

Chapter 2

Preliminaries

Abstract This chapter introduces some basic definitions and results on graph theory, consensus decomposition of linear space theory, matrix theory, linear system theory, and singular system theory, which will be used in the following chapters. First, the definitions of directed graph, spanning tree, and Laplacian matrix, etc. are given, and properties of Laplacian matrix are addressed. Then the concepts of consensus subspace, complement consensus subspace, and state/output space decomposition are defined. Third, the properties of Kronecker product and Schur complement lemma are introduced. Moreover, the definitions and criteria for controllability, observability, and stability of linear time-invariant systems are summarized, and some results on partial stability of linear systems are also given. Finally, the definitions and results on the regularity, equivalent form, admissibility, and controllability of singular systems are introduced.

2.1 Graph Theory

A graph $G = (\mathcal{V}(G), \mathcal{E}(G))$ consists of a node set $\mathcal{V}(G) = \{v_1, v_2, \dots, v_N\}$ and a edge set $\mathcal{E}(G) \subseteq \{(v_i, v_j), i \neq j; v_i, v_j \in \mathcal{V}(G)\}$. Denote by (v_i, v_j) and e_{ij} the edge from node v_i to node v_j , where v_i is called the *parent node* and v_j is called the *child node*. If a graph $G_0 = (\mathcal{V}_0(G_0), \mathcal{E}_0(G_0))$ satisfies $\mathcal{V}_0(G_0) \subseteq \mathcal{V}(G)$ and $\mathcal{E}_0(G_0) \subseteq \mathcal{E}(G)$, then G_0 is called a subgraph of G . If for any $e_{ij} \in \mathcal{E}(G)$, $e_{ji} \in \mathcal{E}(G)$, then G is an *undirected graph*, otherwise, G is a *directed graph*. A directed path from node v_{i_1} to v_{i_l} is a sequence of ordered edges with the form of $(v_{i_k}, v_{i_{k+1}})$, where $v_{i_k} \in \mathcal{V}(G)$ ($k = 1, 2, \dots, l - 1$). For a directed graph G , if for any two different nodes v_i and v_j , there exists a directed path from node v_i to node v_j , then G is said to be strongly connected. If for any two different nodes v_i and v_j , there exists a node v_k that has directed paths to node v_i and v_j , then G is said to be weakly connected. For undirected graphs, weakly connected and strongly connected are equivalent, which can all be called as connected. A directed graph is said to have a *spanning tree* if there exists at least one node having a directed path to all the other nodes.

Define the *adjacency matrix* of G as the nonnegative matrix $\mathcal{W} = [w_{ij}] \in \mathbb{R}^{N \times N}$ where w_{ij} represents the weight of edge e_{ji} with $w_{ij} > 0 \Leftrightarrow e_{ji} \in \mathcal{E}(G)$. Node v_i is called a *neighbor* of node v_j if there exists an edge e_{ij} . Denote by $N_i = \{v_j \in \mathcal{V}(G) : (v_j, v_i) \in \mathcal{E}(G)\}$ the neighbor set of node v_i . The in-degree and out-degree of node v_i are represented by $\deg_{in}(v_i) = \sum_{j=1}^N w_{ij}$ and $\deg_{out}(v_i) = \sum_{j=1}^N w_{ji}$. A graph G is balanced if for any node v_i , the in-degree is equal to the out-degree. Define the in-degree matrix of G by diagonal matrix \mathcal{D} the elements of which are the in-degrees of nodes. The *Laplacian matrix* of G is defined as $L = \mathcal{D} - \mathcal{W}$.

The following lemmas show basic properties of the Laplacian matrix L .

Lemma 2.1 ([1, 2]) *For a directed graph G with N nodes, it holds that*

(i) *L has at least one 0 eigenvalue, and $\mathbf{1}$ is the associated eigenvector; that is, $L\mathbf{1} = 0$;*

(ii) *If G has a spanning tree, then 0 is a simple eigenvalue of L , and all the other $N - 1$ eigenvalues have positive real parts;*

(iii) *If G does not have a spanning tree, then L has at least two 0 eigenvalues with the geometric multiplicity being not less than 2.*

Lemma 2.2 ([3]) *For an undirected graph G with N nodes, it follows that*

(i) *L has at least one 0 eigenvalue, and $\mathbf{1}$ is the associated eigenvector satisfying $L\mathbf{1} = 0$;*

(ii) *If G is connected, then 0 is a simple eigenvalue of L , and all the rest $N - 1$ eigenvalues are positive.*

2.2 Consensus Decomposition of Linear Space

Define λ_i ($i = 1, 2, \dots, N$) as the eigenvalues of $L \in \mathbb{R}^N$, where the associated eigenvector of $\lambda_1 = 0$ is $\bar{u}_1 = \mathbf{1}$. Define a nonsingular matrix $U = [\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N] \in \mathbb{C}^{N \times N}$. Let $c_k \in \mathbb{R}^v$ ($k = 1, 2, \dots, v$) be linearly independent vectors and $p_j = \bar{u}_i \otimes c_k$ ($j = (i - 1)v + k$; $i = 1, 2, \dots, N$; $k = 1, 2, \dots, v$). A *consensus subspace* is defined as the subspace $\mathbb{C}(U)$ spanned by $p_k = \bar{u}_1 \otimes c_k = \mathbf{1} \otimes c_k$ ($k = 1, 2, \dots, v$), and a *complement consensus subspace* is defined as the subspace $\bar{\mathbb{C}}(U)$ spanned by $p_{v+1}, p_{v+2}, \dots, p_{Nv}$. Note that p_j ($j = 1, 2, \dots, Nv$) are linearly independent. The following conclusion can be obtained.

Lemma 2.3 $\mathbb{C}(U) \oplus \bar{\mathbb{C}}(U) = \mathbb{C}^{Nv}$.

Remark 2.1 From Lemma 2.3, one sees that any \mathbb{C}^{Nv} can be uniquely projected onto $\mathbb{C}(U)$ and $\bar{\mathbb{C}}(U)$. In the following chapters, the value of v will be determined by the dimension of the state or output of each agent in the swarm systems. The decomposition of the state space or output space of the swarm system is called the *state space decomposition* or *output space decomposition*.

2.3 Matrix Theory

For matrices $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$, the *Kronecker product* can be defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{(mp) \times (nq)};$$

and the direct sum is defined as

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathbb{R}^{(m+p) \times (n+q)}.$$

For matrices A, B, C , and D with appropriate dimensions, the Kronecker product has the following properties [4]:

- (i) $A \otimes (B + C) = A \otimes B + A \otimes C$;
- (ii) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;
- (iii) $(A \otimes B)^T = A^T \otimes B^T$; and
- (iv) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Definition 2.1 For matrix $A \in \mathbb{C}^{n \times n}$, if all the eigenvalues of A have negative real parts, then A is called a *Hurwitz matrix* or *stable matrix*.

Lemma 2.4 (Schur complement [5]) For a given matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$, where

$S_{11} \in \mathbb{R}^{r \times r}$, the following statements are equivalent:

- (i) $S < 0$;
- (ii) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$; and
- (iii) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

2.4 Linear System Theory

Consider the following linear time-invariant system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \quad (2.1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{q \times n}$, and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$ is the state, control input and output, respectively.

Definition 2.2 For any initial state $x(0)$, if there exists control input $u(t)$ such that the state $x(t)$ of system (2.1) can converge to the origin in a finite time, then system (2.1) is called *controllable* or (A, B) is controllable.

Lemma 2.5 ([6]) If $\text{rank}[B, AB, \dots, A^{n-1}B] = n$, then (A, B) is controllable.

Lemma 2.6 (Popov-Belevitch-Hautus (PBH) test for controllability [6]) If $\text{rank}[sI - A, B] = n$ ($\forall s \in \mathbb{C}$), then (A, B) is controllable.

Definition 2.3 If any initial state $x(0)$ of system (2.1) can be uniquely determined by the control input $u(t)$ and output $y(t)$ in a finite time, then system (2.1) is called *observable* or (C, A) is observable.

Lemma 2.7 ([6]) If $\text{rank}\begin{bmatrix} C^T, A^T C^T, \dots, (A^{n-1})^T C^T \end{bmatrix}^T = n$, then (C, A) is observable.

Lemma 2.8 (PBH test for observability [6]) If $\text{rank}\begin{bmatrix} C^T, sI - A^T \end{bmatrix}^T = n$ ($\forall s \in \mathbb{C}$), then (C, A) is observable.

Definition 2.4 If matrix A is Hurwitz, then system (2.1) is *asymptotically stable*.

Lemma 2.9 For system (2.1), the following statements are equivalent:

- (i) System (2.1) is asymptotically stable;
- (ii) For any given positive matrix R , the Lyapunov function $A^T P + PA + R = 0$ has positive definite solution P ;
- (iii) There exists a positive definite matrix R such that the Lyapunov function $A^T P + PA + R = 0$ has unique positive definite solution P ; and
- (iv) There exists a positive definite matrix P such that $A^T P + PA < 0$.

Definition 2.5 If there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is Hurwitz, then system (2.1) is *stabilizable* or (A, B) is stabilizable.

Lemma 2.10 ([7]) System (2.1) is stabilizable if and only if $\text{rank}[sI - A, B] = n$ ($\forall s \in \bar{\mathbb{C}}^+$), where $\bar{\mathbb{C}}^+ = \{s | s \in \mathbb{C}, \text{Re}(s) \geq 0\}$ represents the closed right complex space.

Definition 2.6 If there exists gain matrix $K \in \mathbb{R}^{n \times q}$ such that $A + KC$ is Hurwitz, then system (2.1) is *detectable* or (C, A) is detectable.

Lemma 2.11 ([7]) System (2.1) is detectable if and only if $\text{rank}\begin{bmatrix} sI - A^T, C^T \end{bmatrix}^T = n$ ($\forall s \in \bar{\mathbb{C}}^+$).

Consider the following linear time-invariant system

$$\begin{cases} \dot{x}(t) = Ax(t), \\ y(t) = Cx(t), \end{cases} \quad (2.2)$$

where $y(t) = \begin{bmatrix} y_o(t) \\ y_{\bar{o}}(t) \end{bmatrix}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $C = [I, 0]$.

Definition 2.7 If for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|y(0)\| < \delta \Rightarrow \|y_o(t)\| < \varepsilon$ ($\forall t \geq 0$), then system (2.2) is said to be stable with respect to $y_o(t)$.

Definition 2.8 If system (2.2) is stable with respect to $y_o(t)$ and $\lim_{t \rightarrow \infty} y_o(t) = 0$, then system (2.2) is said to be asymptotically stable with respect to $y_o(t)$.

Lemma 2.12 ([8]) *If (A_{22}, A_{12}) is completely observable, then system (2.2) is asymptotically stable with respect to $y_o(t)$ if and only if A is Hurwitz.*

If (A_{22}, A_{12}) is not completely observable, then there always exists a nonsingular matrix T such that

$$(T^{-1}A_{22}T, A_{12}T) = \left(\begin{bmatrix} D_1 & 0 \\ D_2 & D_3 \end{bmatrix}, [E_1, 0] \right), \quad T^{-1}A_{21} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},$$

where (D_1, E_1) is completely observable. The following results can be obtained.

Lemma 2.13 ([8]) *If (A_{22}, A_{12}) is not completely observable, then system (2.2) is asymptotically stable with respect to $y_o(t)$ if and only if*

$$\begin{bmatrix} A_{11} & E_1 \\ F_1 & D_1 \end{bmatrix}$$

is Hurwitz.

2.5 Singular System Theory

Consider the following high-order LTI singular system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (2.3)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $E \in \mathbb{R}^{n \times n}$ satisfying $\text{rank}(E) = r \leq n$, $x(t) \in \mathbb{R}^n$ is the state and $u_i(t) \in \mathbb{R}^m$ is the control input. In the following, the definitions and criteria for regularity, equivalent form, admissibility, and controllability of singular system (2.3) are summarized.

Definition 2.9 If there exists constant s_0 such that $\det(s_0 E - A) \neq 0$, then system (2.3) is said to be *regular* or (E, A) is regular.

If (E, A) is regular, then there always exist nonsingular matrices P and Q such that

$$PEQ = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $N \in \mathbb{R}^{(n-r) \times (n-r)}$ represents the nilpotent matrix with nilpotent index l . Let $Q^{-1}x(t) = [x_1^T(t), x_2^T(t)]^T$. Then singular system (2.3) can be decomposed into

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u(t), \quad (2.4)$$

$$N \dot{x}_2(t) = x_2(t) + B_2 u(t). \quad (2.5)$$

This decomposition is often called as the *first equivalent form* of singular system (2.3) [9]. Subsystems (2.4) and (2.5) are said to be the slow subsystem and fast subsystem of singular system (2.3). Denote by $u^{(i)}(t)$ the i th derivative of $u(t)$. If the initial state $x(0)$ satisfies

$$x(0) = Q \begin{bmatrix} x_1(0) \\ -\sum_{i=0}^{l-1} N^i B_2 u^{(i)}(0) \end{bmatrix},$$

then $x(0)$ is said to be *admissible*.

Lemma 2.14 ([10]) *For given admissible initial state $x(0)$, singular system (2.3) has unique solution if and only if it is regular.*

Definition 2.10 If $\deg(\det(sE - A)) = \text{rank}(E)$ ($\forall s \in \mathbb{C}$), then singular system (2.3) is called *impulse-free* or (E, A) is impulse-free.

Lemma 2.15 ([10]) *Singular system (2.3) is impulse-free if and only if*

$$\text{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \text{rank}(E).$$

Definition 2.11 If (E, A) is regular, impulse-free, and asymptotically stable, then singular system (2.3) is said to be *admissible*.

Definition 2.12 If for any $x_T \in \mathbb{R}^n$, $x(0) \in \mathbb{R}^n$ and $T > 0$, there exists admissible control input $u(t)$ such that $x(T) = x_T$, then singular system (2.3) is *controllable* or (E, A, B) is controllable.

Definition 2.13 If singular system (2.3) is controllable in \mathbb{R} , then it is called *\mathbb{R} -controllable*.

Lemma 2.16 ([11]) *Singular system (2.3) is \mathbb{R} -controllable if and only if the slow subsystem (2.4) is controllable or stabilizable.*

Lemma 2.17 ([11]) *Singular system (2.3) is controllable if and only if the slow subsystem (2.4) is controllable and*

$$\text{rank} \begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = n + \text{rank}(E).$$

2.6 Conclusions

In this chapter, the basic definitions and results on graph theory, consensus decomposition of linear space theory, matrix theory, linear system theory, and singular system theory were introduced. These definitions and results are the research foundation of the following chapters.

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