KYP) lemma have been proposed [19–21]. From the view of modern robustness theory, absolute stability theory can be considered as the first approach to robust stability of nonlinear uncertain systems [22].

Absolute stability theory guarantees stability of feedback systems whose forward path contains a dynamic linear time-invariant system and whose feedback path contains a memoryless (possibly time-varying) nonlinearity. These stability criteria are generally stated in terms of the linear system and apply to every element of a specified class of nonlinearities. Hence, absolute stability theory provides sufficient conditions for robust stability with a given class of uncertain elements. The literature on absolute stability is extensive. A convenient way to distinguish these results is to focus on the allowable class of feedback nonlinearities. Specifically, the small-gain, positivity, and circle theorems guarantee stability for arbitrarily time-varying nonlinearities, whereas the Popov criterion does not. This is not surprising since the Lyapunov function upon which the small-gain, positivity, and circle theorems are based is a fixed quadratic Lyapunov function which permits arbitrary time variation of the nonlinearity. Alternatively, the Popov criterion is based on a Lur'e–Postnikov Lyapunov function which explicitly depends on the nonlinearity, thereby restricting its allowable time variation.

#### (5) *Complete stability*

A system is said to be completely stable if each trajectory of the system converges toward an equilibrium point, as  $t \rightarrow \infty$ . Complete stability of neural networks is one of the most important dynamical properties in view of practical applications to solve a large number of signal processing tasks, including image processing, pattern recognition, and optimization problems.

The standard Lyapunov–Krasovskii functional methods or Lyapunov–Razumikhin function methods are usually used for global asymptotic stability analysis of delayed neural networks with a unique equilibrium point. For the analysis of complete stability, the Lyapunov method and the classic LaSalle approach are no longer effective

because of the multiplicity of attractors [23]. For example, for PWL neuron activations a nonstrict energy is present both in the generic case where the neural network equilibrium points are isolated, as well as in the degenerate case where there are infinite nonisolated equilibrium points [24]. It is well known that to prove complete stability using LaSalle approach for nonstrict energy functions, it is required to characterize the invariant sets contained in the set where the energy is constant on orbits. As a matter of fact, such a characterization seems hard to accomplish for the NNs with PWL function or cellular neural network (CNN), since the sets involved have a complex structure even in the simplest case where there are finitely many equilibrium points. The situation is further complicated in degenerate cases where there are infinitely many nonisolated equilibrium points. In [25], an example shows that the existence of a stable equilibrium point does not imply complete stability of a CNN.

In [24], a new method is proposed to study the complete stability of cellular neural network (CNN). It allows one to completely sidestep the analysis of the neural network invariant sets. The method is based on a fundamental limit theorem for the length of the neural network output trajectories. Namely, it has been shown that the symmetry of the CNN interconnection matrix implies that the total length of the CNN output trajectories is necessarily finite. This in turn ensures convergence of the outputs, and also the state variables, toward an equilibrium point. Furthermore, this result is true regardless of the nature of the set of the CNN equilibrium points, so that complete stability is naturally proved not only for isolated equilibrium points, but also when the equilibrium points are not isolated.

#### (6) Input-to-state stability (ISS)

The *input to state stability* property provides a natural framework in which to formulate notions of stability with respect to input perturbations. It is generally known that ISS and set-ISS are powerful tools in the analysis of the stability and robustness of control systems [26]. Seminal works on ISS and set-ISS include [27] which provides a converse Lyapunov theory for set stability, and [28–31] which introduce ISS, extend the notion to noncompact sets, and generalize to arbitrary closed invariant sets, respectively. Input-to-state stability was introduced in [32], and has proved to be a very useful paradigm in the study of nonlinear stability, see for instance [33–45], as well as its variants such as integral ISS and input/output stability [46–56]. The notion of ISS takes into account the effect of initial states in a manner fully compatible with Lyapunov stability, and incorporates naturally the idea of “nonlinear gain” functions. Roughly speaking, a system is ISS provided that, no matter what is the initial state, if the inputs are small, then the state must eventually be small. Dualizing this definition one arrives at the notion of detectability which is the main subject of input/output-to-state stability (IOSS). A system  $\dot{x} = f(x, u)$  with measurement “output” map  $y = h(x)$  is IOSS if there are some functions  $\beta \in \mathcal{KL}$  and  $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$  such that the estimate:

$$|x(t)| \leq \max\{\beta(x(0), t), \gamma_1(\|u_{[0,t]}\|), \gamma_2(\|y_{[0,t]}\|)\},$$

holds for any initial state  $x(0)$  and any input  $u(\cdot)$ , where  $x(\cdot)$  is the ensuing trajectory and  $y(t) = h(x(t))$  the respective output function. (States  $x(t)$ , input values  $u(t)$ , and output values  $y(t)$  lie in appropriate Euclidean spaces. We use  $|\cdot|$  to denote Euclidean norm and  $\|\cdot\|$  for supremum norm.) The terminology IOSS is self-explanatory: formally there is “stability from the I/O data to the state.” The term was introduced in the paper [57], but the same notion had appeared before: it represents a natural combination of the notions of “strong” observability [32] and ISS, and was called simply “detectability” in [58, 59] and was called “strong unboundedness observability” in [35].

Roughly, a system is *output stable* if, for any initial state, the output converges to zero as  $t \rightarrow \infty$ . Inputs may influence this stability in different ways, for instance, one may ask that output approaches to zero only for those inputs for which input approaches to zero, or just that output remains bounded whenever input is bounded. The notion of output stability is also related to that of stability with respect to two measures [60]. As in the corresponding ISS paper [61], there are close relationships between output stability with respect to inputs, and robustness of stability under output feedback. This suggests the study of yet another property, which is obtained by a “small gain” argument from IOS: there must exist some  $\mathcal{X} \in \mathcal{KL}$  so that  $|y(t)| \leq \beta(x(0), t)$  if  $|u(t)| \leq \mathcal{X}(|y(t)|)$ ,  $\forall t$ . Combining the traditional stability definition and input/output stability conception, there may form many new stability conceptions such as output Lagrange stability, output-Lagrange input-to-output stable, state-independent IOS, robustly output stable, and so on. Such behavior is of central interest in control theory [60]. For the stability research of neural networks, these concepts are also very important.

The relation between *stability and ISS* can be briefly stated as follows. There are two very conceptually different ways of formulating the notion of stability of control systems. One of them, which we may call the input/output approach, relies on operator-theoretic techniques. In this approach, a “system” is a causal operator  $F$  between spaces of signals, and “stability” is taken to mean that  $F$  maps bounded inputs into bounded outputs, or finite-energy inputs into finite-energy outputs. More stringent typical requirements are that the gain of  $F$  be finite (in more classical mathematical terms that the operator be bounded), or that it have finite incremental gain (mathematically, that it be globally Lipschitz). The input/output approach has been extremely successful in the robustness analysis of linear systems subject to nonlinear feedback and mild nonlinear uncertainties, and in general in the area that revolves around the various versions of the small-gain theorem. Moreover, geometric characterizations of robustness (gap metric and the like) are elegantly carried out in this framework. Finally, I/O stability provides a natural setting in which to study the classification and parameterization of dynamic controllers. On the other hand, there is the model-based, or state-space approach to systems and stability, where the basic object is a forced dynamical system, typically described by differential or difference equations. In this approach, there is a standard notion of stability, namely Lyapunov asymptotic stability of the unforced system. Associated to such a system, there is an operator  $F$  mapping inputs (forcing functions) into state trajectories (or into outputs, if partial measurements on states are of interest). It becomes of interest then to ask

to what extent Lyapunov-like stability notions for a state-space system are related to the stability of the associated operator  $F$ . It is well known that [62], in contrast to the case of linear systems, where there is an equivalence between state-space and I/O stability, for nonlinear systems the two types of properties are not so closely related. Even for the very special and comparatively simple case of feedback linearizable systems, this relation is far more subtle than it might appear at first sight: if one first linearizes a system and then stabilizes the equivalent linearization, in terms of the original system one does not in general obtain a closed-loop system that is input/output stable in any reasonable sense. However, it is always possible to make a choice of a feedback law that achieves such stability, in the linearizable case as well as for all other stabilizable systems [63].

A system that is ISS exhibits low overshoot and low total energy response when excited by uniformly bounded or energy- bounded signals, respectively. These are highly desirable qualitative characteristics. However, it is sometimes the case that feedback design does not render ISS behavior, or that only a weaker property than ISS is verified in a step in recursive design. Input-to-state stability concept gives a link between the two alternative paradigms of stability, I/O and state space. This notion differs fundamentally from the operator-theoretic ones that have been classically used in control theory, first of all because it takes account of initial states in a manner fully compatible with Lyapunov stability. Second, *boundedness* (finite gain) is far too strong a requirement for general nonlinear operators, and it must be replaced by nonlinear gain estimates, in which the norms of output signals are bounded by a nonlinear function of the norms of inputs, the definition of ISS incorporates such gains in a natural way. The iss notion was originally introduced in [32] and has since been employed by several authors in deriving results on control of nonlinear systems. It can be stated in several equivalent manners, which indicates that it is at least a mathematically natural concept: dissipation, robustness margins, and classical Lyapunov-like definitions. The dissipation characterizations are closely related to the pioneering work of Willems in 1976, who introduced an abstract concept of energy *dissipation* in order to unify I/O and state space stability, and in particular with the purpose of understanding conceptually the meaning of Kalman–Yakubovich positive-realness (passivity), and frequency domain stability theorems in a general nonlinear context. Four natural definitions of input-to-state stability are proposed, that is, from GAS to ISS, from lyapunov to dissipation, gain margins and estimates, all these case are equivalent for some kind of nonlinear system. More details can refer to famous reference [63].

#### (7) Practical stability

For a practical system, engineers concern not only stability in the sense of Lyapunov but also boundedness properties of the system responses. This is because a system might be stable or asymptotically stable in theory, while it is actually unstable in practice because the stable domain of the desired attractor is not large enough. On the other hand, sometimes the desired states of a system may be mathematically unstable, and yet the system may oscillate sufficiently near its state such that its performance is acceptable. That it, it is stable in practice. Taking into this fact, researchers have introduced the notion of *practical stability* [64–68]. Some important

results in practical stability analysis over a finite time interval has been obtained in [68], which is related to the stability definition over a finite time. From this point of view, *finite-time stability* is also investigated for dynamical systems [69–72].

#### (8) Estimation of *domain of attraction*

The study of the determination of stable region for nonlinear systems is one of the most interesting aspects [36, 73, 74]. For this reason in the last 20 years several efforts have been made on the subject, generally arising from Lyapunov stability theory [75–78]. Among these studies, to construct a Lyapunov function to estimate domain of attraction is usually adopted. Genesio, etc., used time reversing method (or trajectory reversing method) and demonstrated the results by a two-dimensional system [76]. Vidyasagar and Vannelli modified the Zubov’s theorem to compute the stable region [78]. They proposed a method to construct a rational function for Lyapunov function with the help of Taylor series expansion approximation. Reference [79] discussed how to maximize the estimation of the domain of attraction by choosing linear state feedback control law for nonlinear control systems.

In general, the origin of a given nonlinear system is not globally asymptotically stable, instead, it is locally asymptotically stable. Thus, it is important to know the stable region (or “domain of attraction”) of the operation point of the system. The domain of attraction of the origin is defined as  $S = \{x_0 : x(t, x_0) \rightarrow 0 \text{ as } t \rightarrow \infty\}$ , where  $x(\cdot, x_0)$  denotes the solution of the system corresponding to the initial condition  $x(0) = x_0$ . In addition, the domain of attraction  $S$  is also called “region of attraction”, “basin” or “stable region (margin).” All trajectories starting within this neighborhood converge to the origin. For technical systems the knowledge of the size of such a region is very important because it contains all the initial states which lead to an asymptotically stable system behavior. Unfortunately, in general, an algebraic description of this region is not available [80, 81].

As is well known, Hopfield-type neural networks are mainly applied as either associative memories (pattern recognition) or optimization solvers. When applied as associative memories, the equilibrium points of the neural networks represent the stored patterns. The attraction domain of each equilibrium point coincides with the region from which the corresponding stored pattern can be retrieved even in the existence of noise, that is, the attraction domain of a stable equilibrium point characterizes the error-correction capability of the corresponding stored pattern. When applied as an optimization solver, the equilibrium points of the neural networks characterize all possible optimal solutions of the optimization problem. The attraction domain of each equilibrium point then coincides with the region that the network, starting from any initial guess in it, will evolve to the optimal solution. Therefore, identifying the attraction domain is important in the application of neural networks.

The approaches extensively used in the existing investigation into this field of neural networks are mainly based on Lyapunov direct method and so depend on the construction of Lyapunov function. However, there is no general rule guiding us to construct an optimal Lyapunov function for a given system, that is, constructing a Lyapunov function requires skill. Meanwhile, in the existing results, the Lyapunov functions used to characterize the attraction domain are mostly constructed by the method of characteristics, which strongly depends on the solutions of system [82].

Therefore, how to propose simple algorithms to estimate the domain of attraction for nonlinear system with and without control is still a challenging topic.

Note that, some contents in above two sections are from the Wikipedia on the internet. Following the trajectories of development of Lyapunov stability theory and other stability methods, some brief comments can be provided by the authors.

(1) Although the *Lyapunov stability theory* was proposed in his Ph.D. dissertation in 1892, its popularization just began in the 1950s, which is a delay over half a century. Before the 1950s, Lyapunov was a famous scientist only in Russia, while after 1950s, Lyapunov was a world famous scientist! This phenomenon is mainly due to the successful application of Lyapunov stability theory to the stability research of aerospace guidance systems which typically contain strong nonlinearities not treatable by other methods in the Cold War (1953–1962) period. Therefore, it is the famous application that promotes the popularization of one potential theoretical achievement, along with the discovery's name.

(2) Almost all theoretical findings and achievements have potential application value in practice. One theoretical achievement can be significantly recognized after a long time, for example, some decades or centuries. This may be attributed to the assonance or synchronization of theory and application. After all, all the theoretical achievements are used to deal with the matters encountered in the real life world. Significant and large-scale application projects may produce or promote the rapid development of some kinds of technical theories. Small scale applications may keep the development of some technical theories gradually. Almost all the theories in the application fields belong to the technical theory, or more accurately the techniques. Therefore, it is not reasonable to expect one theoretical achievement to have significant scientific value in real world in a short time. How to evaluate the theoretical achievement should be systematically considered, for example, by the peer review of authorities and public evaluations.

(3) No matter what the theoretical researchers do, their projects or interests must have some relations to the reality, for example, social science and natural science. Therefore, theoretical research achievement may not be a direct application in industrial fields, but it can be very useful in social science. Keeping in mind, research or problem is doomed to come from the reality, the mixed world of livings and nature. There are many phenomena to be discovered and explained. How to build or find a relation or bridge to the real word is a key way to demonstrate the value of the theoretical research.

(4) If one researcher has no a specific research direction in his speciality, he can trace the plan and demand of the state's project to find a topic, which is better to relate to the field of his research interest. Therefore, with the development of the state's project, theoretical problems occurred in the real world are indispensable and need to be solved. Under such kind of background, some theories associated with the projects or the problems may be emphasized gradually. Thus, it is not difficult to understand that much theoretical research has its historical roles in the developments of human society.

(5) Along with the same line as above arguments, we can consider the importance of the stability research of recurrent neural networks. In the 1980s, the digital and

electronic computers encountered many difficulties to be dealt with. In this case, analog computer began to appear, and neural computer as one of typical representatives began to be studied intensively. One of the fundamental problem of neural computer is the calculation ability. In this circumstance, Hopfield studied a kind of additive neural networks model and found their calculating capability. Then in 1984, Hopfield designed a circuit implementation of neural optimizer, which lay a solid foundation for the analog computer. Because the neural optimizer is a dynamical system, and its dynamics effect the stability of the equilibrium point, which is directly related to the optimal solution of the corresponding optimization problems, the stability analysis of recurrent neural networks began to be developed since 1982 or 1983. Up to now, it has gone thirty years for the research of stability theory of recurrent neural networks. There are many different achievements to be obtained.

(6) For a given equilibrium point of a dynamical system (i.e., the considered equilibrium point must be known in advance), how the initial values or the boundary values of the dynamical systems affect the stability property of the given equilibrium point is the main topic of Lyapunov's stability theory. Interestingly, the size of the stability degree  $\epsilon$  or the initial size of neighbor  $\delta$  in the definition of stability is not deterministic. Therefore, there are more space to define different kind of stability concept. In contrast, Hopfield studied the stability problem of the so-called Hopfield neural networks in the sense of *Hopfield's stability* instead of Lyapunov's stability (for example, the energy function introduced by Hopfield is not a standard Lyapunov function, which has been pointed by X. Liao in [83]). In fact, no matter what kind of definitions of stability, the common purpose of the definition is to solve the practical problems both in engineering and theory. Therefore, there are many different kind of stability concepts being proposed such as structure stability, practical stability, connective stability, synchronization stability, periodic stability, and so on. Keeping in mind, there is no fixed form of stability definition (correspondingly, the stability theory), all the theoretical researches must adapt to the different demands of practical applications.

(7) *Lyapunov second method* in essence is the gradient descent method of solving the numerical computation. This method is fundamental in the numerical analysis, which can be equivalent to the Euler method or the tangents method. All these methods are the local methods, which means the boundedness of the initial space or universe. According to the construction of Lyapunov's energy function, many optimization problems can be solved based on the gradient descent algorithms. Hopfield neural network itself is just a kind of gradient dynamical system, which can be directly applied to the optimization problems. How to build the relationship between dynamical neural network model and its energy function is the key problems. It is Hopfield who creatively proposed the *energy function* in the sense of Hopfield's definition, and gave the stability analysis of the concerned neural network in the sense of Hopfield's stability. Note that, as pointed out in Chap. 1, Professor Grossberg used the concept of energy function in neural networks field more earlier than hopfield did. However, since Professor Hopfield used a kind of additive neural networks to solve the optimization problems and, meanwhile, used the energy function concept to analyze the stability property of the concerned neural networks successfully in



1982–1984, researches on neural networks began to recover. It is the Hopfield's pioneering work that initiated the new era of optimization computation in neural network community. As far as this point is concerned, Hopfield is more excellent than Grossberg in control and optimization engineering fields. For convenience to mention the work by Hopfield, the additive neural network model studied by Hopfield is called Hopfield neural networks, while the energy function used by Hopfield is called Hopfield energy function, and the corresponding stability concept is also called Hopfield stability by the later researchers. Due to the introduction of Hopfield's energy function into recurrent neural networks, optimization problems based on RNNs in many engineering fields such as mechanics, dynamic engineering, architecture, operation research, computational science, and so on have been solved or promoted significantly.

(8) In *Lyapunov stability theory*, the concerned system is in the absence of inputs, i.e.,  $\dot{x}(t) = f(x(t))$ , and the equilibrium point is usually assumed to be zero, that is,  $f(0) = 0$ . This requirement in fact is the most important assumption in the application of Lyapunov method. How to guarantee the zero solution being the equilibrium point was not discussed in the Lyapunov stability theory. After all, in 1892, the considered system was only limited to the *isolated systems* (respect to the concept of complex systems or complex networks at present), and the stability of the concerned system on its own is existent or objective. There is no need to discuss the existence and uniqueness of the equilibrium point of the concerned systems. Therefore, from the viewpoint of energy, a system must be stable when the energy approached to the minimum, i.e., zero. Therefore, zero, as an objective existence, is believed to be a natural way to understand the world. However, with the emergence of complex systems or complex phenomena, how to determine the stable states is not easy, how to find the minimal point is not easy either. Equivalence is not necessarily equivalent. Hence, Lyapunov's stability theory falls into the scope of Newton mechanism, which is based on the reference frame or *reference coordinate system*. Different selection of reference frame may have different influence on the stability analysis of the concerned system. On the Earth, it is natural to choose the Earth as the reference frame. However, when studying the relative motion among different objects, how to select the frame is important for the considered problems. For example, *synchronization stability* as an extension of classical stability, is now intensively emphasized by the researchers due to the emergence of world wide web, interconnection networks, internet of things, and so on. In fact, in the research of synchronization problem, Lyapunov stability theory is still valid. Recalling the Lyapunov stability theory again, we can find that the nonlinear systems studied by Lyapunov is just an *error system*. This is the underlying reason why Lyapunov stability theory can be used in the observer design, filter design and synchronization! Therefore, Lyapunov stability is about the stability of error system. Since the intrinsic equilibrium point of the nonlinear system  $\dot{x}(t) = f(x(t))$  is zero, then the error system between the state and the intrinsic equilibrium point is just the nonlinear system itself  $\dot{x}(t) = f(x(t))$ . The zero in  $f(0) = 0$  is just the relative distance or the error between one state and an reference point. The reference point can be the intrinsic equilibrium point of the nonlinear system itself or the desired target out side the nonlinear system. If the *relative motion*

is static or the zero solution of error system is stable, it is the fundamental meaning of Lyapunov stability theory (i.e., *relative stability* or stability is just the relatively static movement). This also implies that idea of relativity has been used in Lyapunov stability theory, which is the fundamental reason of the universality of Lyapunov stability theory. When the input is considered in the system, i.e.,  $\dot{x}(t) = f(x(t), u(t))$  and  $y(t) = h(x(t))$ , stability with respect to different variables may form different stability concepts [84], for example input-state stability, integral input-to-state stability, and so on. Therefore, in the new environments, Lyapunov stability theory should be kept with the times, which can make the idea or thoughts of Lyapunov stability theory be carried forward and further developed. For example, Lyapunov synchronization stability theory (LSST) should be regarded as an upgrade of classical Lyapunov stability theory (LST) in the networks era. The most outstanding features of LSST include: 1) The equilibrium point of each node system is not required. The synchronous state or synchronous target can be the dynamics of any node system or their combinations, or the external specified target. 2) Synchronization stability is a relative stability. Many existing stability definitions such as Lyapunov stability, ISS, IOSS, stability of fixed points, stability of orbits, stability of sets can be unified in the framework of synchronization stability. 3) Many synthesis problems such as regulation/tracking problems, observer, filter, master-slave synchronization, drive-response synchronization, state/parameter estimation, system/parameter identification can be unified in the frameworks of synchronization stability theory. While classical Lyapunov stability theory has already promoted the development of automatic control theory for isolated systems, Lyapunov synchronization stability theory would be sure to promote the development of automatic control theory for complex interconnected networks.

(9) Referring to the stability definition in the sense of Lyapunov, there may have many different stability definitions existing in practice according to different applications. One of remarkable stability concepts is the stability of living systems. This kind of system has many stably periodic trajectories, limited cycles or other complex dynamics, except the fixed equilibrium points. For example, the heart rate and biological cycle in a living creature are all sinusoidal wave. This kind of sinusoidal signal may include two parts: frequency and amplitude. Therefore, stability definition may be a mixed concept of time domain, frequency domain, and space domain. Inspired by the *Lyapunov stability theory*, many different kinds of stability definition can be proposed. So does the stability theory of complex neural networks.

(10) In the qualitative research of dynamical systems, most efforts are placed on the external evolutionary dynamics of dynamical systems, for example, the infinite state behavior of dynamical systems as time approaches to infinity. This leads to the different definitions of stability and their corresponding stability results. It is well known that most fixed point or equilibrium point is locally stable. Starting from different *domain of attraction* of equilibrium may lead to different dynamical trajectories. This feature of initial domain is especially important in the learning of neural networks when choosing the initial weight values, as well as in the associative memory and pattern formation. Therefore, there are two directions in the qualitative analysis of dynamical systems. One is concerned with the ultimate state behavior

as time approached to infinity, such as many kinds of qualitative characteristic researches as global asymptotical stability, input-to-state stability, passivity, domain of attraction of equilibrium and dissipativity. The other is concerned with the initial condition set as time just begins, which is a inverse mapping of the infinite dynamical behavior of a dynamical system. The origin or the beginning of the initial condition is the definition domain of the concerned problem, from which the ultimate dynamic behavior of the concerned system begins to evolve as time approaches to infinity. In general, a good initial condition may trigger a better solution of the concerned problem and a less cost of the design procedure.

(11) Nowadays, more emphasis is placed on the stability of dynamical systems. This kind of system can be modeled by differential equation, difference equation, or other kind of recurrent forms. However, in the real world, too many systems can not be defined or modeled by the mathematical analytical equations. For this kind of nonanalytical systems, the results for the dynamical systems can not be used. For example, for a complex *data-driven system*, it is impossible to model it by mathematical model accurately. Any approximated description method including differential equation and difference equation can be suitable for the case of small scale systems, and not suitable for the large-scale complex systems. This evolution is similar to the principle of using distributed delay to replace the discrete delay for the large-scale circuits. Therefore, how to establish data-driven stability theory for the large-scale complex system is an urgent project in the contemporary era. This will be similar to the emergence of Lyapunov stability theory in the 1900s.

In brief, doing research with great concentration is a basic principle. One cannot expect his findings to be popularized in a short time. A good attitude for a researcher is a prerequisite for his great achievement, no matter when his achievements have been recognized and popularized.

## 2.5 Applications of Dynamical Systems Theory

In the following, some application fields are listed for the significant contribution of dynamical systems theory.

### (1) In biomechanics

In sports biomechanics, dynamical systems theory has emerged in the movement sciences as a viable framework for modeling athletic performance. From a dynamical systems perspective, the human movement system is a highly intricate network of codependent subsystems (e.g., respiratory, circulatory, nervous, skeletomuscular, perceptual) that are composed of a large number of interacting components (e.g., blood cells, oxygen molecules, muscle tissue, metabolic enzymes, connective tissue and bone). In dynamical systems theory, movement patterns emerge through generic processes of self-organization found in physical and biological systems.

## (2) In cognitive science

*Dynamical system theory* has been applied in the field of neuroscience and cognitive development, especially in the neo-Piagetian theories of cognitive development. It is the belief that cognitive development is best represented by physical theories rather than theories based on syntax and artificial intelligence (AI). It is also believed that differential equations are the most appropriate tool for modeling human behavior. These equations are interpreted to represent an agent's cognitive trajectory through state space. In other words, dynamicists argue that psychology should be the description (via differential equations) of the cognitions and behaviors of an agent under certain environmental and internal pressures. The language of chaos theory is also frequently adopted. In it, the learner's mind reaches a state of disequilibrium where old patterns have broken down. This is the phase transition of cognitive development. Self-organization (the spontaneous creation of coherent forms) sets in as activity levels link to each other. Newly formed macroscopic and microscopic structures support each other, speeding up the process. These links form the structure of a new state of order in the mind through a process called scalloping (the repeated building up and collapsing of complex performance). This new state is progressive, discrete, idiosyncratic, and unpredictable. Dynamical systems theory has recently been used to explain a long-unanswered problem in child development referred to as the A-not-B error.

## (3) In human development

Dynamical systems theory is a psychological theory of human development. Unlike dynamical systems theory, which is a mathematical construct, dynamical systems theory is primarily nonmathematical and driven by qualitative theoretical propositions. This psychological theory does, however, apply metaphors derived from the mathematical concepts of dynamical systems theory to attempt to explain the existence of apparently complex phenomena in human psychological and motor development.

As it applies to developmental psychology, this psychological theory was developed by Esther Thelen, Ph.D. at Indiana University Bloomington [85]. Thelen became interested in developmental psychology through her interest and training in behavioral biology. She wondered if "fixed action patterns," or highly repeatable movements seen in birds and other animals, were also relevant to the control and development of human infants.

According to Miller [86], dynamical systems theory is the broadest and most encompassing of all the developmental theories. Theory attempts to encompass all the possible factors that may be in operation at any given developmental moment, i.e., it considers development from many levels (from molecular to cultural) and timescales (from milliseconds to years). Development is viewed as constant, fluid, emergent or nonlinear, and multidetermined. Dynamical systems theory's greatest impact lies in early sensorimotor development. However, researchers working in fields closely related to (developmental) psychology such as linguistics have built upon Thelen's work in order to, for example, model the development of language

in an individual using dynamic systems theory by linking language development to overall cognitive development.

Esther Thelen believed that development involved a deeply embedded and continuously coupled dynamic system. It is unclear, however, if her utilization of the concept of “dynamic” refers to the conventional dynamics of classical mechanics or to the metaphorical representation of “something that is dynamic” as applied in the colloquial sense in common speech, or both. The typical view presented by R.D. Beer showed that information from the world goes to the nervous system, which directs the body, which in turn interacts with the world. Esther Thelen instead offers a developmental system that has continual and bidirectional interaction between the world, nervous system and body. The exact mechanisms for such interaction, however, remain unspecified.

The dynamical systems view of development has three critical features that separate it from the traditional input–output model. Firstly, the system must be multiply causal and self-organizing. This means that behavior is a pattern formed from multiple components in cooperation with none being more privileged than another. The relationship between the multiple parts is what helps provide order and pattern to the system. Why this relation would provide such order and pattern, however, is unclear. Secondly, a dynamic system depends on time making the current state a function of the previous state and the future state a function of the current state. The third feature is the *relative stability* of a dynamic system. For a system to change, a loose stability is needed to allow for the components to reorganize into a different expressed behavior. What constitutes a stability as being loose or not loose, however, is not specified. Parameters that dictate what constitutes one state of organization versus another state are also not specified, as a generality, in dynamical systems theory. The theory contends that development is a sequence of times where stability is low allowing for new development and where stability is stable with less pattern change. The theory contends that to make these movements, you must scale up on a control parameter to reach a threshold (past a point of stability). Once that threshold is reached, the muscles begin to form the different movements. This threshold must be reached before each muscle can contract and relax to make the movement. The theory can be seen to present a variant explanation for muscle length-tension regulation but the extrapolation of a vaguely outlined argument for muscle action to a grand theory of human development remains unconvincing and unvalidated.

## 2.6 Notations and Discussions on Some Stability Problems

This section is divided into two parts. One is for the symbol notations, basic lemmas, basic definition of stability and equilibrium point. The other is for the discussions on some stability concepts, which will show the diversity of stability definitions.

### 2.6.1 Notations and Preliminaries

Throughout this book, the following notations are used if no confusion occurs.

Let  $\mathbb{R}^n$  denote the  $n$ -dimensional Euclidean space, and  $\mathbb{R}$  denote the real space. Let  $A = [a_{ij}]_{n \times n}$  denote an  $n \times n$  matrix. Let  $W^T$ ,  $W^{-1}$  denote the transpose and the inverse of a square matrix  $W$ , respectively. Let  $W > 0 (< 0)$  denote a positive (negative) definite symmetric matrix.  $I$  denote an identity matrix with compatible dimension.  $A = \text{diag}(a_i)$  denotes the diagonal matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The symbol  $*$  is used to denote a matrix which can be inferred by symmetry. If  $A$  is a matrix,  $\|A\|$  denotes its operator norm or Euclidean norm.  $\lambda_{\max}(A)$ ,  $\lambda_{\min}(A)$  or  $\lambda_M(A)$  and  $\lambda_m(A)$  mean the maximum/largest and minimum/smallest eigenvalue of  $A$  respectively.  $\text{Re}(\lambda) < 0$  means the real part of eigenvalue  $\lambda$  is negative, where  $\lambda$  is the eigenvalue of a square matrix  $A$ . For  $h > 0$ ,  $\mathbb{C}([-h, 0]; \mathbb{R}^n)$  denotes the family of continuous functions  $\phi$  from  $[-h, 0]$  to  $\mathbb{R}^n$  with the norm  $\|\phi\| = \sup_{-h \leq s \leq 0} |\phi(s)|$ , where  $|\cdot|$  is the Euclidean norm in  $\mathbb{R}^n$ .

**Definition 2.1** (*Positive (semi)definite function*) A function  $V : \mathcal{D} \rightarrow \mathbb{R}$ , where  $\mathcal{D} \subseteq \mathbb{R}^n$  is said to be positive semidefinite if  $V(0) = 0$  and for every  $x \in \mathcal{D}$  it holds that  $V(x) \geq 0$ . It is called positive definite if additionally for every  $x \in \mathcal{D} - \{0\}$  it is true that  $V(x) > 0$ . A function  $V$  is called negative (semi)definite if  $-V$  is positive (semi)definite.

**Definition 2.2** (*Positive (semi)definite matrix*) A matrix  $Q \in M_n(\mathbb{R})$  is called *positive (semi)definite* if the corresponding quadratic function  $V(x) = x^T Q x$  is positive (semi)definite.

An algebraic criterion exists to test whether a given symmetric matrix is positive definite or positive semidefinite.

**Definition 2.3** (*Criterion for positive semidefiniteness*) Let  $Q$  be a symmetric matrix with appropriate dimension. Then, it is *positive (semi)definite* if and only if all its eigenvalues are (nonnegative)positive.

**Definition 2.4** (*Derivative along the trajectories of a system, see [87]*) Given a dynamical system  $\Sigma : \dot{x}(t) = f(x(t))$ , where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ , we define the derivative of  $V$  along the trajectories of the system  $\Sigma$  as:  $\frac{dV}{dt}|_{\Sigma} \equiv \frac{dV}{dt}|_{x(t)} = \frac{\partial V}{\partial z} f(z)|_{z=x(t)}$ .  $(L_f V)(x)$  or  $\dot{V}(x)$  is widely used to denote the derivative of  $V$ .

**Lemma 2.5** (Lyapunov stability theorem, see [87]) Let  $x = 0$  be an equilibrium point for the system  $\dot{x} = f(x)$  with  $x \in \mathbb{R}^n$  and  $\mathcal{D} \subseteq \mathbb{R}^n$  with  $\mathcal{D} \ni 0$ . The vector field  $f$  is locally Lipschitz (so that the differential equation admits unique solutions). Let  $V : \mathcal{D} \rightarrow [0, \infty)$  be a continuously differentiable function such that: 1)  $V$  is positive definite in  $\mathcal{D}$ . 2)  $L_f V$  is negative semidefinite in  $\mathcal{D}$ . Then  $x = 0$  is stable. If additionally  $L_f V$  is negative definite in  $\mathcal{D}$ , then the origin is locally asymptotically stable.

**Lemma 2.6** (Krasovskii–LaSalle principle) *The Krasovskii–LaSalle principle (also known as the invariance principle) is a criterion for the asymptotic stability of an autonomous (possibly nonlinear) dynamical system.*

*The global Krasovskii–LaSalle principle: given a representation of the system  $\dot{x} = f(x)$ , where  $x$  is the vector of variables, with  $f(0) = 0$ . If a  $C^1$  function  $V(x)$  can be found such that (1)  $V(x) > 0$  for all  $x \neq 0$  (positive definite); (2)  $\dot{V}(x) \leq 0$  for all  $x$  (negative seminegative); (3)  $V(x) \rightarrow \infty$  if  $x \rightarrow \infty$  and  $V(0) = \dot{V}(0) = 0$  (Such functions can be thought of as “energy-like”).*

*Let  $\mathcal{I}$  be the union of complete trajectories contained entirely in the set  $\{x : \dot{V}(x) = 0\}$ . Then the set of accumulation points of any trajectory is contained in  $\mathcal{I}$ . In particular, if  $\mathcal{I}$  contains no trajectory of the system except the trivial trajectory  $x(t) = 0$  for  $t \geq 0$ , then the origin is globally asymptotically stable.*

*Local version of the Krasovskii–LaSalle principle: if  $V(x) > 0$  when  $x \neq 0$ ,  $\dot{V}(x) \leq 0$  holds only for  $x$  in some neighborhood  $D$  of the origin, and the set  $\{\dot{V}(x) = 0\} \cap D$  does not contain any trajectories of the system besides the trajectory  $x(t) = 0$ ,  $t \geq 0$ , then the local version of the Krasovskii–LaSalle principle states that the origin is locally asymptotically stable.*

*Relation to Lyapunov theory. If  $\dot{V}(x)$  is negative definite, the global asymptotic stability of the origin is a consequence of Lyapunov’s second theorem. The Krasovskii–LaSalle principle gives a criterion for asymptotic stability in the case when  $\dot{V}(x)$  is only negative semidefinite.*

**Definition 2.7** (Lyapunov–Krasovskii functional, see [88–91]) Generalizations of the Lyapunov method to delay differential equations have been found, notably by Krasovskii [90]. As an example in [91] (see [Sect. 5.3, Corollary 3.1]), for a general system

$$\dot{x} = f(x_t), f : \mathcal{C}([-\tau, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n, \quad (2.6)$$

under usual regularity assumptions, the existence of a so-called Lyapunov–Krasovskii functional  $V : \mathcal{C}([-\tau, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$  and of  $\alpha_1, \alpha_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $\alpha_1$  is unbounded, and  $\alpha_2$  is positive definite, such that

$$\alpha_1(|x(t)|) \leq V(x_t), \frac{dV(x_t)}{dt} \leq \alpha_2(|x(t)|), \quad (2.7)$$

along the trajectories, ensures asymptotic stability of the origin. Here, one defines as usual  $x_t(s) = x(t+s)$ ,  $\tau \leq s \leq 0$ .

In particular, simple quadratic Lyapunov–Krasovskii functionals of the type

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau}^t x^T(\eta)Qx(\eta)d\eta, \quad (2.8)$$

for positive definite matrices  $P, Q \in \mathbb{R}^{n \times n}$  have been used early [91, 92].



**Definition 2.8** (*infinitesimal generator, see [93]*) Consider the stochastic system

$$dx(t) = f(x(t))dt + g(x(t))d\omega, \quad (2.9)$$

where  $x(t) \in \mathbb{R}^n$  is the system state,  $\omega$  is a  $r$ -dimensional standard Wiener process, and  $f(\cdot), g(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are locally Lipschitz functions and satisfy  $f(0) = g(0) = 0$ .

For any given  $V(x) \in \mathcal{C}^2$ , associated with the stochastic system (2.9), the *infinitesimal generator*  $\mathcal{L}$  is defined as follows:

$$\mathcal{L}V(x) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2} \text{Tr} \left\{ g^T(x) \frac{\partial^2 V}{\partial x^2} g(x) \right\}, \quad (2.10)$$

where  $\text{Tr}(A)$  is the trace of a matrix  $A = (a_{ij})_{n \times n}$ , i.e.,  $\text{Tr}(A) = \sum_{i=1}^n a_{ii}$ ,  $a_{ii}$  is the element on the main diagonal of square matrix  $A$ .

**Lemma 2.9** (Schur Complement, see [94]) For a given symmetric matrix  $S \in \mathbb{R}^{n \times n}$ , and  $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$ , where  $S_{ij} \in \mathbb{R}^{n_i \times n_j}$  are matrix blocks with appropriate dimensions, the following statements are equivalent: (1)  $S < 0$ ; (2)  $S_{11} < 0$ ,  $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$ ; (3)  $S_{22} < 0$ ,  $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$ .

**Definition 2.10** (*Nonsingular M-matrix, see [95]*) An  $n \times n$  matrix  $P$  with nonpositive offdiagonal elements is called a *nonsingular M-matrix* if all its principal minors are positive.

**Definition 2.11** (*Nonsingular M-matrix, see [95]*)  $P$  is a nonsingular *M-matrix* if and only if there exists a positive diagonal matrix  $D$  such that  $PD$  is a diagonally dominant matrix.

**Lemma 2.12** (see [95]) Let  $D_0$  be an  $n \times n$  positive diagonal matrix and  $P$  be an  $n \times n$  matrix with  $P = (p_{ij})_{n \times n}$ . If  $D_0 - |P|$  is a nonsingular *M-matrix* with  $|P| = (|p_{ij}|)_{n \times n}$ , then  $D_0 + P$  is nonsingular.

In mathematics and, specifically, real analysis, the Dini derivatives are a class of generalizations of the derivative. They were introduced by Ulisse Dini (1845–1918), who was an Italian mathematician, and is known for his contribution to real analysis [96].

**Definition 2.13** (*Dini derivative*)

(1) The upper *Dini derivative*, which is also called an upper right-hand derivative of a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , is denoted by  $f_+'$  and defined by

$$D^+ f(t) = f_+'(t) \triangleq \limsup_{h \rightarrow 0_+} \frac{f(t+h) - f(t)}{h},$$

where  $\limsup$  is the supremum limit.

(2) The lower Dini derivative  $f_-'$  is defined by

$$D_- f(t) = f_-'(t) \triangleq \liminf_{h \rightarrow 0^+} \frac{f(t+h) - f(t)}{h},$$

where  $\liminf$  is the infimum limit.

(3) If  $f(t)$  is defined on a vector space, then the upper Dini derivative at  $t$  in the direction  $d$  is defined by

$$D^+ f(t) = f_+'(t) \triangleq \limsup_{h \rightarrow 0^+} \frac{f(t+hd) - f(t)}{h},$$

where  $\limsup$  is the supremum limit.

(4) If  $f(t)$  is locally Lipschitz, then  $f_+'(t)$  is finite. If  $f(t)$  is differentiable at  $t$ , then the Dini derivative at  $t$  is the usual derivative at  $t$ .

Also,

$$\begin{aligned} D^- f(t) &\triangleq \limsup_{h \rightarrow 0^-} \frac{f(t+h) - f(t)}{h}, \\ D_- f(t) &\triangleq \liminf_{h \rightarrow 0^-} \frac{f(t+h) - f(t)}{h} \end{aligned}$$

are used to denote the upper Dini derivative and lower Dini derivative, respectively.

Therefore, when using the  $D$  notation of the Dini derivatives, the plus or minus sign indicates the left-hand or right-hand limit, and the placement of the sign indicates the infimum or supremum limit.

**Lemma 2.14** (Schauder fixed-point theorem) *The Schauder fixed-point theorem is an extension of the Brouwer fixed-point theorem to topological vector spaces, which may be of infinite dimension. It asserts that if  $K$  is a convex subset of a topological vector space  $V$  and  $T$  is a continuous mapping of  $K$  into itself so that  $T(K)$  is contained in a compact subset of  $K$ , then  $T$  has a fixed point.*

*A consequence, called Schaefer's fixed-point theorem, is particularly useful for proving existence of solutions to nonlinear partial differential equations. Schaefer's theorem is in fact a special case of the far reaching Leray–Schauder theorem which was discovered earlier by Juliusz Schauder and Jean Leray. The statement is as follows: Let  $T$  be a continuous and compact mapping of a Banach space  $X$  into itself, such that the set*

$$\{x \in X : x = \lambda Tx \text{ for some } 0 \leq \lambda \leq 1\}$$

*is bounded. Then  $T$  has a fixed point.*

**Lemma 2.15** (Brouwer's fixed-point theorem) *Brouwer's fixed-point theorem is a fixed-point theorem in topology, named after Luitzen Brouwer. It states that for any*

continuous function  $f$  mapping a compact convex set into itself there is a point  $x_0$  such that  $f(x_0) = x_0$ . The simplest forms of Brouwer's theorem are for continuous functions  $f$  from a closed interval  $I$  in the real numbers to itself or from a closed disk  $D$  to itself. A more general form than the latter is for continuous functions from a convex compact subset  $K$  of Euclidean space to itself.

Among hundreds of fixed-point theorems, Brouwer's is particularly well known, due in part to its use across numerous fields of mathematics. In its original field, this result is one of the key theorems characterizing the topology of Euclidean spaces, along with the Jordan curve theorem, the hairy ball theorem and the Borsuk–Ulam theorem. This gives it a place among the fundamental theorems of topology. The theorem is also used for proving deep results about differential equations and is covered in most introductory courses on differential geometry. It appears in unlikely fields such as game theory. In economics, Brouwer's fixed-point theorem and its extension, the Kakutani fixed-point theorem, plays a central role in the proof of existence of general equilibrium in market economies as developed in the 1950s by economics Nobel prize winners Kenneth Arrow and Grard Debreu.

**Lemma 2.16** (Contraction mapping principle) *In mathematics, a contraction mapping, or contraction or contractor, on a metric space  $(M, d)$  is a function  $f$  from  $M$  to itself, with the property that there is some nonnegative real number  $0 \leq k < 1$  such that for all  $x$  and  $y$  in  $M$ ,*

$$d(f(x), f(y)) \leq kd(x, y).$$

*The smallest such value of  $k$  is called the Lipschitz constant of  $f$ . Contractive maps are sometimes called Lipschitzian maps. If the above condition is instead satisfied for  $k \leq 1$ , then the mapping is said to be a non-expansive map.*

*More generally, the idea of a contractive mapping can be defined for maps between metric spaces. Thus, if  $(M, d)$  and  $(N, d^1)$  are two metric spaces, and  $f : M \rightarrow N$ , then there is a constant  $k < 1$  such that*

$$d^1(f(x), f(y)) \leq kd(x, y),$$

*for all  $x$  and  $y$  in  $M$ . Every contraction mapping is Lipschitz continuous and hence uniformly continuous (for a Lipschitz continuous function, the constant  $k$  is no longer necessarily less than 1).*

*A contraction mapping has at most one fixed point. Moreover, the Banach fixed-point theorem states that every contraction mapping on a nonempty complete metric space has a unique fixed point, and that for any  $x$  in  $M$  the iterated function sequence  $x, f(x), f(f(x)), f(f(f(x))), \dots$ , converges to the fixed point. This concept is very useful for iterated function systems where contraction mappings are often used. Banach's fixed-point theorem is also applied in proving the existence of solutions of ordinary differential equations, and is used in one proof of the inverse function theorem.*

In mathematics, in the field of differential equations, an initial value problem (also called the Cauchy problem by some authors) is an ordinary differential equation together with a specified value, called the initial condition, of the unknown function at a given point in the domain of the solution. In physics or other sciences, modeling a system frequently amounts to solving an initial value problem; in this context, the differential equation is an evolution equation specifying how, given initial conditions, the system will evolve with time.

**Definition 2.17** (*Initial value problem*) An initial value problem is a differential equation  $y'(t) = f(t, y(t))$  with  $f : \Omega \subset \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , where  $\Omega$  is an open set of  $\mathbb{R} \times \mathbb{R}^n$ , together with a point in the domain of  $f(t_0, y_0) \in \Omega$  called the initial condition.

A solution to an initial value problem is a function  $y$ , that is, a solution to the differential equation and satisfies  $y(t_0) = y_0$ .

In higher dimensions, the differential equation is replaced with a family of equations  $y'_i = f_i(t, y_1(t), y_2(t), \dots)$  and  $y(t)$  is viewed as the vector  $(y_1(t), \dots, y_n(t))$ . More generally, the unknown function  $y$  can take values on infinite dimensional spaces, such as Banach spaces or spaces of distributions. Initial value problems are extended to higher orders by treating the derivatives in the same way as an independent function, e.g.  $y''(t) = f(t, y(t), y'(t))$ .

Brouwer's fixed-point theorem, Schauder fixed-point theorem, contraction mapping principle and initial value problem are from Wikipedia, the free encyclopedia on the internet.

**Definition 2.18** (*Solution in the sense of Caratheodory, see [10]*) Consider the following ordinary differential equation (ODE) in  $\mathbb{R}^n$ ,

$$\dot{x} = f(x, t), t \geq 0, x(0) = x_0. \quad (2.11)$$

By a solution of Eq. (2.11) we mean a continuously differentiable function of time  $x(t)$  satisfying

$$x(t) = x_0 + \int_0^t f(x(s), s) ds. \quad (2.12)$$

Such a solution to Eq. (2.11) is called a *solution in the sense of Caratheodory*.

**Definition 2.19** (*Local existence and uniqueness, see [10]*) Consider the system (2.11). Assume that  $f(x, t)$  is continuous in  $t$  and  $x$ , and that there exist  $T, r, k, h$  such that for all  $t \in [0, T]$ , we have

$$\begin{aligned} |f(x, t) - f(y, t)| &\leq k|x - y|, \forall x, y \in B(x_0, r), \\ |f(x_0, t)| &\leq h, \end{aligned} \quad (2.13)$$

with  $B(x_0, r) = B_r = \{x \in \mathbb{R}^n : |x - x_0| \leq r\}$  is a ball of radius  $r$  centered at  $x_0$ . Then Eq.(2.11) has exactly one solution of the form of (2.12) on  $[0, \delta]$  for  $\delta$  sufficiently small.

**Definition 2.20** (*Global existence and uniqueness, see [10]*) Consider the system (2.11) and assume that  $f(x, t)$  is piecewise continuous with respect to  $t$  and for each  $T \in [0, \infty)$  there exist finite constants  $k_T, h_T$  such that for all  $t \in [0, T]$ , we have

$$\begin{aligned} |f(x, t) - f(y, t)| &\leq k_T |x - y|, \forall x, y \in \mathbb{R}^n, \\ |f(x_0, t)| &\leq h_T. \end{aligned} \quad (2.14)$$

Then Eq.(2.11) has exactly one solution on  $[0, T]$  for all  $T < \infty$ .

**Definition 2.21** (*Continuous dependence on initial conditions, see [10]*) Consider the system (2.11) and let  $f(x, t)$  satisfy the hypothesis (2.14). Let  $x(\cdot), y(\cdot)$  be two solutions of this system starting from  $x_0$  and  $y_0$  respectively. Then for given  $\epsilon > 0$ , there exists  $\delta(\epsilon, T)$  such that

$$|x_0 - y_0| \leq \delta \Rightarrow |x(\cdot) - y(\cdot)| \leq \epsilon. \quad (2.15)$$

**Definition 2.22** (*Lipschitz continuous, see [10]*) The function  $f$  is said to be locally Lipschitz continuous in  $x$  if for some  $h > 0$  there exists  $L \geq 0$  such that

$$|f(x_1, t) - f(x_2, t)| \leq L|x_1 - x_2|, \quad (2.16)$$

for all  $x_1, x_2 \in B_h, t \geq 0$ . The constant  $L$  is called the Lipschitz constant. A definition for globally Lipschitz continuous functions follows by requiring Eq.(2.16) to hold for  $x_1, x_2 \in \mathbb{R}^n$ . The definition of semi-globally Lipschitz continuous functions hold as well by requiring that Eq.(2.16) hold in  $B_h$  for arbitrary  $h$  but with  $L$  possibly a function of  $h$ . The Lipschitz property is by default assumed to be uniform in  $t$ .

If  $f$  is Lipschitz continuous in  $x$ , it is continuous in  $x$ . On the other hand, if  $f$  has bounded partial derivatives in  $x$ , then it is Lipschitz. Formally, if  $D_1 f(x, t) \triangleq \left[ \frac{\partial f_i}{\partial x_j} \right]$  denotes the partial derivative matrix of  $f$  with respect to  $x$  (the subscript 1 stands for the first argument of  $f(x, t)$ ), then  $|D_1 f(x, t)| \leq L$  implies that  $f$  is Lipschitz continuous with Lipschitz constant  $L$  (again locally, globally, or semi-globally depending on the region in  $x$  that the bound on  $|D_2 f(x, t)|$  is valid).

Definition 2.5 provides a universal definition on the stability and asymptotic stability. However, to meet the needs of this book, we will use the following delayed neural networks (DNNs),

$$\dot{x}(t) = -Ax(t) + Bg(x(t)) + Cg(x(t - \tau(t))) + U, \quad (2.17)$$

to present some stability definitions for convenience, where  $x(t) = (x_1(t), \dots, x_n(t))^T \in \mathbb{R}^n$ ,  $g(x(t)) = g_1(x_1(t)), \dots, g_n(x_n(t))^T \in \mathbb{R}^n$ ,  $g_i(x_i(t))$  is the neuronal

activation function,  $A = \text{diag}(a_1, \dots, a_n)$ ,  $a_i > 0$ ,  $B$  and  $C$  are connection matrices with appropriate dimensions,  $\tau(t)$  is a time-varying delay,  $0 \leq \tau(t) \leq \tau$ ,  $\dot{\tau}(t) \leq \mu$ ,  $\tau$  and  $\mu$  are positive constants,  $U = (U_1, \dots, U_n)^T \in \mathbb{R}^n$  is the external constant input vector,  $x_0(s) = \phi(s) \in \mathbb{C}$ ,  $i = 1, \dots, n$ . As usual, the solution  $x(t, x_0)$  is also called a trajectory of (2.17).

Let  $\mathcal{D} \subset \mathbb{R}^n$  be a subset.  $\mathcal{D}$  is said to be *invariant* under the system (2.17) if  $x_0 \in \mathcal{D}$  implies  $\Gamma_1(x_0) \subseteq \mathcal{D}$ , where  $\Gamma_1(x_0)$  is the trajectory of system (2.17) through  $x_0$ . A point  $x^*$  is called a  $\omega$ -*limit point* of  $\Gamma_1(x_0)$  if there is a subsequence  $\{t_i\}$  such that  $x^* = \lim_{i \rightarrow \infty} x(t_i, x_0)$ . All the  $\omega$ -limit points constitute the  $\omega$ -limit set  $\omega(\Gamma_1(x_0))$  of  $\Gamma_1(x_0)$ . The  $\omega$ -limit set is invariant under the dynamics. Recall that a constant vector  $x^*$  is said to be an *equilibrium state* of the system (2.17) if  $x^*$  is a zero point of operator  $F_1(x(t))$  defined by [97, 98],

$$F_1(x^*) = -Ax^* + Bg(x^*) + Cg(x^*) + U = 0. \quad (2.18)$$

The *equilibrium state*  $x^*$  is said to be *stable* if any trajectory of (2.17) can stay within a small neighborhood of  $x^*$  whenever the initial  $x_0$  is close to  $x^*$ , and is said to be *attractive* if there is a neighborhood  $\mathcal{E}(x^*)$ , called the attraction basin of  $x^*$ , such that any trajectory of (2.17) initialized from a state in  $\mathcal{E}(x^*)$  will approach to  $x^*$  as time goes to infinity. An equilibrium state  $x^*$  is said to be *asymptotically stable* if it is both stable and attractive, whilst the equilibrium state  $x^*$  is said to be *exponentially stable* if there exist a constant  $\alpha > 0$  and a strictly increasing function  $M : \mathbb{R} \rightarrow \mathbb{R}^+$  with  $M(0) = 0$  such that the following inequality holds,

$$\|x(t, x_0) - x^*\| \leq M(\|x(t, x_0) - x^*\|)e^{-\alpha t}. \quad (2.19)$$

Further,  $x^*$  is said to be *globally asymptotically stable* if it is asymptotically stable, and  $\mathcal{E}(x^*) = \mathbb{R}^n$ . System (2.17) is said to be *globally convergent* if  $x(t, x_0)$  converges to an equilibrium state of (2.17) for every initial point (the limit of  $x(t, x_0)$  may not be the same for different  $x_0$ ), whilst it is said to be *exponentially convergent* if it is globally convergent with  $x(t, x_0)$ , and its limit  $x^*$  satisfying (2.19) [97].

Note that, some exponential stability definitions are required to satisfy the form  $\|x(t, x_0) - x^*\| \leq \bar{M}(\|x(t, x_0) - x^*\|)e^{-\alpha t}$  for some positive value  $\bar{M} > 0$  and  $\alpha > 0$ . However, some scholars define  $\bar{M} \geq 1$  instead of  $\bar{M} > 0$ . Although this is a trivial problem, when  $0 < \bar{M} < 1$ , the exponential characteristic of the dynamical behavior is difficult to achieve. According to the *contraction mapping principle*, the state may be convergent but may not be exponentially convergent. As far as the exponential form of stability is concerned, many stability definitions satisfy this form. Therefore, strictly speaking, to the authors' knowledge, it is better to define  $\bar{M} \geq 1$  in the exponential stability definition.

### 2.6.2 Discussions on Some Stability Definitions

In this subsection, we will introduce the evolution of equilibrium from fixed, stable flow to invariant set, along with different stability understandings and stability definitions. This will help us to understand different kinds of stability definitions in depth.

**Definition 2.23** (*Absolute stability, see [22]*) The neural network (2.17) is said to be *robustly absolutely stable* in the sector  $[K_1, K_2]$  if the system is globally uniformly asymptotically stable for any nonlinear function  $g(x)$  satisfying  $(g(x(t)) - K_1 x(t))^T (g(x(t)) - K_2 x(t)) \leq 0$ .

**Definition 2.24** (*Absolute stability*) There exists a unique equilibrium point for DNNs (2.17) attracting all trajectories in phase space and, moreover, that this property is valid for all neuron activations within a specific class of nonlinear functions and for all constant input stimuli to the networks.

**Definition 2.25** (*Delay-independent condition, see [99]*) The system (2.17) with  $g(x(t)) = x(t)$  and  $U = 0$  is said to be stable independent of delay if it is asymptotically stable for every  $\tau(t) \in [0, \tau]$ . In this case one says that the system (2.17) is *absolutely stable*.

**Definition 2.26** The dynamical system defined by (2.17) is *globally asymptotically stable* (GAS) if there exists a unique equilibrium point,  $x^*$ , which is stable and to which every system trajectory converges.

The concept of absolute stability for DNNs (2.17) concerns the persistence of GAS when the input  $U$  is varied. (In general, one also varies the activation functions  $g(\cdot)$ , but the DNNs model assumes the activation functions belonging to a form of specific Class). When the types of activation function are given in (2.17), the following ABST definition is given in [100].

**Definition 2.27** (*see [100]*) The dynamical system described by (2.17) with fixed activation function is said to be *absolutely stable* if it possesses a GAS equilibrium point for each input  $U \in \mathbb{R}^n$ .

**Definition 2.28** (*Absolute stability, see [101]*) Consider the system (2.17), if we can find a Lyapunov function  $V(x(t))$  such that  $\dot{V}(x(t)) < 0$  for any initial condition  $x(t_0) = \phi(t) \in \mathbb{C}$ , then the system (2.17) is absolutely stable.

Neural network is called absolutely stable (ABST), i.e., that it possesses a globally asymptotically stable (GAS) equilibrium point for every neuron activation function and for every constant input vector. It is easily realized that the property of absolute stability is really desirable in view of solving many signal processing tasks. Consider, for example, self-organizing neural networks. Absolute stability guarantees that convergence is preserved even during the learning phase when parameters are slowly adjusted in an unpredictable way. Moreover, absolute stability ensures convergence



of the neural network also when, for some parameter values, there are infinitely many *nonisolated equilibrium points* (e.g., a *manifold of equilibria*). This feature is useful in practical problems, requiring convergence even in the presence of nonisolated equilibria. For instance, concerning minimization of multilinear polynomials with nonisolated minima, it was noted that the natural choice of the neural network parameters leads to a network with nonisolated equilibria. Another significant case is related to the neural network for the solution of linear or quadratic programming problems with infinite solutions. Once more, the design procedure naturally leads to the presence of manifolds of equilibria. A third case is that of gradient systems, which possess manifolds of equilibria in the generic case. It is also worth to mention that there are interesting problems where it is explicitly required to implement an associative memory, which is able to store and retrieve some pattern within a set of infinitely many nonisolated equilibrium patterns.

**Definition 2.29** (*Complete stability-I, see [95]*) DNNs (2.17) is said to be *completely stable* if for any initial continuous function  $\phi(t)$ , the solution  $x(t, \phi(t))$  of (2.17) satisfies  $\lim_{t \rightarrow \infty} x(t, \phi) = \text{constant}$ .

**Definition 2.30** (*Complete stability-II, see [95, 102]*) DNNs (2.17) is said to be *completely stable* if for any initial value starting from  $x_0$  at  $t = t_0$ , the trajectory  $x(t; t_0, x_0)$  of (2.17) satisfies

$$\lim_{t \rightarrow \infty} \|x(t; t_0, x_0) - x^*\| = 0 \quad (2.20)$$

where  $x^*$  is an equilibrium point of neural networks (2.17).

Above definitions of complete stability mean that each trajectory converges toward an equilibrium point (a *stationary state*), possibly within a set of many equilibrium points. In [103], the property of complete stability is referred to as *global pattern formation*, in order to highlight the ability of the neural networks to produce a steady-state pattern, i.e.,  $\lim_{t \rightarrow \infty} x(t)$  in response to any input pattern  $U$  and initial activity pattern  $x_0$ .

It is shown that several classes of real-value neural networks (RVNNs) can be completely stable by combining energy minimization and the *LaSalle invariant principle*. The equilibrium points of such networks were required to be isolated. However, under the framework of combining the energy minimization method and the Cauchy convergence principle to study complete stability for RVNNs, the equilibrium points of such networks were no longer required to be isolated. Meanwhile, complete stability for continuous-time and discrete-time RVNNs was further considered by combining the energy minimization method and the Cauchy convergence principle, respectively. Furthermore, as the extensive versions of the existing complete stability results for RVNNs, complete stability for discrete-time complex-value neural networks (CVNNs) was investigated by the energy minimization method. Note that GAS implies complete stability, but not vice versa [95].

**Definition 2.31** (*Strict Lyapunov function, see [24]*) An energy function  $V$  is said to be strict if and only if the set  $\{x \in \mathbb{R}^n | \dot{V}(x) = 0\}$  coincides with the set of equilibrium points. This is equivalent to the fact that the energy is strictly decreasing along nonstationary solutions.

*Remark 2.32* It is necessary to give some comparisons among absolute stability, complete stability, asymptotical stability, and global stability.

(1) As one of the basic enabling properties of multistable neural networks, complete stability, which allows each trajectory converges toward an equilibrium point, possibly within a set of many equilibrium points. In contrast, absolute stability has a unique equilibrium point.

(2) Complete stability of (2.17) can hold both in the generic case where (2.17) has finitely many equilibrium points, as well in degenerate situations where there are infinitely many *nonisolated equilibrium points* [24].

(3) Consider the neural networks (2.17) with symmetric interconnection matrices, and neuron activation is a continuous, nondecreasing, and bounded piecewise linear (PWL) function, such that  $g(0) = 0$ . Then, within this class global pattern formation is absolutely stable, i.e., complete stability holds for any choice of the parameters defining  $A, B, C, U$  and  $g(\cdot)$  in neural network [24]. Few results are reported on absolute stability of global pattern formation [24]. In some constraints, complete stability and absolute stability can be equivalent.

(4) In the analysis of complete stability, the Lyapunov method and the classic LaSalle approach are no longer effective because of the multiplicity of attractors, some new method must be proposed, for example, a convergence theorem of Gauss–Seidel method [23–25]. While for the *absolute stability*, the Lyapunov method and the classic LaSalle approach are very effective.

(5) Both absolute stability and asymptotical stability are concerned with the neural networks with unique equilibrium point. That is, for a specified equilibrium point, how to determine the stability property of the unique/specific equilibrium point. By contrast, complete stability and global stability is about the total dynamics of the concerned networks, not for a specific equilibrium state. Therefore, for neural networks with given equilibrium point, the Lyapunov stability theory can be effective, while for the nonisolated equilibrium point, *Lyapunov stability theory* can not be applied.

(6) *Global stability* of neural network (2.17) has been extensively investigated in the context of absolute stability theory, i.e., global absolute stability. When the input nonlinearities satisfy the sector constraints for only some finite range of their arguments, the network can only be guaranteed to be locally asymptotically stable. By contrast, complete stability itself is the *global dynamics*. Therefore, there is no global/local complete stability concept.

**Definition 2.33** (*Global dynamic behavior, see [104]*) A stable system equilibrium point of the neural network (2.17) is defined to be the state vector with all its components consisting of stable equilibrium states.

**Definition 2.34** (*Global stability, see [13]*) If every nonequilibrium solution of neural network (2.17) converges to an equilibrium, then neural network (2.17) is said to be *global stable*. In order to ensure that neural network (2.17) is globally stable, one requires that the set of equilibria for the networks (2.17) is discrete set. Thus, any point in the limit set of state is an equilibrium of networks (2.17) as  $t \rightarrow \infty$ , and this point approaches to some equilibrium of networks (2.17).

It follows from the above definition that a cellular neural network is always at one of its stable system equilibrium point after the transient has decayed to zero. From the dynamical systems theory point of view, the transient of a cellular neural network is simply the trajectory starting from some initial state and ending at an equilibrium point of the system. Since any stable equilibrium point of a cellular neural network is a *limit point* of a set of trajectories of the corresponding differential equations (2.17), such an attracting limit point has a basin of attraction, namely, the union of all trajectories converging to this point. Therefore, the state space of a cellular neural network can be partitioned into a set of basins centered at the stable system equilibrium points.

**Definition 2.35** Neural network (2.17) is said to be *bounded* if its each trajectory are bounded.

**Definition 2.36** (*Globally attractive set, see [98]*) Let  $S$  be a compact subset of  $\mathbb{R}^n$ . Denote the  $\epsilon$ -neighborhood of  $S$  by  $S_\epsilon$ . A compact set  $S$  is called a *globally attractive set* of neural network (2.17) if, for any  $\epsilon > 0$ , all the trajectories of neural network (2.17) ultimately enter and remain in  $S_\epsilon$ .

As a special class of RNNs, cellular neural networks have many outstanding features with stable dynamics when it is applied in pattern recognition and storage memory. Generally, cellular neural networks can be characterized by a large system of ordinary differential equations. Since all of the cells are arranged in a regular array, one can exploit many spatial properties, such as regularity, sparsity, and symmetry in studying the dynamics of cellular neural networks. There are two mathematical models which can characterize dynamical systems having these spatial properties. One is partial differential equation and the other is cellular automata. Partial differential equation, cellular automata, and cellular neural networks share a common property, namely, their dynamic behavior depend only on their spatial local interactions.

In general, the *limit set* of a complex nonlinear system is very difficult, if not impossible, to determine, either analytically or numerically. Although, for piecewise linear circuit, it is possible to find all dc solutions by using either a brute force algorithm or some more efficient ones, it is nevertheless very time consuming for large systems. For a cellular neural network, in view of the nearest neighbor interactive property, one can solve for all system equilibrium points by first determining the stable cell equilibrium states, and then using the neighbor interactive rules to find the corresponding system equilibrium. As presented above, the *dynamic behavior* of a cellular neural network with zero control operators and nonzero feedback operators is reminiscent of a two-dimensional cellular automaton. Both of them have the parallel signal processing capability and are based on the nearest neighbor interactive

dynamic rules. The main difference between a cellular neural network and a cellular automata machine is in their dynamic behaviors. The former is a continuous time while the latter is a discrete-time dynamical system. Because the two systems have many similarities, one can use cellular automata theory to study the *steady-state behavior* of cellular neural networks. Another remarkable distinction between them is that while the cellular neural networks will always settle to stable equilibrium points in the steady state, a cellular automata machine is usually imbued with a much richer dynamical behavior, such as periodic, chaotic, and even more complex phenomena. Of course, one can train a cellular neural network by choosing a sigmoid nonlinearity. If one chooses some other nonlinearity for the nonlinear elements, many more complex phenomena will also occur in cellular neural networks.

**Definition 2.37** ( *$\mathcal{K}$ -class function, see [63]*) A function  $\Phi : [0, a] \rightarrow [0, +\infty)$  is said to be positive if  $\Phi(s) > 0$  for all  $s > 0$  and  $\Phi(0) = 0$ . A continuous function  $\alpha : [0, a] \rightarrow [0, +\infty)$  is said to belong to class  $\mathcal{K}$  if it is positive, strictly increasing and  $\alpha(0) = 0$ . It is said to belong to class  $\mathcal{K}_\infty$  if  $a = \infty$  and  $\alpha(r) \rightarrow +\infty$  as  $r \rightarrow \infty$ . Similarly, the continuous function  $\beta : [0, a] \times [0, \infty) \rightarrow [0, \infty)$  is said to belong to class  $\mathcal{KL}$  if, for each fixed  $s$ , the mapping  $\beta(r, s)$  belongs to class  $\mathcal{K}$  with respect to  $r$  and, for each fixed  $r$ , the mapping  $\beta(r, s)$  is decreasing with respect to  $s$  and  $\beta(r, s) \rightarrow 0$  as  $s \rightarrow \infty$ .

An example of a class  $\mathcal{K}_\infty$  function is  $\alpha(r) = r^c$  with  $c > 0$ . An example of a class  $\mathcal{KL}$  function is  $\beta(r, s) = r^c e^{-s}$  with  $c > 0$ .

The system

$$\dot{x}(t) = f(x(t), u(t)), y(t) = h(x(t)), \quad (2.21)$$

is said to be *forward complete* if for every initial state  $x_0 = x(0) = \xi$  and for every input  $u$  defined on  $\mathbb{R}^+$ ,  $t_{\max} = +\infty$ . The corresponding output is denoted by  $y(t; \xi, u) = h(x(t; \xi, u))$  on the domain of definition of the solution, where the input means a measurable and locally essentially bounded function,  $x(t; \xi, u)$  denotes the unique maximal solution of the *initial value problem* of (2.21) with  $x_0 = x(0) = \xi$ .

**Definition 2.38** (see [105]) Let  $u = 0$ . For some region  $\mathcal{D} \subseteq \mathbb{R}^n$ , if for any  $\epsilon > 0$ , there exists a  $\delta = \delta(\epsilon)$  such that when  $x_0 \in \mathcal{D}$  and  $\|x_0\| \leq \delta$ , the following inequality holds:  $\|x(t, x_0)\| < \epsilon$ ,  $\forall t \geq 0$ . Then, system (2.21) is said to be *uniformly stable* (US) in  $\mathcal{D}$  and  $\mathcal{D}$  is called to be a stable region of system (2.21).

**Definition 2.39** (see [105]) Let  $u = 0$ . system (2.21) is said to be *uniformly asymptotically stable* (UAS) in  $\mathcal{D}$  if it is US in  $\mathcal{D}$ , and moreover the equality holds:  $\lim_{t \rightarrow \infty} \|x(t, x_0)\| = 0$ ,  $\forall t \geq 0$ .

**Definition 2.40** (see [105]) Let  $u = 0$ . If there exist positive constants  $\alpha > 0$ ,  $K \geq 1$  such that for any  $x_0 \in \mathcal{D}$ ,  $\|x(t, x_0)\| < K \|x_0\| e^{-\alpha t}$ ,  $\forall t \geq 0$ . Then system (2.21) is said to be *uniformly exponentially stable* (UES) in  $\mathcal{D}$  and  $\mathcal{D}$  is called to be an exponential stable region of system (2.21).

**Definition 2.41** (see [60]) A forward complete system (2.21) is

(1) *input-to-output stable* (IOS) if there exist a  $\mathcal{KL}$ -function  $\beta$  and a  $\mathcal{K}$ -function  $\gamma$  such that

$$|y(t; \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|u\|), \forall t \geq 0; \quad (2.22)$$

(2) *output-Lagrange input-to-output stable* (OLIOS) if it is IOS and there exist some  $\mathcal{K}$ -functions  $\sigma_1, \sigma_2$  such that

$$|y(t; \xi, u)| \leq \max\{\sigma_1(|h(\xi)|), \sigma_2(\|u\|)\}, \forall t \geq 0; \quad (2.23)$$

(3) *state-independent IOS* (SI IOS) there exist some  $\beta \in \mathcal{KL}$  and some  $\gamma \in \mathcal{K}$  such that

$$|y(t; \xi, u)| \leq \beta(|h(\xi)|, t) + \gamma(\|u\|), \forall t \geq 0. \quad (2.24)$$

**Definition 2.42** (see [59]) The system (2.21) without  $u(t)$  is *output-to-state stable* (OSS) if there exist some  $\beta \in \mathcal{KL}$  and some  $\gamma \in \mathcal{K}$  such that

$$|x(t, \xi)| \leq \max\{\beta(|\xi|, t), \gamma(\|y_\xi|_{[0,t]}\|)\}, \quad (2.25)$$

for all  $\xi \in \mathcal{X}$  and all  $t \in [0, t_{\max}]$ , where  $|\xi|$  indicates the Euclidean norm, and  $\|y_\xi|_{[0,t_0]}\|$  is the sup-norm of the restriction of  $y_\xi$  to real interval  $[0, t_0]$ , that is  $\sup_{t \in [0, t_0]} y_\xi(t)$ .

**Definition 2.43** (see [110]) The system (2.21) is *globally asymptotically stable* (GAS) if there exists a function  $\beta(s, t) \in \mathcal{KL}$ , such that, with the control  $u = 0$ , given any initial state  $\xi$ , the solution exists for all  $t > 0$  and it satisfies the estimate

$$|x(t)| \leq \beta(|\xi|, t), \quad (2.26)$$

for all  $t \geq 0$ .

**Definition 2.44** (see [110]) The system (2.21) is *input-to-state stable* (ISS) if there exist functions  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}_\infty$ , such that for each measurable essentially bounded control  $u(t)$  and each initial state  $\xi$ , the solution exists for each  $t \geq 0$ , and furthermore it satisfies

$$|x(t)| \leq \beta(|\xi|, t) + \gamma(|u(t)|), \quad (2.27)$$

for all  $t \geq 0$ .

The above definition of GAS is of course equivalent to the usual one (stability plus attractivity) but it is much more elegant and easier to work with. The definition of an ISS system is a natural generalization of this. Since  $\gamma(0) = 0$ , an ISS system is necessarily GAS. For linear systems  $\dot{x} = Ax + Bu$  with asymptotically stable

matrix  $A$ , an estimate (2.27) is obtained from the variation of parameters formula, but in general, GAS does not imply ISS. The notion of ISS is somewhat related to the classical *total stability* notion, but total stability typically studies only the effect of small perturbations (or controls), while ISS concerns with the bounded behavior for arbitrary bounded controls [110].

**Definition 2.45** (*Integral input-to-state stability (iISS)*, see [49]) System (2.21) is iISS if there exist functions  $\alpha \in \mathcal{K}_\infty$ ,  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$ , such that, for all  $\xi \in \mathbb{R}^n$  and all  $u$ , the solution  $x(t, \xi, u)$  is defined for all  $t \geq 0$ , and

$$\alpha(|x(t, \xi, u)|) \leq \beta(|\xi|, t) + \int_0^t \gamma(|u(s)|) ds, \quad (2.28)$$

for all  $t \geq 0$ .

Observe that a system is iISS if and only if there exist functions  $\beta \in \mathcal{KL}$  and  $\gamma_1, \gamma_2 \in \mathcal{K}$  such that

$$\alpha(|x(t, \xi, u)|) \leq \beta(|\xi|, t) + \gamma_1\left(\int_0^t \gamma_2(|u(s)|) ds\right), \quad (2.29)$$

for all  $t \geq 0$ , all  $\xi \in \mathbb{R}^n$ , and all  $u$ . Also note that if system (2.21) is iISS, then it is 0-GAS, that is, the 0-input system  $\dot{x} = f(x, 0)$  is globally asymptotically stable (GAS). (That is, the *zero solution* of this system is globally asymptotically stable.)

**Definition 2.46** (see [49]) A continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is called an iISS-Lyapunov function for system (2.21) if there exist functions  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  and  $\sigma \in \mathcal{K}$ , and a continuous positive definite function  $\alpha_3$ , such that

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|) \quad (2.30)$$

for all  $\xi \in \mathbb{R}^n$ , and

$$DV(\xi)f(\xi, u) \leq -\alpha_3(|\xi|) + \sigma(|u|), \quad (2.31)$$

for all  $\xi \in \mathbb{R}^n$ , and all  $u \in \mathbb{R}^m$ .

Note that the estimate (2.30) amounts to the requirement that  $V$  must be *positive definite* (i.e.,  $V(x) > 0$  for all  $x \neq 0$  and  $V(0) = 0$ ), and *proper* (i.e., radially unbounded, namely, as  $V(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ ).

Stability so far is studied with respect to equilibrium points. Stability, however, can be studied with respect to invariant sets.

**Definition 2.47** (*Stability of Sets*, see [87]) A set  $M$  is said to be stable for the system  $\dot{x} = f(x)$  if for every  $\epsilon > 0$ , there is a  $\delta = \delta(\epsilon) > 0$  such that  $\text{dist}_M(x) < \delta \Rightarrow \text{dist}_M(x(t, x_0)) < \epsilon$  for all  $t \geq 0$ . Where  $\text{dist}_M(x)$  denotes the distance of  $x$  from the set  $M$  defined as  $\text{dist}_M(x) = \inf_{z \in M} \|x - z\|$ .

**Definition 2.48** (*Asymptotic stability of Sets*, see [87]). A set  $M$  is said to be locally asymptotically stable for the system  $\dot{x} = f(x)$  if it is stable and additionally  $\lim_{t \rightarrow \infty} \text{dist}_M(x(t, x_0)) = 0$  for all  $x$  such that  $\text{dist}_M(x(t, x_0)) < \eta$  for some  $\eta > 0$ . If the limitation holds true for all  $x \in \mathbb{R}^n$ , then  $M$  is called a globally asymptotically stable set.

**Definition 2.49** (*LaSalle's Principle*, see [87]). Let  $\Omega \subset \mathcal{D}$  be a compact set that is positively invariant with respect to  $\dot{x} = f(x)$ . Let  $V : \mathcal{D} \rightarrow \mathbb{R}$  be a continuously differentiable and positive definite function such that the derivative of  $V$ , i.e.,  $L_f V$ , is negative semidefinite. Define the set:  $E \triangleq \{x \in \mathcal{D}, (L_f V)(x) = 0\}$ . Let  $C \subset E$  be a maximal invariant set in  $E$ . Then  $M$  is globally asymptotically stable.

Aforementioned stability definitions are mainly for the nonlinear autonomous systems. In the following, we will consider a kind of systems that may be represented by equations of the form [84],

$$\frac{dx}{dt} = f(x, t), \quad (2.32)$$

where  $x \in \mathbb{R}^n$ ,  $x = (x_1, \dots, x_n)^T$ ,  $F : \mathbb{R}^n \times J \rightarrow \mathbb{R}$  are considered,  $J = [t_0, \infty)$ ,  $t_0$  is an finite initial time instant. It is assumed that  $f$  is continuous on  $\mathbb{R}^n \times J$ , which denote their Cartesian product. The solutions of (2.32) are denoted by  $x(t; x_0, t_0)$  with  $x(t_0; x_0, t_0) = x_0$ . In general it is not required that  $f(0, t) = 0$ .

Let  $S(t) \in \mathbb{R}^n$  for all  $t \in J$ . Assume that  $S(t)$  is a connected open region. Let  $\bar{S}(t)$  denote the closure of  $S(t)$  and let  $\partial S(t)$  denote the boundary of  $S(t)$ . Assume that  $S(t)$  is bounded for all  $t \in J$ , and  $\lim_{t \rightarrow t_a} S(t)$  exists for all  $t_a \in J$ , and that  $\lim_{t \rightarrow t_a} S(t) = S(t_a)$ . Henceforth, whenever the symbol  $S(t)$  (with appropriate subscripts) is used to denote a set, it is assumed that this set possesses the properties described above. Let  $[S(t) - S_0(t)] = \{x \in \mathbb{R}^n : x \in S(t), x \notin S_0(t)\}$ ,  $B(a) = \{x \in \mathbb{R}^n : \|x\| < a\}$ ,  $\partial B(a) = \{x \in \mathbb{R}^n : \|x\| \leq a\}$ . Let  $t_i$  be any point (initial time) in  $J$ , and let  $x_i = x(t_i; x_i, t_i)$ . Then the following definitions are presented in [84].

**Definition 2.50** (*Uniformly stable*) System (2.32) is *stable* with respect to  $\{S_0(t), S(t), t_0\}$ , if  $x_0 \in S_0(t_0)$  implies  $x(t; x_0, t_0) \in S(t)$  for all  $t \in J$ . System (2.32) is *uniformly stable* with respect to  $(S_0(t), S(t))$ , if for all  $t_i \in J$ ,  $x_i \in S_0(t_i)$  implies that  $x(t; x_i, t_i) \in S(t)$  for all  $t \in [t_i, \infty)$ .

**Definition 2.51** (*Unstable*) System (2.32) is *unstable* with respect to  $\{S_0(t), S(t), t_0\}$ ,  $S_0(t_0) \subseteq S(t_0)$ , if  $x_0 \in S_0(t_0)$  and a  $t_c \in J$  such that  $x(t_c; x_0, t_0) \in \partial S(t_c)$ .

**Definition 2.52** (*Uniformly asymptotically stable*) System (2.32) is *asymptotically stable* with respect to  $\{S_0(t), S(t), S_f, t_0\}$ , if it is stability respect to  $\{S_0(t), S(t), t_0\}$ , and if in addition,  $x_0 \in S_0(t_0)$  implies  $x(t; x_0, t_0) \rightarrow S_f$  as  $t \rightarrow \infty$  ( $d(x(t; x_0, t_0), S_f) \rightarrow 0$  as  $t \rightarrow \infty$ , where  $d(x, S_f) = \inf_{y \in S_f} \|y - x\|$ ). System (2.32) is *uniformly asymptotically stable* with respect to  $\{S_0(t), S(t), S_f\}$ , if it is uniformly stable with respect to  $\{S_0(t), S(t)\}$ , and if in addition, for all  $t_i \in J$ ,  $x_i \in S_0(t_i)$  implies  $x(t; x_i, t_i) \rightarrow S_f$ , as  $t \rightarrow \infty$ .



**Definition 2.53** (*Practically stable*) If in Definition 2.50, if  $S(t) \equiv B(\beta) = \{x \in \mathbb{R}^n : \|x\| < \beta\}$ ,  $S_0(t) \equiv S_0(t_0) = B(\alpha)$ ,  $\alpha \leq \beta$ , then system (2.32) is said to be *practically stable* with respect to  $\alpha, \beta, t_0, \|\cdot\|$ , and uniformly practically stable with respect to  $\alpha, \beta, \|\cdot\|$ . If in Definition 2.52,  $S(t) \equiv B(\beta)$ ,  $S_0(t) \equiv S_0(t_0) = B(\alpha)$ ,  $S_f = B(0)$ ,  $\alpha \leq \beta$ , then system (2.32) is said to be *practically asymptotically stable* with respect to  $(\alpha, \beta, t_0, \|\cdot\|)$ , and uniformly practically asymptotically stable with respect to  $(\alpha, \beta, \|\cdot\|)$ . Finally, system (2.32) is said to be *practically exponentially stable* with respect to  $(\alpha, \beta, \gamma, t_0, \|\cdot\|)$ ,  $\alpha \leq \beta, \gamma > 0$ , if  $\|x_0\| < \alpha$  implies  $\|x(t; x_0, t_0)\| \leq \beta e^{-\gamma(t-t_0)}$  for all  $t \in J$ .

In the following, suppose  $f : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^n$  is continuous and consider the following retarded functional differential equation or delayed nonlinear system,

$$\dot{x} = f(t, x_t), \quad (2.33)$$

where  $x_t(\theta) = x(t + \theta)$  for  $\theta \in [-r, 0]$ . The function  $f$  will be supposed to be completely continuous and to satisfy enough additional smoothness conditions to ensure the solution  $x(\sigma, \phi)(t)$  through  $(\sigma, \phi)$  is continuous in  $(\sigma, \phi, t)$  in the domain of definition of the function. A function  $x$  is said to be a *solution* of Eq. (2.33) on  $[\sigma - r, \sigma + q]$  if there are  $\sigma \in \mathbb{R}$  and  $q > 0$  such that  $x \in \mathbb{C}([\sigma - r, \sigma + q], \mathbb{R}^n)$ ,  $(t, x_t) \in \mathbb{D}$ , and  $x(t)$  satisfies equation (2.33) for  $t \in [\sigma, \sigma + q]$ , where  $\mathbb{C} = \mathbb{C}([-r, 0], \mathbb{R}^n)$ , which is the Banach space of linear functions mapping the interval  $[-r, 0]$  into  $\mathbb{R}^n$  with the topology of uniform convergence. For given  $\sigma \in \mathbb{R}$ ,  $\phi \in \mathbb{C}$ , we say  $x(\sigma, \phi, f)$  is a solution of Eq. (2.33) with initial value  $\phi$  at  $\sigma$  or simply a solution through  $(\sigma, \phi)$ . Finding a solution of Eq. (2.33) through  $(\sigma, \phi)$  is equivalent to solving the integral equation

$$\begin{aligned} x_\sigma &= \phi \\ x(t) &= \phi(0) + \int_\sigma^t f(s, x_s) ds, \quad t \geq \sigma. \end{aligned} \quad (2.34)$$

**Definition 2.54** (*Uniformly asymptotically stable, see [91]*) Suppose  $f(t, 0) = 0$  for all  $t \in \mathbb{R}$ . The solution  $x = 0$  of system (2.33) is said to be *stable* if for any  $\sigma \in \mathbb{R}, \epsilon > 0$ , there is a  $\delta = \delta(\epsilon, \sigma)$  such that  $\phi \in \mathcal{B}(0, \delta)$  implies  $x_t(\sigma, \phi) \in \mathcal{B}(0, \epsilon)$  for  $t \geq \sigma$ . The solution  $x = 0$  of system (2.33) is said to be *asymptotically stable* if it is stable and there is a  $b_0 = b_0(\sigma) > 0$  such that  $\phi \in \mathcal{B}(0, b_0)$  implies  $x(\sigma, \phi)(t) \rightarrow 0$  as  $t \rightarrow \infty$ . The solution  $x = 0$  of system (2.33) is said to be *uniformly stable* if the number  $\delta$  in the definition is independent of  $\sigma$ . The solution  $x = 0$  of system (2.33) is said to be *uniformly asymptotically stable* if it is uniformly stable and there is a  $b_0 > 0$  such that, for every  $\eta > 0$ , there is a  $t_0(\eta)$  such that  $\phi \in \mathcal{B}(0, b_0)$  implies  $x_t(\sigma, \phi) \in \mathcal{B}(0, \eta)$  for  $t \geq \sigma + t_0(\eta)$  for every  $\sigma \in \mathbb{R}$ .

If  $y(t)$  is any solution of system (2.33), then  $y$  is said to be *stable* if the solution  $z = 0$  of the equation

$$\dot{z}(t) = f(t, z_t + y_t) - f(t, y_t), \quad (2.35)$$

is stable.

**Definition 2.55** (Uniformly ultimately bounded, see [91]) A solution  $x(\sigma, \phi)$  of system (2.33) is *bounded* if there is a  $\beta(\sigma, \phi)$  such that  $|x(\sigma, \phi)(t)| < \beta(\sigma, \phi)$  for  $t \geq \sigma - r$ ,  $r > 0$  is the maximal bound of time delay. The solutions are *uniformly bounded* if, for any  $\alpha > 0$ , there is a  $\beta = \beta(\alpha) > 0$  such that, for all  $\sigma \in \mathbb{R}$ ,  $\phi \in \mathbb{C}$ , and  $|\phi| \leq \alpha$ , we have  $|x(\sigma, \phi)(t)| \leq \beta(\alpha)$  for all  $t \geq \sigma$ . The solutions are *ultimately bounded* if there is a constant  $\beta$  such that, for any  $(\sigma, \phi) \in \mathbb{R} \times \mathbb{C}$ , there is a constant  $t_0(\sigma, \phi)$  such that  $|x(\sigma, \phi)(t)| < \beta$  for  $t \geq \sigma + t_0(\sigma, \phi)$ . The solutions are *uniformly ultimately bounded* if there is a  $\beta > 0$  such that, for any  $\alpha > 0$ , there is a constant  $t_0(\alpha) > 0$  such that  $|x(\sigma, \phi)(t)| \leq \beta$  for  $t \geq \sigma + t_0(\alpha)$  for all  $\sigma \in \mathbb{R}$ ,  $\phi \in \mathbb{C}$ ,  $|\phi| \leq \alpha$ .

If  $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a given positive definite continuously differentiable function, the derivative of  $V$  along a solution of system (2.33) is given by

$$\dot{V}(x(t)) = \frac{\partial V(x(t))}{\partial x} f(t, x_t). \quad (2.36)$$

In order for  $\dot{V}$  to be nonpositive for all initial data, one would be forced to impose very severe restrictions on the function  $f(\phi)$ . In fact, the point  $\phi(0)$  must play a dominant role and therefore, the results will apply only to equations which are very similar to ordinary differential equations.

A few moments of reflection in this proper direction indicates that it is unnecessary to require that Eq. (2.36) be nonpositive for all initial data in order to have stability. In fact, if a solution of the Eq. (2.33) begins in a ball and is to leave this ball at some time  $t$ , then  $|x_t| = |x(t)|$ , that is,  $|x(t+s)| \leq |x(t)|$  for all  $[s \in [-r, 0]]$ . Consequently, one need only consider initial data satisfying the latter property. This is the basic idea of *Razumikhin-type stability theorem*.

**Lemma 2.56** (Razumikhin uniform stability theorem, see [91]) Suppose  $f : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^n$  takes  $\mathbb{R} \times$  bounded sets of  $\mathbb{C}$  into bounded sets of  $\mathbb{R}^n$  and consider the system (2.33). Suppose  $u, v, w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are continuous, nondecreasing functions,  $u(s), v(s)$  positive for  $s > 0$ ,  $u(0) = v(0) = 0$ . If there is a continuous function  $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\begin{aligned} u(|x|) \leq V(t, x) \leq v(|x|), \quad t \in \mathbb{R}, x \in \mathbb{R}^n, \\ \dot{V}(t, \phi(0)) \leq -w(|\phi(0)|) \text{ if } V(t+s, \phi(s)) \leq V(t, \phi(0)), s \in [-r, 0], \end{aligned} \quad (2.37)$$

then the solution  $x = 0$  of system (2.33) is uniformly stable.

**Lemma 2.57** (Razumikhin uniform asymptotical stability theorem, see [91]) Suppose all of the conditions of Lemma 2.56 are satisfied and in addition  $w(s) > 0$  if  $s > 0$ . If there is a continuous nondecreasing function  $p(s) > s$  for  $s > 0$  such that

$$\begin{aligned} u(|x|) \leq V(t, x) \leq v(|x|), \quad t \in \mathbb{R}, x \in \mathbb{R}^n, \\ \dot{V}(t, \phi(0)) \leq -w(|\phi(0)|) \text{ if } V(t+s, \phi(s)) \leq p(V(t, \phi(0))), s \in [-r, 0], \end{aligned} \quad (2.38)$$

then the solution  $x = 0$  of system (2.33) is uniformly asymptotically stable. If  $u(s) \rightarrow \infty$  as  $s \rightarrow \infty$ , then the solution  $x = 0$  is also a global attractor for the system (2.33).

**Lemma 2.58** (Razumikhin uniformly ultimately bounded theorem, see [91]) Suppose  $f : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^n$  takes  $\mathbb{R} \times$  bounded sets of  $\mathbb{C}$  into bounded sets of  $\mathbb{R}^n$  and consider the system (2.33). Suppose  $u, v, w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are continuous nondecreasing functions,  $u(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . If there is a continuous nondecreasing function  $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ , a continuous nondecreasing function  $p : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $p(s) > s$  for  $s > 0$ , and a constant  $H \geq 0$  such that

$$\begin{aligned} u(|x|) &\leq V(t, x) \leq v(|x|), \quad t \in \mathbb{R}, x \in \mathbb{R}^n, \\ \dot{V}(t, \phi(0)) &\leq -w(|\phi(0)|) \\ \text{if } |\phi(s)| &\geq H, \quad V(t+s, \phi(s)) \leq p(V(t, \phi(0))), \quad s \in [-r, 0], \end{aligned} \quad (2.39)$$

then the solutions of system (2.33) are uniformly ultimately bounded.

With the coupling of neural networks, complex neural networks (CNN) have become the hot topic in the scientific community. For this kind of CNN, new dynamics such as synchronization have been proposed and studied. For this purpose, *synchronization stability* is necessary to be introduced.

Colloquially, synchronization means correlated in-time behavior between different processes [106, 107]. Indeed, the Oxford Advanced dictionary, 12 defines synchronization as “to agree in time” and “to happen at the same time.” From this intuitive definition we propose that synchronization requires the following four tasks [108, 109]: (1) Separating the dynamics of a large dynamical system into the dynamics of subsystems. (2) Measuring properties of the subsystems. (3) Comparing properties of the subsystems. (4) Determining whether the properties agree in time. If the properties agree then the systems are synchronized.

Consider the following dynamical system:

$$\begin{cases} \dot{x} = F_1(x, y, t), \\ \dot{y} = F_2(x, y, t), \end{cases} \quad (2.40)$$

where  $x \in X \subset \mathbb{R}^{d_1}$ ,  $y \in Y \subset \mathbb{R}^{d_2}$ , and  $t \in \mathbb{R}$ . The space of all trajectories is defined by  $Z = X \times Y$ , and the global trajectory is denoted by  $\Phi(z)$ ,  $z = (x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ ,  $d_1$  and  $d_2$  are positive integers, respectively. The trajectory properties of each subsystem are defined by the functionals:  $g_x : X \times \mathbb{R} \rightarrow \mathbb{R}^k$ ,  $g_y : Y \times \mathbb{R} \rightarrow \mathbb{R}^k$ . An example of functional commonly used is  $g_x = x(t)$ . The comparison of the functionals is made by the function  $h : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ , that is called comparison function. By using these functionals two synchronization definitions can be given.

**Definition 2.59** (*Synchronization-I*) [108, 109]: The subsystems of equation (2.40) are synchronized on the trajectory  $\Phi(z)$ , with respect to the properties  $g_x, g_y$  and synchronization norm  $\|\cdot\|_s$  if there is a time independent mapping  $h$  such that  $\|h(g_x, g_y)\|_s = 0$ .

**Definition 2.60** (*Synchronization-2*) [108, 109]: The subsystems of equation (2.40) are synchronized with respect to the properties  $g_x, g_y$  and synchronization norm  $\|\cdot\|_s$  if there is a time independent mapping  $h$  such that  $\|h(g_x, g_y)\|_s = 0$ .

The synchronization definition 2.60 is what many papers in the literature call synchronization. However, synchronization depends strongly on the trajectory. Two subsystems can be synchronized on some trajectories and not synchronized on other trajectories. Therefore, the trajectory dependence in synchronization definition 2.59 cannot be ignored.

The stability of synchronous motion is another issues raised by these two definitions. Specifically, stability is not required by the synchronization definition 2.59. This definition only requires that  $\|h\| = 0$  exists for properties measured on the trajectory. If the trajectory of one of the sub-systems is perturbed then this condition may no longer hold. As a trivial example of this, consider two uncoupled identical Lorenz systems with parameter values that produce chaotic trajectories. If both systems have the same initial condition then they will follow the same trajectory and their motion will clearly be synchronous. However, because of chaos, this type of synchronization is very unstable. In contrast, the second synchronization definition 2.60 implicitly requires the notion of stability because it requires  $\|h\| = 0$  for all trajectories.

A strength of the definition is that the properties and comparison functions are not specified, a priori. Due to different synchronous target, different applications require different properties and comparison functions. Those that are suitable for one application are often completely unsuitable for another. This also implies that synchronization stability is a relative stability with respect to specified or virtual target.

The major point about this definition is the appropriated choice of the functionals  $g_x, g_y$  and the synchronization norm. The appropriated choice of the functional and synchronization norm depends upon the required type of synchronization.

In the majority of chaotic applications the required synchronization is the one referred as identical. In this case the functional are given by  $g_x = x(t)$  and  $g_y = y(t)$ . The synchronization stability of identical subsystems coupled in a master-slave configuration exhibiting chaotic behavior is now presented, based on a criterion that assures based on a criterion that assures the stability of the synchronous motion under small perturbations.

**Definition 2.61** (*Synchronization stability*): The subsystems of equation (2.40) are synchronous stable with respect to the properties  $g_x = x(t), g_y = y(t)$  and synchronization norm  $\|\cdot\|_s$  if there is a time independent mapping  $h$  such that  $\|h(g_x, g_y)\|_s = \|x(t) - y(t)\| = 0$ .

When  $\|x(t) - y(t)\| = 0$ , it implies  $x(t) = y(t) = s(t)$ , where  $s(t)$  is the synchronous state, and  $s(t)$  can be an equilibrium point, a limit cycle, an aperiodic orbit, or a chaotic orbit.

Based on above different definitions of stability, we can present a brief review on the evolution process of *stability theory*.

(1) In the original stability definition, it is about a fixed equilibrium point of system  $\dot{x}(t) = f(x(t))$  under some restrictions on the nonlinear function. In the phase, asymptotical stability, exponential stability, attractivity, and so on have been studied. For the multiple fixed or discrete equilibrium point, global stability, complete stability, convergence, etc., have been developed. When the solution of  $\dot{x}(t) = f(x(t))$  is periodic or chaotic, such concepts as invariant sets and limit sets have been proposed. In general, stability in the sense of Lyapunov is about the nonlinear system without input, or the system with zero input and nonzero initial states.

(2) With the development of nonlinear system theory, some external actions can affect the dynamics of the autonomous system  $\dot{x}(t) = f(x(t))$ . In this case, such systems as  $\dot{x}(t) = f(x(t), u(t))$  have been studied. In this phase, the external control input has direct influence on the nonlinear system. In order to more precisely describe the qualitative behavior of nonlinear system, such concepts as ISS, iISS, OSS, IOS, SIiOS, passive, dissipative, and so on have been proposed, and the relations among global asymptotical stability and ISS, OSS have been discussed. These concepts have generalized the stability conception in the sense of Lyapunov.

(3) Besides the internal states and certain input, a system always suffers from the external disturbance. Therefore, a more accurate description of the concerned nonlinear system is  $\dot{x}(t) = f(x(t), u(t), w(t))$ , where  $w(t)$  can represent uncertain input such as disturbance, noise, fault, and so on. In this phase, generalized input  $(u(t), w(t))$  can be used to evaluate the stability in the sense of *boundedness*. Meanwhile, in the aspect of control performances, such concepts as  $\mathcal{L}_2$  performance and  $\mathcal{H}_\infty$  performance have been proposed. All above three-phased, the concerned nonlinear systems are isolated or are not connected to other systems (e.g., node system with respect to complex system with couplings).

(4) For the complex systems with couplings, such as *complex networks* and multi-agent systems, such concepts as *synchronization* used consensus have been proposed. Synchronization and consensus stresses the identical behavior among all the node dynamics. Even though the complex network is synchronous, the global dynamics of this complex networks can not be stable. This is a significant difference between synchronization and stability in the sense of Lyapunov (or called *stability of fixed point*). The *reference frame* of stability of fixed point is itself, or the origin of the system without any input. Stability of fixed point is the internal dynamical behavior of a system, while synchronization is the external dynamical behavior among different systems. The reference frame of synchronization can be the dynamics of any node systems or other external reference target. Therefore, synchronization is the upgrade of stability in the sense of Lyapunov. The essence of stability of fixed point is a *relative stability* with the inherent equilibrium of the concerned system. For the system  $\dot{x}(t) = f(x(t))$ , its inherent equilibrium is naturally the origin. This reflects that  $f(0) = 0$  is the fundamental assumption on the nonlinear function. Or in another way, one can say that  $\dot{x}(t) = f(x(t))$  with  $f(0) = 0$  is just an *error system* required in the research of stability of fixed point, state estimation, system identification, regulation, tracking, and synchronization. Therefore, where there is the research of relative motion such as tracking and regulation, where there is the Lyapunov stability theory.

(5) The systems concerned in [84] is an interconnected large-scale systems, in which  $f(0, t) = 0$  is not required. However, there is no any explanation on this features. Although some similar ideas to the present book may contain, to the best of the authors' knowledge, it is not explicitly noted in his series of studies on the stability of dynamical systems. Except the work in [84], above stability definitions are about the *total variables* of the systems  $\dot{x}(t) = f(x(t))$  or  $x(t) = f(x(t), u(t))$ . That is to say, Lyapunov stability theory, synchronization, ISS, and its variants are all about the total variables or the whole systems. The global dynamics of the whole system ultimately have the same characteristics, for example, asymptotic stability, or synchronization, or chaos. These *qualitative concepts* are relative to the whole systems. In fact, there may coexist many different dynamics in a system, for example, periodic solution and stable equilibrium point. This phenomenon relates to the concept of *partial stability*. Partial stability is an important variants and complements to the original Lyapunov and Lagrange stability concepts. For a given motion of a dynamical system, say  $x(t, x_0, t_0) = (y(t, x_0, t_0), z(t, x_0, t_0))$ , partial stability concerns the qualitative behavior of the  $y$ -component of the motion, relative to disturbances in the entire initial vector  $x(t_0, x_0, t_0) = x_0 = (y_0, z_0)$ , or relative to disturbances in the initial component  $y_0$ . In the former case one speaks simply of  $y$ -stability, while in the latter case, one speaks more explicitly of  $y$ -stability under arbitrary  $z$ -perturbations [111–114]. We note in passing that problems concerning partial stability of dynamical systems are closely related to problems of stability with respect to two measures [115].

Thus, stability is *relative* to some reference frame. If the reference frame is the total variable, then the stability result is global; while if the reference frame is partial variables, then the stability result is partial.

## 2.7 Summary

This chapter is mainly concerned with the preliminaries of the dynamical systems and stability theory, since neural network is also a special kind of dynamical systems. For the stability analysis of dynamical systems, the famous Lyapunov stability concept is emphasized here. From the background of the different stability definitions, we can find that all the theory researches must conform to the contemporary requirement including the industry and information technology. Practical production demands need the scientific innovation and technical revolution. Therefore, by understanding the history of different stability concepts and analysis methods, one may have a deep insight into the research of science and technology, and arousing the study interests and enthusiasm of researchers. Note that, some contents about the research background and partial stability theory and definitions of this chapter are from the Wikipedia—the free encyclopedia on the internet, while some original comments for the development of stability theory and stability concepts are presented by the authors.

## References

1. G. Birkhoff, Proof of Poincaré's geometric theorem. *Trans. Amer. Math. Soc.* **14**, 14–22 (1913)
2. A.M. Lyapunov, The general problem of the stability of motion (in Russian), Doctoral Dissertation, University of Kharkov, 1892 (English translation version: (1) *Stability of Motion*, (Academic Press, New York, 1966). (2) *The General Problem of the Stability of Motion* (Taylor and Francis, London, 1992))
3. A.M. Letov, *Stability of Nonlinear Control Systems* (Russian), Moscow 1955 (English translation, Princeton, 1961)
4. R.E. Kalman, J.F. Bertram, Control system analysis and design via the second method of Lyapunov: I-continuous-time systems. *J. Fluids Eng.* **82**(2), 371–394 (1960)
5. J.P. LaSalle, S. Lefschetz, *Stability by Lyapunov's Second Method with Applications* (Academic Press, New York, 1961)
6. P.C. Parks, Lyapunov's method in automatic control theory. *Control* **5**(3), 102–105 (1962)
7. R.E. Kalman, Lyapunov functions for the problem of Lurie in automatic control. *Proc. Natl Acad. Sci. USA* **49**(2), 201–207 (1963)
8. M.J. Smith, M.B. Wisten, A continuous day-to-day traffic assignment model and the existence of a continuous dynamic user equilibrium. *Ann. Oper. Res.* **60**(1), 59–79 (1995)
9. B. Liu, W. Lu, T. Chen, New criterion of asymptotic stability for delay systems with time-varying structures and delays. *Neural Netw.* **54**, 103–111 (2014)
10. S. Sastry, *Nonlinear System, Analysis, Stability and Control* (Springer, New York, 1999)
11. X. Liao, L. Wang, P. Yu, *Stability of Dynamical Systems* (Elsevier, The Netherlands, 2007)
12. C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos* (CRC Press, Boca Raton, 1998)
13. A.N. Michel, L. Hou, D. Liu, *Stability of Dynamical Systems: Continuous, Discontinuous, and Discrete Systems* (Birkhauser, Boston, 2008)
14. A.A. Andronov, L.S. Pontryagin, *Systèmes grossiers*. *Dokl. Akad. Nauk SSSR* **14**(5), 247–250 (1937)
15. F. Diacu, P. Holmes, *Celestial Encounters* (Princeton University Press, Princeton, 1996)
16. G. Birkhoff, *Dynamical Systems* (Amer. Math. Soc., Providence, 1927)
17. S. Smale, Differentiable dynamical systems. *Bull. Am. Math. Soc.* **73**, 747–817 (1967)
18. A.I. Lur'e, V.N. Postnikov, On the theory of stability of control systems. *Prikladnaya Matematika Mehkanika* **8**, 246–248 (1944)
19. H.K. Khalil, *Nonlinear Systems* (Prentice Hall, Upper Saddle River, 1996)
20. V. Kolmanovskii, A. Myshkis, *Applied Theory of Functional Differential Equations* (Kluwer Academic Press, Boston, 1992)
21. K. Gu, V.L. Kharitonov, J. Chen, *Stability of Time-Delay Systems* (Birkhauser, Boston, 2003)
22. Q. Han, Y. Tian, D. Han, Robust absolute stability criteria for a class of uncertain Lur'e systems of neutral type, in *Proceedings of the 6th World Congress on Intelligent Control and Automation*, Dalian, China, June 21–23, pp. 2339–2343 (2006)
23. Z. Zeng, J. Wang, Complete stability of cellular neural networks with time-varying delays. *IEEE Trans. Circuits Syst.-I: Regul. Pap.* **53**(4), 944–955 (2006)
24. M. Forti, Some extensions of a new method to analyze complete stability of neural networks. *IEEE Trans. Neural Netw.* **13**(5), 1230–1238 (2002)
25. N. Takahashi, L.O. Chua, On the complete stability of nonsymmetric cellular neural networks. *IEEE Trans. Circuits Syst.-I: Fundam. Theory Appl.* **45**(7), 754–758 (1998)
26. E.D. Sontag, Input to state stability: basic concepts and results. *Lect. Notes Math.* **1932**, 163–220 (2008)
27. Y. Lin, E.D. Sontag, Y. Wang, A smooth converse Lyapunov theorem for robust stability. *SIAM J. Control Optim.* **34**(1), 124–160 (1996)
28. Y. Lin, E. Sontag, Y. Wang, Various results concerning set input-to-state stability, in *Proceedings of the 34th IEEE Conference on Decision and Control*, vol. 2, pp. 1330–1335 (1995)



29. E.D. Sontag, Y. Lin, Stabilization with respect to noncompact sets: Lyapunov characterizations and effect of bounded inputs, in *Nonlinear Control Systems Design 1992*, IFAC Symposia Series, pp. 43–49 (1992)
30. E.D. Sontag, Smooth stabilization implies coprime factorization. *IEEE Trans. Autom. Control* **34**(4), 435–443 (1989)
31. E.D. Sontag, Y. Wang, On characterizations of input-to-state stability with respect to compact sets, in *Proceedings of IFAC Non-Linear Control Systems Design Symposium (NOLCOS'95)*, Tahoe City, CA, pp. 226–231 (1995)
32. E.D. Sontag, Smooth stabilization implies coprime factorization. *IEEE Trans. Autom. Control* **34**, 435–443 (1989)
33. A. Isidori, *Nonlinear Control Systems II* (Springer, London, 1999)
34. Z.-P. Jiang, L. Praly, Preliminary results about robust Lagrange stability in adaptive nonlinear regulation. *Int. J. Adapt. Control Signal Process.* **6**, 285–307 (1992)
35. Z.-P. Jiang, A. Teel, L. Praly, Small-gain theorem for ISS systems and applications. *Math. Control Signals Syst.* **7**, 95–120 (1994)
36. H.K. Khalil, *Nonlinear Systems*, 2nd edn. (Prentice-Hall, Upper Saddle River, 1996)
37. M. Krichman, E.D. Sontag, A version of a converse Lyapunov theorem for input output to state stability, in *Proceedings of IEEE Conference Decision and Control*, Tampa, Dec. 1998, IEEE Publications, pp. 4121–4126 (1998)
38. M. Krstic, H. Deng, *Stabilization of Uncertain Nonlinear Systems* (Springer, London, 1998)
39. M. Krstic, I. Kanellakopoulos, P.V. Kokotovic, *Nonlinear and Adaptive Control Design* (Wiley, New York, 1995)
40. D. Liberzon, E. Sontag, Y. Wang, On integral input-to-state stabilization, in *Proceedings of American Control Conference*, June, San Diego, pp. 1598–1602 (1999)
41. M. Arcak, A. Teel, Input-to-state stability for a class of Lurie systems. *Automatica* **38**, 1945–1949 (2002)
42. H. Edwards, Y. Lin, Y. Wang, On input-to-state stability for time varying nonlinear systems, in *Proceedings of 39th IEEE Conference Decision and Control*, Sydney, Australia, pp. 3501–3506 (2000)
43. N.C.S. Fah, Input-to-state stability with respect to measurement disturbances for one-dimensional systems. *Control Optim. Calc. Var.* **4**, 99–121 (1999)
44. L. Grune, Input-to-state dynamical stability and its Lyapunov function characterization. *IEEE Trans. Autom. Control* **47**, 1499–1504 (2002)
45. Z.-P. Jiang, Y. Wang, Input-to-state stability for discrete-time nonlinear systems. *Automatica* **37**, 857–869 (2001)
46. B. Ingalls, E.D. Sontag, A small-gain lemma with applications to input/output systems, incremental stability, detectability, and interconnections. *J. Frankl. Inst.* **339**, 211–229 (2002)
47. J.P. Hespanha, D. Liberzon, A.S. Morse, Supervision of integral-input-to-state stabilizing controllers. *Automatica* **38**, 1327–1335 (2002)
48. D. Angeli, Intrinsic robustness of global asymptotic stability. *Syst. Control Lett.* **38**, 297–307 (1999)
49. D. Angeli, E.D. Sontag, Y. Wang, A characterization of integral input to state stability. *IEEE Trans. Autom. Control* **45**, 1082–1097 (2000)
50. D. Angeli, An almost global notion of input-to-state stability. *IEEE Trans. Autom. Control* **49**, 866–874 (2004)
51. D. Angeli, B. Ingalls, E.D. Sontag, Y. Wang, Separation principles for input-output and integral-input to state stability. *SIAM J. Control Optim.* **43**, 256–276 (2004)
52. D. Angeli, B. Ingalls, E.D. Sontag, Y. Wang, Uniform global asymptotic stability of differential inclusions. *J. Dyn. Control Syst.* **10**, 391–412 (2004)
53. D. Angeli, E.D. Sontag, Y. Wang, Input-to-state stability with respect to in-puts and their derivatives. *Int. J. Robust Nonlinear Control* **13**, 1035–1056 (2003)
54. D. Angeli, E.D. Sontag, Y. Wang, Further equivalences and semiglobal versions of integral input to state stability. *Dyn. Control* **10**, 127–149 (2000)

55. M. Arcak, D. Angeli, E.D. Sontag, A unifying integral ISS framework for stability of nonlinear cascades. *SIAM J. Control Optim.* **40**, 1888–1904 (2002)
56. R.A. Freeman, Global internal stabilizability does not imply global external stabilizability for small sensor disturbances. *IEEE Trans. Autom. Control* **40**, 2119–2122 (1996)
57. E.D. Sontag, Y. Wang, Output-to-state stability and detectability of nonlinear systems. *Syst. Control Lett.* **29**, 279–290 (1997)
58. E.D. Sontag, Some connections between stabilization and factorization, in *Proceedings of IEEE Conference Decision and Control*, Tampa, pp. 990–995, Dec. (1989)
59. E.D. Sontag, Y. Wang, Output-to-state stability and detectability of nonlinear systems. *Syst. Control Lett.* **29**, 279–290 (1997)
60. E.D. Sontag, Y. Wang, Notions of input to output stability. *Syst. Control Lett.* **38**, 235–248 (1999)
61. E.D. Sontag, Y. Wang, On characterizations of the input-to-state stability property. *Syst. Control Lett.* **24**, 351–359 (1995)
62. P.P. Varaiya, R. Liu, Bounded-input bounded-output stability of nonlinear time varying differential systems. *SIAM J. Control* **4**, 698–704 (1966)
63. E.D. Sontag, On the input-to-state stability property. *Eur. J. Control* **1**(1), 24–36 (1995)
64. L.T. Grujic, On practical stability. *Int. J. Control* **17**(4), 881–887 (1973)
65. A.R. Teel, J. Peuteman, D. Aeyels, Semi-global practical asymptotic stability and averaging. *Syst. Control Lett.* **37**, 329–334 (1999)
66. L. Moreau, D. Aeyels, Practical stability and stabilization. *IEEE Trans. Autom. Control* **45**(8), 1554–1558 (2000)
67. X. Xu, G. Zhai, Practical stability and stabilization of hybrid and switched systems. *IEEE Trans. Autom. Control* **50**(11), 1897–1903 (2005)
68. L. Weiss, E.F. Infante, Finite time stability under perturbing forces and on product spaces. *IEEE Trans. Autom. Control* **12**(1), 54–59 (1967)
69. S. Bhat, D. Bernstein, Finite-time stability of continuous autonomous systems. *SIAM J. Control Optim.* **38**, 751–766 (2000)
70. Y. Hong, Z.-P. Jiang, Finite-time stabilization of nonlinear systems with parametric and dynamic uncertainties. *IEEE Trans. Autom. Control* **49**, 1950–1956 (2006)
71. Y.G. Hong, Z.-P. Jiang, G. Feng, Finite-time input-to-state stability and applications to finite-time control design. *SIAM J. Control Optim.* **48**(7), 4395–4418 (2010)
72. W.S. Chen, L.C. Jiao, Finite-time stability theorem of stochastic nonlinear systems. *Automatica* **46**(12), 2105–2108 (2010)
73. Y.Y. Cao, Z. Lin, D.G. Ward, An antiwindup approach to enlarging domain of attraction for linear systems subject to actuator saturation. *IEEE Trans. Autom. Control* **47**(1), 140–145 (2002)
74. A. Isidori, *Nonlinear Control Systems: A Introduction* (Springer, Berlin, 1989)
75. E.J. Davison, E.M. Kurak, A computational method for determining quadratic Lyapunov functions for nonlinear systems. *Automatica* **7**, 627–636 (1971)
76. R. Genesio, A. Vicino, New techniques for constructing asymptotic stability regions for nonlinear systems. *IEEE Trans. Circuits Syst.* **31**, 574–581 (1984)
77. A. Tesi, F. Villorresi, R. Genesio, On the stability domain estimation via a quadratic Lyapunov function: convexity and optimality properties for polynomial systems. *IEEE Trans. Autom. Control* **41**(1), 1650–1657 (1996)
78. A. Vannelli, M. Vidyasagar, Maximal Lyapunov function and domain of attraction for autonomous nonlinear systems. *Automatica* **21**, 69–80 (1985)
79. K. Ichikawa, R. Ortega, On stabilization of nonlinear systems with enlarged domain of attraction. *Automatica* **28**, 623–626 (1992)
80. D. Liaw, C. Lee, An approach to estimate domain of attraction for nonlinear control systems, in *Proceedings of the First International Conference on Innovative Computing, Information and Control (ICICIC'06)*, pp. 146–149 (2006)
81. S. Mayer, B. Tibken, S. Warthenpfohl, Parallel computation of domains of attraction for nonlinear dynamic systems, in *Proceedings of 2011 Sixth International Symposium on Parallel Computing in Electrical Engineering*, pp. 87–92 (2011)

82. D. Jin, J. Peng, A new approach for estimating the attraction domain for Hopfield-type neural networks. *Neural Comput.* **21**, 101–120 (2009)
83. X. Liao, Stability of Hopfield-type neural networks: I. *Sci. China: Ser. A* **38**(4), 407–418 (1995)
84. A.N. Michel, Simple and interconnected systems: stability, boundedness and trajectory behavior. *IEEE Trans. Circuits Syst.* **17**(3), 292–301 (1970)
85. E. Thelen, E. Bates, Connectionism and dynamic systems: are they really different? *Dev. Sci.* **6**(4), 378–391 (2003)
86. P.H. Miller, *Theories of Developmental Psychology*, 4th edn. (Worth Publishers, New York, 2002)
87. P. Sopasakis, The Realm of Linear State Space Control Systems, <http://users.ntua.gr/chvng/en> (2011)
88. P.-A. Bliman, LMI characterization of the strong delay-independent stability of linear delay systems via quadratic Lyapunov-Krasovskii functionals. *Syst. Control Lett.* **43**, 263–274 (2001)
89. P.-A. Bliman, Lyapunov-Krasovskii method and strong delay-independent stability of linear delay systems, in *Proceedings of the Second IFAC Workshop on Linear Time Delay Systems*, Ancona, Italy, pp. 5–9 (2000)
90. N.N. Krasovskii, *Stability of Motion* (Stanford University Press, Stanford, 1963)
91. J.K. Hale, *Theory of Functional Differential Equations* (Springer, New York, 1977)
92. E.I. Verriest, A.F. Ivanov, Robust stabilization of systems with delayed feedback, in *Proceedings of the Second International Symposium on Implicit and Robust Systems*, Warsaw, Poland, pp. 190–193 (1991)
93. Q. Zhou, P. Shi, S. Xu, H. Li, Observer-based adaptive neural network control for nonlinear stochastic systems with time delay. *IEEE Trans. Neural Netw. Learn. Syst.* **24**(1), 71–80 (2013)
94. S. Boyd, L.E.I. Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequality in System and Control Theory* (SIAM, Philadelphia, 1994)
95. W. Chen, W. Zheng, A new method for complete stability analysis of cellular neural networks with time delay. *IEEE Trans. Neural Netw.* **21**(7), 1126–1139 (2010)
96. W.B. Ford, *A brief account of the life and work of the late professor Ulisse Dini*. *Bull. AM. Math. Soc.* **26**(4), 173–177 (1920)
97. H. Qiao, J. Peng, Z. Xu, B. Zhang, A reference model approach to stability analysis of neural networks. *IEEE Trans. Syst. Man Cybern.-Part B: Cybern.* **33**(6), 925–936 (2003)
98. W. Zhou, J. Zurada, Discrete-time recurrent neural networks with complex-valued linear threshold neurons. *IEEE Trans. Circuits Syst. II Exp. Br.* **56**(8), 669–673 (2009)
99. J. Louisell, Absolute stability in linear delay-differential systems: ill-posedness and robustness. *IEEE Trans. Autom. Control* **40**(7), 1288–1291 (1995)
100. M.P. Joy, On the absolute stability of delayed cellular neural networks, in *1998 Fifth IEEE International Workshop on Cellular Neural Networks and their Applications*, London, England, 14–17, April, pp. 94–99 (1998)
101. R.Q. Lu, W.N. Zhou, H.Y. Su, J. Chu, A new absolute stability and stabilization conditions for a class of Lurie uncertain time-delay systems. in *IEEE International Conference on Systems, Man and Cybernetics*, vol. 2, pp. 1916–1921 (2003)
102. B. Zhou, Q. Song, Boundedness and complete stability of complex-valued neural networks with time delay. *IEEE Trans. Neural Netw. Learn. Syst.* **24**(8), 1227–1238 (2013)
103. M. Cohen, S. Grossberg, Absolute stability of global pattern formation and parallel memory storage by competitive neural networks, *IEEE Trans. Syst. Man Cybern.* **13**(5), 815–826 (1983)
104. L.O. Chua, L. Yang, Cellular neural networks: theory. *IEEE Trans. Circuits Syst.* **35**(10), 1257–1272 (1988)
105. B. Liu, D.J. Hill, Uniform stability and ISS of discrete-time impulsive hybrid systems. *Nonlinear Anal. Hybrid Syst.* **4**, 319–333 (2010)

- 106. H. Fujisaka, T. Yamada, Stability theory of synchronized motion in coupled oscillator systems. *Progress Theoret Phys*, **1**, 32–47 (1983)
- 107. L.O. Chua, M. Itoh, Chaos synchronization in Chua's circuit. *J. Circuit Syst. Comp.* **1**, 93–108 (1993)
- 108. R. Brown, A unifying definition of synchronization for dynamical systems, *Chaos*, **10**(2), 344–349 (2000)
- 109. U.F. Moreno, P.L. Peres, I.S. Bonatti, Synchronization stability analysis of chaotic oscillators, in *Proceedings of the IEEE Conference on Decision and Control*, vol. 5, pp. 4833–4838 (2000)
- 110. E.D. Sontag, Smooth stabilization implies coprime factorization. *IEEE Trans. Autom. Control* **34**, 435–443 (1989)
- 111. Y. Sun, A.P. Molchanov, A.N. Michel, Partial stability of dynamical systems, in *Proceedings of 15th International Symposium Mathematical Theory Networks Systems*, University of Notre Dame (2002)
- 112. A.N. Michel, A.P. Molchanov, Y. Sun, Partial stability and boundedness of discontinuous dynamical systems. *Nonlinear Stud.* **9**(3), 225–247 (2002)
- 113. A.N. Michel, A.P. Molchanov, Y. Sun, Partial stability and boundedness of general dynamical systems on metric spaces. *Nonlinear Anal. TMA* **52**(4), 1295–1316 (2003)
- 114. V.I. Vorotnikov, *Partial Stability and Control* (Birkhauser, Boston, 1998)
- 115. V. Lakshmikantham, X. Liu, *Stability Analysis in Terms of Two Measures* (World Scientific, Singapore, 1993)

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