Chapter 2
Exponential Stability and Synchronization Control of Neural Networks

In this chapter, we are concerned with exponential stability analysis for neural networks with fuzzy logical BAM and Markovian jump and synchronization control problem of stochastically coupled neural networks.

2.1 Global Exponential Stability of NN with Fuzzy Logical BAM and Markovian Jump

2.1.1 Introduction

It is well known that the bidirectional associative memory (BAM) neural networks have been deeply investigated in recent years due to its applicability in solving some image processing, signal processing, optimization, pattern recognition problems, and other areas. Many researchers have been attracted by this new class of artificial neural networks and a great deal of research has been done since fuzzy logical BAM neural networks are introduced by Kosko in [10–12]. Especially, since the global stability is one of the most desirable dynamic properties of neural networks, there have been growing research interests on the stability analysis and synthesis for BAM neural networks. For example, in [2] authors analyzed the global asymptotic stability of a BAM neural networks with constant time delays and the exponential stability of periodic solution to Cohen-Grossberg-type BAM neural networks with time-varying delays has been investigated in [36].

In recent years, the concept of incorporating fuzzy logic into neural networks has developed into an extensive research topic. Among various method developed for the analysis and synthesis of complex nonlinear systems, fuzzy logic control is an attractive and effective rule-based one. Therefore, fuzzy neural networks receive great attention since they are the hybrid of fuzzy logic and traditional neural networks. In many of the model-based fuzzy control approaches, the well-known Takagi-Sugeno

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(T-S) fuzzy model is recognized as an convenient and efficient tool in functional approximations. During the last decades, sufficient attention has been paid to the stability analysis and control synthesis of T-S fuzzy BAM neural networks [1, 19, 21]. In [20], researchers discuss the global asymptotic stability problem of T-S fuzzy BAM neural networks with time-varying delays. Moreover, the robust stability problem for uncertain fuzzy BAM neural networks with Markovian jumping and time-varying interval delays is investigated in [3]. However, in [4], a new class of fuzzy logical bidirectional associative memory (FLBAM) neural networks is introduced and analyzed. This model not only varies from the traditional BAM neural networks, but also is different from the T-S fuzzy BAM neural networks. In [37], the authors discussed the exponential stability and periodic solution for fuzzy logical BAM neural networks with time-varying delays.

In this section, we are concerned with the development of the exponential stability of fuzzy logical BAM neural networks with Markovian jumping parameters. Most scholars investigated the global stability of T-S fuzzy BAM neural networks with Markovian jumping parameters. However, the global stability of FLBAM neural networks with Markovian jumping parameters is seldom researched. The main purpose of this section is to derive some sufficient conditions for the exponential stability of fuzzy logical BAM neural networks with Markovian jumping parameters by constructing a Lyapunov functional and utilizing the linear matrix inequality (LMI) method.

2.1.2 System Description and Preliminaries

Consider the following FLBAM neural networks:

$$
\begin{cases}
\dot{u}_i(t) = -a_i(t)u_i(t) + \wedge_{j=1}^{n} b_{ij}(t) f_j(v_j(t)) + \vee_{j=1}^{n} c_{ij}(t) f_j(v_j(t)) \\
\qquad\qquad + \wedge_{j=1}^{n} \alpha_{ij}(t) g_j(t) + \vee_{j=1}^{n} \beta_{ij}(t) g_j(t) + I_i(t), \\
\dot{v}_j(t) = -d_j(t)v_j(t) + \wedge_{i=1}^{m} e_{ji}(t) f_i(u_i(t)) + \vee_{j=1}^{m} w_{ij}(t) f_i(u_i(t)) \\
\qquad\qquad + \wedge_{i=1}^{m} \gamma_{ji}(t) h_i(t) + \vee_{i=1}^{m} \delta_{ji}(t) h_i(t) + J_j(t),
\end{cases}
\tag{2.1}
$$

for $i = \{1, 2, \ldots, n\}$, $j = \{1, 2, \ldots, n\}$, $t \geq 0$, where $u_i(t)$ and $v_j(t)$ denote the activations of the ith neurons and jth neurons, $g_j(t)$ and $h_i(t)$ denote the state, respectively; $a_i(t)$ and $d_j(t)$ are positive constants while f_k $(k = 1, 2, \ldots, \max(m, n))$ is the activation functions; $b_{ij}(t)$ and $e_{ji}(t)$, $c_{ij}(t)$, and $w_{ji}(t)$ are elements of fuzzy feedback MIN template, and fuzzy feedback MAX template; $\alpha_{ij}(t)$ and $\gamma_{ji}(t)$, $\beta_i(t)$, and $\delta_{ji}(t)$ stand for fuzzy feed-forward MIN template and fuzzy feed-forward MAX template at the time t; $\wedge$ and $\vee$ denote the fuzzy AND and fuzzy OR operations, respectively; I_i and J_j denote the external inputs. To draw our conclusion, we proposed following assumption.

Assumption 2.1 The neuron activation functions in (2.1) satisfy that $f_z(0) = 0$ and f_z are globally Lipschitz continuous, i.e., there exist positive constants λ_z fulfilling

$$|f_z(x) - f_z(y)| \leq \lambda_Z |x - y|,$$

for all $x, y \in \mathbb{R}$ and $Z = 1, 2, \ldots, \max(m, n)$.

Now, based on the fuzzy logical BAM neural networks of model (2.1), we discuss the exponential stability of fuzzy logical BAM neural networks with Markovian jumping parameters.

In this section, we consider the following fuzzy logical neural networks with Markovian jumping parameters, which is actually a modification of (2.1):

$$\begin{cases} \dot{u}_i(t,r(t)) = -a_i(r(t))u_i(t) + \wedge_{j=1}^{n} b_{ij}(r(t)) f_j(v_j(t)) \\ \qquad + \vee_{j=1}^{n} c_{ij}(r(t)) f_j(v_j(t)) + \wedge_{j=1}^{n} \alpha_{ij}(r(t)) g_j(t) \\ \qquad + \vee_{j=1}^{n} \beta_{ij}(t) g_j(t) + I_i(t), \\ \dot{v}_j(t,r(t)) = -d_j(r(t))v_j(t) + \wedge_{i=1}^{m} e_{ji}(r(t)) f_i(u_i(t)) \\ \qquad + \vee_{j=1}^{m} w_{ji}(r(t)) f_i(u_i(t)) + \wedge_{i=1}^{m} \gamma_{ji}(r(t)) h_i(t) \\ \qquad + \vee_{i=1}^{m} \delta_{ji}(t) h_i(t) + J_j(t). \end{cases} \tag{2.2}$$

where $\{r(t), t \geq 0\}$ is a homogeneous finite-state Markovian process with right-continuous trajectories on the probability space which takes values in the finite space $\mathbb{S} = \{1, 2, \ldots, S\}$ with its generator $\Gamma = (\theta_{\eta\eta'})$ $(\eta, \eta' \in \mathbb{S})$. Then, we shall work on the network model $r(t) = \eta$ for each $\eta \in \mathbb{S}$.

Suppose the vector

$$L(t) = (l_1(t), l_2(t), \ldots, l_{m+n}(t))^T = (u_1(t), u_2(t), \ldots, u_m(t), v_1(t)), \ldots, v_n(t))^T.$$

For any $L \in \mathbb{R}^{m+n}$, we define the norm

$$||L(t)|| = \max_{1 \leq i \leq m, 1 \leq j \leq n} (\sup_{t \in \mathbb{R}} |u_i(t)|, \sup_{t \in \mathbb{R}} |v_i(t)|).$$

Set $B = \{L | L = (u_1, \ldots, u_m, v_1, \ldots, v_n)^T\}$. For any $L \in B$, we define its induced model as

$$||L|| = ||L(t)|| = \max_{1 \leq i \leq m, 1 \leq j \leq n} (\sup_{t \in \mathbb{R}} |u_i(t)|, \sup_{t \in \mathbb{R}} |v_i(t)|).$$

where B is a Banach space.

For any $\phi, \varphi \in B$, we denote the solutions of system (2.2) through $(0, \phi)$ and $(0, \varphi)$ as follows:

$$L(t, r(t), \phi) = (u_1(t, \eta, \phi), u_2(t, \eta, \phi), \ldots, u_m(t, \eta, \phi), \\ v_1(t, \eta, \phi), \ldots, v_n(t, \eta, \phi))^T,$$

$$L(t, r(t), \varphi) = (u_1(t, \eta, \varphi), u_2(t, \eta, \varphi), \ldots, u_m(t, \eta, \varphi),$$
$$v_1(t, \eta, \varphi), \ldots, v_n(t, \eta, \varphi))^T,$$

where $r(t) = \eta \in \mathbb{S}$, respectively.

Definition 2.2 The system (2.2) is globally exponentially stable if there existing positive constants k and σ satisfying

$$||L(t, \eta, \phi) - L(t, \eta, \phi)|| \geq \sigma||\phi - \varphi||e^{-kt},$$

for all $r(t) = \eta \in \mathbb{S}$ and $t \geq 0$.

Lemma 2.3 *Suppose l and l' are two states of system (2.2), then the following inequalities are established for all $r(t) = \eta \in \mathbb{S}$:*

$$| \wedge_{j=1}^{n} \tau_{ij} f_j(l_j) - \wedge_{j=1}^{n} \tau_{ij} f_j(l'_j)| \leq \sum_{j=1}^{n} |\tau_{ij}|||f_j(l_j) - f_j(l'_j)||,$$
$$| \vee_{j=1}^{n} \zeta_{ij} f_j(l_j) - \vee_{j=1}^{n} \zeta_{ij} f_j(l'_j)| \leq \sum_{j=1}^{n} |\zeta_{ij}|||f_j(l_j) - f_j(l'_j)||.$$

2.1.3 Main Results

In this section, we will discuss the global exponential stability of fuzzy logical BAM neural networks with Markovian jumping parameters. A new sufficient criterion will be proposed to prove the exponential stability of the model.

Theorem 2.4 *If there exist a positive scalar $k > 0$ and a position definite matrix $P_\eta > 0$ such that the following linear matrix inequality holds:*

$$kP_\eta - P_\eta W_\eta + G_\eta E_\eta P_\eta < 0, \tag{2.3}$$

then the system of (2.2) is global exponential stable for any $r(t) = \eta$ $(\forall \eta \in \mathbb{S})$, where $G_\eta = \operatorname{diag}(\lambda_1, \ldots, \lambda_{m+n})$, $W_\eta = \operatorname{diag}(a_1(\eta), \ldots, a_m(\eta), d_1(\eta), \ldots, d_n(\eta))$, $E_1 = (|b_{ij}(\eta)| + |c_{ij}(\eta)|)_{n\times m}$, $E_2 = (|e_{ji}(\eta)| + |w_{ji}(\eta)|)_{n\times m}$, $E = \begin{bmatrix} 0 & E_2 \\ E_1 & 0 \end{bmatrix}$.

Proof To prove our conclusion, we denote that

$$l(t, r(t)) = L(t, r(t), \phi) - L(t, r(t), \varphi),$$

then we can obtain from (2.2) that

$$\begin{cases} \dot{l}_i(t, r(t)) = -a_i(r(t))l_i(t, r(t)) \\ \quad + \wedge_{j=1}^{n} b_{ij}(r(t)) f_j(v_j(t, \phi)) - \wedge_{j=1}^{n} b_{ij}(r(t)) f_j(v_j(t, \varphi)) \\ \quad + \vee_{j=1}^{n} c_{ij}(r(t)) f_j(v_j(t, \phi)) - \vee_{j=1}^{n} c_{ij}(r(t)) f_j(v_j(t, \varphi)), \\ \dot{l}_{m+j}(t, r(t)) = -d_j(r(t))l_{m+j}(t, r(t)) \\ \quad + \wedge_{i=1}^{m} e_{ji}(r(t)) f_i(u_i(t, \phi)) - \wedge_{i=1}^{m} e_{ji}(r(t)) f_i(u_i(t, \varphi)) \\ \quad + \vee_{j=1}^{m} w_{ji}(r(t)) f_i(u_i(t, \phi)) - \vee_{j=1}^{m} w_{ji}(r(t)) f_i(u_i(t, \varphi)). \end{cases} \tag{2.4}$$

For the sake of discussing the global exponentially stability of system (2.2), we consider the following Lyapunov-Krasovskii functional:

$$V(t, l(t), \eta) = e^{2kt}\left(\sum_{i=1}^{m} P_i(\eta)l_i^2(t) + \sum_{j=1}^{m} P_{m+j}(\eta)l_{m+j}^2(t)\right).$$

Let $\mathcal{L}$ be the weak infinitesimal generator of random process $\{l(t), r(t), t \geq 0\}$. Then, for each $r(t) = \eta \in \mathbb{S}$ we can obtain that

$$\begin{aligned} \mathcal{L}V(t, l(t), \eta) &= 2ke^{2kt}\sum_{i=1}^{m} P_i(\eta)l_i^2(t) + 2e^{2kt}\sum_{i=1}^{m} P_i(\eta)l_i(t)\dot{l}_i(t) \\ &+ 2e^{2kt}\left(k\sum_{j=1}^{n} P_{m+j}(\eta)l_{m+j}^2(t) + \sum_{j=1}^{n} P_{m+j}(\eta)l_{m+j}(t)\dot{l}_{m+j}(t)\right) \\ &+ \sum_{\eta'=1}^{S} \theta_{\eta\eta'} e^{2kt}\left(\sum_{i=1}^{m} P_i(\eta)l_i^2(t) + \sum_{j=1}^{m} P_{m+j}(\eta)l_{m+j}^2(t)\right) \\ &= 2ke^{2kt}\sum_{i=1}^{m+n} P_i(\eta)l_i^2(t) + 2e^{2kt}\sum_{i=1}^{m} P_i(\eta)l_i(t)\{-a_i(\eta)l_i(t) \\ &+ [\wedge_{j=1}^{n} b_{ij}(\eta) f_j(v_j(t, \phi)) - \wedge_{j=1}^{n} b_{ij}(\eta) f_j(v_j(t, \varphi))] \\ &+ [\vee_{j=1}^{n} c_{ij}(\eta) f_j(v_j(t, \phi)) - \vee_{j=1}^{n} c_{ij}(\eta) f_j(v_j(t, \varphi))]\} \\ &+ 2e^{2kt}\sum_{j=1}^{n} P_{m+j}(\eta)l_{m+j}(t)\{-d_j(\eta)l_{m+j}(t) \\ &+ [\wedge_{i=1}^{m} e_{ji}(\eta) f_i(u_i(t, \phi)) - \wedge_{i=1}^{m} e_{ji}(\eta) f_i(u_i(t, \varphi))] \\ &+ [\vee_{i=1}^{m} w_{ji}(\eta) f_i(u_i(t, \phi)) - \vee_{i=1}^{m} w_{ji}(\eta) f_i(u_i(t, \varphi))]\} \\ &= 2ke^{2kt}\sum_{i=1}^{m+n} P_i(\eta)l_i^2(t) \end{aligned}$$

$$
\begin{aligned}
&+2e^{2kt}\left\{-\sum_{i=1}^{m} a_i(\eta)P_i(\eta)l_i^2(t)-\sum_{j=1}^{n} d_j(\eta)P_{m+j}(\eta)l_{m+j}^2(t)\right.\\
&+\sum_{i=1}^{m} P_i(\eta)l_i(t)[\wedge_{j=1}^{n} b_{ij}(\eta)f_j(v_j(t,\phi))\\
&\qquad\qquad -\wedge_{j=1}^{n} b_{ij}(\eta)f_j(v_j(t,\varphi))]\\
&+\sum_{i=1}^{m} P_i(\eta)l_i(t)[\vee_{j=1}^{n} c_{ij}(\eta)f_j(v_j(t,\phi))\\
&\qquad\qquad -\vee_{j=1}^{n} c_{ij}(\eta)f_j(v_j(t,\varphi))]\\
&+\sum_{j=1}^{n} P_{m+j}(\eta)l_{m+j}(t)[\wedge_{i=1}^{m} e_{ji}(\eta)f_i(u_i(t,\phi))\\
&\qquad\qquad -\wedge_{i=1}^{m} e_{ji}(\eta)f_i(u_i(t,\varphi))]\\
&+\sum_{j=1}^{n} P_{m+j}(\eta)l_{m+j}(t)[\vee_{i=1}^{m} w_{ji}(\eta)f_i(u_i(t,\phi))\\
&\qquad\qquad \left.-\vee_{i=1}^{m} w_{ji}(\eta)f_i(u_i(t,\varphi))]\right\}\\
&\le 2ke^{2kt}\sum_{i=1}^{m+n} P_i(\eta)l_i^2(t)\\
&+2e^{2kt}\left\{-\sum_{i=1}^{m} a_i(\eta)P_i(\eta)l_i^2(t)-\sum_{j=1}^{n} d_j(\eta)P_{m+j}(\eta)l_{m+j}^2(t)\right.\\
&+\sum_{i=1}^{m} P_i(\eta)l_i(t)\left[\sum_{j=1}^{n}(|b_{ij}(\eta)|+|c_{ij}(\eta)|)\right.\\
&\qquad\qquad \left.\cdot|f_j(v_j(t,\phi))-f_j(v_j(t,\varphi))|\right]\\
&+\sum_{j=1}^{n} P_{m+j}(\eta)l_{m+j}(t)\left[\sum_{i=1}^{m}(|e_{ji}(\eta)|+|w_{ji}(\eta)|)\right.\\
&\qquad\qquad \left.\left.\cdot\,|f_i(u_i(t,\phi))-f_i(u_i(t,\varphi))|\right]\right\}\\
&\le 2ke^{2kt}\sum_{i=1}^{m+n} P_i(\eta)l_i^2(t)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2e^{2kt} \Bigg\{ - \sum_{i=1}^{m} a_i(\eta) P_i(\eta) l_i^2(t) - \sum_{j=1}^{n} d_j(\eta) P_{m+j}(\eta) l_{m+j}^2(t) \\
&+ \sum_{i=1}^{m} P_i(\eta) l_i(t) \left[\sum_{j=1}^{n} (|b_{ij}(\eta)| + |c_{ij}(\eta)|) \cdot \lambda_{m+j} \cdot |l_{m+j}(t)| \right] \\
&+ \sum_{j=1}^{n} P_{m+j}(\eta) l_{m+j}(t) \left[\sum_{i=1}^{m} (|e_{ji}(\eta)| + |w_{ji}(\eta)|) \cdot \lambda_i \cdot |l_i(t)| \right] \Bigg\} \\
&\leq 2e^{2kt} |l^T(t)| (kP_\eta - P_\eta W_\eta + G_\eta E_\eta P_\eta) |l(t)|.
\end{aligned}
$$

Since $kP_\eta - P_\eta W_\eta + G_\eta E_\eta P_\eta < 0$, then we have

$$\mathcal{L}V(t, l(t), r(t) = \eta) < 0.$$

That is to say, for each $r(t) = \eta \in \mathbb{S}$, we can conclude that

$$V(l(t)) \leq V(l(0)) = l^T(0) P_\eta l(0) \leq \lambda_M(P_\eta) ||\phi - \varphi||^2,$$

where $\lambda_M(P_\eta) = \max\{\lambda_1, \lambda_2, \ldots, \lambda_{m+n}\}$.

On the other hand, it can be shown that the following inequality is established for each $r(t) = \eta \in \mathbb{S}$:

$$V(t, l(t), r(t) = \eta) \geq e^{2kt} \lambda_m(P_\eta) ||l(t)||^2,$$

where $\lambda_m(P_\eta) = \min\{\lambda_1, \lambda_2, \ldots, \lambda_{m+n}\}$.

Hence, we have

$$e^{2kt} \lambda_m(P_\eta) ||l(t)||^2 \leq \lambda_M(P_\eta) ||\phi - \varphi||^2,$$

which is equivalent to

$$||L(t, \eta, \phi) - L(t, \eta, \varphi)|| \leq \sqrt{\frac{\lambda_M(P_\eta)}{\lambda_m(P_\eta)}} ||\phi - \varphi|| e^{-kt}.$$

By the Definition 2.2, we can draw the conclusion that the system (2.2) is globally exponentially stable for all $r(t) = \eta \in \mathbb{S}$ and $t \geq 0$.

Remark 2.5 The conclusion is just content under the Assumption 2.1, that is to say the activation functions must meet Lipschitz conditions. The FLBAM model is different from T-S fuzzy BAM model, which has been investigated in [3].

Remark 2.6 Note that (2.3) is a linear matrix inequality, which can be solved by using the Matlab LMI toolbox. The matrix is relatively simple on account of that we haven't thought of the time delay. General, time-delay exists in many systems, while in our model we ignore the time-delay for convenience.

2.1.4 Numerical Examples

In this section, a numerical example will be given to demonstrate the feasible of the proposed results.

Consider the following fuzzy logical BAM neural networks with Markovian jumping parameters:

$$\begin{cases} \dot{u}_i(t,\eta) = -a_i(\eta)u_i(t) + \wedge_{j=1}^{2} b_{ij}(\eta) f_j(v_j(t)) \\ \qquad + \vee_{j=1}^{2} c_{ij}(\eta) f_j(v_j(t)) + \wedge_{j=1}^{2} \alpha_{ij}(\eta) g_j(t) \\ \qquad + \vee_{j=1}^{2} \beta_{ij}(\eta) g_j(t) + I_i(t), \\ \dot{v}_j(t,\eta) = -d_j(\eta)v_j(t) + \wedge_{i=1}^{2} e_{ji}(\eta) f_i(u_i(t)) \\ \qquad + \vee_{j=1}^{2} w_{ji}(\eta) f_i(u_i(t)) + \wedge_{i=1}^{2} \gamma_{ji}(\eta) h_i(t) \\ \qquad + \vee_{i=1}^{m} \delta_{ji}(\eta) h_i(t) + J_j(t). \end{cases}$$

where $a_1 = a_2 = d_1 = d_2 = 4.5$, $b = c = e = w = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\alpha = \beta = \gamma = \delta = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$.

We take the activation functions as follows:

$$f_i(x) = \frac{1}{2}(|x+1| - |x-1|), \quad (i = 1, 2).$$

To comfort the Assumption 2.1, we take $\lambda_i = 0$ $(i = 1, 2, 3, 4)$. Thus, through the numerical values mentioned above, we can obtain the matrices W_η, G_η, and E_η as follows:

$$W_\eta = \begin{bmatrix} 4.5 & 0 & 0 & 0 \\ 0 & 4.5 & 0 & 0 \\ 0 & 0 & 4.5 & 0 \\ 0 & 0 & 0 & 4.5 \end{bmatrix}, \quad G_\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_\eta = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

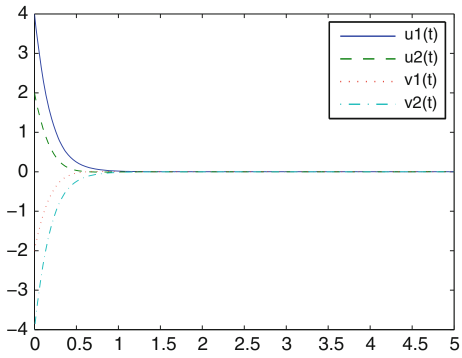


Fig. 2.1 State trajectory of the system with initial conditions (4, 2, −2, −4)

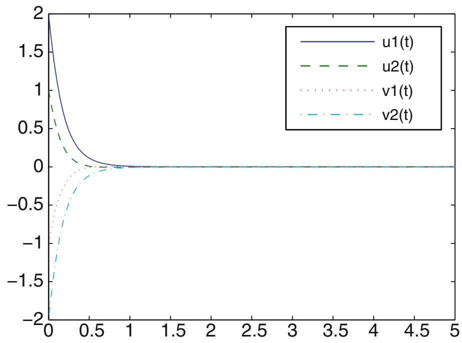


Fig. 2.2 State trajectory of the system with initial conditions (2, 1, −1, −2)

By using Matlab LMI Toolbox, we can solve the LMI (2.3), where the solutions are as follows:

$$k = 14.1, \ \ P_\eta = \begin{bmatrix} 15.3 & 5.6 & 8.0 & 8.0 \\ 5.6 & 15.3 & 8.0 & 8.0 \\ 8.0 & 8.0 & 15.3 & 5.6 \\ 8.0 & 8.0 & 5.6 & 15.3 \end{bmatrix}.$$

By Theorem 2.4, the system is global exponential stable. For this example, the figures below are the trajectories of the system with different initial conditions. The initial conditions of Fig. 2.1 is (4, 2, −2, −4) while Fig. 2.2 is (2, 1, −1, −2). The simulation results show that the system is global exponential stable.

2.1.5 Conclusion

In this section, we have investigated the global exponential stability of fuzzy logical BAM neural networks with Markovian jumping parameters, which have not been focus enough attentions on. Based on the Lyapunov functional approach and linear matrix inequality, a new sufficient stability criteria has been derived, which can be tested by using the Matlab LMI Toolbox. A numerical example is developed to demonstrate our proposed results.

2.2 Synchronization Control of Stochastically Coupled DNN

2.2.1 Introduction

In the past two decades, Delayed neural networks (DNNs) have received considerable attention from researchers in different fields. As is known, DNNs always present complex and unpredictable behaviors in practice, besides the traditional stability and periodic oscillation that have got a great deal of investigated in the past years. Recently, the synchronization problem of complex dynamical networks [5–9, 13, 17, 18, 27, 35, 38], like the synchronization of DNNs, is becoming the latest focus of attention.

Thanks to the tireless efforts of the former researchers, several results on neural network synchronization have been proposed in the literature. For example, in Ref. [24], synchronization of coupled delayed neural networks was released the first time. Then, some further studies in this field have appeared in recent years [14–16, 22, 26, 30, 34]. Wang and Cao studied synchronization in an array of linearly coupled networks with time-varying delay [27], and synchronization in an array of linearly stochastically coupled networks with time delays [7], respectively. In Ref. [6], via Lyapunov functional method and LMI approach, synchronization control of stochastic neural networks with time-varying delays has been researched and the estimation gains of controller that can ensure the synchronization have been obtained. In addition, in Ref. [16], the global exponential synchronization of coupled connected neural networks with delays was investigated and a sufficient condition was derived by using the LMI approaching. Meanwhile, through the stability theory for impulsive functional differential equations, some new criteria to guarantee the robust synchronization of coupled networks via impulsive control were derived in Ref. [26]. And, in Ref. [30], on the basis of Lyapunov stability theory, time-delay feedback control and other techniques, the exponential synchronization problem of a class of stochastic perturbed chaotic delayed neural networks was considered.

It is well known that, time-delays are often encountered in many kinds of neural networks, which can be the sources of oscillation and instability of neural networks [25, 28, 29, 31–33]. However, from the literature mentioned above, we can find that only discrete time-delay has been considered. Another important time-delay, namely, distributed time-delay, has not attracted wide attention of the researchers. Ref. [31] pointed out that there is usually a spatial extent in neural networks due to the presence of many parallel pathways with a variety of axon sizes and lengths, so, a distribution of propagation delays will appear over a period of time. Although the signal transmission is sometimes immediate and can be modeled with discrete delays, it may be distributed during a certain time period [29]. Hence, it is often the case that modeling a realistic neural network with both discrete and distributed delays [23].

Cao and Wang [7] investigated the synchronization in linearly stochastically coupled networks via a simple adaptive feedback control scheme considering the noises' influence and the discrete time delays. In Ref. [6], synchronization of stochastic neural networks with discrete time-delays was researched by using LMI approach. Motivated by these recently literatures and for the sake of modeling a more realistic and comprehensive networks, we consider the synchronization of linearly stochastically coupled networks with both discrete and distributed time-delays.

In this section, we aim to study the synchronization problem in an array of linearly stochastically coupled neural networks with discrete and distributed time delays. By employing the Lyapunov-Krasovskii functional method and LMI approach, we give several new criterions that can ensure the complete synchronization of the system. At the same time, the estimation gains of the delayed feedback controller are obtained. Then, an illustrative example is provided to prove the effectiveness of our results. Finally, we make a conclusion for the section.

2.2.2 Problem Formulation

In Ref. [7], an array of linearly stochastically coupled identical neural networks with time delays has been considered by Cao and Wang as follows:

$$\begin{aligned} dx_i(t) = [-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau))]dt + c_i \sum_{j=1}^{N} G_{ij}\Gamma x_j(t)dW_{i1}(t) \\ + d_i \sum_{j=1}^{N} G_{ij}\Gamma_\tau x_j(t-\tau)dW_{i2}(t) + U_i dt,\ i = 1, 2, \ldots, N, \end{aligned} \tag{2.5}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n (i = 1, 2, \ldots, N)$ is the state vector associated with the ith DNNs; $f(x_i(t)) = [f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t))]^T \in \mathbb{R}^n$ is the activation functions of the neurons with $f(0) = 0$; $C = \mathrm{diag}\{c_1, c_2, \ldots, c_n\} > 0$ is a diagonal matrix that shows the rate of the ith unit resetting its potential to the resting state in isolation when disconnected from the external inputs and the network; $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ stand for, respectively, the connection weight matrix and the discretely delayed connection weight matrix; $W_i = [W_{i1}, W_{i2}]^T$ are two-dimensional Brownian motions; $\Gamma \in \mathbb{R}^{n\times n}$ and $\Gamma_\tau \in \mathbb{R}^{n\times n}$ denotes the internal coupling of the network at time t and $t-\tau$, where $\tau > 0$ is the time-delay; c_i and d_i indicate the intensity of the noise; U_i is the input of the controller; $G = (G_{ij})_{N\times N}$ describes the topological structure and the coupling strength of the networks, and it meet the following conditions [27]:

$$G_{ii} = -\sum_{j=1, j\neq i}^{N} G_{ij}. \tag{2.6}$$

Though the linearly stochastically coupled neural networks has been investigated in-depth comparatively, only the discrete time delay was considered. So, in order to model a more realistic and comprehensive stochastically coupled DNNs, a novel model is presented as follows:

$$
\begin{aligned}
dx_i(t) = & \left[-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau)) + W \int_{t-\tau}^{t} f(x_i(s))ds \right] dt \\
& + c_i \sum_{j=1}^{N} G_{ij} \Gamma x_j(t) dW_{i1}(t) + d_i \sum_{j=1}^{N} G_{ij} \Gamma_\tau x_j(t-\tau) dW_{i2}(t) \\
& + U_i dt, \ i = 1, 2, \ldots, N
\end{aligned}
\tag{2.7}
$$

where $W = (w_{ij})_{n \times n}$ is the distributive delayed connection weight matrix. Then, we give the form of initial states corresponds with model (2.7) as follows:

For any $\phi_i \in \mathbb{L}^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, we have $x_i(t) = \varphi_i(t), i = 1, 2, \ldots, N$, where $-\tau \le t \le 0$.

Remark 2.7 It is obvious to see that both the discrete and distributed time delays are considered in the new model (2.7). Thus, the model will be more realistic and comprehensive than (2.5). To the best of the authors' knowledge, it is the first time that the synchronization problem of stochastically coupled identical neural networks with discrete and distributed time delays is proposed. In order to achieve our results, the following necessary assumption is made:

Assumption 2.8 The activation functions $f_i(u)$ are bounded and satisfy the Lipschitz condition:

$$
|f_i(u) - f_i(v)| \le \beta_i \, |u - v| \, , \ \forall u, v \in \mathbb{R}, i = 1, 2, \ldots, n,
\tag{2.8}
$$

where $\beta_i > 0$ is a constant.

Remark 2.9 Throughout this literature $f_i(u)$, the activation functions of the neurons, are always supposed to be continuous, differentiable and nondecreasing. And we only need the Lipschitz condition and boundedness to be satisfied. Actually, we can see this type of activation functions in many papers, such as Refs. [7, 28] etc.

Definition 2.10 Suppose that $x_i(t; t^*, X^*)$ is the solution of model (2.7), where $X^* = (x_1^*, x_2^*, \ldots, x_N^*)$, and $r(t) \in \mathbb{R}^n$ is the response of an isolated node

$$
dr(t) = \left[-Cr(t) + Af(r(t)) + Bf(r(t-\tau)) + W \int_{t-\tau}^{t} f(r(\eta))d\eta \right] dt.
\tag{2.9}
$$

If there exits a nonempty subset $\Psi \subseteq \mathbb{R}^n$, with $x_i^* \in \Psi$, and for any $t \ge 0$, we have $x_i(t; t^*, X^*) \in \mathbb{R}^n$ and

$$
\lim_{t \to \infty} E \left\| x_i(t; t^*, X^*) - r(t; t^*, x_0) \right\|^2 = 0,
\tag{2.10}
$$

where $i = 1, 2, \ldots, N$, and $x_0 \in \mathbb{R}^n$, then, it can be said that the DNNs model (2.7) achieve synchronization.

Next, we denote $e_i(t) = x_i(t) - r(t)$, which indicates the error signal. From (2.7), (2.9) and (2.6), the error signal system can be easily obtained as follows:

$$de_i(t) = \left[-Ce_i(t) + Ag(e_i(t)) + Bg(e_i(t-\tau)) + W \int\limits_{t-\tau}^{t} g(e_i(s))ds \right] dt$$

$$+ c_i \sum_{j=1}^{N} G_{ij} \Gamma e_j(t) dW_{i1}(t) + d_i \sum_{j=1}^{N} G_{ij} \Gamma_\tau e_j(t-\tau) dW_{i2}(t) + U_i dt, \ i = 1, 2, \ldots, N, \tag{2.11}$$

where $g(e_i(t)) = f(e_i(t) + r(t)) - f(r(t))$ and $g(e_i(t-\tau)) = f(e_i(t-\tau) + r(t-\tau)) - f(r(t-\tau))$. From (2.8) and $g(0) = 0$, it is obvious to see that

$$\|g(e_i(t))\| \le \|\mathrm{M} e_i(t)\| \tag{2.12}$$

where $\mathrm{M} = \mathrm{diag}\{\beta_1, \beta_2, \beta_3, \ldots, \beta_n\} > 0$ is a known constant matrix.

Considering make the controller more appropriate and realistic, we design a delayed feedback controller of the following form:

$$U_i = K_1 e_i(t) + K_2 e_i(t-\tau) \tag{2.13}$$

where $K_1 \in \Re^{n\times n}$ and $K_2 \in \Re^{n\times n}$ are constant gain matrices.

Remark 2.11 As Ref. [6] proposed, in many real applications, the memoryless state-feedback controller $U_i = Ke_i(t)$ is more popular, since it has an advantage of easy implementation, but its performance is not better than (2.13). Though $U_i = Ke_i(t) + \int_{t-\tau}^{t} K_1 e_i(s)ds$ is a more general form of delayed feedback controller, it is difficult for us to handle all the initial states of $e_i(t)$. However, the controller (2.13) is a compromise between better performance and simple implementation. Hence, in our section, we design the controller as (2.13) shows.

Definition 2.12 If the error signal satisfies that

$$\lim_{t\to\infty} E\|e_i(t)\|^2 = 0, \ i = 1, 2, \ldots, N \tag{2.14}$$

then, the error signal system (2.11) is globally asymptotically stable in mean square.

2.2.3 Main Results and Proofs

In this section, by using a properly designed delayed feedback controller, we will present a new criteria for the synchronization of stochastically coupled neural networks with discrete and distributed time delays on the basis of the Lyapunov-Krasovskii functional approach.

In order to simplify the description, we denote:

$$\Pi_{11} = P(-C + K_1) + (-C + K_1)^T P + Q_1 + (1 - \sigma_i)^{-1}\tau^2 \mathrm{M}^T \mathrm{M} + cN^2 \Lambda\lambda_{\max}\Gamma^T\Gamma, \tag{2.15}$$

$$\Pi_{22} = \mathrm{M}^T\mathrm{M} + dN^2\Lambda\lambda_{\max}\Gamma_\tau^T\Gamma_\tau - Q_1, \tag{2.16}$$

$$\Omega = PAA^T P + \mathrm{M}^T\mathrm{M} + PBB^T P + PWW^T P. \tag{2.17}$$

Theorem 2.13 *Let* $0 < \sigma_i < 1 (i = 1, 2, \ldots, N)$ *be any given constants. If there exit positive definite matrices* $P = (p_{ij})_{n\times n}$ *and* $Q_1 = (q_{ij})_{n\times n}$, *such that the following matrix inequality*

$$N = \begin{bmatrix} \Pi_{11} & PK_2 & PA & \mathrm{M}^T & PB & PW \\ * & \prod_{22} & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{2.18}$$

holds, where Π_{11} *and* Π_{22} *are defined in (2.15) and (2.16) respectively, then the error signal model (2.11) is globally asymptotically stable in mean square.*

Proof Define the following Lyapunov-Krasovskii functional candidate $V(t, e_i(t))$ by

$$V(t, e_i(t)) = \sum_{i=1}^{N} e_i^T(t)Pe_i(t) + \sum_{i=1}^{N} \int_{t-\tau}^{t} e_i^T(s)Q_1e_i(s)ds + \sum_{i=1}^{N} \int_{-\tau}^{0} \int_{t+s}^{t} e_i^T(\eta)Q_2e_i(\eta)d\eta ds \tag{2.19}$$

where $P = (p_{ij})_{n\times n}$, $Q = (q_{ij})_{n\times n}$ are positive definite matrices that to be determined, and $Q_2 \geq 0$ is given by

$$Q_2 = (1 - \sigma_i)^{-1}\tau \mathrm{M}^T\mathrm{M}. \tag{2.20}$$

By $It\hat{o}$ differential formula, the stochastic derivative of $V(t, e_i(t))$ along error system (2.11) can be obtained as follows:

$$
\begin{aligned}
dV(t, e_i(t)) = \mathcal{L}V(t, e_i(t))dt + \sum_{i=1}^{N} 2e_i^T(t)P\Bigg[c_i \sum_{j=1}^{N} G_{ij}\Gamma x_j(t)dW_{i1}(t) \\
+ d_i \sum_{j=1}^{N} G_{ij}\Gamma_\tau x_j(t-\tau)dW_{i2}(t)\Bigg],
\end{aligned} \tag{2.21}
$$

where the weak infinitesimal operator $\mathcal{L}V$ of the stochastic process is given by

$$
\begin{aligned}
\mathcal{L}V(t, e_i(t)) = \sum_{i=1}^{N} 2e_i^T(t)P\Bigg[&-Ce_i(t) + Ag(e_i(t)) + Bg(e_i(t-\tau)) \\
&+ K_1 e_i(t) + K_2 e_i(t-\tau) + W\int_{t-\tau}^{t} g(e_i(s))ds\Bigg] \\
&+ \sum_{i=1}^{N}\Bigg[e_i^T(t)(Q_1 + \tau Q_2)e_i(t) - e_i^T(t-\tau)Q_1 e_i(t-\tau) \\
&- \int_{t-\tau}^{t} e_i^T(s)Q_2 e_i(s)ds\Bigg] \\
&+ c_i^2 \sum_{i=1}^{N}\Bigg[\sum_{j=1}^{N} G_{ij}\Gamma e_j(t)\Bigg]^T\Bigg[\sum_{j=1}^{N} G_{ij}\Gamma e_j(t)\Bigg] \\
&+ d_i^2 \sum_{i=1}^{N}\Bigg[\sum_{j=1}^{N} G_{ij}\Gamma_\tau e_j(t-\tau)\Bigg]^T\Bigg[\sum_{j=1}^{N} G_{ij}\Gamma_\tau e_j(t-\tau)\Bigg] \\
= \sum_{i=1}^{N}\Bigg\{&2[e_i^T(t)P(-C + K_1)e_i(t) + e_i^T(t)PK_2 e_i(t-\tau) \\
&+ e_i^T(t)PAg(e_i(t)) + e_i^T(t)PBg(e_i(t-\tau)) + e_i^T(t)PW \\
&\times \int_{t-\tau}^{t} g(e_i(s))ds] + e_i^T(t)(Q_1 + \tau Q_2)e_i(t) - e_i^T(t-\tau)Q_1 e_i(t-\tau)
\end{aligned}
$$

$$
\begin{aligned}
&- \int\limits_{t-\tau}^{t} e_i^T(s) Q_2 e_i(s) ds + c_i^2 \left[\sum_{j=1}^{N} G_{ij} \Gamma e_j(t) \right]^T \left[\sum_{j=1}^{N} G_{ij} \Gamma e_j(t) \right] \\
&+ d_i^2 \left[\sum_{j=1}^{N} G_{ij} \Gamma_\tau e_j(t-\tau) \right]^T \left[\sum_{j=1}^{N} G_{ij} \Gamma_\tau e_j(t-\tau) \right] \Bigg\}. \qquad (2.22)
\end{aligned}
$$

Then, following from the relation (2.12) and Lemma 1.13, we can obtain

$$
\begin{aligned}
e_i^T(t) P A g(e_i(t)) &\le \frac{1}{2} e_i^T(t) P A A^T P e_i(t) + \frac{1}{2} g^T(e_i^T(t)) g(e_i(t)) \\
&\le \frac{1}{2} e_i^T(t) P A A^T P e_i(t) + \frac{1}{2} e_i^T(t) \mathrm{M}^T \mathrm{M} e_i(t) \qquad (2.23)
\end{aligned}
$$

$$
\begin{aligned}
e_i^T(t) P B g(e_i(t-\tau)) &\le \frac{1}{2} e_i^T(t) P B B^T P e_i(t) + \frac{1}{2} g^T(e_i^T(t-\tau)) g(e_i(t-\tau)) \\
&\le \frac{1}{2} e_i^T(t) P B B^T P e_i(t) + \frac{1}{2} e_i^T(t-\tau) \mathrm{M}^T \mathrm{M} e_i(t-\tau) \qquad (2.24)
\end{aligned}
$$

$$
\begin{aligned}
e_i^T(t) P W \int\limits_{t-\tau}^{t} g(e_i(s)) ds &\le \frac{1}{2} e_i^T(t) P W W^T P e_i(t) \\
&+ \frac{1}{2} \left(\int\limits_{t-\tau}^{t} g(e_i(s)) ds \right)^T \left(\int\limits_{t-\tau}^{t} g(e_i(s)) ds \right) \qquad (2.25)
\end{aligned}
$$

where $\mathrm{M} = \mathrm{diag}\{\beta_1, \beta_2, \ldots, \beta_n\}$ is a known constant matrix. Moreover, it can be seen from Lemma 1.20, (2.12) and (2.20) that

$$
\begin{aligned}
\frac{1}{2} \left(\int\limits_{t-\tau}^{t} g(e_i(s)) ds \right)^T \left(\int\limits_{t-\tau}^{t} g(e_i(s)) ds \right) &\le \frac{1}{2} \tau \int_{t-\tau}^{t} g^T(e_i(s)) g(e_i(s)) ds \\
&\le \frac{1}{2} \tau \int_{t-\tau}^{t} e_i^T(s) \mathrm{M}^T \mathrm{M} e_i(s) ds = \frac{1}{2} (1-\sigma_i) \int\limits_{t-\tau}^{t} e_i^T(s) Q_2 e_i(s) ds. \qquad (2.26)
\end{aligned}
$$

Hence, from (2.25) and (2.26), we have

$$
e_i^T(t) P W \int\limits_{t-\tau}^{t} g(e_i(s)) ds \le \frac{1}{2} e_i^T(t) P W W^T P e_i(t) + \frac{1}{2} (1-\sigma_i) \int\limits_{t-\tau}^{t} e_i^T(s) Q_2 e_i(s) ds \qquad (2.27)
$$

Next, we can estimate the two following terms by

$$\left[\sum_{j=1}^{N} G_{ij}\Gamma e_j(t)\right]^T \left[\sum_{j=1}^{N} G_{ij}\Gamma e_j(t)\right] \le NG_{ij}^2 \sum_{j=1}^{N} e_j^T(t)\Gamma^T\Gamma e_j(t)$$
$$\le NG_{ij}^2 \lambda_{\max}(\Gamma^T\Gamma) \sum_{j=1}^{N} e_j^T(t)e_j(t), \tag{2.28}$$

$$\left[\sum_{j=1}^{N} G_{ij}\Gamma_\tau e_j(t-\tau)\right]^T \left[\sum_{j=1}^{N} G_{ij}\Gamma_\tau e_j(t-\tau)\right]$$
$$\le NG_{ij}^2 \sum_{j=1}^{N} e_j^T(t-\tau)\Gamma_\tau^T\Gamma_\tau e_j(t-\tau)$$
$$\le NG_{ij}^2 \lambda_{\max}(\Gamma_\tau^T\Gamma_\tau) \sum_{j=1}^{N} e_j^T(t-\tau)e_j(t-\tau) \tag{2.29}$$

Therefore, applying (2.23), (2.24), (2.27)–(2.29) to (2.22), one yields

$$\begin{aligned}\mathcal{L}V(t, e_i(t)) \le \sum_{i=1}^{N} & 2e_i^T(t)P(-C+K_1)e_i(t) + 2e_i^T(t)PK_2e_i(t-\tau)\\ &+ e_i^T(t)PAA^TPe_i(t) + e_i^T(t)\mathrm{M}^T\mathrm{M}e_i(t) + e_i^T(t)PBB^TPe_i(t)\\ &+ e_i^T(t-\tau)\mathrm{M}^T\mathrm{M}e_i(t-\tau) + e_i^T(t)PWW^TPe_i(t)\\ &+ (1-\sigma_i)\int_{t-\tau}^{t} e_i^T(s)Q_2e_i(s)ds + e^T(t)(Q_1+\tau Q_2)e_i(t)\\ &- e_i^T(t-\tau)Q_1e_i(t-\tau) - \int_{t-\tau}^{t} e_i^T(s)Q_2e_i(s)ds + cN\Lambda\lambda_{\max}(\Gamma^T\Gamma)\\ &\times \sum_{j=1}^{N} e_j^T(t)e_j(t) + dN\Lambda\lambda_{\max}(\Gamma_\tau^T\Gamma_\tau)\sum_{j=1}^{N} e_j^T(t-\tau)e_j(t-\tau)\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \Bigg\{ e_i^T(t)[2P(-C+K_1) + PAA^TP + \mathrm{M}^T\mathrm{M} + PBB^TP \\
&\quad + PWW^TP + Q_1 + \tau Q_2 + cN^2\Lambda\lambda_{\max}(\Gamma^T\Gamma)]e_i(t) \\
&\quad + 2e_i^T(t)PK_2e_i(t-\tau) + e_i^T(t-\tau)[\mathrm{M}^T\mathrm{M} + dN^2\Lambda\lambda_{\max}(\Gamma_\tau^T\Gamma_\tau) \\
&\quad - Q_1]e_i(t-\tau) - \sigma_i \int_{t-\tau}^{t} e_i^T(s)Q_2e_i(s)ds \Bigg\} \\
&\le \sum_{i=1}^{N} \left\{ \begin{bmatrix} e_i(t) & e_i(t-\tau) \end{bmatrix} \begin{bmatrix} \prod_{11} + \Omega & PK_2 \\ K_2^TP & \prod_{22} \end{bmatrix} \begin{bmatrix} e_i(t) \\ e_i(t-\tau) \end{bmatrix} \right\} \\
&= \sum_{i=1}^{N} \left\{ \begin{bmatrix} e_i(t) & e_i(t-\tau) \end{bmatrix} N \begin{bmatrix} e_i(t) \\ e_i(t-\tau) \end{bmatrix} \right\}
\end{aligned} \tag{2.30}
$$

where $N = \begin{bmatrix} \prod_{11} + \Omega & PK_2 \\ K_2^TP & \prod_{22} \end{bmatrix}$, $\Lambda = \max_{1 \le i,j \le N} \left\{ G_{ij}^2 \right\}$, $c = \max_{1 \le i \le N} \left\{ c_i^2 \right\}$ and $d = \max_{1 \le i \le N} \left\{ d_i^2 \right\}$.

From Lemma 1.21, the form of $N = \begin{bmatrix} \prod_{11} + \Omega & PK_2 \\ K_2^TP & \prod_{22} \end{bmatrix} < 0$ can be transformed to (2.18), and the two forms are equivalent. It is obvious to see from (2.29) and *Itô* rule that

$$
\mathbb{E}V(t, e_i(t)) - \mathbb{E}V(t_0, e_i(t_0)) = \mathbb{E} \int_{t_0}^{t} \mathcal{L}V(s, e_i(s))ds \tag{2.31}
$$

For the positive constant $\eta_i > 0 (i = 1, 2, \ldots, N)$, it can be concluded that

$$
\begin{aligned}
\eta_i \mathbb{E}\|e_i(t)\|^2 &\le \mathbb{E}V(t, e_i(t)) \le \mathbb{E}V(t_0, e_i(t_0)) + \mathbb{E} \textstyle\int_{t_0}^{t} \mathcal{L}V(s, e_i(s))ds \\
&\le \mathbb{E}V(t_0, e_i(t_0)) + \lambda_{\max}\mathbb{E} \int_{t_0}^{t} \|e_i(t)\|^2 ds
\end{aligned} \tag{2.32}
$$

where $\lambda_{\max} < 0$ indicates the maximal eigenvalue of N. Therefore, from all the above proofs and results (2.32), together with the study in Ref. [30], we can conclude that the error signal model (2.11) is globally asymptotically stable in mean square. This completes the proof.

Remark 2.14 As it is presented in Theorem 2.13, the synchronization of an array of linearly stochastically coupled identical neural networks with discrete and distributed time delays can be guaranteed if the matrix inequality (2.18) is feasible. Since (2.18) is linear with $P > 0$ and $Q_1 > 0$, by utilizing the Matlab LMI toolbox, we can check the feasibility of (2.18) directly. Meanwhile, the estimate gain matrix K_1 and K_2 can also be obtained.

Remark 2.15 In this section, for the sake of simplifying the description, we are concerned with the constant time delay. As for time-varying delay, we can derive the similar results without difficulties, which will be more realistic and comprehensive.

Corollary 2.16 *Let $0 < \sigma_i < 1 (i = 1, 2, \ldots, N)$ be any given constants. If there exits a positive definite matrix $Q_1 = (q_{ij})_{n\times n}$, such that the following matrix inequality:*

$$N_1 = \begin{bmatrix} \Xi_{11} & \rho K_2 & \rho A & \mathrm{M}^T & \rho B & \rho W \\ * & \Xi_{22} & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{2.33}$$

holds, where
$\Xi_{11} = \rho(-C + K_1) + \rho(-C + K_1)^T + Q_1 + (1 - \sigma_i)^{-1}\tau^2\mathrm{M}^T\mathrm{M} + cN^2\Lambda\lambda_{\max}\Gamma^T\Gamma$ *and* $\Xi_{22} = \mathrm{M}^T\mathrm{M} + dN^2\Lambda\lambda_{\max}\Gamma_\tau^T\Gamma_\tau - Q_1$ *then the error signal model (2.11) is globally asymptotically stable in mean square.*

Proof Let $P = \rho I$, where ρ is a positive constant and I is the identity matrix. From Theorem 2.13 we can obtain Corollary 2.16 immediately.

For the sake of presenting the designed estimate gain matrix K_1 and K_2 by using the LMI toolbox in Matlab conveniently, we made a simple transformation. Then, the following theorem can be easily derived.

Theorem 2.17 *Let $0 < \sigma_i < 1 (i = 1, 2, \ldots, N)$ be any given constants. If there exits positive definite matrices $P = (p_{ij})_{n\times n}$ and $Q_1 = (q_{ij})_{n\times n}$, such that the following matrix inequality*

$$N_2 = \begin{bmatrix} \Omega_{11} & K_2^* & PA & \mathrm{M}^T & PB & PW \\ * & \Omega_{22} & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{2.34}$$

holds, where $\Omega_{11} = -PC + K_1^* - C^T P + K_1^{*T} + Q_1 + (1 - \sigma_i)^{-1}\tau^2\mathrm{M}^T\mathrm{M} + cN^2\Lambda\lambda_{\max}\Gamma^T\Gamma$ and $\Omega_{22} = \mathrm{M}^T\mathrm{M} + dN^2\Lambda\lambda_{\max}\Gamma_\tau^T\Gamma_\tau - Q_1$, furthermore, $K_1^* = PK_1$ and $K_2^* = PK_2$, then the error signal model (2.11) is globally asymptotically stable in mean square.

Proof In Theorem 2.13, let $K_1 = P^{-1}K_1^*$ and $K_2 = P^{-1}K_2^*$. Then Theorem 2.17 can be derived directly.

Remark 2.18 The method in Theorem 2.17 of solving the estimate gain matrix K_1 and K_2 once be used in Ref. [6], and it is very useful to design the controller that can ensure the system (2.7) achieve synchronization.

Corollary 2.19 *Let $0 < \sigma_i < 1 (i = 1, 2, \ldots, N)$ be any given constants. If there exits a positive definite matrix $Q_1 = (q_{ij})_{n\times n}$, such that the following matrix inequality:*

$$N_3 = \begin{bmatrix} \Delta_{11} & K_2^* & \rho A & \mathrm{M}^T & \rho B & \rho W \\ * & \Delta_{22} & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{2.35}$$

holds, where $\Delta_{11} = -\rho C + K_1^* - \rho C^T + K_1^{*T} + Q_1 + (1-\sigma_i)^{-1}\tau^2 \mathrm{M}^T \mathrm{M} + cN^2 \Lambda\lambda_{\max} \Gamma^T \Gamma$ and $\Delta_{22} = \mathrm{M}^T \mathrm{M} + dN^2 \Lambda\lambda_{\max} \Gamma_\tau^T \Gamma_\tau - Q_1$, furthermore, $K_1^* = \rho K_1$ and $K_2^* = \rho K_2$, then, the error signal model (2.11) is globally asymptotically stable in mean square.

Proof Let $P = \rho I$ in Theorem 2.17, where ρ is a positive constant and I is the identity matrix. Then, we can obtain Corollary 2.19 immediately.

Remark 2.20 Through Corollaries 2.16 and 2.19, it is obvious to see that our main result in Theorem 2.13 is general enough to contain some special cases, such as $P = \rho I$.

2.2.4 Illustrative Example

In this section, our main purpose is to authenticate the global asymptotical stability of the error signal model (2.11). In order to illustrate the effectiveness of our results, an example is presented here.

Example
Consider, the following chaotic DNNs with discrete and distributed time delays:

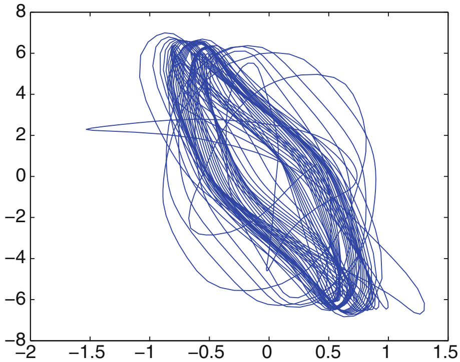
$$\mathrm{d}x(t) = \left[-Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + W \int_{t-\tau}^{t} f(x(s))ds \right] dt \tag{2.36}$$

where $x(t) = [x_1(t), x_2(t)]^T$ is the state vector of the single node in the DNNs, $f(x(t)) = [\tanh(x_1(t)), \tanh(x_2(t))]^T$, $\tau = 1$,

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -0.1 \\ -4.8 & 4.5 \end{bmatrix}, B = \begin{bmatrix} -1.7 & -0.1 \\ -0.3 & -4.1 \end{bmatrix}, W = \begin{bmatrix} -1.2 & -0.3 \\ -0.4 & -3.2 \end{bmatrix}.$$

In the condition that the initial value is chosen as $x_1(t) = 0.4$, $x_2(t) = 0.6$, $\forall t \in [-1, 0]$, the chaotic phase trajectories can be easily obtained as Fig. 2.3 shows.

Fig. 2.3 Chaotic phase trajectories



In order to verify the effectiveness of our results that can make the model (2.11) achieve synchronization, we just need to test the global asymptotical stability of the error signal model as the following shows:

$$\begin{aligned} de_i(t) = &\left[-Ce_i(t) + Ag(e_i(t)) + Bg(e_i(t-\tau)) + W\int_{t-\tau}^{t} g(e_i(s))ds\right]dt \\ &+ c_i\sum_{j=1}^{N} G_{ij}\Gamma e_j(t)dW_{i1}(t) + d_i\sum_{j=1}^{N} G_{ij}\Gamma_\tau e_j(t-\tau)dW_{i2}(t) \\ &+ [K_1e_i(t) + K_2e_i(t-\tau)]dt, \end{aligned} \tag{2.37}$$

where $i = 1, 2, \ldots, N$, $e_i(t) = [e_{i1}(t), e_{i2}(t)]^T$. Let $c_i = \sqrt{0.1}$, $d_i = \sqrt{0.1}$, $N = 4$, $\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Gamma_\tau = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the coupling matrix $G_{ij} = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}_{4\times 4}$.

The constant matrix M referred in (2.12) is chosen as $\mathrm{M} = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.2 \end{bmatrix}$. Then according to Theorem 2.13 and by utilizing the Matlab LMI toolbox, the following feasible results are derived:

$$P = \begin{bmatrix} 4.9114 & 0.4327 \\ 0.4327 & 1.4143 \end{bmatrix}, \ Q_1 = \begin{bmatrix} 21.4806 & 0.1140 \\ 0.1140 & 20.5590 \end{bmatrix},$$

$$K_1^* = \begin{bmatrix} -45.4784 & 0.0047 \\ 0.0047 & -45.5166 \end{bmatrix}, \ K_2^* = \begin{bmatrix} -7.0969 & -0.0766 \\ -0.0766 & -6.4766 \end{bmatrix}$$

Next, from $K_1 = P^{-1}K_1^*$ and $K_2 = P^{-1}K_2^*$, we can obtain the estimate gain matrix

Table 2.1 Initial states of model (2.37)

i	1	2	3	4
$e_{i1}(t)$	0.4	1.4	2.4	3.4
$e_{i2}(t)$	−1.6	−0.6	0.4	1.4

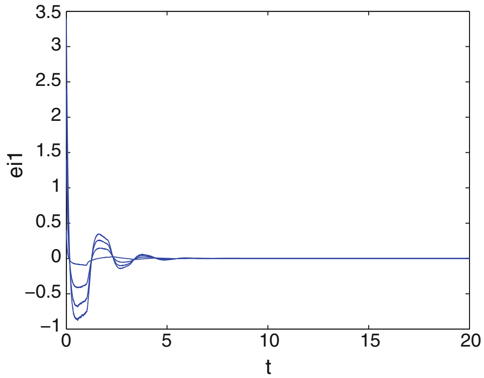


Fig. 2.4 Synchronization error of e_{i1}

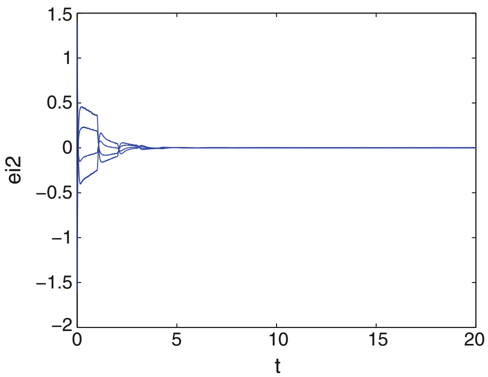


Fig. 2.5 Synchronization error of e_{i2}

$$K_1 = \begin{bmatrix} -9.5166 & 2.9151 \\ 2.9151 & -33.0759 \end{bmatrix}, \text{ and } K_2 = \begin{bmatrix} -1.4801 & 0.3987 \\ 0.3987 & -4.7022 \end{bmatrix}$$

immediately.

Under the Initial states as given in Table 2.1 applying the above-mentioned results to the error signal model (2.37), we can derive the wave diagrams of the error signal $e_{i1}(t)$ and $e_{i2}(t)$ as Figs. 2.4 and 2.5 show, respectively ($i = 1, 2, 3, 4$).

In Figs. 2.4 and 2.5, it is obvious to see that the error signal model (2.37) or (2.11) is globally asymptotically stable. That is to say, from our simulation results, it can be found that the synchronization of an array of linearly stochastically coupled identical neural networks with discrete and distributed time delays is achieved by using the delayed feedback controller that we designed. Thus, our theoretical results have been tested to be true by the simulations, and we can conclude that our study in the synchronization control problem of stochastically coupled neural networks with discrete and distributed time delays is practical and effective.

2.2.5 Conclusion

The synchronization control problem for an array of coupled DNNs has been thoroughly studied in this section. Several sufficient conditions to guarantee the synchronization have been obtained by constructing a Lyapunov-Krasovskii functional and using the LMI approach. Especially, the discrete and distributed time delay terms have been considered in the model, together with the stochastic coupling term. The delayed feedback controller gains have been gained based on the stability condition of error system. Finally, an illustrative example has been given to verify the theoretical analysis. The results are novel, because there are few works about the synchronization of system with both discrete and distributed time delays. At the same time, it is possible to apply the results to the realistic systems in practice.

References

1. P.B. Ali, M. Syed, Stability analysis of Takagi-Sugeno fuzzy Cohen-Grossberg BAM neural networks with discrete and distributed time-varying delays. Math. Comput. Model. **53**(1), 151–160 (2011)
2. S. Arik, V. Tavsanoglu, Global asymptotic stability analysis of bidirectional associative memory neural networks with constant time delays. Neurocomputing **68**, 161–176 (2005)
3. P. Balasubramanian, R. Rakkiyappan, R. Sathy, Delay dependent stability results for fuzzy BAM neural networks with Markovian jumping parameters. Expert Syst. Appl. **38**(1), 121–130 (2011)
4. R. Belohlavek, Fuzzy logical bidirectional associative memory. Inf. Sci. **128**(1), 91–103 (2000)
5. J. Cao, P. Li, W. Wang, Global synchronization in arrays of delayed neural networks with constant and delayed coupling. Phys. Lett. A **353**(4), 318–325 (2006)
6. J. Cao, X. Li, Stability in delayed Cohen-Grossberg neural networks: LMI optimization approach. Phys. D **212**(1), 54–65 (2005)
7. J. Cao, Z. Wang, Y. Sun, Synchronization in an array of linearly stochastically coupled networks with time-delays. Phys. A: Stat. Mech. Appl. **385**(2), 718–728 (2007)
8. G.R. Chen, J. Zhou, Z.R. Liu, Global synchronization of coupled delayed neural networks and applications to chaotic CNN model. Int. J. Bifurc. Chaos **14**(7), 2229–2240 (2004)
9. M. Chen, D. Zhou, Synchronization in uncertain complex networks. Chaos: an interdisciplinary. J. Nonlinear Sci. **16**(1), 013101 (2006)
10. B. Kosko, Adaptive bi-directional associative memories. Appl. Opt. **26**(23), 4947–4960 (1987)
11. B. Kosko, Bi-directional associative memories. IEEE Trans. Syst., Man Cybern **18**(1), 49–60 (1988)
12. B. Kosko, *Neural Networks and Fuzzy Systems—A Dynamical System Approach to Machine Intelligence* (Prentice-Hall, Englewood Cliffs, 1992)
13. C. Li, S. Li, X. Liao, J. Yu, Synchronization in coupled map lattices with small-world delayed interactions. Phys. A **335**(3), 365–370 (2004)
14. C.G. Li, G.R. Chen, Synchronization in general complex dynamical networks with coupling delays. Phys. A **343**, 263–278 (2004)

15. P. Li, J. Cao, Z. Wang, Robust impulsive synchronization of coupled delayed neural networks with uncertainties. Phys. A **373**, 261–272 (2007)
16. Z. Li, G. Chen, Robust adaptive synchronization of uncertain dynamical networks. Phys. Lett. A **324**(2), 166–178 (2004)
17. W. Lin, G. Chen, Using white noise to enhance synchronization of coupled chaotic systems. Chaos **16**(1), 013133–013134 (2006)
18. B. Liu, X.Z. Liu, G.R. Chen, Robust impulsive synchronization of uncertain dynamical networks. IEEE Trans. Circuits Syst. I **52**(7), 1431–1441 (2005)
19. B. Liu, P. Shi, Delay-range-dependent stability for fuzzy BAM neural networks with time-varying delays. Phys. Lett. A **373**(21), 1830–1838 (2009)
20. X. Lou, B. Cui, Robust asymptotic stability of uncertain fuzzy BAM neural networks with time-varying delays. Fuzzy Sets Syst. **158**(24), 2746–2756 (2007)
21. X. Lou, B. Cui, Stochastic exponential stability for Markovian jumping BAM neural networks with time-varying delays. IEEE Trans. Syst. **37**(3), 713–719 (2007)
22. W. Lu, T. Chen, Synchronization of coupled connected neural networks with delays. IEEE Trans. Circuits Syst. I **51**(12), 2491–2503 (2004)
23. J. Lv, G. Chen, A time-varying complex dynamical network model and its controlled synchronization criteria. IEEE Trans. Autom. Control **50**(6), 841–846 (2005)
24. L.M. Pecora, T.L. Carroll, G. Johnson, D. Mar, K.S. Fink, Synchronization stability in coupled oscillator arrays: solution for arbitrary configurations. Int. J. Bifurc. Chaos **10**(2), 273–290 (2000)
25. S. Ruan, R. Filfil, Dynamics of a two-neuron system with discrete and distributed delays. Phys. D **191**(3), 323–342 (2004)
26. Y. Sun, J. Cao, Z. Wang, Exponential synchronization of stochastic perturbed chaotic delayed neural networks. Neurocomputing **70**(13), 2477–2485 (2007)
27. W. Wang, J. Cao, Synchronization in an array of linearly coupled networks with time-varying delay. Phys. A **366**, 197–211 (2006)
28. Z. Wang, J. Fang, X. Liu, Global stability of stochastic high-order neural networks with discrete and distributed delays. Chaos Solitons Fractals **36**(2), 388–396 (2008)
29. Z. Wang, S. Lauria, J. Fang, X. Liu, Exponential stability of uncertain stochastic neural networks with mixed time-delays. Chaos Solitons Fractals **32**(1), 62–72 (2007)
30. Z. Wang, Y. Liu, F. Karl, X. Liu, Stochastic stability of uncertain Hopfield neural networks with discrete and distributed delays. Phys. Lett. A **354**(4), 288–297 (2006)
31. Z. Wang, Y. Liu, X. Liu, On global asymptotic stability of neural networks with discrete and distributed delays. Phys. Lett. A **345**(4), 299–308 (2005)
32. Z. Wang, H. Shu, J. Fang, X. Liu, Robust stability for stochastic Hopfield neural networks with time delays. Nonlinear Anal.: Real World Appl. **7**(5), 1119–1128 (2006)
33. Z. Wang, H. Shu, Y. Liu, D.W.C. Ho, X. Liu, Robust stability analysis of generalized neural networks with discrete and distributed time delays. Chaos Solitons Fractals **30**(4), 886–896 (2006)
34. C.W. Wu, Perturbation of coupling matrices and its effect on the synchronizability in arrays of coupled chaotic systems. Phys. Lett. A **319**(5–6), 495–503 (2003)
35. C.W. Wu, Synchronization in array of coupled nonlinear system with delay and nonreciprocal time-varying coupling. IEEE Trans. Circuits Syst. **52**(5), 282–286 (2005)
36. H. Xiang, J. Cao, Exponential stability of periodic solution to Cohen-Grossberg-type BAM networks with time-varying delays. Neurocomputing **72**(7), 1702–1711 (2009)
37. H. Xiang, J. Wang, Exponential stability and periodic solution for fuzzy BAM neural networks with time varying delays. Appl. Math. J. Chin. Univ. **24**(2), 157–166 (2009)
38. W. Yu, J. Cao, Synchronization control of stochastic delayed neural networks. Phys. A **373**, 252–260 (2007)

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