Chapter 2
Part 2 Solutions

2.1 Chapter 1

2.1.1 Example 1.1

The displacement x of the mass is defined as
$x = x_0 \cdot \sin(\omega t)$ where $\omega = 2\pi/T$ and T the time period for one cycle.
Dissipated energy W_d during the period T is

$$W_d = \int_0^T F_\mathrm{d} \cdot \mathrm{d}x; \text{ with } \mathrm{d}x = \dot{x} \cdot \mathrm{d}t \text{ and the dissipative force}$$

$$F_d = Q \cdot \dot{x} \Rightarrow W_d = \int_0^T \dot{x}^2 \cdot Q \cdot \mathrm{d}t = \frac{Qx_o^2 \cdot \omega^2 T}{2}$$

(a) Viscous damping Eq. (1.3) $\Rightarrow Q = c$

$$W_d = c \cdot x_o^2 \omega \pi$$

(b) Structural damping Eq. (1.4) $\Rightarrow Q = \frac{\alpha}{\pi\omega}$

$W_d = \alpha x_o^2 \Rightarrow$ Dissipated energy independent of frequency for hysteretic or structural damping.

2.1.2 Example 1.2

Assuming viscous losses the equation of motion for the mass is

$$m\ddot{x} + k_0 x + c\dot{x} = F \tag{2.1.2.1}$$

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where F is the external force required to maintain the motion. The displacement x is

$$x = x_0 \cdot \sin(\omega t) \tag{2.1.2.2}$$

The Eqs. (2.1.2.1) and (2.1.2.2) give

$$x_0 \sin(\omega t)[k_0 - m\omega^2] + c\omega x_0 \cos(\omega t) = F \tag{2.1.2.3}$$

Since $x_0 \cdot \sin(\omega t) = x$ it follows that

$$x_0 \cdot \cos(\omega t) = \pm x_0 \sqrt{(1 - \sin^2(\omega t))} = \pm\sqrt{x_0^2 - x^2} \tag{2.1.2.4}$$

The Eqs. (2.1.2.3) and (2.1.2.4) yield

$$F - (k_0 - m\omega^2)x = \pm\omega c\sqrt{(x_0^2 - x^2)}$$

After quadration this expression is rewritten as

$$F^2 + (k_0 - m\omega^2)^2 x^2 - 2Fx(k_0 - m\omega^2) = (\omega c)^2(x_0^2 - x^2) \tag{2.1.2.5}$$

The expression (2.1.2.5) is an equation for an ellipse. In Fig. 2.1 F is shown as a function of x. The arrows indicate the path followed by increasing t.

For frictional damping the equations governing the motion of the mass are

$$m\ddot{x} + kx + F_d = F \quad \text{for } \dot{x} > 0 \tag{2.1.2.6}$$

$$m\ddot{x} + kx - F_d = F \quad \text{for } \dot{x} < 0 \tag{2.1.2.7}$$

The insertion of Eq. (2.1.2.2) in Eqs. (2.1.2.6) and (2.1.2.7) gives

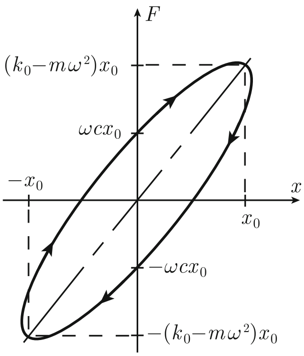


Fig. 2.1 Viscous damping

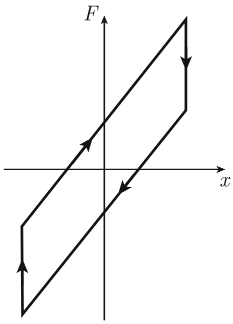


Fig. 2.2 Frictional damping

$$x(k - m\omega^2) + F_d = F \quad \text{for } \dot{x} > 0 \tag{2.1.2.8}$$

$$x(k - m\omega^2) - F_d = F \quad \text{for } \dot{x} < 0 \tag{2.1.2.9}$$

Figure 2.2 shows the force F as function of the displacement x.

2.1.3 Example 1.3

The displacement $x(t)$ of a 1-DOF system with frictional damping is shown in the Fig. 2.3.

The velocity is zero for $t = t_1, t_2$ etc. The frictional force is constant, its direction counteracts the motion of the mass.

In the time interval $0 \leqslant t \leqslant t_1$ the equation of motion is, assuming a frictional force F_0

$$m\ddot{x}_1 + kx_1 - F_0 = 0 \tag{2.1.3.1}$$

The initial conditions are $\dot{x}_1(0) = 0$; $x_1(0) = x_0$. Define ω_0 as $\omega_0 = (k/m)^{1/2}$.

The general solution to Eq. (2.1.3.1) is

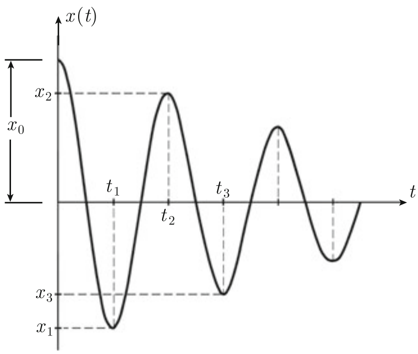


Fig. 2.3 Time decay of amplitude of a 1-DOF system having frictional damping

$$x_1(t) = F_0/k + A_1 \cdot \cos(\omega_0 t) + B_1 \cdot \sin(\omega_0 t) \tag{2.1.3.2}$$

The initial conditions and Eq. (2.1.3.2) yield for $0 \leqslant t \leqslant t_1$

$$x_1(t) = F_0/k + (x_0 - F_0/k)\cos(\omega_0 t) \tag{2.1.3.3}$$

At the turning point at $t = t_1$ the velocity $\dot{x}$ is equal to zero. Thus from Eq. (2.1.3.3) it follows that $\omega_0 t_1 = \pi$. The displacement $x_1(t)$ at $t = t_1$ is consequently

$$x_1(t_1) = 2F_0/k - x_0 \tag{2.1.3.4}$$

In the time interval $t_1 \leqslant t \leqslant t_2$ the equation of motion is, assuming a frictional force F_0

$$m\ddot{x}_2 + kx_2 + F_0 = 0 \tag{2.1.3.5}$$

The initial conditons are $x_2(t_1) = x_1(t_1)$ and $\dot{x}_2(t_1) = \dot{x}_1(t_1) = 0$. The general solution to Eq. (2.1.3.5) is

$$x_2(t) = -F_0/k + A_2 \cdot \cos(\omega_0 t) + B_2 \cdot \sin(\omega_0 t) \tag{2.1.3.6}$$

The initial conditions and Eq. (2.1.3.6) yield

$$x_2 = -F_0/k + (x_0 - 3F_0/k)\cos(\omega_0 t) \tag{2.1.3.7}$$

$$x_2(t_2) = x_0 - 4F_0/k; \quad \dot{x}_2(t_2) = 0 \tag{2.1.3.8}$$

$$\omega_0 t_2 = 2\pi \tag{2.1.3.9}$$

In a similar way, the response x_3 in the time domain $t_2 \leqslant t \leqslant t_3$ with $\omega_0 t_3 = 3\pi$ the response is obtained as

$$x_3 = -F_0/k + (x_0 - 5F_0/k)\cos(\omega_0 t)$$

etc.

From the results it follows that the difference in amplitude between two consecutive maxima is constant and equal to

$$x_1(0) - x_2(t_2) = x_0 - (x_0 - 4F_0/k) = 4F_0/k \tag{2.1.3.10}$$

The time difference between two maxima is

$$t_2 - t_0 = 2\pi/\omega_0 \tag{2.1.3.11}$$

Compare Fig. 1.9 and the discussion in Sect. 1.2, Vol. 1.

2.1.4 Example 1.4

The displacement of a critically damped 1-DOF is according to Eq. (1.17)

$$x(t) = e^{-\beta t} \cdot [x_0 + t(v_0 + \beta x_0)] \tag{2.1.4.1}$$

The displacement $x(t)$ is equal to zero for $t = t_0$ where according to (2.1.4.1)

$$t_0 = -\frac{x_0}{v_0 + \beta x_0} \tag{2.1.4.2}$$

t_0 must be larger than zero. This is only possible if $x_0 < 0$ and $v_0 > -\beta x_0$

or $x_0 > 0$ and $v_0 + \beta x_0 < 0$

Equations (2.1.4.1) and (2.1.4.2) give

$$x(t) = e^{-\beta t} (x_0 - x_0 t / t_0) = e^{-\beta t} x_0 \left(\frac{t_0 - t}{t_0} \right) \tag{2.1.4.3}$$

For $t > t_0$, $x(t) \neq 0 \Rightarrow x(t)$ can only equal zero once.

2.1.5 Example 1.5

The response due to an impulse I at $t = 0$ is, from Eq. (1.35)

$$x_0 = I e^{-\beta t} \sin(\omega_r t)/(m\omega_r) \tag{2.1.5.1}$$

The response due to an impulse at $t = -N \cdot t$ for $N = 1, 2, \ldots$ is

$$x_N = I e^{-\beta \cdot (t+NT)} \sin[\omega_r(t + NT)] \Big/ (m\omega_r) \tag{2.1.5.2}$$

Based on the principle of superpositioning the total response for $0 \leqslant t \leqslant T$ is

$$x(t) = \sum_N x_N = I \sum_N e^{-\beta(t+NT)} \sin[\omega_r(t + NT)]/(m\omega_r)$$

Using the identity $\sin(\varphi) = \frac{1}{2i} \left(e^{i\varphi} - e^{-i\varphi} \right)$ the result is

$$x(t) = \frac{I e^{-\beta t}}{2im\omega_r} \cdot \left\{ J_1 e^{i\omega_r t} - J_2 e^{-i\omega_r t} \right\} \tag{2.1.5.3}$$

$$J_1 = \sum_{N=0}^{\infty} e^{-NT(\beta - i\omega_\mathrm{r})}; \quad J_2 = \sum_{N=0}^{\infty} e^{-NT(\beta + i\omega_r)}$$

J_1 and J_2 are geometric series. Thus

$$J_1 = \frac{1}{1 - e^{-T(\beta - i\omega_r)}}; \quad J_2 = \frac{1}{1 - e^{-T(\beta + i\omega_r)}} \tag{2.1.5.4}$$

The response $x(t)$ is obtained from Eqs. (2.1.5.3) and (2.1.5.4) as

$$x(t) = \frac{Ie^{-\beta t}}{m\omega_r} \cdot \left\{ \frac{\sin(\omega_r t) - e^{-\beta T} \sin[\omega_r (t - T)]}{1 - 2e^{-\beta t} \cos(\omega_r T) + e^{-2\beta T}} \right\} \quad \text{for } 0 \leqslant t \leqslant T$$

$$x(t + T) = x(t) \tag{2.1.5.5}$$

Note that $\lim_{T\to\infty} x(t) = \dfrac{Ie^{-\beta t} \sin(\omega_r t)}{m\omega_r}$. This expression is equal to Eq. (2.1.5.1).

2.1.6 Example 1.6

The respose $x(t)$ of a 1-DOF system excited by a force $F(t)$ is according to Eq. (1.38) given by

$$x(t) = \int_0^t \mathrm{d}\xi \cdot F(\xi) \cdot h(t - \xi) \tag{2.1.6.1}$$

The function $h(t)$ is given in Eq. (1.36) as

$$h(t) = e^{-\beta t} \sin(\omega_r t)/(m\omega_r) \tag{2.1.6.2}$$

For $F(t) = F_0$ for $0 \leqslant t \leqslant T$ the response is obtained from (2.1.6.1) and (2.1.6.2) as

$$x_1(t) = \frac{F_0}{m\omega_r} \left[\omega_r - \beta e^{-\beta t} \cdot \sin(\omega_r t) - \omega_r e^{-\beta t} \cdot \cos(\omega_r t) \right] / (\beta^2 + \omega_r^2) \tag{2.1.6.3}$$

The velocity for $0 \leqslant t \leqslant T$ is

$$\dot{x}_1(t) = \frac{F_0}{m\omega_r} \sin(\omega_r t) e^{-\beta t} \tag{2.1.6.4}$$

For $t \geqslant T$ with $F(t) = 0$ the response is given by Eq. (1.51) as

$$x_2(t) = e^{-\beta(t-T)} \left\{ x_0 \cos[\omega_r(t-T)] + \frac{v_0 + \beta x_0}{\omega_r} \sin[\omega_r(t-T)] \right\} \qquad (2.1.6.5)$$

where $x_0 = x_1(T)$ and $v_0 = \dot{x}_1(T)$ given by Eqs. (2.1.6.3) and (2.1.6.4).
Case A $\omega_r T = 2\pi$. From Eq. (2.1.6.3) the response $x_0 = x_1(T)$ is obtained as

$$x_1(T) = \frac{F_0\omega_n}{m\omega_r(\beta^2 + \omega_r^2)}(1 - e^{-\beta T}) \approx 0 \quad \text{for } \beta T \ll 1 \qquad (2.1.6.6)$$

According to Eq. (2.1.6.4) $v_0 = \dot{x}_1(T) = 0$. Thus, for small losses the mass is almost at rest for $t > T$ since $x_0(T) = x_1(T) \approx 0$ and $v_0 = \dot{x}_1(T) = 0$
Case B $\omega_r T = \pi$

$$x_1(T) \approx \frac{2F_0}{m\omega_r^2} \quad \text{for } \beta \ll 1$$

$$\dot{x}_1(T) = 0 \Rightarrow$$

$$x_1(t) = e^{-\beta(t-T)} \frac{2F_0 \cos[\omega_n(t-T)]}{m\omega_r^2} \quad \text{for } t > T$$

Case C $\omega_r T = \pi/2$

$$x_1(T) \approx \frac{F_0}{m\omega_r^2} \quad \text{for } \beta \ll 1$$

$$\dot{x}_1(T) \approx \frac{F_0}{m\omega_r} \Rightarrow$$

$$x_1(t) = e^{-\beta\cdot(t-T)} \frac{F_0\sqrt{2}}{m\omega_r^2} \sin[\omega_r(t-T) + \pi/4] \quad \text{for } t > T$$

2.1.7 Example 1.7

The response of the 1-DOF system described in Problem 1.6 is for $t < T$

$$x_1(t) = \frac{F_0}{m\omega_r}\left[\omega_r - \beta e^{-\beta t} \cdot \sin(\omega_r t) - \omega_r e^{-\beta t} \cdot \cos(\omega_r t)\right] \Big/ (\beta^2 + \omega_r^2) \quad (2.1.7.1)$$

The corresponding velocity is

$$\dot{x}_1(t) = \frac{F_0}{m\omega_r} \sin(\omega_r t) e^{-\beta t} \qquad (2.1.7.2)$$

x_1 has a maximum when $\dot{x}_1(t) = 0$, i.e. for $t_1 = \pi/\omega_\mathrm{r}$ if $T > t_1$

$$(x_1)_{\max} = \frac{F_0}{m \cdot \omega_n^2} \cdot 2 \quad \text{if } \beta \text{ and } \beta t_1 \ll 1 \tag{2.1.7.3}$$

The displacement for $t > T$ is according to Eq. (2.1.7.5), Problem 1.6 given by

$$x_2 = e^{-\beta(t-T)} \left\{ x_0 \cos[\omega_r(t-T)] + \frac{v_0 + \beta x_0}{\omega_r} \sin[\omega_r(t-T)] \right\} \tag{2.1.7.4}$$

The corresponding velocity is

$$\dot{x}_2 = e^{-\beta(t-T)} \left\{ v_0 \cos[\omega_r(t-T)] - \sin[\omega_r(t-T)] \frac{\omega_r^2 x_0 + \beta^2 x_0 + \beta v_0}{\omega_r} \right\} \tag{2.1.7.5}$$

The amplitude has maxima when $\dot{x}_2 = 0$ or when for $\beta T \ll 1$ and $\beta \ll 1$

$$\tan[\omega_r(t-T)] = \frac{v_0}{\omega_r x_0} \tag{2.1.7.6}$$

Equation (2.1.7.6) gives

$$\sin[\omega_r(t-T)] = \frac{v_0}{\sqrt{v_0^2 + \omega_r^2 x_0^2}} \text{ and } \cos[\omega_r(t-T)] = \frac{\omega_n v_0}{(v_0^2 + \omega_r^2 x_0^2)^{1/2}}$$

These two expressions inserted in Eq. (2.1.7.4) give the maximum amplitude for x_2 in the time domain $t > T$. The displacement x_0 is obtained from Eq. (2.1.7.1) as $x_0 = x_1(T)$ and v_0 from Eq. (2.1.7.2) as $v_0 = \dot{x}_1(T)$. The maximum value of x_2 is thus for $t > T$

$$\begin{aligned}(x_2)_{\max} &= e^{-\beta(t-T)} \left\{ \omega_r^2 \left(\frac{F_0}{m\omega_r^2} \right)^2 [1 - \cos(\omega_r T)]^2 + \left(\frac{F_0}{m\omega_r^2} \right)^2 \sin^2(\omega_r T) \right\} \\ &\approx \frac{F_0}{m\omega_r^2} \sqrt{2 - 2\cos(\omega_r T)}\end{aligned}$$

The absolute maximum is obtained when $\cos(\omega_r T) = -1 \Rightarrow T = \pi/\omega_r$. The maximum amplitude is then $2F_0/(m\omega_r^2)$ or the same amplitude as given by Eq. (2.1.7.3).

The time T_0 for one cycle of the undamped system is $T_0 = 2\pi/\omega_r$. This absolute maximum is obtained when the length of the rectangular pulse is equal to $T_0/2$. For a half sine pulse the corresponding value is $0, 8.\ T_0$ as discussed in Sect. 1.3 and presented in Fig. 1.18.

2.1.8 Example 1.8

A periodic function $x(t)$, period T, is expanded in a Fourier series as

$$x(t) = a_0/2 + \sum_{n=1} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)] \tag{2.1.8.1}$$

where $\omega_n = 2\pi n/T$. Multiply Eq. (2.1.8.1) first by $\cos(\omega_k t)$, $\omega_k = 2\pi k/T$, and integrate with respect to time over one period. The result is

$$\begin{aligned}\int_0^T x(t)\cdot\cos(\omega_k t)\mathrm{d}t &= \int_0^T (a_0/2)\cos(\omega_k t)\mathrm{d}t \\ &+ \sum_{n=1}\int_0^T \cos(\omega_k t)\,[a_n\cos(\omega_n t) + b_n \sin(\omega_n t)]\mathrm{d}t\end{aligned} \tag{2.1.8.2}$$

For $k = 0$ and $\cos(\omega_k t) = 0$ Eq. (2.1.8.2) gives

$$\int_0^T x(t)\mathrm{d}t = \int_0^T (a_0/2)\mathrm{d}t = a_0 T/2 \tag{2.1.8.3}$$

For $k > 0$

$$\int_0^T \cos(\omega_n t)\cos(\omega_k t)\mathrm{d}t = 0 \quad \text{for } n \neq k \tag{2.1.8.4}$$

$$\int_0^T \cos(\omega_n t)\cos(\omega_k t)\mathrm{d}t = T/2 \quad \text{for } n = k \tag{2.1.8.5}$$

$$\int_0^T \sin(\omega_n t)\cos(\omega_k t)\mathrm{d}t = 0 \tag{2.1.8.6}$$

The results (2.1.8.2) and (2.1.8.4)–(2.1.8.6) give

$$a_n = \frac{2}{T}\cdot\int_0^T x(t)\cdot\cos(\omega_n t)\mathrm{d}t \quad \text{for } n = 0, 1, 2, \ldots \tag{2.1.8.7}$$

The coefficients b_n are obtained in a similar way by multiplying Eq. (2.1.8.1) by $\sin(\omega_k t)$, $\omega_k = 2\pi k/T$. The resulting expression is integrated with respect to time over one period. The result is

$$b_n = \frac{2}{T}\cdot\int_0^T x(t)\cdot\sin(\omega_n t)\mathrm{d}t$$

2.1.9 Example 1.9

The equation of motion for a 1-DOF system with viscous losses is

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{2.1.9.1}$$

Since $F(t) = F(t+T)$ then, excluding transient vibrations, the response must be periodic i.e.

$$x(t) = A_0/2 + \sum A_n \cos(\omega_n t) + \sum B_n \sin(\omega_n t) \tag{2.1.9.2}$$

$$\omega_n = 2\pi n/T \tag{2.1.9.3}$$

The amplitudes A_n and B_n are according to (1.67) given by

$$A_n = \frac{2}{T}\int_0^T x(t)\cdot\cos(\omega_n t)\mathrm{d}t \tag{2.1.9.4}$$

$$B_n = \frac{2}{T}\int_0^T x(t)\cdot\sin(\omega_n t)\mathrm{d}t \tag{2.1.9.5}$$

Equation (2.1.9.1) is multiplied by $\cos(\omega_n t)$and integrated over time. Thus

$$\int_0^T \mathrm{d}t m\ddot{x}\cdot\cos(\omega_n t) + \int_0^T \mathrm{d}t c\dot{x}\cdot\cos(\omega_n t) + k\int_0^T x\cos(\omega_n t)\mathrm{d}t$$
$$= \int_0^T \mathrm{d}t F(t)\cdot\cos(\omega_n t) \Rightarrow I_1 + I_2 + I_3 = I_4 \tag{2.1.9.6}$$

where $I_1, \ldots, I_4$ denote the integrals.

From Eq. (2.1.9.2) it follows that

$$I_3 = \frac{kT}{2}A_n \tag{2.1.9.7}$$

The expression I_2 is integrated by parts

$$I_2 = [cx\cos(\omega_n t)]_0^T + \int_0^T c\omega_n x\sin(\omega_n t)\mathrm{d}t$$

$x(T) = x(0)$ and $\omega_n T = n2\pi \Rightarrow$ from (2.1.9.2) it follows that

$$I_2 = \frac{c\omega_n T}{2}B_n \tag{2.1.9.8}$$

In the same way I_1 is obtained as:

$$
\begin{aligned}
I_1 &= [c\dot{x}\cos(\omega_n t)]_0^T + m\omega_n \int_0^T \mathrm{d}t\,\dot{x}\sin(\omega_n t) \\
&= [m\dot{x}\cos(\omega_n t) + xm\omega_n\sin(\omega_n t)]_0^T - m\omega_n^2 \int_0^T x\cos(\omega_n t)\mathrm{d}t
\end{aligned}
$$

Since $\dot{x}(0) = \dot{x}(T)$ and $\omega_n T = 2\pi n$

$$
I_1 = -\frac{m\omega_n^2 T}{2} A_n \tag{2.1.9.9}
$$

For a force $F(t)$ given by

$F(t) = G_0/2 + \sum G_n\cos(\omega_n t) + \sum H_n\sin(\omega_n t)$ the integral I_4 is obtained as

$$
I_4 = \int_0^T \mathrm{d}t\,F(t)\cdot\cos(\omega_n t) = TG_n/2 \tag{2.1.9.10}
$$

The Eqs. (2.1.9.7)–(2.1.9.10) yield

$$
(-m\omega_n^2 + k)A_n + \omega_n c B_n = G_n \tag{2.1.9.11}
$$

Next multiply Eq. (2.1.9.1) by $\sin(\omega_n \cdot t)$ and repeat the procedure described above. The result is

$$
(-m\omega_n^2 + k)B_n - \omega_n c A_n = H_n \tag{2.1.9.12}
$$

The solutions to Eqs. (2.1.9.11) and (2.1.9.12) are

$$
A_\mathrm{n} = \frac{(-m\omega_n^2 + k)G_n - \omega_n c H_n}{[(-m\omega_n^2 + k)^2 + (\omega_n c)^2]} \tag{2.1.9.13}
$$

$$
B_n = \frac{\omega_n c G_\mathrm{n} + (-m\omega_n^2 + k)H_n}{[(-m\omega_n^2 + k)^2 + (\omega_n c)^2]} \tag{2.1.9.14}
$$

The displacement of the mass is given by inserting (2.1.9.13) and (2.1.9.14) in Eq. (2.1.9.2).

2.1.10 Example 1.10

The force $F(t)$ is periodic with the period T. Thus

$$F(t+T) = F(t) = I\delta(t) \tag{2.1.10.1}$$

The periodic function $F(t)$ is also written

$$F(t) = G_0/2 + \sum G_n \cos(\omega_n t) + \sum H_n \sin(\omega_n t) \tag{2.1.10.2}$$

where

$$G_n = \frac{2}{T}\int_0^T F(t)\cdot\cos(\omega_n t)\mathrm{d}t; \quad H_n = \frac{2}{T}\int_0^T F(t)\cdot\sin(\omega_n t)\mathrm{d}t \tag{2.1.10.3}$$

Thus

$$G_n = 2I/T; \quad H_n = 0 \tag{2.1.10.4}$$

The response $x(t)$ is also periodic

$$x(t) = A_0/2 + \sum A_n \cos(\omega_n t) + \sum B_n \sin(\omega_n t) \tag{2.1.10.5}$$

The parameters A_n and B_n are derived as in Problem 1.8. Equations (2.1.10.11) and (2.1.10.12) in Problem 1.8 give

$$A_n = \frac{2I}{T}\frac{(-m\omega_n^2 + k)}{[(-m\omega_n^2 + k)^2 + (\omega_n c)^2]} \tag{2.1.10.6}$$

$$B_n = \frac{2I}{T}\frac{\omega_n c}{[(-m\omega_n^2 + k)^2 + (\omega_n c)^2]} \tag{2.1.10.7}$$

The summation can be carried out as discussed in Example 1.5. The result (2.1.10.7) is also obtained if the solution in Example 1.6 is expanded in a Fourier series.

2.1.11 Example 1.11

The equation of motion for the 1-DOF system is

$$m\ddot{x} + kx = F; \quad k = k_0(1 + i\delta) \tag{2.1.11.1}$$

Let $x = x_0 \cdot e^{i\omega t}$ and $F = F_0 \cdot e^{i\omega t}$. Equation (2.1.11.1) gives

$$x = F_0 \cdot e^{i\omega t}/(k - m\omega^2) \tag{2.1.11.2}$$

The velocity is

$$v = \dot{x} = i\omega \cdot F_0 \cdot e^{i\omega t}/(k - m\omega^2) \tag{2.1.11.3}$$

The time average of the potential energy of the system is

$$\bar{\mathcal{U}} = \text{Re}\left[k\bar{x}^2/2\right] = k_0 x x^*/4 = \frac{k_0}{4} \cdot \frac{|F_0|^2}{(k_0 - m\omega^2)^2 + (k_0\delta)^2} \tag{2.1.11.4}$$

The time average of the kinetic energy is

$$\bar{\mathcal{T}} = m\bar{\dot{x}}^2/2 = mvv^*/4 = \frac{m\omega^2}{4} \cdot \frac{|F_0|^2}{(k_0 - m\omega^2)^2 + (k_0\delta)^2} \tag{2.1.11.5}$$

The time average of the input power to the system is

$$\bar{\Pi} = \frac{1}{2} \cdot \text{Re}(F \cdot v^*) = \frac{1}{2} \cdot \frac{|F_0|^2\, \omega k_0 \delta}{(k_0 - m\omega^2)^2 + (k_0\delta)^2} \tag{2.1.11.6}$$

Since $\delta = \omega c/k_0$ it follows from Eq. (2.1.11.6) that

$$\bar{\Pi} = \frac{1}{2} \cdot \frac{|F_0|^2\, \omega^2 c}{(k_0 - m\omega^2)^2 + (k_0\delta)^2} = \frac{2\bar{\mathcal{T}} \cdot c}{m} \tag{2.1.11.7}$$

For viscous losses the parameter c is constant. The time average of the input power to the system is thus proportional to the kinetic energy of the system. For structural damping $c = \alpha/(\pi\omega)$ which means that for harmonic excitation the power input is proportional neither to the kinetic nor to the potential energy.

2.1.12 Example 1.12

Two differential equations are given

$$m\ddot{h} + c\dot{h} + k_0 h = \delta(t - \tau) \tag{2.1.12.1}$$

$$m\ddot{x} + c\dot{x} + k_0 x = F(t) \tag{2.1.12.2}$$

$$h = h(t - \tau) \Rightarrow \frac{\partial h}{\partial t} = -\frac{\partial h}{\partial \tau}.$$

Thus Eq. (2.1.12.1) is also written

$$m\frac{\partial^2 h}{\partial\tau^2} - c\frac{\partial h}{\partial\tau} + k_0 h = \delta(t-\tau) \tag{2.1.12.3}$$

Multiply (2.1.12.3) by $x(\tau)$and integrate over τ. The result is

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \left[m\frac{\partial^2 h}{\partial\tau^2}x - c\frac{\partial h}{\partial\tau}x + k_0 hx \right] = \int_{-\infty}^{\infty} \mathrm{d}\tau\delta(t-\tau)x(\tau) = x(t) \tag{2.1.12.4}$$

Partial integration of the first expression inside the bracket gives

$$\left[m\frac{\partial h}{\partial\tau}x \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \mathrm{d}\tau \left[-m\frac{\partial h}{\partial\tau}\frac{\partial x}{\partial\tau} - c\frac{\partial h}{\partial\tau}x + k_0 hx \right] = x(t) \tag{2.1.12.5}$$

However, x tends to zero as $t \to \pm\infty$. Equation (2.1.12.5) is reduced to

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \left[-m\frac{\partial h}{\partial\tau}\frac{\partial x}{\partial\tau} - c\frac{\partial h}{\partial\tau}x + k_0 hx \right] = x(t) \tag{2.1.12.6}$$

By setting $x = x(\tau)$ Eq. (2.1.12.2) is written

$$m\frac{\partial^2 x}{\partial\tau^2} + c\frac{\partial x}{\partial\tau} + k_0 x = F(\tau)$$

This equation is multiplied by $h(t-\tau)$ and integrated over τ resulting in

$$\int_{-\infty}^{\infty} \mathrm{d}\tau (mh\frac{\partial^2 x}{\partial\tau^2} + ch\frac{\partial x}{\partial\tau} + k_0 hx) = \int_{-\infty}^{\infty} \mathrm{d}\tau F(\tau)h(t-\tau)$$

Following the same procedure as above the expression is reduced to

$$\int_{-\infty}^{\infty} \mathrm{d}\tau (- m\frac{\partial h}{\partial\tau}\frac{\partial x}{\partial\tau} + ch\frac{\partial x}{\partial\tau} + k_0 hx) = \int_{-\infty}^{\infty} \mathrm{d}\tau F(\tau)h(t-\tau) \tag{2.1.12.7}$$

Subtracting (2.1.12.7) from (2.1.12.6) gives

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \left[-c\frac{\partial h}{\partial\tau}x - c\frac{\partial x}{\partial\tau}h \right] = x(t) - \int_{-\infty}^{\infty} \mathrm{d}\tau F(\tau)h(t-\tau) \tag{2.1.12.8}$$

The integral on the left-hand side is

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \left[-c\frac{\partial (hx)}{\partial\tau} \right] = -c\,[hx]_{-\infty}^{\infty} = 0$$

Thus Eq. (2.1.12.8) is simplified to

$$x(t) = \int_{-\infty}^{\infty} \mathrm{d}\tau F(\tau) h(t-\tau) \tag{2.1.12.9}$$

$h(t-\tau) = 0$ for $\tau \geqslant t$. Thus

$$x(t) = \int_{-\infty}^{t} \mathrm{d}\tau F(\tau) h(t-\tau)$$

2.1.13 Example 1.13

In the first case the equation of motion for the 1-DOF system is

$$m\ddot{x} + kx = F; \quad k = k_0(1 + i\delta) \tag{2.1.13.1}$$

Let $x = x_0 \cdot e^{i\omega t}$ and $F = F_0 \cdot e^{i\omega t}$. Multiply Eq. (2.1.13.1) by $\dot{x}^*$ and derive the time average of the input power as

$$\bar{\Pi} = \frac{1}{2}\mathrm{Re}(Fv^*) = \frac{1}{2}\mathrm{Re}\left[m\ddot{x}\dot{x}^* + k_0(1 + i\delta)x\dot{x}^*\right] \tag{2.1.13.2}$$

Since $\dot{x} = i\omega x$ it follows that

$$\bar{\Pi} = \frac{\omega k\delta \, |x|^2}{2} = \omega k\delta \, |\bar{x}|^2 = 2\omega\delta\bar{U} \tag{2.1.13.3}$$

In the second case, the equation of motion for the 1-DOF system is

$$m\ddot{x} + c\dot{x} + k_0 x = F \tag{2.1.13.4}$$

Let $x = x_0 \cdot e^{i\omega t}$ and $F = F_0 \cdot e^{i\omega t}$. Multiply Eq. (2.1.13.4) by $\dot{x}^*$ and derive the time average of the input power as

$$\bar{\Pi} = \frac{1}{2}\mathrm{Re}(Fv^*) = \frac{1}{2}\mathrm{Re}\left[m\ddot{x}\dot{x}^* + c\dot{x}\dot{x}^* + k_0 x\dot{x}^*\right] \tag{2.1.13.5}$$

Again $\dot{x} = i\omega x$ etc. Thus,

$$\bar{\Pi} = \frac{1}{2}\mathrm{Re}(Fv^*) = \frac{c\,|\dot{x}|^2}{2} = c\left|\bar{\dot{x}}\right|^2 = \frac{c}{m}\bar{\mathcal{T}} \tag{2.1.13.6}$$

where $\bar{\mathcal{T}}$ is the kinetic energy of the system.

2.2 Chapter 2

2.2.1 Example 2.1

The Fourier Transform (FT) of $h(t)$ is according to Eq. (2.3)

$$H(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \cdot h(t) \cdot e^{-i\omega t} = \frac{1}{m\omega_0} \int_0^{\infty} \mathrm{d}t e^{-\beta t - i\omega t} \sin(\omega_0 t) \tag{2.2.1.1}$$

The function $\sin(\omega_0 t)$ is written

$$\sin(\omega_n t) = \frac{1}{2i} \left(e^{i\omega_n t} - e^{-i\omega_n t} \right) \tag{2.2.1.2}$$

Equations (2.2.1.1) and (2.2.1.2) give

$$\begin{aligned} H(\omega) &= \frac{1}{2im\omega_n} \int_0^{\infty} \mathrm{d}t \cdot \left\{ e^{-t(\beta - i\omega_n + i\omega)} - e^{-t(\beta + i\omega_n + i\omega)} \right\} \\ &= \frac{1}{2im\omega_n} \left[\frac{2i\omega_n}{\beta^2 + \omega_n^2 - \omega^2 + 2i\omega\beta} \right] \end{aligned} \tag{2.2.1.3}$$

By setting $\beta = \omega_0^2 \delta/(2\omega)$ and $\omega_n^2 = \omega_0^2 - \beta^2$ in Eq. (2.2.1.3) $H(\omega)$ is obtained as

$$H(\omega) = \frac{1}{m} \cdot \left[\frac{1}{\omega_0^2 - \omega^2 + i\omega_0^2 \delta} \right] \tag{2.2.1.4}$$

This is the FT of $h(t)$ and also according to Eq. (2.15) the transfer or frequency response function $H(\omega) = \hat{F}(\omega)/\hat{x}(\omega)$ of a 1-DOF system described by the equation of motion

$$m\ddot{x} + c\dot{x} + kx = F(t) \quad \text{with } c = 2m\beta$$

2.2.2 Example 2.2

$F(t)$ is a periodic function with the period T, i.e. $F(t + T) = F(t)$. According to Eq. (1.66) $F(t)$ can be expanded in a Fourier series as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \omega_n t + b_n \sin \omega_n t] \tag{2.2.2.1}$$

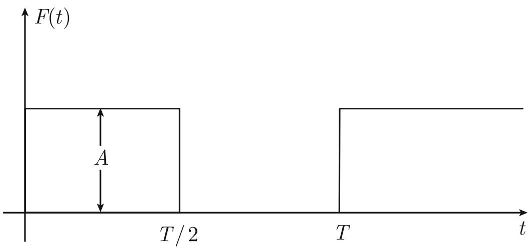


Fig. 2.4 Periodic force as function of time

where $\omega_n = 2\pi n/T$. The parameters a_n and b_n are obtained from

$$a_n = \frac{2}{T}\int_0^T \mathrm{d}t \cdot F(t)\cos\omega_n t; \quad b_n = \frac{2}{T}\int_0^T \mathrm{d}t \cdot F(t)\sin\omega_n t; \tag{2.2.2.2}$$

The function $F(t)$ is shown in Fig. 2.4.

The parameters a_n and b_n are obtained from Eq. (2.2.2.2) as

$$\begin{aligned} a_n &= \frac{2A}{T}\int_0^{T/2} \mathrm{d}t \cdot \cos\omega_n t = \frac{2A}{T}\left[\frac{\sin\omega_n t}{\omega_n}\right]_0^{T/2} = 0 \\ a_0 &= \frac{2A}{T}\frac{T}{2} = A \\ b_n &= \frac{2A}{T}\left[\frac{\cos\omega_n t}{\omega_n}\right]_{T/2}^{0} = \frac{2A}{T\omega_n}\left[1-\cos(\pi n)\right] \\ b_n &= \frac{2A}{2\pi n}\cdot 2 \quad \text{for } n \text{ odd} \\ b_n &= 0 \quad \text{for } n \text{ even} \end{aligned} \tag{2.2.2.3}$$

The autocorrelation function $R_{FF}(\tau)$ is for the periodic function $F(t)$ defined as

$$R_{FF}(\tau) = \frac{1}{T}\int_0^T F(t)F(t+\tau)\mathrm{d}t \tag{2.2.2.4}$$

where the function $F(t)$ is given by Eq. (2.2.2.1). The integration of cross terms like $\sin(\omega_n t)\cos(\omega_n t)$ gives:

$$\frac{1}{T}\int_0^T \mathrm{d}t \cdot b_m \sin(\omega_m t) b_n \sin\left[\omega_n(t+T)\right] = \begin{cases} 0 & \text{for } m \neq n \\ \dfrac{b_n^2}{2}\cos(\omega_m \tau) & \text{for } m = n \end{cases} \tag{2.2.2.5}$$

Further,

$$\frac{1}{T}\int_0^T \mathrm{d}t \cdot \frac{a_0}{2} b_n \sin\omega_n t = 0 \tag{2.2.2.6}$$

Thus

$$R_{FF}(\tau) = \left(\frac{a_0}{2}\right)^2 + \sum_{n \text{ odd}} \frac{b_n^2}{2} \cos \omega_n t$$
$$= \left(\frac{A}{2}\right)^2 + \sum_{n \text{ odd}} \left(\frac{2A}{\pi n}\right)^2 \frac{1}{2} \cos \omega_n \tau \tag{2.2.2.7}$$

The two-sided power spectral density $S_{FF}(\omega)$ is defined as

$$S_{FF}(\omega) = \int_{-\infty}^{\infty} R_{FF}(\tau) \cdot e^{-i\omega\tau} \mathrm{d}\tau \tag{2.2.2.8}$$

where $R_{FF}(\tau)$ is given in Eq. (2.2.2.4). From Eq. (2.4) it follows that

$$\int_{-\infty}^{\infty} \left(\frac{A}{2}\right)^2 \cdot e^{-i\omega\tau} \mathrm{d}\tau = 2\pi \left(\frac{A}{2}\right)^2 \delta(\omega) \tag{2.2.2.9}$$

Equations (2.40) and (2.41) give

$$\int_{-\infty}^{\infty} \left(\frac{2A}{\pi n}\right)^2 \frac{1}{2} \cos \omega_n \tau \cdot e^{-i\omega\tau} \mathrm{d}\tau = \frac{\pi}{2} \left(\frac{2A}{\pi n}\right)^2 [\delta(\omega - \omega_n) + \delta(\omega + \omega_n)] \tag{2.2.2.10}$$

Consequently, the Eqs. (2.2.2.7)–(2.2.2.10) yield for $\omega_n = 2\pi n/T$

$$S_{FF}(\omega) = 2\pi \left(\frac{A}{2}\right)^2 \delta(\omega) + 2\pi \sum_{n \text{ odd}} \left(\frac{A}{n\pi}\right)^2 [\delta(\omega - \omega_n) + \delta(\omega + \omega_n)]$$

2.2.3 Example 2.3

The inverse FT of $H(\omega)$ is given in Eq. (2.2) as

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \cdot e^{i\omega t} \mathrm{d}\omega \tag{2.2.3.1}$$

The frequency response function $H(\omega)$ is defined as

$$H(\omega) = \frac{1}{m[(\omega_0^2 - \omega^2) + i\omega_0^2\delta]} \tag{2.2.3.2}$$

The function $H(\omega)$ has poles when $\omega_0^2 - \omega^2 + i\omega_0^2\delta = 0$ or when $\omega_{1,2} = \pm\omega_0\sqrt{1 + i\delta} \approx \pm\omega_0(1 + i\delta/2)$ for $\delta \ll 1$. According to Eq. (1.81) $\delta = 2\omega\beta/\omega_0^2$.

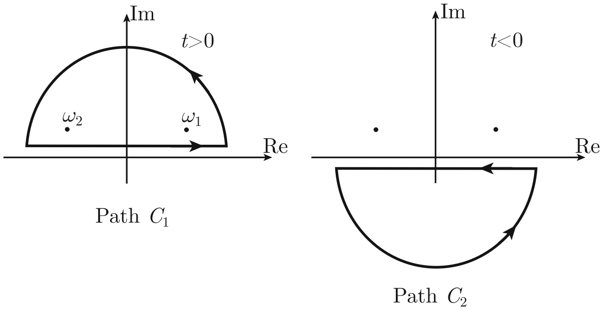


Fig. 2.5 Paths of integration

The loss factor δ is negative for $\omega < 0$ and positive for $\omega > 0$. Considering this the poles are:

$$\omega_1 = \omega_0\,(1 + i\delta/2)\,; \quad \omega_1 = -\omega_0(1 - i\delta/2); \quad \delta = \left|2\omega\beta/\omega_0^2\right|$$

The function $h(t)$ as defined in Eq. (2.2.3.1) is derived by a contour integration in the complex plane. The choice of path depends on t as indicated in Fig. 2.5.

When the radius of the semicircle goes to infinity, the integral along the curved path of C_1 approaches zero since for $t > 0$

$$e^{i\omega t} \to 0 \quad \text{as } |\omega| \to \infty \text{ and } \mathrm{Im}(\omega) > 0$$

Thus for $t > 0$

$$h(t) = \oint_{C_1} H(\omega) \cdot e^{i\omega t} \mathrm{d}\omega \tag{2.2.3.3}$$

The procedure of the contour integration is discussed in Sect. 2.7. For $t > 0$ the poles are $\omega_1 = \omega_0(1 + i\delta/2)$ and $\omega_2 = -\omega_0(1 - i\delta/2)$:

$$\begin{aligned}
h(t) &= 2\pi i \sum_{n=1}^{2} \frac{e^{i\omega_n t}}{2\pi m(-2\omega_n)} \\
&= \frac{i}{m}\left\{\frac{e^{i\omega_0(1+i\delta)t/2}}{-2\omega_0(1+i\delta/2)} + \frac{e^{i\omega_0(1-i\delta)t/2}}{2\omega_0(1-i\delta/2)}\right\} \\
&= \frac{i}{2m\omega_0} \cdot e^{-\omega\delta t/2}\left\{\frac{e^{-i\omega_0 t}}{(1-i\delta/2)} - \frac{e^{i\omega_0 t}}{(1+i\delta/2)}\right\} \\
&= \frac{i}{2m\omega_0} \cdot e^{-\omega\delta t/2}\left\{\frac{e^{-i\omega_0 t} - e^{i\omega_0 t}}{1+(\delta/2)^2} + \frac{\mathrm{i}\delta}{1+(\delta/2)^2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)\right\}
\end{aligned}$$

$$= \frac{i}{2m\omega_0} \cdot e^{-\omega\delta t/2} \left\{ \frac{-2i \cdot \sin\omega_0 t}{1 + (\delta/2)^2} + \frac{i\delta 2 \cos\omega_0 t}{1 + (\delta/2)^2} \right\}$$

$$\cong \frac{1}{m\omega_0} \cdot e^{-\omega\delta t/2} \sin(\omega_0 t) \tag{2.2.3.4}$$

for $|\delta| \ll 1$ and $t > 0$. For $t < 0$ the contour integration is along the path C_2 to ensure that the integration along the semicircle goes to zero as the radius increases. The contour includes no poles. For $t < 0$

$$h(t) = -\int_{C_2} H(\omega) \cdot e^{i\omega t} \mathrm{d}\omega = 0$$

Thus for $t < 0,\ h(t) = 0$.

2.2.4 Example 2.4

The equation of motion for the 1-DOF system is

$$m\ddot{x} + kx = F \quad \text{with } k = k_0(1 + i\delta)$$

According to Example 2.2 the force F can be expanded in a Fourier series as

$$F(t) = \frac{A}{2} + \sum_{\substack{n \\ \text{odd}}} \frac{2A}{\pi n} \cdot \sin(\omega_n t); \quad \omega_n = 2\pi n/T$$

The displacement $x(t)$ is expanded in a similar way. Thus

$$x(t) = X_0 + \sum_{\substack{n \\ \text{odd}}} X_n \cdot \sin(\omega_n t)$$

The coefficients X_n are obtained as

$$X_0 = A/(2k); \quad X_n = \frac{2A}{n\pi(k - m\omega_n^2)}$$

The velocity v is

$$v = \dot{x} = \sum \omega_n X_n \cdot \cos(\omega_n t)$$

The time average of the velocity squared is consequently given by

$$\left|\bar{v}^2\right| = \frac{1}{T}\cdot\int_0^T \mathrm{d}t\cdot\left|v^2\right| = \sum_{\substack{n\\ \text{odd}}}\left(\frac{4A}{T}\right)^2\cdot\frac{1}{2m^2}\left\{\frac{1}{(\omega_0^2-\omega_n^2)^2+\omega_0^4\delta^2}\right\};\quad \omega_0^2=\frac{k}{m}$$

Alternative method
The FT of the displacement of the system is written

$$\hat{x} = \hat{F}\cdot H$$

The FT of the velocity is consequently $\hat{v} = \hat{\dot{x}} = i\omega\hat{F}\cdot H$ where for $k = k_0(1+i\delta)$ and $k_0/m = \omega_0^2$ the frequency response function is

$$H = \frac{1}{k - m\omega_n^2};\quad |H|^2 = \frac{1}{m^2[(\omega_0^2-\omega^2)^2+\omega_0^4\delta^2]}$$

The power spectral density S_{vv} is according to Eq. (2.53)

$$S_{vv} = \omega^2|H|^2\cdot S_{FF};\quad S_{FF} \text{ from Example 2.3}$$
$$\left|\bar{v}^2\right| = \int_{-\infty}^{\infty}\mathrm{d}f\cdot S_{vv}(f) = \sum_{\substack{n\\ \text{odd}}}\left(\frac{4A}{T}\right)^2\cdot\frac{1}{2m^2}\left\{\frac{1}{(\omega_0^2-\omega_n^2)^2+\omega_0^4\delta^2}\right\}$$

2.2.5 *Example 2.5*

In the first case the two-sided power spectral density is given as

$$G_{xx}(\omega) = a \quad \text{for } f_1 = f_0 - (B/2) \leqslant f \leqslant f_0 + B/2 = f_2 \Rightarrow$$

$$S_{xx}(\omega) = a/2 \quad \text{for } -f_2 \leqslant f \leqslant -f_1 \text{ and } f_1 \leqslant f \leqslant f_2 \qquad (2.2.5.1)$$

The autocorrelation function is defined in Eq. (2.34). Thus

$$\begin{aligned} R_{xx}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\cdot e^{i\omega\tau}\mathrm{d}\omega \\ &= \frac{a}{4\pi}\left[\int_{-\omega_2}^{-\omega_1} e^{i\omega\tau}\mathrm{d}\omega + \int_{\omega_1}^{\omega_2} e^{i\omega\tau}\mathrm{d}\omega\right] \\ &= \frac{a}{4\pi}\int_{\omega_1}^{\omega_2}\mathrm{d}\omega\left[e^{i\omega\tau}+e^{-i\omega\tau}\right] = \frac{a}{2\pi}\int_{\omega_1}^{\omega_2}\mathrm{d}\omega\cdot\cos(\omega\tau) \\ &= \frac{a}{2\pi\tau}\left[\sin(\omega\tau)\right]_{\omega_1}^{\omega_2} = \frac{a}{2\pi\tau}\left[\sin(\omega_0\tau+\pi B\tau)-\sin(\omega_0\tau-\pi B\tau)\right] \end{aligned}$$

$$= \frac{a}{\pi\tau}\cos(\omega_0\tau)\cdot\sin(\pi B\tau)$$
$$= \frac{aB\sin(\pi B\tau)}{\pi B\tau}\cdot\cos(\omega_0\tau) \tag{2.2.5.2}$$

In the second case, $G_{xx}(\omega) = a$ for $0 \leqslant f \leqslant B$. This expression is obtained by setting $f_0 = B/2$ in Eq. (2.2.5.1). The autocorrelation function is given by Eq. (2.2.5.2) as

$$R_{xx}(\tau) = a\cdot B\frac{\sin(\pi B\tau)\cdot\cos(\omega_0\tau)}{\pi B\tau} \tag{2.2.5.3}$$

But $\omega_0 = 2\pi f_0 = \pi B$. This inserted in Eq. (2.2.5.3) gives the autocorrelation function

$$R_{xx}(\tau) = a\cdot B\frac{\sin(2\pi B\tau)}{2\pi B\tau} \tag{2.2.5.4}$$

2.2.6 Example 2.6

The force $F(t)$ exciting the mass of a simple 1-DOF system is given by

$$F(t) = A\cdot\sin(\omega_1 t) + \xi(t) = f(t) + \xi(t) \tag{2.2.6.1}$$

The force $\xi(t)$ is random. According to Eq. (2.24) the auto correlation function $R_{FF}(\tau)$ can be written

$$R_{FF}(\tau) = R_{ff}(\tau) + R_{\xi\xi}(\tau) \tag{2.2.6.2}$$

The power spectral density $G_{FF}(\omega)$ is

$$G_{FF}(\omega) = G_{ff}(\omega) + G_{\xi\xi}(\omega) \tag{2.2.6.3}$$

According to Eq. (2.41) and Eq. (2.2.6.3) the power spectral density $G_{FF}(\omega)$ is

$$G_{FF}(\omega) = \pi A^2\delta(\omega-\omega_1) + G_{\xi\xi} = \pi A^2\delta(\omega-\omega_1) + \frac{A^2}{2\omega_0} \tag{2.2.6.4}$$

The frequency response function $H(\omega)$ is given in Eq. (2.17). Thus

$$|H|^2 = \frac{1}{m^2[(\omega_0^2-\omega^2)^2 + (\omega_0^2\delta)^2]} \tag{2.2.6.5}$$

From (2.53) the one-sided power spectral density for the velocity is given by

$$G_{vv}(\omega) = \omega^2 \left|H^2\right| \cdot G_{FF}(\omega) \tag{2.2.6.6}$$

Further, from Eq. (2.54) the time average of the velocity squared is

$$\bar{v}^2 = \frac{1}{2\pi}\int_0^\infty G_{vv}(\omega)\mathrm{d}\omega \tag{2.2.6.7}$$

The Eqs. (2.2.6.4)–(2.2.6.6) inserted in Eq. (2.2.6.7) give

$$\begin{aligned}
\bar{v}^2 &= \frac{A^2}{2}\int_0^\infty \mathrm{d}\omega \cdot \frac{\omega^2\delta(\omega-\omega_1)}{m^2[(\omega^2-\omega_0^2)^2+(\omega_0^2\delta)^2]} \\
&\quad + \frac{A^2}{4\pi}\int_0^\infty \mathrm{d}\omega \cdot \frac{\omega^2}{\omega_0 m^2[(\omega^2-\omega_0^2)^2+(\omega_0^2\delta)^2]} \\
&= \frac{A^2}{2}\left\{\frac{\omega_1^2}{m^2[(\omega_1^2-\omega_0^2)^2+(\omega_0^2\delta)^2]}+\frac{1}{4\delta k_0 m}\right\}
\end{aligned}$$

The second integral is solved as described in Sect. 2.7 using

$$\int_{-\infty}^\infty \mathrm{d}\omega \cdot \frac{g(\omega)}{(\omega^2-\omega_0^2)^2+(\delta\omega_0^2)^2} = \frac{\pi g(\omega_0)}{\omega_0^3\delta} \text{ and } \omega_0^2 = \frac{k_0}{m}$$

2.2.7 *Example 2.7*

The equation of motion for the simple 1-DOF system is

$$m\ddot{x} + c\dot{x} + kx = F_0 \cdot \sin(\omega_1 t) \tag{2.2.7.1}$$

From the Eqs. (1.57) and (1.60) the displacement $x(t)$ is obtained as

$$x(t) = A_0 \sin(\omega_1 t + \varphi); \quad A_0 = \frac{F_0}{m[(\omega_0^2-\omega^2)^2+(2\beta\omega)^2]^{1/2}} \tag{2.2.7.2}$$

The time average of the potential energy is

$$\bar{U} = \frac{1}{2}k \cdot \frac{1}{T}\int_0^T x^2\mathrm{d}t = \frac{kA_0^2}{4} \tag{2.2.7.3}$$

The time average of the kinetic energy is

$$\bar{\mathcal{T}} = \frac{1}{2}m \cdot \frac{1}{T}\int_0^T \left|\dot{x}^2\right| \mathrm{d}t = \frac{m\omega_1^2 \cdot A_0^2}{4} \tag{2.2.7.4}$$

The results (2.2.7.3) and (2.2.7.4) give

$$\begin{aligned} \bar{\mathcal{T}} &= \left(\frac{\omega_1}{\omega_o}\right)^2 \cdot \bar{\mathcal{V}}; \quad \omega_0^2 = \frac{k}{m} \\ \bar{\mathcal{T}} &= \bar{\mathcal{V}} \quad \text{only when } \omega_0 = \omega_1 \end{aligned} \tag{2.2.7.5}$$

2.2.8 Example 2.8

A lightly damped 1-DOF system is excited by a force with the FT $\hat{F}$. The FT of the response of the system is $\hat{x}$. The system is described by the frequency response function $H(\omega)$. Thus according to Eq. (2.15)

$$\hat{x} = H \cdot \hat{F} \tag{2.2.8.1}$$

The FT of the velocity is

$$\hat{v} = i\omega H\hat{F} \tag{2.2.8.2}$$

The one-sided power spectral density for the velocity is from Eq. (2.53) given by

$$G_{vv} = \omega^2 \left|H\right|^2 \cdot G_{FF} \tag{2.2.8.3}$$

For the problem discussed G_{FF} is defined as

$$G_{FF} = 4a/(a^2 + \omega^2) \tag{2.2.8.4}$$

The frequency response function for a 1-DOF is given by Eq. (2.17). Thus

$$\left|H\right|^2 = \frac{1}{m^2[(\omega^2 - \omega_0^2)^2 + (\omega_0^2\delta)^2]} \tag{2.2.8.5}$$

The time average of the velocity squared is given by (2.54).

By inserting the Eqs. (2.2.8.4) and (2.2.8.5) in this expression the time average of the velocity squared is obtained as

$$\bar{v}^2 = \frac{1}{2\pi}\int_0^\infty G_{vv} d\omega = \frac{1}{4\pi}\int_{-\infty}^\infty G_{vv}(\omega)\mathrm{d}\omega$$
$$= \frac{a}{\pi}\cdot\int_{-\infty}^\infty \frac{\omega^2 \mathrm{d}\omega}{(a^2+\omega^2)\cdot[(\omega^2-\omega_0^2)^2+(\omega_0^2\delta)^2]} \tag{2.2.8.6}$$

The integral is solved using Eq. (2.63). Thus the result is

$$\bar{v}^2 = \frac{a}{\omega_0\delta m^2(a^2+\omega_0^2)}$$

2.2.9 Example 2.9

Equation of motion

$$m\ddot{x} + k(x-y) = F \tag{2.2.9.1}$$

The force on foundation is (Fig. 2.6)

$$F_f = k(x-y) \tag{2.2.9.2}$$

The FT of the displacements x and y are obtained by inserting in Eqs. (2.2.9.1) and (2.2.9.2) the substitutions $x \rightarrow \hat{x}\cdot\exp(i\omega t)$, $y \rightarrow \hat{y}\cdot\exp(i\omega t)$,$F \rightarrow \hat{F}\cdot\exp(i\omega t)$ and $F_f \rightarrow \hat{F}_f\hat{x}\cdot\exp(i\omega t)$. The point mobility of the foundation is $Y_f = \hat{y}/\hat{F}$. Thus

$$i\omega\hat{y} = \hat{F}_f \cdot Y_f \tag{2.2.9.3}$$

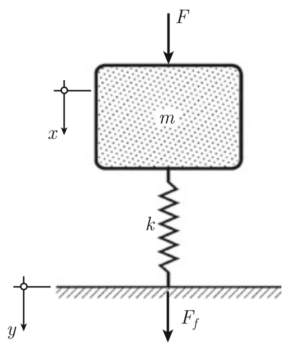


Fig. 2.6 Mass-spring system mounted on a foundation having the point mobility Y_f

The point mobility of the mass at the excitation point is:

$$Y = \hat{\dot{x}}/\hat{F} = i\omega\hat{x}/\hat{F} \tag{2.2.9.4}$$

The elimination of y and F by inserting (2.2.9.2) and (2.2.9.3) in (2.2.9.1) and by using the substitutions $x \to \hat{x} \cdot \exp(i\omega t)$ etc. the point mobility Y at the excitation point is

$$Y = \frac{i\omega + kY_f}{k - m\omega^2 + im\omega kY_f} \tag{2.2.9.5}$$

2.2.10 Example 2.10

Equation of motion for the mass

$$m\ddot{x} + k(x - y) = F \tag{2.2.10.1}$$

By making the substitutions $x \to \hat{x} \cdot e^{i\omega t}$, $y \to \hat{y} \cdot e^{i\omega t}$ and $F \to \hat{F} \cdot e^{i\omega t}$ in Eq. (2.2.10.1) the result is

$$\hat{x}(-m\omega^2 + k) - k\hat{y} = \hat{F} \tag{2.2.10.2}$$

The FT of the force on the foundation is $\hat{F}_f = k(\hat{x} - \hat{y})$. The FT of the velocity of the foundation is $\hat{v}_y = i\omega\hat{y} = \hat{F}_f Y_f$. The combination of these expressions gives

$$i\omega\hat{y} = Y_f k(\hat{x} - \hat{y}) \tag{2.2.10.3}$$

The elimination of $\hat{x}$ from Eqs. (2.2.10.2) and (2.2.10.3) gives

$$\hat{y} = \hat{F} \cdot \frac{Y_f k}{-im\omega^3 + i\omega k - m\omega^2 kY_f} \tag{2.2.10.4}$$

The FT of the velocity is thus

$$\hat{v} = i\omega\hat{y} = \frac{\hat{F} \cdot Y_f k}{-m\omega^2 + k + im\omega kY_f} \tag{2.2.10.5}$$

According to definition $\hat{v} = \hat{F}_f Y_f$ or

$$\hat{F}_f = \frac{\hat{v}_y}{Y_f} \tag{2.2.10.6}$$

According to Eq. (2.55) the time average of the input power to a structure, in this case the foundation, is written

$$\bar{\Pi} = \frac{1}{2\pi}\int_0^{\infty} \mathrm{Re}(G_{Fv})\mathrm{d}\omega \tag{2.2.10.7}$$

The one-sided power spectral density G_{Fv} is

$$G_{Fv} = 2\lim_{T\to\infty}\frac{\hat{F}^{*}\cdot\hat{v}}{T} = 2\lim_{T\to\infty}\frac{\left|\hat{v}\right|^2}{T}\cdot\frac{1}{Y_f^{*}} = G_{vv}\cdot\frac{1}{Y_f^{*}} \tag{2.2.10.8}$$

The point mobility Y_f of the foundation is in this case defined as being real.

From Eq. (2.2.10.5) G_{vv} is obtained as

$$\begin{aligned} G_{vv} &= \lim_{T\to\infty} 2\frac{\left|\hat{v}\right|^2}{T} = \lim_{T\to\infty} 2\frac{\left|\hat{F}\right|^2}{T}\cdot\frac{\left|Y_f k\right|^2}{(k-m\omega^2)^2+(m\omega Y_f k)^2} \\ &= G_{FF}\cdot\frac{\left|Y_f k\right|^2}{m^2[(\omega_0^2-\omega^2)^2+(\omega Y_f k)^2]} = G_{FF}\cdot\frac{\left|Y_f\right|^2\cdot\omega_0^4}{(\omega^2-\omega_0^2)^2+(\omega Y_f k)^2} \end{aligned} \tag{2.2.10.9}$$

where $\omega_0 = \sqrt{k/m}$.

For Y_f real the Eqs. (2.2.10.8) and (2.2.10.9) give

$$G_{Fv} = G_{FF}\cdot\frac{Y_f\cdot\omega_0^4}{(\omega^2-\omega_0^2)^2+(\omega Y_f k)^2} \tag{2.2.10.10}$$

The time average of the power to the foundation is from Eq. (2.2.10.7).

For white noise excitation of the mass G_{FF} is constant. For Y_f independent of frequency the result is

$$\bar{\Pi} = \frac{G_{FF}\cdot Y_f\omega_0^4}{2\pi}\int_0^{\infty}\frac{\mathrm{d}\omega}{(\omega^2-\omega_0^2)^2+(\omega Y_f k)^2}$$

For $\left|\omega Y_f k\right| \ll 1$

$$\bar{\Pi} = \frac{G_{FF}\cdot Y_f\omega_0^4}{4\omega_0^3(Y_f k/\omega_0)} = \frac{G_{FF}\cdot\omega_0^2}{4k} = \frac{G_{FF}}{4m}$$

This means that the power fed into the foundation only depends on the mass and the force if the mobility of the foundation is low. Compare the result given in Eq. (2.67).

2.2.11 Example 2.11

The total displacement of the mass is defined as $z = x + y$ where x and y are uncorrelated. The power spectral G_{zz} density is thus

$$G_{zz} = G_{xx} + G_{yy} \tag{2.2.11.1}$$

The FT of the displacement x is

$$\hat{x} = \hat{F} \cdot H \tag{2.2.11.2}$$

where $\hat{F}$ is the FT of the force exciting the system and H the frequency response function of the system. From Eq. (2.2.11.2) it follows that

$$G_{xx} = G_{FF}\,|H|^2 \tag{2.2.11.3}$$

The cross-spectral density G_{Fz} is

$$G_{Fz} = G_{Fx} + G_{Fy} \tag{2.2.11.4}$$

Since F and y are uncorrelated it follows that $G_{Fy} = 0$. The cross-spectral density G_{Fx} is obtained from Eq. (2.2.11.2) as

$$G_{Fx} = H \cdot G_{FF} \tag{2.2.11.5}$$

For $G_{Fy} = 0$ Eqs. (2.2.11.4) and (2.2.11.5) give

$$G_{Fz} = H \cdot G_{FF} \tag{2.2.11.6}$$

The coherence function as defined in Eq. (2.45) is

$$\gamma_{Fx}^2 = \frac{|G_{Fz}|^2}{G_{FF} \cdot G_{zz}}$$

By inserting Eqs. (2.2.11.1) and (2.2.11.6) in this definition the result is

$$\gamma_{Fx}^2 = \frac{|G_{Fz}|^2}{G_{FF} \cdot G_{zz}} = \frac{|H|^2 \cdot |G_{FF}|^2}{G_{FF} \cdot (G_{FF}\,|H|^2 + G_{yy})} \tag{2.2.11.7}$$

For G_{FF} real, $G_{FF}^2 = |G_{FF}|^2$. Considering this Eq. (2.2.11.7) is written

$$\gamma_{Fx}^2 = \frac{|H|^2 \cdot G_{FF}^2}{|H|^2 \cdot G_{FF}^2 + G_{FF} \cdot G_{yy}} = \frac{1}{1 + G_{yy}/(|H|^2 \cdot G_{FF})} \leqslant 1 \tag{2.2.11.8}$$

Equality holds only when $G_{yy} = 0$ i.e. for no external noise.

2.2.12 Example 2.12

The time average of the power input to system is using Eq. (1.81), $2\omega\beta = \omega_0^2\delta$

$$\bar{\Pi} = \int_{-\infty}^{\infty} \frac{S_{FF}(\omega)\omega\delta\omega_0^2 \mathrm{d}\omega}{2\pi m\left[(\omega_0^2 - \omega^2)^2 + (\omega_0^2\delta)^2\right]} \tag{2.2.12.1}$$

Considering the result (2.63) the solution to Eq. (2.2.12.1) is

$$\bar{\Pi} = \frac{S_{FF}(\omega_0)}{2m} \tag{2.2.12.2}$$

2.2.13 Example 2.13

A function $x(t)$ is according to Eq. (1.66) expanded in a Fourier series in the time interval $-T/2 \leqslant t \leqslant T/2$. The series is written

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\omega_n t + b_n \sin\omega_n t); \quad \omega_n = \frac{2\pi n}{T} \tag{2.2.13.1}$$

Since $\cos\varphi = \left(e^{i\varphi} + e^{-i\varphi}\right)/2$ and $\sin\varphi = \left(e^{i\varphi} - e^{-i\varphi}\right)/2i$ it follows that (2.2.12.1) can be written

$$x(t) = \sum_{n=-\infty}^{\infty} A_n e^{2\pi nit/T} \tag{2.2.13.2}$$

Equation (2.2.12.2) is multiplied by $e^{-2\pi ikt/T}$, k an integer, and integrated with respect to time. Thus,

$$\int_{-T/2}^{T/2} \mathrm{d}t \cdot x(t)e^{-2\pi ikt/T} = \sum_{n=-\infty}^{\infty} \int_{-T/2}^{T/2} \mathrm{d}t \cdot A_n \cdot e^{2\pi it(n-k)/T} \tag{2.2.13.3}$$

The integral on the right-hand side is $\int_{-T/2}^{T/2} \mathrm{d}t \cdot A_n \cdot e^{2\pi it(n-k)/T} = T$, for $n = k$ otherwise zero. Thus in combination with (2.2.13.3) the parameters A_n in (2.2.13.2) are

$$\int_{-T/2}^{T/2} \mathrm{d}t \cdot x(t)e^{-2\pi nit/T} = A_n \cdot T \tag{2.2.13.4}$$

As $T \to \infty$ the summation in (2.2.13.2) tends to an integral. Define $\omega = 2\pi n/T$ and $\mathrm{d}n = T\mathrm{d}\omega/(2\pi)$ and $A_n T = \hat{x}(\omega)$ and insert in (2.2.13.2) and (2.2.13.4). The result is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \cdot \hat{x}(\omega)e^{i\omega t}$$

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \cdot x(t)e^{-i\omega t}$$

2.2.14 Example 2.14

According to definition

$$\begin{aligned}
&R_{xx}(\tau) = E\left[x(t)x(t+\tau)\right] = E\left[x(t-\tau)x(t)\right] \Rightarrow \\
&\frac{\mathrm{d}R_{xx}}{\mathrm{d}\tau} = E\left[x(t)\dot{x}(t+\tau)\right] = E\left[x(t-\tau)\dot{x}(t)\right] \Rightarrow \\
&\frac{\mathrm{d}^2 R_{xx}}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}E\left[x(t-\tau)\dot{x}(t)\right] = -E\left[\dot{x}(t-\tau)\dot{x}(t)\right] \Rightarrow \left[\frac{\mathrm{d}^2 R_{xx}}{\mathrm{d}\tau^2}\right]_{\tau=0} = -E[\dot{x}^2(t)]
\end{aligned}$$

2.3 Chapter 3

2.3.1 Example 3.1

The displacement along the beam or the x-axis is given by

$$\xi = A\sin(\omega t - k_l x) \tag{2.3.1.1}$$

For a thin beam σ_y and σ_z are assumed to be zero across the beam. Inserting $\sigma_y = \sigma_z = 0$ in Eq. (3.6) gives

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\partial\xi}{\partial x} \tag{2.3.1.2}$$

For $\xi = A\sin(\omega t - k_l x)$ it follows that

$$\varepsilon_y = \frac{\partial\eta}{\partial y} = -\nu\frac{\partial\xi}{\partial x} = \nu k_l A\cos(\omega t - k_l x) \tag{2.3.1.3}$$

By integrating (2.3.1.3) the displacement η is obtained as

$$\eta = \nu k_l A \cdot y \cdot \cos(\omega t - k_l x) \tag{2.3.1.4}$$

The displacement at the surface, i.e. for $y = h/2$, is according to Eq. (2.3.1.4)

$$\eta\,(h/2) = \frac{\nu k_l A h}{2} \cdot \cos(\omega t - k_l x) \tag{2.3.1.5}$$

In a similar way; $\varepsilon_z = \dfrac{\partial \zeta}{\partial z}$ and

$$\zeta = \nu k_l A \cdot z \cdot \cos(\omega t - k_l x) \text{ and } \zeta\,(b/2) = \frac{\nu k_l A b}{2} \cdot \cos(\omega t - k_l x) \tag{2.3.1.6}$$

2.3.2 Example 3.2

The kinetic energy per unit volume is

$$T_v = \frac{\rho}{2}\left\{\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right\}$$

Inserting Eqs. (2.3.1.1), (2.3.1.4), and (2.3.1.6) from Example 3.1 in this expression gives

$$\begin{aligned}
\mathcal{T}_v &= \frac{\rho}{2}\left\{\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right\} \\
&= \frac{\rho\omega^2 A^2}{2}\{\cos^2(\omega t - k_l x) + \nu^2 (y k_l)^2 \sin^2(\omega t - k_l x) \\
&\quad + \nu^2 (z k_l)^2 \sin^2(\omega t - k_l x)\}
\end{aligned}$$

The time average of kinetic energy per unit length

$$\mathcal{T}_l = \frac{1}{T}\int_0^T \mathrm{d}t \int_{-h/2}^{h/2} \mathrm{d}y \int_{-b/2}^{b/2} \mathrm{d}z \cdot \mathcal{T}_\nu = \frac{\rho\omega^2 A^2 bh}{4}\left\{1 + \frac{\nu k_l^2 (h^2 + b^2)}{12}\right\}$$

For $k_l h \ll 1$ and $k_l b \ll 1$, $\mathcal{T}_l$ is

$$\mathcal{T}_l = \frac{\rho\omega^2 A^2 bh}{4}$$

2.3.3 Example 3.3

The displacement along the x-axis is defined as

$$\xi = f(x - c_l t); \quad c_l = \sqrt{E/\rho} \tag{2.3.3.1}$$

The intensity I_x is

$$I_x = -\sigma_x \cdot \dot{\xi} \tag{2.3.3.2}$$

Equation (2.3.3.1) gives

$$\sigma_x = E \cdot \frac{\partial \xi}{\partial x} = E \cdot f'(x - c_l t) \tag{2.3.3.3}$$

$$\dot{\xi} = \frac{\partial \xi}{\partial t} = -c_l f'(x - c_l t) \tag{2.3.3.4}$$

Equations (2.3.3.3) and (2.3.3.4) inserted in (2.3.3.2) give

$$I_x = E c_l \cdot \left[f'(x - c_l t)\right]^2 \tag{2.3.3.5}$$

The energy flow is

$$P_x = I_x \cdot S = E c_l S \left[f'(x - c_l t)\right]^2 \tag{2.3.3.6}$$

where S is the cross-sectional area of the beam. For a thin beam and neglecting contraction, the kinetic and potential energies per unit length are

$$\begin{aligned} \mathcal{T}_l &= \frac{S\rho}{2} \cdot \left(\frac{\partial \xi}{\partial t}\right)^2 = \frac{S\rho c_l^2}{2} \cdot \left[f'(x - c_l t)\right]^2 \\ &= \frac{SE}{2} \cdot \left[f'(x - c_l t)\right]^2 \end{aligned} \tag{2.3.3.7}$$

$$\mathcal{U}_l = \frac{SE}{2} \cdot \left(\frac{\partial \xi}{\partial x}\right)^2 = \frac{SE}{2} \cdot \left[f'(x - c_l t)\right]^2 \tag{2.3.3.8}$$

The energy flow is from Eqs. (2.3.3.6)–(2.3.3.8) obtained as

$$\Pi = c_l(\mathcal{T}_l + \mathcal{U}_l) \tag{2.3.3.9}$$

The energy flow Π is equal to the total energy per unit length times the speed of propagation.

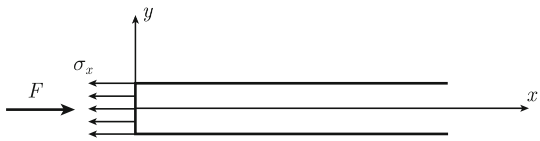


Fig. 2.7 Semi infinite beam excited by a force at $x = 0$

2.3.4 Example 3.4

The force induces a wave propagating along the positive x-axis. See Fig. 2.7. The displacement ξ of the wave can according to Sect. 3.4 be written as

$$\xi = f(x - c_l t) \tag{2.3.4.1}$$

The normal stress σ_x in the beam is

$$\sigma_x = E \cdot \frac{\partial \xi}{\partial x} = E \cdot f'(x - c_l t) \tag{2.3.4.2}$$

At the boundary $x = 0$, $-\sigma_x \cdot S = F$. Thus from Eq. (2.3.4.2) and for $x = 0$

$$F(t) = SE \cdot f'(x - c_l t) \tag{2.3.4.3}$$

Let $-c_l t = \zeta$. Inserting this in Eq. (2.3.4.3) gives

$$F\left(-\zeta/c_l\right) = SE \cdot \frac{\partial f}{\partial \zeta} \quad \text{or}$$

$$f(\zeta) = \frac{1}{SE} \int \mathrm{d}\zeta \cdot F(-\zeta/c_l) \tag{2.3.4.4}$$

For $F(t) = F_0 \sin(\omega t) \Rightarrow$

$$f(\zeta) = \frac{F_0}{SE} \int \mathrm{d}\zeta \cdot \sin(-\omega\zeta/c_l) = \frac{F_0 c_l}{SE\omega} \cos(\omega\zeta/c_l)$$

For $\zeta = x - c_l t$,

$$\xi(x, t) = f(x - c_l t) = \frac{F_0 c_l}{SE\omega} \cos\left[\omega(x/c_l - t)\right]$$

2.3.5 Example 3.5

The torsional angle in the shaft is

$$\theta = \theta_0 \sin(k_t x - \omega t); \quad k_t = \omega\sqrt{\rho/G} \tag{2.3.5.1}$$

The torsion τ in the shaft is according to Eq. (3.59)

$$\tau = r \cdot G \cdot \frac{\partial \theta}{\partial x} \tag{2.3.5.2}$$

The potential energy per unit length of the shaft is

$$\mathcal{U}_l = \int \mathrm{d}S \cdot G \cdot \gamma^2/2 \tag{2.3.5.3}$$

with $\mathrm{d}S = 2\pi r \mathrm{d}r$ and $\gamma = r \cdot \dfrac{\partial \theta}{\partial x}$. The potential energy is rewritten as

$$\begin{aligned}\mathcal{U}_l &= \frac{2\pi}{2}\int_0^R \mathrm{d}r \cdot r^3 G \cdot \left(\frac{\partial \theta}{\partial x}\right)^2 = \frac{\pi}{4} G R^4 \cdot \left(\frac{\partial \theta}{\partial x}\right)^2 \\ &= \frac{\pi}{4} R^4 \cdot \theta_0^2 \cdot k_t^2 \cdot \cos^2(k_t x - \omega t) \cdot G \\ &= \frac{\pi}{4} R^4 \cdot \theta_0^2 \omega^2 \rho \cos^2(k_t x - \omega t)\end{aligned} \tag{2.3.5.4}$$

In a similar way, the kinetic energy per unit length is

$$\begin{aligned}\mathcal{T}_l &= \int_0^R ds \cdot \rho \frac{(r\dot{\theta})^2}{2} = \frac{\pi}{4} \cdot \rho R^4 \theta_0^2 \cdot \left(\dot{\theta}\right)^2 \\ &= \frac{\pi}{4} \cdot \rho R^4 \omega^2 \cdot \theta_0^2 \cos^2(k_t x - \omega t) = \mathcal{U}_l\end{aligned} \tag{2.3.5.5}$$

The intensity along the shaft is

$$I_x = -\tau \cdot r \cdot \dot{\theta} = -r^2 G \cdot \frac{\partial \theta}{\partial x} \cdot \frac{\partial \theta}{\partial t} \tag{2.3.5.6}$$

The resulting energy flow is

$$\begin{aligned}\Pi &= \int_0^R 2\pi r \mathrm{d}r \cdot I_x = 2\pi G \int_0^R \mathrm{d}r \cdot r^3 \frac{\partial \theta}{\partial x} \cdot \frac{\partial \theta}{\partial t} \\ &= \frac{\pi \mathrm{G} r^4}{2} \cdot \theta_0^2 \cdot k\omega \cos^2(k_l x - \omega t)\end{aligned}$$

$$= \cos^2(kx - \omega t)\frac{\pi R^4 \theta_0^2 \omega^2}{2}\sqrt{\frac{G}{\rho}}$$
$$= \frac{\pi R^4 \theta_0^2}{2} c_{\text{torsion}} \cdot \cos^2(kx - \omega t)$$
$$= c_{\text{torsion}}(\mathcal{T}_l + \mathcal{U}_l)$$

where the speed of propagation is $c_{\text{torsion}} = \sqrt{G/\rho}$. Compare the result of Example 3.3.

2.3.6 Example 3.6

The potential energy per unit volume of a solid is according to Eq. (3.17)

$$\mathcal{U}_v = G\left\{\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{\nu}{1-2\nu}(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + (\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2)/2\right\} \tag{2.3.6.1}$$

Shear effects neglected means that $\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$. Further for a thin beam $\sigma_y = \sigma_z = 0$ which as discussed in Example 3.1 gives

$$\varepsilon_y = -\nu\varepsilon_x; \quad \varepsilon_z = -\nu\varepsilon_x \tag{2.3.6.2}$$

From Eq. (3.72)

$$\varepsilon_x = -z \cdot \frac{\partial^2 w}{\partial x^2} \tag{2.3.6.3}$$

By inserting Eqs. (2.3.6.2) and (2.3.6.3) in (2.3.6.1) and neglecting shear the result is

$$\mathcal{U}_v = \frac{E}{2(1+\nu)} \cdot \varepsilon_x^2\left\{1 + 2\nu^2 + \frac{\nu}{1-2\nu}(1-2\nu)^2\right\}$$
$$= \frac{E}{2} \cdot z^2 \cdot \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \tag{2.3.6.4}$$

The potential energy per unit length is

$$\mathcal{U}_l = \int \mathrm{d}y\mathrm{d}z\,\mathcal{U}_v = \int_{-b/2}^{b/2} \mathrm{d}y \int_{-h/2}^{h/2} \mathrm{d}z \cdot \frac{E}{2} \cdot z^2 \cdot \left(\frac{\partial^2 w}{\partial x^2}\right)^2$$
$$= \frac{E}{2} \cdot \frac{bh^3}{12} \cdot \left(\frac{\partial^2 w}{\partial x^2}\right)^2 = \frac{D'}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2$$

Width of beam is b and height h. The result is the same as given by Eq. (3.84).

2.3.7 Example 3.7

The equation governing the lateral displacement η of a string is

$$\frac{\partial^2 \eta}{\partial x^2} - \frac{1}{c_s^2} \cdot \frac{\partial^2 \eta}{\partial t^2} = 0 \tag{2.3.7.1}$$

The general solution to this equation is

$$\eta = f(x - c_s t) + g(x + c_s t) \tag{2.3.7.2}$$

The velocity $\dot{\eta}$ of the string is

$$\dot{\eta}(x, t) = -c_s \cdot f'(x - c_s t) + c_s \cdot g'(x + c_s t) \tag{2.3.7.3}$$

The initial conditions are

$$\eta(x, 0) = f(x) + g(x) = \cos(\pi x/L) \tag{2.3.7.4}$$
$$\dot{\eta}(x, 0) = 0 \Rightarrow f'(x) = g'(x) \Rightarrow f(x) = g(x) \tag{2.3.7.5}$$

Considering the symmetry Eqs. (2.3.7.4) and (2.3.7.5) give

$$f(x) = g(x) = \frac{1}{2} \cos(\pi x/L) \tag{2.3.7.6}$$

Thus $f(x - c_s t) = 1/2 \cdot \cos[\pi(x - c_s t)/L]$ and $g(x + c_s t) = 1/2 \cdot \cos[\pi(x + c_s t)/L]$ and from (2.3.7.2)

$$\begin{aligned} \eta(x, t) &= 1/2 \cdot \{\cos[\pi(x - c_s t)/L] + \cos[\pi(x + c_s t)/L]\} \\ &= \cos(\pi x/L) \cdot \cos(\pi c_s t/L) \end{aligned}$$

2.3.8 Example 3.8

The wave equation for flexural waves in a thin homogeneous beam is according to Eq. (3.77) for $F' = 0$

$$\frac{\partial^4 w}{\partial x^4} + \frac{m'}{D'} \cdot \frac{\partial^2 w}{\partial t^2} = 0 \tag{2.3.8.1}$$

At $t = 0$ the beam is at rest but has a certain displacement. The initial conditions are

$$w(x, 0) = e^{-(x/2a)^2}; \quad \dot{w}(x, 0) = 0 \tag{2.3.8.2}$$

The spatial FT of $w(x, t)$ is defined as

$$\widetilde{w}(k, t) = \int_{-\infty}^{\infty} w(x, t) \cdot e^{-ikx} \mathrm{d}x \tag{2.3.8.3}$$

Thus

$$w(x, t) = \frac{1}{2\pi} \int \widetilde{w}(x, t) \cdot e^{ikx} \mathrm{d}k \tag{2.3.8.4}$$

The Eqs. (2.3.8.4) and (2.3.8.1) yield

$$\widetilde{w}k^4 + \frac{m'}{D'} \cdot \frac{\partial^2 w}{\partial t^2} = 0 \tag{2.3.8.5}$$

The general solution to the Eq. (2.3.8.5) is

$$\widetilde{w}(k, t) = A \cdot \sin(\Omega t) + B \cdot \cos(\Omega t) \tag{2.3.8.6}$$

$$\Omega = k^2 \sqrt{D'/m'} = k^2 \beta; \quad \beta = \sqrt{D'/m'} \tag{2.3.8.7}$$

According to the initial condition (2.3.8.2), the velocity is equal to zero for $t = 0$ or $\dot{w}(x, 0) = 0$. Thus according to Eq. (2.3.8.3) $\dot{\widetilde{w}} = 0$. The parameter A defined in Eq. (2.3.8.6) is consequently equal to zero. From (2.3.8.6) it also follows that $B = \widetilde{w}(k, 0)$. The Eqs. (2.3.8.2) and (2.3.8.3) give

$$B = \widetilde{w}(k, 0) = \int_{-\infty}^{\infty} \mathrm{d}x \cdot e^{-(x/2a)^2 - ikx} \tag{2.3.8.8}$$

The exponent is rewritten as

$$-\left(\frac{x}{2a}\right)^2 - ikx = -\left[\left(\frac{x}{2a}\right) + ika\right]^2 - (ka)^2 \tag{2.3.8.9}$$

Equation (2.3.8.9) inserted in (2.3.8.8) gives

$$\widetilde{w}(k, 0) = e^{-(ka)^2} \cdot \int_{-\infty}^{\infty} e^{-[x/(2a) + ika]^2} \mathrm{d}x \tag{2.3.8.10}$$

However, $\int_{-\infty}^{\infty} \mathrm{d}x \cdot e^{-qx^2} = \sqrt{\frac{\pi}{q}}$, thus

$$\widetilde{w}(k, 0) = 2a\sqrt{\pi} \cdot e^{-(ka)^2} = B \tag{2.3.8.11}$$

Equations (2.3.8.11) and (2.3.8.6) in (2.3.8.4) $\Rightarrow$

$$w(x,t) = \frac{2a\sqrt{\pi}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \cdot e^{-(ka)^2+ikx} \cdot \left[\frac{e^{ik^2\beta t} + e^{-ik^2\beta t}}{2}\right] \tag{2.3.8.12}$$

The expression inside the bracket is equal to

$$\cos(k^2\beta t) = \cos(\Omega t)$$

The exponent in the integral is rewritten as

$$-k^2\left[a^2 + i\beta t\right] + ikx = -\left[k\sqrt{a^2+i\beta t} - \frac{ix}{\sqrt{2(a^2+i\beta t)}}\right]^2 - \frac{x^2}{4(a^2+i\beta t)} \tag{2.3.8.13}$$

Equations (2.3.8.12) and (2.3.8.13) $\Rightarrow$

$$w(x,t) = \frac{a}{2\sqrt{\pi}}\left[e^{-\frac{x^2}{4(a^2+i\beta t)}} \cdot \frac{\sqrt{\pi}}{\sqrt{a^2+i\beta t}} + e^{-\frac{x^2}{4(a^2-i\beta t)}} \cdot \sqrt{\frac{\pi}{a^2-i\beta t}}\right] \tag{2.3.8.14}$$

The expression $(a^2 - i\beta t)$ is written as

$$\begin{aligned} a^2 - i\beta t &= \sqrt{a^4 + (t\beta)^2} \cdot e^{i\varphi} \\ \tan\varphi &= -\beta t/a^2 \end{aligned} \tag{2.3.8.15}$$

The Eqs. (2.3.8.14) and (2.3.8.15) give the displacement of the beam as

$$w = \frac{a}{2} \cdot \frac{1}{\sqrt[4]{a^4+(\beta t)^2}} \cdot \left[e^{-\frac{i\varphi}{2}} \cdot e^{-\frac{x^2(a^2-i\beta t)}{4[a^4+(\beta t)^2]}} + e^{\frac{i\varphi}{2}} \cdot e^{-\frac{x^2(a^2+i\beta t)}{4[a^4+(\beta t)^2]}}\right]$$

This expression is simplified to

$$w(x,t) = \frac{e^{-\frac{a^2x^2}{4[a^4+(\beta t)^2]}}}{\sqrt[4]{1+\frac{(\beta t)^2}{a^4}}} \cdot \cos\left[\frac{\beta t x^2}{4(a^4+\beta^2 t^2)} - \frac{\varphi}{2}\right]$$

with $\varphi = \arctan\left(\beta t/a^2\right)$ and $\beta = \sqrt{\dfrac{D'}{m'}}$.

2.3.9 Example 3.9

Case 1:

A simple bending wave is propagating along the beam. The displacement $w(x, t)$ is

$$w = A \cdot e^{i(\omega t - \kappa x)} \tag{2.3.9.1}$$

According to Eq. (3.91), the energy flow in the beam is

$$\bar{\Pi} = \omega D' \kappa^3 |A|^2 \tag{2.3.9.2}$$

The time average of the velocity squared is

$$\bar{v}^2 = \frac{1}{2}\omega^2 |w|^2 = \frac{\omega^2 |A|^2}{2} \tag{2.3.9.3}$$

The expressions (2.3.9.2) and (2.3.9.3) give

$$\bar{\Pi} = \frac{2}{\omega} D' \kappa^3 \bar{v}^2 \tag{2.3.9.4}$$

The energy flow is correctly measured by means of one accelerometer if the evanescent and reflected waves can be neglected.

Case 2:

The wave field is composed of a propagating wave and an evanescent wave and given by

$$w = A \cdot e^{i\omega t} \cdot (e^{-i\kappa x} - i \cdot e^{-\kappa x}) \tag{2.3.9.5}$$

The energy flow is

$$\bar{\Pi}_1 = \omega D' \kappa^3 |A|^2 \tag{2.3.9.6}$$

The time average of the velocity squared is

$$\begin{aligned} \bar{v}^2 &= \frac{\omega^2 |A|^2}{2} \cdot \left\{1 + e^{-2\kappa x} + 2\mathrm{Re} \cdot [i \cdot \cos(\kappa x) - i \cdot [i \cdot \sin(\kappa x)]]\right\} \\ &= \frac{|A|^2}{2}\omega^2 \cdot \left\{1 + e^{-2\kappa x} + 2\sin(\kappa x) \cdot e^{-\kappa x}\right\} \end{aligned} \tag{2.3.9.7}$$

The measured energy flow based on the measured velocity by means of just one accelerometer is

$$\bar{\Pi}_2 = \frac{2}{\omega} D' \kappa^3 \bar{v}^2 \tag{2.3.9.8}$$

where $\bar{v}^2$ is defined in Eq. (2.3.9.7). The ratio between the true and the measured power flow is thus

$$\frac{\bar{\Pi}_1}{\bar{\Pi}_2} = \frac{1}{1 + e^{-2\kappa x} + 2\sin(\kappa x) \cdot e^{-\kappa x}} \tag{2.3.9.9}$$

Case 3:

A propagating and reflected wave are given by

$$w = A \cdot e^{i\omega t} \cdot (e^{-i\kappa x} + X \cdot e^{i\kappa x}) \tag{2.3.9.10}$$

The energy flow in positive direction of the beam is

$$\bar{\Pi}_1 = \omega D' \kappa^3 \left|A\right|^2 \tag{2.3.9.11}$$

The velocity squared is

$$\bar{v}^2 = \omega^2 \left|A^2\right| [1 + 2\mathrm{Re}(X \cdot e^{2i\kappa x}) + \left|X^2\right|] \tag{2.3.9.12}$$

The ratio between the energy flow in the positive direction and the energy flow measured by just one accelerometer is

$$\frac{\bar{\Pi}_1}{\bar{\Pi}_2} = \frac{1}{1 + 2\mathrm{Re}(X \cdot e^{2i\kappa x}) + \left|X^2\right|} \tag{2.3.9.13}$$

2.3.10 Example 3.10

The bending moment M'_{xy} is defined by Eq. (3.112) as

$$M'_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z \mathrm{d}z \tag{2.3.10.1}$$

where, according to Eqs. (3.127) and (3.129)

$$\tau_{xy} = G_{xy}\gamma_{xy} \text{ and } G_{xy} \approx \frac{\sqrt{E_x E_y}}{2\left(1 + \sqrt{\nu_x \nu_y}\right)} \tag{2.3.10.2}$$

The shear γ_{xy} is defined in Eq. (3.111) as

$$\gamma_{xy} = -2z \cdot \frac{\partial^2 w}{\partial x \partial y} \tag{2.3.10.3}$$

Eqs. (2.3.10.1) through (2.3.10.3) give

$$M'_{xy} = -\int_{-h/2}^{h/2} \frac{\sqrt{E_x E_y}}{(1+\sqrt{\nu_x \nu_y})} \cdot z^2 \frac{\partial^2 w}{\partial x \partial y} = -\frac{h^3 \sqrt{E_x E_y}}{12(1+\sqrt{\nu_x \nu_y})} \tag{2.3.10.4}$$

The bending stifnesses D_x and D_y are defined in Eq. (3.131). This equation is rewritten as

$$E_x h^3/12 = D_x(1-\nu_x \nu_y) \text{ and } E_y h^3/12 = D_y(1-\nu_x \nu_y) \tag{2.3.10.5}$$

Equations (2.3.10.4) and (2.3.10.5) give

$$M'_{xy} = -\frac{\sqrt{D_x D_y}(1-\nu_x \nu_y)}{(1+\sqrt{\nu_x \nu_y})} \frac{\partial^2 w}{\partial x \partial y} = -\sqrt{D_x D_y}(1-\sqrt{\nu_x \nu_y})\frac{\partial^2 w}{\partial x \partial y} \tag{2.3.10.6}$$

2.3.11 Example 3.11

The propagation of L-waves in the beam cause the displacement ξ along the axis of the beam which is oriented along the x-axis of a coordinate system. The displacement is defined as

$$\xi(x,t) = A \cdot \exp\left[i(\omega t - k_l x)\right]; \quad k_l = \omega \sqrt{\frac{\rho}{E}} \tag{2.3.11.1}$$

where k_l is the wavenumber for L-waves. The time average of the intensity is according to Eq. (3.58) and using (2.3.11.1) obtained as

$$\bar{I}_x = -\frac{1}{2}\mathrm{Re}\left\{E\frac{\partial \xi}{\partial x} \cdot \frac{\partial \xi^*}{\partial t}\right\} = -\frac{1}{2}\mathrm{Re}\left\{EA(-ik_l)(-i\omega)A^*\right\} = \frac{\omega k_l E_0 |A|^2}{2} \tag{2.3.11.2}$$

The time average of the energy flow $\bar{\Pi}_x$ in the beam with the cross-sectional area S is

$$\bar{\Pi}_x = S\bar{I}_x = SE_0 k_l \omega/2 = c_l(SE_0 k_l^2/2) = c_l S\omega^2 \rho/2; \quad c_l = \omega/k_l \tag{2.3.11.3}$$

where c_l is the speed of propagation of longitudinal waves.

The time average of the total energy $\bar{\mathcal{E}}_l$ per unit length of the beam is according to Eq. (3.58) and using (2.3.11.1) written as

$$\begin{aligned} \bar{\mathcal{E}}_l = \bar{\mathcal{T}}_l + \bar{\mathcal{U}}_l &= \frac{S}{2}\left[\frac{\rho}{2}\left|\frac{\partial \xi}{\partial t}\right|^2 + \frac{E}{2}\left|\frac{\partial \xi}{\partial x}\right|^2\right] \\ &= \frac{S}{4}\left[\rho\omega^2 |A|^2 + Ek_l^2 |A|^2\right] = \frac{S}{2}\rho\omega^2 |A|^2 \end{aligned} \tag{2.3.11.4}$$

The Eqs. (2.3.11.3) and (2.3.11.4) give

$$\bar{\Pi}_x = c_l \cdot \bar{\mathcal{E}}_l \tag{2.3.11.5}$$

2.3.12 Example 3.12

The total energy $\mathcal{E}_v$ per unit volume is

$$\mathcal{E}_v = \frac{E_x}{2}\left(\frac{\partial\xi}{\partial x}\right)^2 + \frac{\rho}{2}\left(\frac{\partial\xi}{\partial t}\right)^2 \tag{2.3.12.1}$$

Thus

$$\frac{\partial}{\partial t}\mathcal{E}_v = E_x\frac{\partial\xi}{\partial x}\left(\frac{\partial^2\xi}{\partial x\partial t}\right) + \rho\frac{\partial\xi}{\partial t}\left(\frac{\partial^2\xi}{\partial t^2}\right) \tag{2.3.12.2}$$

The displacement ξ should satisfy the wave equation for L-waves or

$$E_x\frac{\partial^2\xi}{\partial x^2} - \rho\frac{\partial^2\xi}{\partial t^2} = 0 \tag{2.3.12.3}$$

Equations (2.3.12.2) and (2.3.12.3) give

$$\frac{\partial}{\partial t}\mathcal{E}_v = E_x\frac{\partial\xi}{\partial x}\left(\frac{\partial^2\xi}{\partial x\partial t}\right) + E_x\frac{\partial\xi}{\partial t}\left(\frac{\partial^2\xi}{\partial x^2}\right) = E_x\frac{\partial}{\partial x}\left[\frac{\partial\xi}{\partial x}\cdot\frac{\partial\xi}{\partial t}\right] \tag{2.3.12.4}$$

According to definition

$$\int \mathrm{d}V\frac{\partial\mathcal{E}_v}{\partial t} + \int \mathrm{d}y\mathrm{d}z I_x = 0 \tag{2.3.12.5}$$

Equation (2.3.12.4) inserted in the first integral of (2.3.12.5) yields

$$\int \mathrm{d}V\frac{\partial\mathcal{E}_v}{\partial t} = \int \mathrm{d}x\mathrm{d}y\mathrm{d}z E_x\frac{\partial}{\partial x}\left[\frac{\partial\xi}{\partial x}\cdot\frac{\partial\xi}{\partial t}\right] = \int \mathrm{d}S E_x\left[\frac{\partial\xi}{\partial x}\cdot\frac{\partial\xi}{\partial t}\right] \tag{2.3.12.6}$$

Thus, Eqs. (2.3.12.5) and (2.3.12.6) give

$$I_x = -E_x\left[\frac{\partial\xi}{\partial x}\cdot\frac{\partial\xi}{\partial t}\right] = -\sigma_x\cdot\frac{\partial\xi}{\partial t}$$

which is the intensity of a plane L-wave propagating in the x-direction of a coordinate system.

2.4 Chapter 4

2.4.1 Example 4.1

The incident T-wave is reflected as T- and L-waves as discussed in Sect. 4.3. The plate is oriented in the x-y-plane. Only waves propagating in this plane are considered. Thus the vector potential governing the T-waves is written $\boldsymbol{\psi} = (0, 0, \psi)$. The incident and reflected waves are shown in Fig. 2.8.

Assume

$$\begin{aligned}\psi =& \exp\{i(\omega t - k_t \cdot \cos\beta \cdot x - k_t \cdot \sin\beta \cdot y)\}\\ &+ B \cdot \exp\{i(\omega t + k_t \cdot \cos\beta \cdot x - k_t \cdot \sin\beta \cdot y)\}\\ \phi =& C \cdot \exp\{i(\omega t + k_l \cdot \cos\alpha \cdot x - k_l \cdot \sin\alpha \cdot y)\}\end{aligned} \tag{2.4.1.1}$$

The wavenumbers for T- and L-waves are k_t and k_l respectively. The angle of incidence for the T-wave is β. The direction of the induced L-wave is given by α. The amplitude of the incident wave is unity. The unknown amplitudes are B and C.

The displacements of the L- and T-waves are according to Eq. (4.24)

$$\xi = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}; \quad \eta = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x} \tag{2.4.1.2}$$

For an infinitely stiff edge, the displacements in both the x- and y-directions are equal to zero. Thus, the boundary conditions are

$$\xi = \eta = 0 \text{ for } x = 0 \tag{2.4.1.3}$$

The boundary conditions in combination with Eqs. (2.4.1.1) and (2.4.1.2) give

$$\begin{cases} k_l \cdot \cos\alpha \cdot C - k_t \cdot \sin\beta - Bk_t \cdot \sin\beta = 0 \\ -k_l \cdot \sin\alpha \cdot C + k_t \cdot \cos\beta - Bk_t \cdot \cos\beta = 0 \end{cases} \tag{2.4.1.4}$$

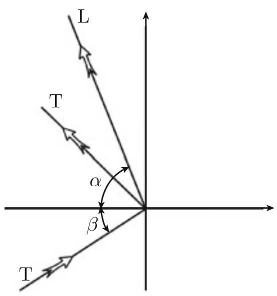


Fig. 2.8 A T-wave is incident on an infinitely stiff edge and reflected as T- and L-waves

The solutions are

$$B = \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)}; \quad C = \frac{2 \sin\alpha \cos\beta}{\cos(\alpha - \beta)} \tag{2.4.1.5}$$

The amplitude of the induced L-wave is $\sqrt{(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2} = k_l C$. The amplitude of the reflected T-wave is $k_t B$. The ratio Γ between the amplitudes of the reflected L-wave and the reflected T wave is obtained from Eq. (2.4.1.5) as

$$\Gamma = \frac{k_l \cdot C}{k_t \cdot B} = \frac{2k_l \sin\alpha \cos\beta}{k_t \cos(\alpha + \beta)} = \frac{2 \sin\beta \cos\beta}{\cos(\alpha + \beta)} \tag{2.4.1.6}$$

The angle α is given by Eq. (4.21) as

$$\sin\alpha = (c_l/c_t) \sin\beta \tag{2.4.1.7}$$

Thus

$$\cos\alpha = \sqrt{\left[1 - (c_l/c_t)^2 \sin^2\beta\right]} \tag{2.4.1.8}$$

The Eqs. (2.4.1.6)–(2.4.1.8) give

$$\Gamma = \frac{\sin(2\beta)}{\cos\beta \left[1 - (c_l/c_t)^2 \sin^2\beta\right]^{1/2} - (c_l/c_t) \sin^2\beta} \tag{2.4.1.9}$$

For $(c_l/c_t) \sin\beta > 1$ $\cos\alpha$, Eq. (2.4.1.8), is imaginary. The reflected L-wave as defined in Eq. (2.4.1.1) is consequently nonpropagating.

2.4.2 Example 4.2

An L-wave is incident on the junction $x = 0$ as shown in Fig. 2.9. L- and T-waves are reflected at the junction in plate 1. L- and T-waves are also transmitted to plate 2. The resulting wave fields are

Plate 1

$$\begin{aligned} \phi_1 = {} & \exp\{i(\omega t - k_l \cdot \cos\alpha \cdot x - k_l \cdot \sin\alpha \cdot y\} \\ & + R \cdot \exp\{i(\omega t + k_l \cdot \cos\alpha \cdot x - k_l \cdot \sin\alpha \cdot y)\} \end{aligned} \tag{2.4.2.1}$$

$$\psi_1 = Z \cdot \exp\{i(\omega t + k_t \cdot \cos\beta \cdot x - k_t \cdot \sin\beta \cdot y)\} \tag{2.4.2.2}$$

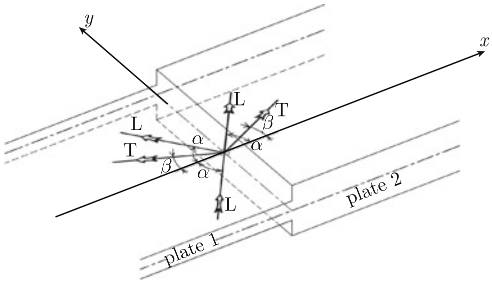


Fig. 2.9 An L-wave incident on a discontinuity and reflected and transmitted as L- and T-waves

Plate 2

$$\phi_2 = T \cdot \exp\{i(\omega t - k_l \cdot \cos\alpha \cdot x - k_l \cdot \sin\alpha \cdot y)\} \tag{2.4.2.3}$$

$$\psi_2 = W \cdot \exp\{i(\omega t - k_t \cdot \cos\beta \cdot x - k_t \cdot \sin\beta \cdot y)\} \tag{2.4.2.4}$$

The wavenumbers for the L- and T-waves are k_l and k_t respectively. Due to the boundary conditions at $x = 0$ and Eq. (4.21) it follows that

$$\sin\alpha = (c_l/c_t)\sin\beta \tag{2.4.2.5}$$

There are four unknown amplitudes $R, Z, T,$ and W in the Eqs. (2.4.2.1) and (2.4.2.2). Thus four boundary conditions are required. Based on the expressions defining the wavefields the displacements and stresses are obtained from Eqs. (3.6), (4.24), and (4.46) as

Displacements

$$\xi = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}; \quad \eta = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x} \tag{2.4.2.6}$$

Stresses

$$\sigma_x = \frac{E}{1-\nu^2} \cdot \left\{\frac{\partial\xi}{\partial x} + \nu\frac{\partial\eta}{\partial y}\right\} = \frac{E}{1-\nu^2} \cdot \left\{\frac{\partial^2\phi}{\partial x^2} + \nu\frac{\partial^2\phi}{\partial y^2} + (1-\nu) \cdot \frac{\partial^2\psi}{\partial x\partial y}\right\} \tag{2.4.2.7}$$

$$\tau_{xy} = G \cdot \left\{\frac{\partial\xi}{\partial y} + \frac{\partial\eta}{\partial x}\right\} = \frac{E}{2(1+\nu)} \cdot \left\{\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial^2\phi}{\partial x\partial y}\right\} \tag{2.4.2.8}$$

At the boundary the displacement in plate 1 must be equal to the displacement in plate 2 along the common junction in both the x- and y-directions. The resulting forces along the junction must also be equal in the x- and y-directions. Thus

Boundary conditions at junction $x = 0$

$$\xi_1 = \xi_2; \tag{2.4.2.9}$$

$$\eta_1 = \eta_2 \tag{2.4.2.10}$$

$$(\sigma_x h)_1 = (\sigma_x H)_2\,; \tag{2.4.2.11}$$

$$\left(\tau_{xy} h\right)_1 = \left(\tau_{xy} H\right)_2\,; \tag{2.4.2.12}$$

The unknown parameters are solved from these boundary conditions.

The incident L-waves in plate 1 are according to Eq. (2.4.2.1) given by

$$\phi_{\text{in}} = \exp\left\{i\left(\omega t - k_l \cdot \cos\alpha \cdot x - k_l \cdot \sin\alpha \cdot y\right)\right\} \tag{2.4.2.13}$$

The resulting incident intensity on the junction in plate 1 is from (3.19)

$$\begin{aligned}
\left(\bar{I}_x\right)_{\text{in}} &= \frac{1}{2}\text{Re}\left\{-\sigma_x \cdot \left(\dot{\xi}\right)^* - \tau_{xy} \cdot (\dot{\eta})^*\right\} \\
&= \frac{1}{2}\text{Re}\left\{\frac{E \cdot |\phi|^2\, k_l^3 \omega}{1-\nu^2} \cdot \cos\alpha(\cos^2\alpha + \nu \cdot \sin^2\alpha) + \frac{E \cdot |\phi|^2\, k_l^3 \omega}{1+\nu} \sin^2\alpha\cos\alpha\right\} \\
&= \frac{1}{2}\text{Re}\left\{\frac{E\omega k_l^3\, |\phi|^2}{1-\nu^2} \cdot \cos\alpha\right\} = \frac{1}{2} \cdot \frac{E\omega k_l^3}{1-\nu^2} \cdot \cos\alpha
\end{aligned} \tag{2.4.2.14}$$

The incident energy flow per unit length is according to Eq. (3.57) and Eq. (2.4.2.14) equal to

$$\left(\bar{\Pi}_x\right)_{\text{in}} = h \cdot \left(\bar{I}_x\right)_{\text{in}} = \frac{1}{2} h \cdot \frac{E\omega k_l^3}{1-\nu^2} \cdot \cos\alpha \tag{2.4.2.15}$$

In a similar way, the transmitted energy flow is

$$\left(\bar{\Pi}_x\right)_{\text{trans}} = H\,\left(\bar{I}_x\right)_{\text{trans}} = \frac{H}{2} \cdot \left(\frac{E\omega k_l^3}{1-\nu^2} \cdot |T|^2 \cdot \cos\alpha + G\omega k_t^3 \cos\beta \cdot |W|^2\right) \tag{2.4.2.16}$$

The ratio between the incident and transmitted energy flow is $(\bar{\Pi}_x)_{\text{in}}/(\bar{\Pi}_x)_{\text{trans}}$.

2.4.3 *Example 4.3*

Equation (4.51) is given as

$$k_x^4 - 2k_x^2 k_0^2 (2 + \nu - \nu^2) - k_0^2 \cdot \frac{(1-\nu^2)12}{h^2} - k_0^4 \cdot C = 0 \tag{2.4.3.1}$$

where

$$k_0^2 = \omega^2 \rho / E \text{ and } C = (1+\nu)^2 (5 - 4\nu)/2 \tag{2.4.3.2}$$

For $h \to 0$ the solution k_x to Eq. (2.4.3.1) is given by the wavenumber for a thin plate under flexure. Thus $k_x \to \left[k_0^2 \cdot 12(1-\nu^2)/h^2\right]^{1/4}$.

If h is sufficiently small, terms which do not include $1/h$ or k_x can be neglected and Eq. (2.4.3.1) is written

$$k_x^4 - 2k_x^2k_0^2(2+\nu-\nu^2) - k_0^2\frac{(1-\nu^2)12}{h^2} \approx 0$$

The solutions to this equation are

$$\begin{aligned} k_x^2 &= k_0^2(2+\nu-\nu^2) \pm \left\{\frac{k_0^2(1-\nu^2)12}{h^2}\right\}^{1/2} + o(h) \\ &\approx k_0^2(2+\nu-\nu^2) \pm \left\{\frac{k_0^2(1-\nu^2)12}{h^2}\right\}^{1/2} \end{aligned} \tag{2.4.3.3}$$

The solutions to the last part of Eq. (2.4.3.3) read

$$k_{x1} = \pm\left\{\left[\frac{k_0^2(1-\nu^2)12}{h^2}\right]^{1/4} + \frac{k_0^2(2+\nu-\nu^2)}{2(k_0/h)^{1/2}\left[(1-\nu^2)12\right]^{1/4}}\right\} \tag{2.4.3.4}$$

$$k_{x2} = \pm i\left\{\left[\frac{k_0^2(1-\nu^2)12}{h^2}\right]^{1/4} - \frac{k_0^2(2+\nu-\nu^2)}{2(k_0/h)^{1/2}\left[(1-\nu^2)12\right]^{1/4}}\right\} \tag{2.4.3.5}$$

The time average of the energy flow for a wave $w = A \cdot \exp\left[i(\omega t - k_{x1}x)\right]$ is

$$\begin{aligned} \bar{\Pi}'_x &= \frac{Eh^3}{12(1-\nu^2)}\omega\,(k_{x1})^3 \cdot |A|^2 \\ &= \left\{\left[\frac{k_0^2(1-\nu^2)12}{h^2}\right]^{1/4} + \frac{k_0^2(2+\nu-\nu^2)}{2(k_0/h)^{1/2}\left[(1-\nu^2)12\right]^{1/4}}\right\}^3 \frac{Eh^3}{12(1-\nu^2)}\omega \cdot |A|^2 \end{aligned}$$

2.4.4 Example 4.4

Equation (4.49) reads

$$\tanh\,(\beta h/2)\left[k_x^2 - k_0^2\,(1+\nu)\right]^2 = k_x^2\alpha\beta\tanh\,(\alpha h/2) \tag{2.4.4.1}$$

The solution k_x to this equation is the wavenumber corresponding to the antiphase motion of the plate as illustrated in Fig. 4.4b. This is also the wavenumber for the quasi longitudinal waves illustrated in Fig. 3.11. A solution to Eq. (2.4.4.1) is for small h obtained by expanding Eq. (2.4.4.1) in a Taylor series and including first-order terms only. For $h \to 0$

$$\tanh(\beta h/2) \to \beta h; \quad \tanh(\alpha h/2) \to \alpha h \tag{2.4.4.2}$$

The Eqs. (2.4.4.1) and (2.4.4.2) give

$$\begin{aligned} &\beta h\left[k_x^2 - k_0^2(1+\nu)\right]^2 = k_x^2\alpha^2\beta h \Rightarrow \\ &k_x^4 - 2k_x^2k_0^2(1+\nu) + k_0^4(1+\nu)^2 = k_x^2\alpha^2 \end{aligned} \tag{2.4.4.3}$$

where according to Eq. (4.38)

$$\alpha^2 = k_x^2 - k_l^2 = k_x^2 - k_0^2(1+\nu)(1-2\nu)/(1-\nu) \tag{2.4.4.4}$$

The solution to Eq. (2.4.4.3) is

$$k_x^2 = k_0^2\left(1-\nu^2\right); \quad k_0 = \omega\sqrt{\frac{\rho}{E}}$$

$$k_x = \omega\sqrt{\frac{\rho\left(1-\nu^2\right)}{E}} \tag{2.4.4.5}$$

This is equal to the wavenumber for quasi-longitudinal waves propagating in a thin plate.

2.4.5 *Example 4.5*

In the low-frequency region, the entire structure is bending like a thin plate. The neutral axis is in the symmetry plane of the beam (Fig. 2.10).

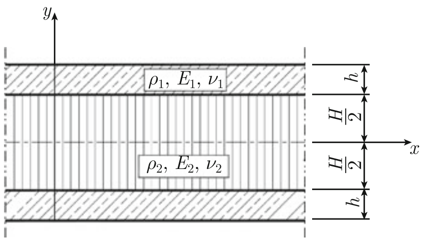


Fig. 2.10 Cross section of a sandwich beam

In the low-frequency region the bending stiffness of the entire structure is according to Eq. (4.71)

$$D_0 = \int \mathrm{d}y \cdot y^2 E(y) = \int_{-H/2}^{H/2} \mathrm{d}y \cdot y^2 \cdot \frac{E_2}{1-\nu_2^2} + 2 \cdot \int_{H/2}^{H/2+h} \mathrm{d}y \cdot y^2 \cdot \frac{E_1}{1-\nu_1^2}$$
$$= \frac{H^3 E_2}{12(1-\nu^2)} + \frac{E_1}{1-\nu^2}\left\{\frac{H^2}{2}h + Hh^2 + \frac{2}{3}h^3\right\}$$
$$\approx \frac{H^3 E_2}{12(1-\nu^2)} + \frac{E_1 h H^2}{(1-\nu_1^2)\cdot 2} \quad \text{for } H \gg h \tag{2.4.5.1}$$

The mass per unit area of the plate is

$$\mu_0 = 2\rho_1 h + \rho_2 H \tag{2.4.5.2}$$

The wavenumber for bending waves is according to Eq. (4.70)

$$\kappa = \left(\frac{\mu_0 \cdot \omega^2}{D_0}\right)^{1/4} \quad \text{for } f \to 0 \tag{2.4.5.3}$$

where D_0 and μ are defined in Eqs. (2.4.5.1) and (2.4.5.2).

In the high-frequency range the laminates vibrate independently of the core. The wavenumber corresponds to the wavenumber governing the bending of one laminate

$$D_\infty = \frac{E_1 h^3}{12(1-\nu_1^2)}; \quad \mu_\infty = \rho_1 h$$

Thus the high frequency limit, $f \to \infty$, is

$$\kappa = \left(\frac{\rho_1 \omega^2 \cdot 12(1-\nu_1^2)}{E_1 h^2}\right)^{1/4}$$

The same result is obtained by setting $\mu = 2\rho_1 h$ and $D = 2D_\infty$.

2.4.6 Example 4.6

The lateral displacement $w(x,t)$ due to the bending of the plate is

$$w(x,t) = \eta_0 \cdot e^{i(\omega t - \kappa x)} \tag{2.4.6.1}$$

The normal stress in the plate is according to Eq. (4.56) equal to

$$\sigma_x = -y \frac{E}{(1-\nu^2)} \frac{\partial^2 w}{\partial x^2} \tag{2.4.6.2}$$

The resulting bending moment is

$$M_y = \int_{-h/2}^{h/2} y\sigma_x \mathrm{d}y = -\frac{Eh^3}{12(1-\nu^2)} \frac{\partial^2 w}{\partial x^2} = -D\frac{\partial^2 w}{\partial x^2} \tag{2.4.6.3}$$

The shear stress is according to Eq. (4.56)

$$\tau_{xy} = -\left(\frac{h^2}{4} - y^2\right) \frac{Eh^3}{2(1-\nu^2)} \frac{\partial^3 w}{\partial x^3} \tag{2.4.6.4}$$

The resulting shear force per unit width of the plate is

$$T_x = \int_{-h/2}^{h/2} \tau_{yx} \mathrm{d}y = -\frac{Eh^3}{12\left(1-\nu^2\right)} \frac{\partial^3 w}{\partial x^3} = -D\frac{\partial^3 w}{\partial x^3} \tag{2.4.6.5}$$

2.4.7 Example 4.7

A flexural plane wave is propagating along the x-axis. The displacement perpendicular to the plate is

$$\eta(x,t) = \eta_0 \cdot \exp\{i(\omega t - \kappa x)\} \tag{2.4.7.1}$$

The displacemt in the x-direction is ξ and in the y-direction, perpendicular to the plate, η. There is no displacement along the z-axis. The time average of the intensity I_x in structure is obtained from Eq. (3.19). The result is

$$\bar{I}_x = \frac{1}{2}\mathrm{Re}\left\{-\sigma_x \dot{\xi}^* - \tau_{xy} \cdot \dot{\eta}^*\right\}; \quad \varsigma = 0 \tag{2.4.7.2}$$

The normal stress in the plate is according to Eq. (4.56) given by

$$\sigma_x = -y \cdot \frac{E}{(1-\nu^2)} \cdot \frac{\partial^2 \eta}{\partial x^2} = y\kappa^2 \cdot \frac{E}{1-\nu^2} \cdot \eta \tag{2.4.7.3}$$

The displacement ξ is obtained from Eq. (3.109) as

$$\xi = -y \cdot \frac{\partial \eta}{\partial x} = i\kappa y \cdot \eta; \quad \dot{\xi}^* = -i\kappa y \cdot \eta^* \tag{2.4.7.4}$$

The shear stress is given by Eq. (4.56) as

$$\tau_{xy} = -\left(\frac{h^2}{4} - y^2\right) \cdot \frac{E}{2(1-\nu^2)} \cdot \frac{\partial^3 \eta}{\partial x^3} = -i\kappa^3 \cdot \left(\frac{h^2}{4} - y^2\right) \cdot \frac{E}{2(1-\nu^2)} \cdot \eta \tag{2.4.7.5}$$

From Eq. (2.4.7.1) it follows that

$$\dot{\eta}^* = -i\omega\eta^* \tag{2.4.7.6}$$

Inserting the expressions (2.4.7.3)–(2.4.7.6) in Eq. (2.4.7.2) gives

$$\begin{aligned} \bar{I}_x &= \frac{1}{2}\mathrm{Re}\left\{\frac{\omega\kappa^3 y^2 E}{1-\nu^2} \cdot |\eta_0|^2 - \frac{\omega\kappa^3 E}{2(1-\nu^2)} \cdot \left(\frac{h^2}{4} - y^2\right) \cdot |\eta_0|^2\right\} \\ &= \frac{1}{2}|\eta_0|^2 \cdot \frac{E\omega\kappa^3}{1-\nu^2} \cdot \left\{y^2 - \frac{1}{2}\left(\frac{h^2}{4} - y^2\right)\right\} \end{aligned} \tag{2.4.7.7}$$

The time average of the energy flow per unit length is using Eq. (2.4.7.7)

$$\bar{\Pi}_x = \int_{-h/2}^{h/2} \bar{I}_x \mathrm{d}y = |\eta_0|^2 \cdot \omega\kappa^3 \cdot \frac{Eh^3}{12(1-\nu^2)} = |\eta_0|^2 \cdot D\omega\kappa^3 \tag{2.4.7.8}$$

This is the same result as given by Eq. (3.91) by exchanging the amplitude from A to η_0.

2.4.8 Example 4.8

The bending of a plate and the corresponding displacements, stresses etc. are given by (4.44) as:

$$\phi = \exp\{i(\omega t - \kappa x)\} B \sinh(\alpha y) \tag{2.4.8.1}$$

$$\psi = \exp\{i(\omega t - \kappa x)\} C \cosh(\beta y) \tag{2.4.8.2}$$

where

$$\alpha = \left\{\kappa^2 - k_0^2(1+\nu)(1-2\nu)/(1-\nu)\right\}^{1/2} \tag{2.4.8.3}$$

$$\beta = \left\{\kappa^2 - k_0^2 2(1+\nu)\right\}^{1/2} \tag{2.4.8.4}$$

The parameters B and C and the wavenumber κ should satisfy (4.46). Thus

$$B\sinh(\alpha h/2)\left\{\kappa^2 - k_0^2(1+\nu)\right\} + i\kappa\beta C\sinh(\beta h/2) = 0 \tag{2.4.8.5}$$

$$-2i\kappa\alpha B\cosh(\alpha h/2)+2C\cosh(\beta h/2)\left\{\kappa^2-k_0^2(1+\nu)\right\}=0 \tag{2.4.8.6}$$

The displacements and stresses are

$$\xi=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y};\quad \psi=\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x} \tag{2.4.8.7}$$

$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\frac{\partial^2\phi}{\partial x^2}+(1-\nu)\frac{\partial^2\phi}{\partial y^2}-(1-2\nu)\frac{\partial^2\psi}{\partial x\partial y}\right\} \tag{2.4.8.8}$$

$$\tau_{xy}=\frac{E}{2(1+\nu)}\left\{2\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right\} \tag{2.4.8.9}$$

Equations (2.4.8.1), (2.4.8.2) plus (2.4.8.7) yield

$$\eta=\exp\left\{i(\omega t-\kappa x)\right\}\left\{B\alpha\cosh(\alpha y)+i\kappa C\cosh(\beta y)\right\}$$

The displacement along the centerline $y=0$ is

$$\eta=\exp\left\{i(\omega t-\kappa x)\right\}\left\{B\alpha+i\kappa C\right\} \tag{2.4.8.10}$$

For $h\to 0$ Eq. (2.4.8.5) yields ($\sinh x\approx x$)

$$B\alpha\left\{\kappa^2-k_0^2(1+\nu)\right\}+i\kappa\beta^2C=0$$

$$i\kappa C=-\frac{B\alpha\left[\kappa^2-k_0^2(1+\nu)\right]}{\beta^2} \tag{2.4.8.11}$$

The result (2.4.8.11) inserted in the expression (2.4.8.10) gives

$$\begin{aligned}\eta&=\exp\left\{i(\omega t-\kappa x)\right\}\cdot B\alpha\left\{1-\frac{\kappa^2-k_0^2(1+\nu)}{\kappa^2-2k_0^2(1+\nu)}\right\}\\&=\exp\left\{i(\omega t-\kappa x)\right\}\cdot B\alpha\cdot\left[-\frac{k_0^2(1+\nu)}{\kappa^2-2k_0^2(1+\nu)}\right]\end{aligned} \tag{2.4.8.12}$$

The displacement along the x-axis is

$$\xi=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}=\exp\left\{i(\omega t-\kappa x)\right\}\cdot\left\{-i\kappa B\sinh(\alpha\cdot y)+\beta C\sinh(\beta\cdot y)\right\}$$

for $y\ll 1$, $\sinh(\alpha\cdot y)=\alpha\cdot y$ etc. $\Rightarrow$

$$\xi = \exp\{i(\omega t - \kappa x)\} \cdot y \cdot \{-i\kappa\} \left\{\alpha B + \frac{iB^2 C}{\kappa}\right\} \tag{2.4.8.13}$$

This result in combination with Eq. (2.4.8.11) gives

$$\xi = \exp\{i(\omega t - \kappa x)\} \cdot y \cdot (-i\kappa\alpha B) \cdot \left\{\frac{k_0^2(1+\nu)}{\kappa^2}\right\} \tag{2.4.8.14}$$

Equations (2.4.8.12) and (2.4.8.14) yield for $\kappa > k_0$

$$\xi = -y \cdot \{-i\kappa\}\dot{\eta} = -y \cdot \frac{\partial \eta}{\partial x} \tag{2.4.8.15}$$

The normal stress σ_x is obtained from Eqs. (2.4.8.1), (2.4.8.2), and (2.4.8.8) as

$$\begin{aligned}\sigma_x =& \frac{E}{(1+\nu)(1-2\nu)} \exp\{i(\omega t - \kappa x)\} \\ &\cdot \left\{\left[\nu\alpha^2 B - \kappa^2 B(1-\nu)\right] \sinh(\alpha y) + (1-2\nu)(-i\kappa)\beta C \sinh(\beta y)\right\}\end{aligned} \tag{2.4.8.16}$$

Since $h \ll 1$ it follows that $\sinh(\alpha y) \approx \alpha y$ etc. By using these approximations in Eq. (2.4.8.16) the result reads

$$\begin{aligned}\sigma_x =& \frac{E}{(1+\nu)(1-2\nu)} \exp\{i(\omega t - \kappa x)\} \\ &\cdot y \left\{\nu\alpha^3 B - \kappa^2 B(1-\nu)\alpha - (1-2\nu)\, i\kappa\beta^2 C\right\}\end{aligned} \tag{2.4.8.17}$$

Equations (2.4.8.11) and (2.4.8.17) $\Rightarrow$

$$\begin{aligned}\sigma_x &= \frac{Ey\alpha\beta}{(1+\nu)(1-2\nu)} \exp\{i(\omega t - \kappa x)\} \\ &\quad \cdot \left\{\left[\nu\alpha^2 - \kappa^2(1-\nu)\right] + (1-2\nu)\left[\kappa^2 - k_0^2(1+\nu)\right]\right\} \\ &= \frac{Ey\alpha\beta}{(1+\nu)(1-2\nu)} \exp\{i(\omega t - \kappa x)\} \cdot \left\{k_0^2 \frac{(1+\nu)(1-2\nu)}{(1-\nu)}\right\} \\ &= \frac{Ey}{(1-\nu)} k_0^2 \alpha B \exp\{i(\omega t - \kappa x)\} = \frac{E}{1-\nu^2} y\kappa^2\eta\end{aligned}$$

The displacement η is defined in Eq. (2.4.8.10). For $\kappa \gg k_0$

$$\sigma_x = \frac{E}{1-\nu^2} y\kappa^2\eta = \frac{E}{1-\nu^2} y \left(-\frac{\partial^2 \eta}{\partial x^2}\right) \tag{2.4.8.18}$$

The shear stress is obtained from Eqs. (2.4.8.1), (2.4.8.2), and (2.4.8.9) as

$$\tau_{xy} = G\left\{-2i\kappa\alpha B\cosh(\alpha y) + C(\beta^2+\kappa^2)\cosh(\beta y)\right\}\cdot\exp\{i\,(\omega t-\kappa x)\} \tag{2.4.8.19}$$

$$\beta^2+\kappa^2 = 2[\kappa^2 - k_0^2(1+\nu)] \tag{2.4.8.20}$$

The parameter C is given in Eq. (2.4.8.6). Equation (2.4.8.19) is rewritten as

$$\tau_{xy} = G\,(-2i\kappa\alpha)\,B\left\{\cosh(\alpha\cdot y) - \frac{\cosh(\alpha h/2)\cosh(\beta y)}{\cosh(\beta h/2)}\right\}\cdot\exp\{i\,(\omega t-\kappa x)\} \tag{2.4.8.21}$$

For $\alpha h/2 \ll 1$ the cosh terms can be expanded in Taylor series as

$$\cosh(\alpha\cdot y) = 1 + \frac{(\alpha\cdot y)^2}{2} + \cdots$$

The expression inside the large bracket of Eq. (2.4.8.21) is using this expansion written as

$$\begin{aligned} J &= \cosh(\alpha\cdot y) - \frac{\cosh(\alpha h/2)\cosh(\beta\cdot y)}{\cosh(\beta h/2)} \\ &\approx 1 + \frac{(\alpha\cdot y)^2}{2} - \frac{\left[1+\dfrac{(\alpha\cdot h)^2}{8}\right]\cdot\left[1+\dfrac{(\beta\cdot y)^2}{2}\right]}{1+\dfrac{(\alpha\cdot h)^2}{8}} \\ &\approx \frac{(\alpha\cdot y)^2}{2} + \frac{(\beta\cdot h)^2}{8} - \frac{(\alpha\cdot h)^2}{8} - \frac{(\beta\cdot y)^2}{2} = \frac{(\alpha^2-\beta^2)}{2}\left\{y^2 - \left(\frac{h}{2}\right)^2\right\} \end{aligned} \tag{2.4.8.22}$$

Equations (2.4.8.3), (2.4.8.4), and (2.4.8.22) give

$$J \approx -\frac{k_0^2(1+\nu)}{(1-\nu)}\cdot\frac{1}{2}\left\{\left(\frac{h}{2}\right)^2 - y^2\right\} \tag{2.4.8.23}$$

The shear stress is consequently given by

$$\begin{aligned} \tau_{xy} = &-\frac{Ek_0^2(1+\nu)}{2(1-\nu^2)}\cdot\left\{\left(\frac{h}{2}\right)^2 - y^2\right\}\cdot\exp\{i\,(\omega t-\kappa x)\} \\ &\cdot\left\{-\frac{k^2(1+\nu)}{\kappa^2}\right\}\cdot\left\{-i\kappa^3\alpha B\right\} \end{aligned} \tag{2.4.8.24}$$

This expression in combination with Eq. (2.4.8.12) finally gives the shear stress inside a thin plate as

$$\tau_{xy} = -\left[\left(\frac{h}{2}\right)^2 - y^2\right] \cdot \frac{E}{2(1-\nu^2)} \cdot \frac{\partial^3 \eta}{\partial x^3} \tag{2.4.8.25}$$

2.4.9 Example 4.9

The wavenumber is according to the text given by

$$k_x = \pm\left\{\frac{1}{2}\left[k_l^2 + k_t^2/T\right] \pm \left[4\kappa^4 + (k_l^2 - k_t^2/T)^2\right]^{1/2}\right\}^{1/2} \tag{2.4.9.1}$$

The wavenumber κ for bending waves is proportional to $\sqrt{f}$ whereas k_l and k_t are proportional to f. In the high-frequency range, for k_l and $k_t \gg \kappa$, the asymptotic solutions to Eq. (2.4.8.1) are

$$k_x = \pm k_l \text{ and } k_x = \pm k_t/\sqrt{T}$$

The first solution represents an L-wave. The second solution should equal the wavenumber k_r for a Rayleigh wave defined in Eq. (4.57). The equality $k_t/T = k_r$ gives according to Eq. (4.57)

$$k_t/\sqrt{T} = X_r\omega\sqrt{\rho/E} \quad \text{where } k_t = \omega\sqrt{2(1+\nu)\rho/E}$$

These expressions give

$$T = 2(1+\nu)/X_r^2 \tag{2.4.9.2}$$

For $\nu = 0.3$ the parameter X_r is given as 1.74 in Table 4.2. The Timoshenko constant T is thus obtained as 0.86 for $\nu = 0.3$.

2.4.10 Example 4.10

According to Eq. (4.46) the normal and shear stresses are

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\frac{\partial^2\phi}{\partial x^2} + (1-\nu)\frac{\partial^2\phi}{\partial y^2} - (1-2\nu)\frac{\partial^2\psi}{\partial x \partial y}\right\} \tag{2.4.10.1}$$

$$\tau_{yx} = \frac{E}{2(1+\nu)} \left\{ 2\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right\} \tag{2.4.10.2}$$

The boundary conditions are $\sigma_y = 0$ and $\tau_{yx} = 0$ for $y = 0$. By inserting the expressions given for ϕ and ψ in Eq. (2.4.10.1), the first boundary condition gives

$$B_1 e^{\alpha h/2} \left[(k_r^2 - k_l^2)(1-\nu) - \nu k_r^2 \right] = (-ik_r)(1-2\nu)\beta C_2 e^{\beta h/2} \tag{2.4.10.3}$$

where

$$k_l^2 = k_0^2 \frac{(1+\nu)(1-2\nu)}{(1-\nu)} \text{ and } k_0^2 = \omega^2 \rho / E \tag{2.4.10.4}$$

Equation (2.4.10.3) is satisfied if

$$\frac{C_2}{B_1} = \frac{i \exp\left[h(\alpha - \beta)/2\right] \left\{ k_r^2 - k_0^2(1+\nu) \right\}}{k_r \beta} \tag{2.4.10.5}$$

The second boundary condition $\tau_{yx} = 0$ for $y = 0$ gives

$$2(-ik_r\alpha) B_1 e^{\alpha h/2} + C_2 e^{\beta h/2} (\beta^2 + k_r^2) = 0 \tag{2.4.10.6}$$

where

$$\beta^2 = k_r^2 - 2k_0^2(1+\nu)$$

The ratio C_2/B_1 is from Eq. (2.4.10.6) obtained as

$$\frac{C_2}{B_1} = \frac{\exp\left[h(\alpha - \beta)/2\right](ik_r\alpha)}{\left[k_r^2 - k_0^2(1+\nu)\right]} \tag{2.4.10.7}$$

The results (2.4.10.5) and (2.4.10.7) must be identical for the boundary conditions to be satisfied. Equality holds when

$$k_r^2 \alpha \beta = \left[k_r^2 - k_0^2(1+\nu) \right]^2 \tag{2.4.10.8}$$

The solution to Eq. (2.4.10.8) is the wavenumber for Rayleigh waves. Consequently, the potentials ϕ and ψ given in the problem satisfy the boundary conditions $\sigma_y = 0$ and $\tau_{yx} = 0$ for $y = 0$, i.e. on the surface of the semi-infinite solid.

2.4.11 Example 4.11

The potentials governing the displacement induced by Rayleigh waves are according to Eq. (4.68) given as

$$\begin{aligned}\phi &= Ae^{\alpha y}e^{i(\omega t - k_r x)} \\ \psi &= iAe^{\beta y}e^{i(\omega t - k_r x)}\left[k_r^2 - k_0^2(1-\nu^2)\right]/(k_r\beta)\end{aligned} \tag{2.4.11.1}$$

The normal and shear stresses are given by Eqs. (4.54) and (4.55) as

$$\begin{aligned}\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\frac{\partial^2\phi}{\partial y^2} + (1-\nu)\frac{\partial^2\phi}{\partial x^2} - (1-2\nu)\frac{\partial^2\psi}{\partial x\partial y}\right\} \\ \tau_{xy} &= \frac{E}{2(1+\nu)}\left\{2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right\}\end{aligned} \tag{2.4.11.2}$$

The displacements in the x- and y-directions are

$$\xi = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}; \quad \eta = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x} \tag{2.4.11.3}$$

For zero displacement in the z-direction, the time average of the intensity in the x-direction is given by

$$\bar{I}_x = \frac{1}{2}\mathrm{Re}\left\{-\sigma_x\dot{\xi} - \tau_{xy}\dot{\eta}\right\} \tag{2.4.11.4}$$

The intensity in the y-direction is

$$\bar{I}_y = \frac{1}{2}\mathrm{Re}\left\{-\sigma_y\dot{\eta}^* - \tau_{xy}\dot{\xi}^*\right\} \tag{2.4.11.5}$$

The intensity component $\bar{I}_y$ is equal to zero since there are no waves propagating along the y-axis.

2.5 Chapter 5

2.5.1 Example 5.1

Notations—see Fig. 5.17.

The junction is hinged—no bending moment transferred from beam 1 to beam 2. Only L-waves in beam 1. No force along the axis of beam 2. Thus only F-waves in beam 2.

Beam 1 Incident and reflected L-waves. Time dependence $e^{i\omega t}$.

$$\xi_1 = e^{-ikx} + A_1 e^{ikx} \tag{2.5.1.1}$$

k-wavenumber for L-waves.
Beam 2 Transmitted F-waves. Time dependence $e^{i\omega t}$.

$$w_2 = A_2 e^{-i\kappa x} + A_3 e^{\kappa y} \tag{2.5.1.2}$$

κ-wavenumber for F-waves.
Boundary conditions $(x = 0;\; y = 0)$
Displacement

$$\xi_1 + w_2 = 0 \tag{2.5.1.3}$$

Bending moments

$$M = -D' \cdot \frac{\partial^2 w_2}{\partial y^2} = 0 \tag{2.5.1.4}$$

Forces, x-direction

$$F_{1x} + F_{2x} = 0 \tag{2.5.1.5}$$

$$F_{1x} = hE \cdot \frac{\partial \xi_1}{\partial x}; \quad F_{2x} = -D' \cdot \frac{\partial^3 w_2}{\partial y^3}$$

There are no forces in the y-direction.
Boundary condition (2.5.1.3) $\Rightarrow$

$$1 + A_1 + A_2 + A_3 = 0 \tag{2.5.1.6}$$

Boundary condition (2.5.1.4) $\Rightarrow$

$$A_2 = A_3 \tag{2.5.1.7}$$

Boundary condition (2.5.1.5) $\Rightarrow$

$$-iSEk\,[1 - A_1] = iD'\kappa^3\,[A_2 + iA_3] \tag{2.5.1.8}$$

Introduce the parameters

$$\beta = \frac{D'\kappa^3}{SEk}; \quad S\text{-cross section area of beam, } S = b \cdot h$$

$$D' = \frac{b \cdot h^3 E}{12}; \quad \kappa = \left\{\frac{\rho b h \cdot \omega^2}{D'}\right\}^{1/4}$$

The incident and transmitted energy flows are

$$\bar{\Pi}_{\text{in}} = \frac{\omega EkS}{2}; \quad \bar{\Pi}_{\text{trans}} = \omega D' \kappa^3 \left|A_2\right|^2 \tag{2.5.1.9}$$

The ratio between the energy flows is

$$\frac{\bar{\Pi}_{\text{in}}}{\bar{\Pi}_{\text{trans}}} = \frac{1}{2\beta \left|A_2\right|^2} \tag{2.5.1.10}$$

The amplitude A_2 is obtained from the Eqs. (2.5.1.6)–(2.5.1.8) as;

$$A_2 = -\frac{2}{\beta + 2 + i\beta} \tag{2.5.1.11}$$

The results (2.5.1.10) and (2.5.1.11) give

$$\frac{\bar{\Pi}_{\text{in}}}{\bar{\Pi}_{\text{trans}}} = \frac{(\beta+2)^2 + \beta^2}{8\beta} \tag{2.5.1.12}$$

2.5.2 Example 5.2

Notations as in Example 5.1.
Beam 1 (incident and reflected F-waves)

$$w_1 = e^{-i\kappa x} + A_1 e^{i\kappa x} + A_2 e^{\kappa x} \tag{2.5.2.1}$$

Beam 2 (transmitted L-wave)

$$\xi_2 = A_3 e^{-iky} \tag{2.5.2.2}$$

Boundary conditions ($x = 0$; $y = 0$)

$$w_1 = \xi_2 \tag{2.5.2.3}$$

$$\frac{\partial^2 w_1}{\partial x^2} = 0 \tag{2.5.2.4}$$

$$F_{1y} = F_{2y} \tag{2.5.2.5}$$

$$F_{1y} = -D' \frac{\partial^3 w_1}{\partial x^3}; \quad F_{2y} = SE \frac{\partial \xi_2}{\partial y} \tag{2.5.2.6}$$

The incident and transmitted energy flows are

$$\bar{\Pi}_{\text{in}} = \omega D' \kappa^3; \quad \bar{\Pi}_{\text{trans}} = \frac{\omega E k S}{2} \cdot |A_3|^2 \tag{2.5.2.7}$$

The amplitude A_3 is obtained from the boundary conditions (2.5.2.3)–(2.5.2.5) as

$$A_3 = -\frac{4\beta}{2 + \beta(i + i)} \tag{2.5.2.8}$$

The results (2.5.2.7) and (2.5.2.8) give

$$\frac{\bar{\Pi}_{\text{in}}}{\bar{\Pi}_{\text{trans}}} = \frac{2\beta}{|A_3|^2} = \frac{(2+\beta)^2 + \beta^2}{8\beta}$$

The same result as for Problem 5.1.

2.5.3 Example 5.3

All plates are equal. Each plate equally excited by bending at the common junction. The same wave field is induced in every plate.

Plate 1

Incident and reflected fields. Time dependence $e^{i\omega t}$.

$$w_1 = e^{-i\kappa x} + A_1 e^{i\kappa x} + A_2 e^{\kappa x} \tag{2.5.3.1}$$

Plate *n*

Transmitted field. Time dependence $e^{i\omega t}$.

$$w_n = B_1 e^{-i\kappa y} + B_2 e^{-\kappa y} \tag{2.5.3.2}$$

No translatory motion at junction.

$$w_1 = 0 \text{ for } x = 0; \quad w_n = 0 \text{ for } y_n = 0 \tag{2.5.3.3}$$

The boundary conditions (2.5.3.3) give

$$1 + A_1 + A_2 = 0 \text{ or } A_2 = -1 - A_1 \tag{2.5.3.4}$$

$$B_1 + B_2 = 0 \text{ or } B_2 = -B_1 \tag{2.5.3.5}$$

Rotation around the junction is the same for every plate. Thus

$$\frac{\partial w_1}{\partial x} = \frac{\partial w_n}{\partial y} \tag{2.5.3.6}$$

This boundary condition in combination with the expressions (2.5.3.1) and (2.5.3.2) give

$$-1 + A_1 - iA_2 = -B_1 + iB_2$$

The results (2.5.3.3)–(2.5.3.6) give

$$A_1 = -B_1 - i \tag{2.5.3.7}$$

The forcing bending moment is equal to the sum of the reacting moments

$$M_1 = \sum_{n=2}^{N} M_n \quad \text{with } M_n = -D \cdot \partial^2 w_n / \partial x^2 \tag{2.5.3.8}$$

Equations (2.5.3.1), (2.5.3.2) and (2.5.3.8) give

$$-1 - A_1 + A_2 = (N-1)(-B_1 + B_2) \tag{2.5.3.9}$$

Use the results (2.5.3.3), (2.5.3.4), (2.5.3.7), and (2.5.3.9) $\Rightarrow$

$$B_1 = \frac{1-i}{N} \tag{2.5.3.10}$$

The incident energy flow is $\bar{\Pi}_{\text{in}} = \omega D \kappa^3$

Transmitted energy flow in beam n is

$$\bar{\Pi}_{\text{trans}} = \omega D \kappa^3 \cdot |B_1|^2 = \bar{\Pi}_{\text{in}} \cdot \frac{2}{N^2} \tag{2.5.3.11}$$

The attenuation R across the junction between beam 1 and beam n is

$$R = 10 \log(\bar{\Pi}_{\text{in}} / \bar{\Pi}_{\text{trans}}) = 10 \log(N^2/2) \tag{2.5.3.12}$$

2.5.4 Example 5.4

Assume a time dependence $\exp(i\omega t)$. Let the wavenumber for L-waves be $k = \omega\sqrt{\rho/E}$ (Fig. 2.11).

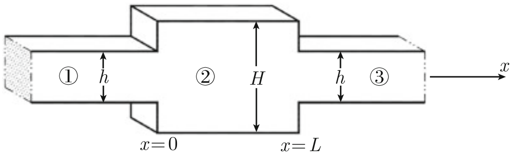


Fig. 2.11 An infinite beam with a discontinuity

Beam 1

Incident and reflected L-waves

$$\xi_1 = e^{-ikx} + R \cdot e^{ikx} \tag{2.5.4.1}$$

Beam 2

Transmitted and reflected L-waves

$$\begin{aligned} \xi_2 &= A' \cdot e^{-ikx} + B' \cdot e^{ikx} \text{ or} \\ \xi_2 &= A \sin(kx) + B \cos(kx) \end{aligned} \tag{2.5.4.2}$$

Beam 3

Transmitted L-wave

$$\xi_3 = T \cdot e^{-ik(x-L)} \tag{2.5.4.3}$$

Boundary conditions

At $x = 0$

Displacement

$$\xi_1 = \xi_2 \tag{2.5.4.4}$$

Forces

$$h \cdot \frac{\partial \xi_1}{\partial x} = H \cdot \frac{\partial \xi_2}{\partial x} \tag{2.5.4.5}$$

At $x = L$

Displacement

$$\xi_1 = \xi_2 \tag{2.5.4.6}$$

Forces

$$H \cdot \frac{\partial \xi_2}{\partial x} = h \cdot \frac{\partial \xi_3}{\partial x} \tag{2.5.4.7}$$

The four boundary conditions give the parameters R, T, A, and B. The parameter T is obtained as

$$T = [\cos\alpha + i\sin\alpha(h/H + H/h)/2]^{-1} \quad \text{where } \alpha = kL$$

The ratio between the time averages of the incident and transmitted energy flows are

$$\frac{\bar{\Pi}_{\text{in}}}{\bar{\Pi}_{\text{trans}}} = \frac{1}{|T|^2} = 1 + \left[\frac{\sin(kL)}{2}\left(\frac{h}{H} - \frac{H}{h}\right)\right]^2 \tag{2.5.4.8}$$

Whenever $\alpha = kL = n\pi$, or when the wavelength is a multiple of the length L, there is no transmission loss across the discontinuity. Maximum transmission loss is obtained for $\alpha = kL = n\pi + \pi/2$.

For $kL \ll 1 \quad \sin(kL) \approx kL$. For this particular case Eq. (2.5.4.8) is approximated by

$$\frac{\bar{\Pi}_{\text{in}}}{\bar{\Pi}_{\text{trans}}} \approx 1 + [(kL/2)\,(h/H - H/h)]^2 = 1 + \frac{\rho}{4E}\cdot\left[\frac{\omega L}{Hh}\cdot\left(H^2 - h^2\right)\right]^2 \tag{2.5.4.9}$$

2.5.5 Example 5.5

Assume a time dependence $\exp(i\omega t)$. Let the wavenumber for L-waves in beam 1 and 3 be $k_1 = \omega\sqrt{\rho_1/E_1}$. The wavenumber for L-waves in beam 2 is $k_2 = \omega\sqrt{\rho_2/E_2}$.

Beam 1

Incident and reflected L-waves

$$\xi_1 = e^{-ik_1x} + R\cdot e^{ik_1x} \tag{2.5.5.1}$$

Beam 2

Transmitted and reflected L-waves

$$\xi_2 = A\sin(k_2x) + B\cos(k_2x) \tag{2.5.5.2}$$

Beam 3

Transmitted L-wave

$$\xi_3 = T\cdot e^{-ik_1(x-L)} \tag{2.5.5.3}$$

Boundary conditions

At $x = 0$

Displacement

$$\xi_1 = \xi_2 \tag{2.5.5.4}$$

Forces

$$h \cdot E_1 \frac{\partial \xi_1}{\partial x} = h \cdot E_2 \frac{\partial \xi_2}{\partial x} \tag{2.5.5.5}$$

At $x = L$

Displacement

$$\xi_1 = \xi_2 \tag{2.5.5.6}$$

Forces

$$h \cdot E_2 \frac{\partial \xi_2}{\partial x} = h \cdot E_1 \frac{\partial \xi_3}{\partial x} \tag{2.5.5.7}$$

The four boundary conditions give the parameters R, T, A, and B. The parameter T is obtained as

$$T = [\cos\alpha + i \sin\alpha(E_1k_1/E_2k_2 + E_2k_2/E_1k_1)/2]^{-1} \quad \text{where } \alpha = k_2L$$

The ratio between the time averages of the incident and transmitted energy flows are

$$\frac{\bar{\Pi}_{\text{in}}}{\bar{\Pi}_{\text{trans}}} = \frac{1}{|T|^2} = 1 + \left[\frac{\sin(k_2L)}{2}\left(\sqrt{\frac{E_1\rho_1}{E_2\rho_2}} - \sqrt{\frac{E_2\rho_2}{E_1\rho_1}}\right)\right]^2 \tag{2.5.5.8}$$

For an elastic interlayer between for example concrete or steel beams the properties of the materials are such that $E_1\rho_1 \gg E_2\rho_2$. If in addition $\alpha \ll 1$, Eq. (2.5.5.8) is simplified to

$$\frac{\bar{\Pi}_{\text{in}}}{\bar{\Pi}_{\text{trans}}} \approx 1 + \left[\frac{\omega L}{2E_2} \cdot \sqrt{E_1\rho_1}\right]^2 \tag{2.5.5.9}$$

Compare Fig. 5.27.

2.5.6 *Example 5.6*

The width of structures is b. Assume $b \ll 1$ (Fig. 2.12)

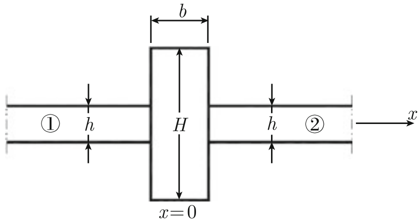


Fig. 2.12 Infinite beam with a blocking mass

$$M = \rho BbH; \quad m' = \rho Bh$$

$$I_\omega = \frac{bBH^3}{12} \cdot \rho = \frac{MH^2}{12};$$

$$D' = \frac{EBh^3}{12}; \quad \kappa = \left[m'\omega^2/D'\right]^{1/4}$$

The mass of the blocking mass is M and its mass moment of inertis is I_ω. The bending stiffness of the beam is D' and the wavenumber for F-waves propagating along the beam is κ. If the blocking mass is sufficiently narrow, $b \ll 1$, the distance between the beams can be neglected in the low frequency range. Assume a time dependence $\exp(i\omega t)$.

Beam 1

Incident and reflected flexural waves

$$w_1 = e^{-i\kappa x} + R \cdot e^{i\kappa x} + X \cdot e^{\kappa x} \tag{2.5.6.1}$$

Beam 2

Transmitted flexural waves

$$w_2 = T \cdot e^{-i\kappa x} + Y \cdot e^{-\kappa x} \tag{2.5.6.2}$$

Boundary conditions at $x = 0$

Displacement

$$w_1 = w_2 \tag{2.5.6.3}$$

Rotation

$$\frac{\partial w_1}{\partial x} = \frac{\partial w_2}{\partial x} \tag{2.5.6.4}$$

Forces

$$F_2 - F_1 = M \cdot \ddot{w}_1 \tag{2.5.6.5}$$

Bending moments

$$M_2 - M_1 = -I_\omega \cdot \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_1}{\partial x}\right) \tag{2.5.6.6}$$

Equation (2.5.6.5) $\Rightarrow$

$$\frac{\partial^3 w_2}{\partial x^3} - \frac{\partial^3 w_1}{\partial x^3} = \frac{\omega^2 M}{D'} \cdot w_1 \tag{2.5.6.7}$$

Equation (2.5.6.6) $\Rightarrow$

$$\omega^2 \cdot I_\omega \cdot \frac{\partial w_2}{\partial x} = -D' \cdot \left\{\frac{\partial^2 w_2}{\partial x^2} - \frac{\partial^2 w_1}{\partial x^2}\right\} \tag{2.5.6.8}$$

The boundary conditions give the amplitudes R, T, X and Y. The ratio between incident and transmitted energy flows is given by

$$\bar{\Pi}_{\text{in}}/\bar{\Pi}_{\text{tr}} = 1/|T|^2 \tag{2.5.6.9}$$

2.5.7 *Example 5.7*

The direction of the energy flow caused by a propagating and an evanescent wave is discussed in Sect. 5.2. Compare the Eqs. (5.35) and (5.36). The energy flow induced by just one evanescent wave is obtained by considering a wave defined as:

$$w = A \cdot \exp\{i\omega t - \kappa x\} \tag{2.5.7.1}$$

The wavenumber is given by $\kappa = \kappa_0(1 - i\eta/4)$. Thus

$$w = A \cdot \exp\{i(\omega t + \kappa_0 x\eta/4) - \kappa_0 x\} \tag{2.5.7.2}$$

Using the expression (2.5.7.2) and considering the discussion in Sect. 5.2, the energy flow is written

$$\begin{aligned}\bar{\Pi} &= \frac{1}{2} \cdot \text{Re}\left\{D' \cdot \left(\frac{\partial^3 w}{\partial x^3}\right)\left(\frac{\partial w}{\partial t}\right)^* - D' \cdot \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial x \partial t}\right)^*\right\} \\ &= \frac{1}{2} \cdot \text{Re}\left\{D' \cdot (-\kappa^3) \cdot w(-i\omega)w^* - D' \cdot \kappa^2 \cdot (i\omega\kappa^*)ww^*\right\} \\ &= \frac{1}{2} \cdot \text{Re}\left\{D' \cdot |w|^2 \cdot \omega\left[i\kappa^3 - i\kappa^2\kappa^*\right]\right\}\end{aligned} \tag{2.5.7.3}$$

For $\eta \ll 1$

$$\kappa^3 \approx \kappa_0^3(1 - i3\eta/4) \tag{2.5.7.4}$$

$$\left.\begin{array}{r}\kappa^2 \approx \kappa_0^2(1 - i\eta/2)\\ \kappa^* \approx \kappa_0(1 + i\eta/4)\end{array}\right\}; \quad \kappa^2\kappa^* \approx \kappa_0^3(1 - i\eta/4) \tag{2.5.7.5}$$

The Eqs. (2.5.7.3)–(2.5.7.5) give

$$\bar{\Pi} = \frac{1}{2} \cdot \mathrm{Re}\left[D' \cdot |w|^2\, \omega\kappa_0^3\, \{i + 3\eta/4 - i - \eta/4\}\right] \tag{2.5.7.6}$$

The bending stiffness is $D' = D_0'(1 + i\eta)$ and $|w|^2 = |A|^2 \cdot e^{-2\kappa_0 x}$. In combination with Eq. (2.5.7.6) these expressions give

$$\bar{\Pi} = \frac{\eta}{4} \cdot D_0' \cdot \omega\kappa_0^3 \cdot |A|^2 \cdot e^{-2\kappa_0 x}$$

The energy flow is positive. $\Rightarrow$ The wave travels along the positive x-axis.

2.5.8 Example 5.8

According to Eq. (5.50) the far field solution is

$$\begin{aligned} w(r,t) &= \sqrt{\frac{2}{\pi\kappa r}}\left\{\frac{(1+i)}{\sqrt{2}}e^{-i\kappa r} - ie^{-\kappa r}\right\} w_0 e^{i\omega t} \\ &\approx \left\{\frac{(1+i)}{\sqrt{\pi\kappa r}}\right\} w_0 \cdot \exp\left[i(\omega t - \kappa r)\right] \end{aligned} \tag{2.5.8.1}$$

for $r \gg 1$. The amplitude w_0 is obtained from Eq. (5.53) as

$$w_0 = -iF_0/(8\kappa^2 D) \tag{2.5.8.2}$$

when F and w are defined positive in the same direction. The far field solution at a distance r from the excitation point is

$$w(r,t) = \frac{F_0}{8\kappa^2 D}\left\{\frac{(1-i)}{\sqrt{\pi\kappa r}}\right\} \cdot \exp\left[i(\omega t - \kappa r)\right] \tag{2.5.8.3}$$

2.5.9 Example 5.9

From Eq. (2.5.9.3) in the previous example, the displacement in far field at a distance r from an excitation point is

$$w(r,t) = \frac{F_0}{8\kappa^2 D}\left\{\frac{(1-i)}{\sqrt{\pi\kappa r}}\right\}\cdot \exp\left[i\left(\omega t - \kappa r\right)\right] \tag{2.5.9.1}$$

The force exciting the plate is $F(t) = F_0 \exp(i\omega t)$. In the far field, the energy flow per unit length is Eq. (5.55)

$$\bar{\Pi}_r = \frac{1}{2}\cdot \mathrm{Re}\left\{D\cdot\left(\frac{\partial^3 w}{\partial r^3}\right)\left(\frac{\partial w}{\partial t}\right)^* - D\cdot\frac{\partial^2 w}{\partial r^2}\left(\frac{\partial^2 w}{\partial r\partial t}\right)^*\right\} \tag{2.5.9.2}$$

The total power passing a circle with radius r is

$$\bar{\Pi}_{\mathrm{tot}} = 2\pi r\cdot\bar{\Pi}_r \tag{2.5.9.3}$$

Using Eq. (2.5.9.3) the derivatives of w are, for $r\to\infty$, given by

$$\frac{\partial w}{\partial r} = (-i\kappa)\,w;\quad \frac{\partial^2 w}{\partial r^2} = -\kappa^2 w;\quad \frac{\partial^3 w}{\partial r^3} = i\kappa^3 w$$

These expressions in combination with Eq. (2.5.9.2) give

$$\bar{\Pi}_r = D\kappa^3\omega\,|w|^2 = \frac{|F_0|^2\,\omega}{32\pi D\kappa^2 r} \tag{2.5.9.4}$$

The total power passing a circle with radius r is from Eq. (2.5.9.3) equal to

$$\bar{\Pi}_{\mathrm{tot}} = \frac{|F_0|^2\,\omega}{16D\kappa^2} = \frac{\left|\bar{F}\right|^2\omega}{8D\kappa^2} \tag{2.5.9.5}$$

According to Eq. (5.54) the point mobility Y for an infinite plate is

$$Y = \frac{1}{8\,(\rho h D)^{1/2}} \tag{2.5.9.6}$$

For a plate exited by a point force $F = F_0\cdot e^{i\omega t}$ the input power to the plate is

$$\bar{\Pi}_{\mathrm{in}} = \frac{1}{2}\mathrm{Re}\left\{F\cdot v^*\right\};\quad v = F\cdot Y \quad\Rightarrow$$

$$\bar{\Pi}_{\mathrm{in}} = \frac{|F_0|^2}{2}\cdot\mathrm{Re}Y = \frac{|F_0|^2}{16\,(\rho h D_0)^{1/2}} = \frac{|F_0|^2}{16 D_0\kappa^2} = \frac{\left|\bar{F}\right|^2}{8D_0\kappa^2} \tag{2.5.9.7}$$

For no losses in the plate $\bar{\Pi}_{\mathrm{tot}} = \bar{\Pi}_{\mathrm{in}}$ as given by Eqs. (2.5.9.5) and (2.5.9.7).

2.5.10 Example 5.10

A plane flexural wave is incident on a straight junction between two semi infinite plates. The angle of incidence is α. The angle is such that $\sin\alpha > \kappa_2/\kappa_1$ where κ_1 and κ_2 are the wavenumbers for flexural waves propagating in the plates 1 and 2. The incident wave is propagating in plate 1.The amplitudes of the reflected and transmitted waves are given in Eq. (5.126) as

$$R = -\frac{U_1 - iU_3}{U_1 - iU_2} \tag{2.5.10.1}$$

$$T = \frac{2i\cos\alpha}{U_1 - iU_2} \tag{2.5.10.2}$$

where according to Eq. (5.125)

$$U_1 = \sqrt{Z^2 + \sin^2\alpha} + Y\sqrt{1 + \sin^2\alpha}; \quad U_2 = Y\cos\alpha - i\sqrt{\sin^2\alpha - Z^2}$$

$$U_3 = -Y\cos\alpha - i\sqrt{\sin^2\alpha - Z^2}; \quad \sin\alpha > \kappa_2/\kappa_1 \tag{2.5.10.3}$$

Equations (2.5.10.1) and (2.5.10.3) give

$$R = -\frac{U_1 - \sqrt{\sin^2\alpha - Z^2} + iY\cos\alpha}{U_1 - \sqrt{\sin^2\alpha - Z^2} - iY\cos\alpha}$$

Thus

$$|R|^2 = \frac{(U_1 - \sqrt{\sin^2\alpha - Z^2})^2 + (Y\cos\alpha)^2}{(U_1 - \sqrt{\sin^2\alpha - Z^2})^2 + (Y\cos\alpha)^2} = 1 \tag{2.5.10.4}$$

The incident energy flow is reflected completely when $\sin\alpha > \kappa_2/\kappa_1$.

2.5.11 Example 5.11

According to Eq. (5.129) the transmission cofficient across the joint is

$$\tau(\alpha) = 1 - |R|^2 \tag{2.5.11.1}$$

The angle of incidence is α. The amplitude of the reflected wave is

$$R = -\frac{U_1 - iU_3}{U_1 - iU_2} \tag{2.5.11.2}$$

with

$$U_1 = \sqrt{Z^2 + \sin^2\alpha} + Y\sqrt{1 + \sin^2\alpha}; \quad U_2 = \sqrt{Z^2 - \sin^2\alpha} + Y\cos\alpha$$
$$U_3 = \sqrt{Z^2 - \sin^2\alpha} - Y\cos\alpha \tag{2.5.11.3}$$

For $\sin\alpha < \kappa_2/\kappa_1$ the functions U_1, U_2 and U_3 are all real. Thus using Eqs. (2.5.11.1) and (2.5.11.2) the transmission coefficient is obtained as

$$\tau(\alpha) = 1 - |R|^2 = \frac{U_2^2 - U_3^2}{U_1^2 + U_2^2} \tag{2.5.11.4}$$

Equations (2.5.11.4) and (2.5.11.3) give

$$\tau(\alpha) = \frac{4Y\cos\alpha\sqrt{Z^2 - \sin^2\alpha}}{U_1^2 + U_2^2} \tag{2.5.11.5}$$

According to Eq. (5.126) $|T|^2$ is

$$|T|^2 = \frac{4\cos^2\alpha}{U_1^2 + U_2^2} \tag{2.5.11.6}$$

This expression in combination with Eq. (2.5.11.5) gives

$$\tau(\alpha) = |T|^2\, Y\frac{\sqrt{Z^2 - \sin^2\alpha}}{\cos\alpha} \tag{2.5.11.7}$$

2.5.12 Example 5.12

The transmission coefficient between the two plates is given by Eqs. (2.5.11.7) and (2.5.11.6) in the previous example. The coupled semi-infinite plates are identical. Thus according to Eq. (5.125) the parameters Y and Z are both equal to unity. For $Y = Z = 1$ the Eqs. (2.5.11.6) and (2.5.11.7), Example 5.11, give

$$\tau(\alpha) = |T|^2 = (\cos\alpha)^2/2 \tag{2.5.12.1}$$

For normal incidence $\alpha = 0$ and $\tau(0) = 1/2$. The average transmission coefficient $\bar{\tau}$ is according to Eq. (5.133)

$$\tau_d = \int_0^{\pi/2} \tau(\alpha)\cos\alpha \mathrm{d}\alpha \tag{2.5.12.2}$$

The Eqs. (2.5.12.1) and (2.5.12.2) give

$$\tau_d = \int_0^{\pi/2} \tau(\alpha)\cos\alpha \mathrm{d}\alpha = \frac{1}{2}\int \mathrm{d}\alpha \left[\cos\alpha - \sin^2\alpha\right]$$
$$= \frac{1}{2}\left[\sin\alpha - (\sin\alpha)^3/3\right]_0^{\pi/2} = 1/3 \tag{2.5.12.3}$$

The ratio between the transmission coefficients for random and normal incidence is thus

$$\tau_d/\tau(0) = 2/3 \tag{2.5.12.4}$$

2.6 Chapter 6

2.6.1 Example 6.1

The wave equation governing L-waves propagating in a beam is

$$\frac{\partial^2\xi}{\partial x^2} - \frac{\rho}{E}\cdot\frac{\partial^2\xi}{\partial t^2} = 0 \tag{2.6.1.1}$$

Assume a solution

$$\xi(x,t) = g(t)\cdot\varphi(x) \tag{2.6.1.2}$$

Thus, as discussed in Sect. 6.1

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + k^2\cdot\varphi = 0 \tag{2.6.1.3}$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + \frac{E}{\rho}\cdot k^2 g = 0 \tag{2.6.1.4}$$

The general solution to Eq. (2.6.1.3) is written

$$\varphi = A\sin(kx) + B\cos(kx) \tag{2.6.1.5}$$

The boundary conditions for clamped edges are

$$\varphi = 0 \quad \text{for } x = 0 \text{ and } x = L \tag{2.6.1.6}$$

The Eqs. (2.6.1.5) and (2.6.1.6) give

$$B = 0; \quad \sin(kL) = 0 \quad \Rightarrow \quad k = n\pi/L \tag{2.6.1.7}$$

Thus

$$\varphi_n = \sin(n\pi x/L) \tag{2.6.1.8}$$

According to Eqs. (6.19) and (6.22) the natural frequencies or eigenfrequencies are

$$f_n = \frac{\omega_{n0}}{2\pi} = \frac{k_n}{2\pi} \cdot \left\{\frac{E_0}{\rho}\right\}^{1/2} = \frac{n}{2L} \cdot \left\{\frac{E_0}{\rho}\right\}^{1/2} \tag{2.6.1.9}$$

The eigenfunctions φ_n are orthogonal since

$$\begin{aligned} \int_0^L \varphi_n(x)\varphi_m(x)\mathrm{d}x &= L/2 \quad \text{for } m = n \\ &= 0 \qquad \text{for } m \neq n \end{aligned} \tag{2.6.1.10}$$

2.6.2 Example 6.2

Following the procedure outlined in Example 6.1, the general solution to the differential equation (2.6.2.3) in the previous example is written

$$\varphi = A\sin(kx) + B\cos(kx) \tag{2.6.2.1}$$

The boundary conditions are

$$\varphi = 0 \quad \text{for } x = 0 \text{ and } \mathrm{d}\varphi/\mathrm{d}x = 0 \text{ for } x = L \tag{2.6.2.2}$$

The Eqs. (2.6.2.1) and (2.6.2.2) give

$$B = 0; \quad \cos(kL) = 0 \quad \Rightarrow \quad k_n = \pi(n + 1/2)/L \tag{2.6.2.3}$$

Thus

$$\varphi_n = \sin(k_n x) \tag{2.6.2.4}$$

According to Eqs. (6.19) and (6.22) the natural frequencies or eigenfrequencies are

$$f_n = \frac{\omega_{n0}}{2\pi} = \frac{k_n}{2\pi} \cdot \left\{\frac{E_0}{\rho}\right\}^{1/2} = \frac{n + 1/2}{2L} \cdot \left\{\frac{E_0}{\rho}\right\}^{1/2} \tag{2.6.2.5}$$

The eigenfunctions φ_n are orthogonal since

$$\begin{aligned}\int_0^L \varphi_n(x)\varphi_m(x)\mathrm{d}x &= L/2 \quad \text{for } m = n \\ &= 0 \qquad \text{for } m \neq n\end{aligned} \tag{2.6.2.6}$$

2.6.3 Example 6.3

The displacement ξ in the resiliently mounted beam is defined as

$$\xi(x, t) = \varphi(x) \cdot g(t) \tag{2.6.3.1}$$

The function φ should satisfy the differential equation

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + k^2\varphi = 0 \tag{2.6.3.2}$$

Let the stiffness of each resilient mount be defined by the spring constant k_s. At each end of the beam, the force due to the normal stress in the beam should equal the spring force. Thus the boundary condition at each end of the beam is

$$k_s\xi = \sigma S = ES\frac{\partial\xi}{\partial x} \quad \text{at } x = 0 \tag{2.6.3.3}$$

At the other end the boundary condition is

$$k_s\xi = -\sigma S = -ES\frac{\partial\xi}{\partial x} \quad \text{at } x = L \tag{2.6.3.4}$$

The cross-sectional area of the beam is S. The Eqs. (2.6.3.1), (2.6.3.3) and (2.6.3.4) give the boundary condition

$$\varphi = \frac{ES}{k_s} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x} \quad \text{at } x = 0 \tag{2.6.3.5}$$

and

$$\varphi = -\frac{ES}{k_s} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x} \quad \text{at } x = L \tag{2.6.3.6}$$

The general solution to Eq. (2.6.3.2) reads

$$\varphi(x) = A\sin(kx) + B\cos(kx) \tag{2.6.3.7}$$

Equations (2.6.3.5) and (2.6.3.7) give for $x = 0$

$$A = k_s B/(SEk) \tag{2.6.3.8}$$

and for $x = L$

$$\frac{SE}{k_s}\left[Ak\cos(kL) - Bk\sin(kL)\right] + A\sin(kL) + B\cos(kL) = 0 \tag{2.6.3.9}$$

The Eqs. (2.6.3.8) and (2.6.3.9) give

$$2\cdot\frac{SEk}{k_s}\cdot\cos(kL) = \left[\left(\frac{SEk}{k_s}\right)^2 - 1\right]\cdot\sin(kL)$$

There is an infinite number of solutions to this equation. Let the solutions be k_n and define α_n as $\alpha_n = k_n L$ as being the solution to

$$\tan(\alpha_n) = \frac{2\cdot\dfrac{SE\alpha_n}{Lk_s}}{\left[\left(\dfrac{SE\alpha_n}{Lk_s}\right)^2 - 1\right]} \tag{2.6.3.10}$$

The eigenfunctions φ_n are obtained from (2.6.3.7) and (2.6.3.8) by replacing k by k_n. Thus

$$\varphi_n = A(SEk_n/k_s)\cos(k_n x) + A\sin(k_n x) \tag{2.6.3.11}$$

By for example setting $A = 1$ the eigenfunction is reduced to

$$\varphi_n(x) = \sin(k_n x + \gamma_n) \quad \tan\gamma_n = \frac{SEk_n}{k_s} \tag{2.6.3.12}$$

The eigenfunctions are orthogonal since the Eq. (6.17) is satisfied.

When the stiffness of the resilient interlayer is made increasingly stiff or when $k_s \to \infty$, then $\tan(\alpha_n) \to 0$ and $\tan(\gamma_n) \to 0$ giving $\alpha_n = \gamma_n = n\pi/L$ and $\varphi_n = \sin(n\pi x/L)$. This is the eigenfunction for a beam with clamped ends. If on the other hand, $k_s \to 0$ the ends tend to be free and $\varphi_n \to \cos(k_n x)$.

2.6.4 Example 6.4

For an eigenfunction to satisfy periodic boundary conditions the requirements are

$$\varphi_n(0) = \varphi_n(L) \tag{2.6.4.1}$$

$$[\partial\varphi_n/\partial x]_{x=0} = [\partial\varphi_n/\partial x]_{x=L} \tag{2.6.4.2}$$

The eigenfunctions should satisfy Eq. (6.11). The general solution to this equation reads

$$\begin{aligned}\varphi_n(x) &= A_n \sin(k_n x) + B_n \cos(k_n x) \quad \text{or}\\ \varphi_n(x) &= C_n \sin(k_n x + \alpha_n)\end{aligned} \tag{2.6.4.3}$$

For φ_n to satisfy Eqs. (2.6.4.1) and (2.6.4.2) k_n must equal $k_n = n\pi/L$. The boundary conditions are then satisfied by Eq. (2.6.4.3) for any A_n, B_n, C_n or α_n. For an eigenfunction $\varphi_n(x) = \sin(k_n x + \alpha_n)$ orthogonality holds since

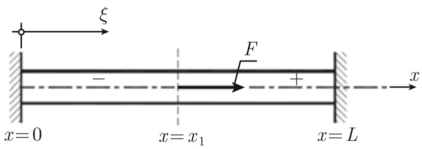
$$\begin{aligned}&\int_0^L \varphi_n(x)\varphi_m(x)\mathrm{d}x = \int_0^L \sin(k_n x + \alpha_n)\sin(k_m x + \alpha_m)\mathrm{d}x\\ &= (1/2)\int_0^L \{\cos[x(k_n - k_m) + (\alpha_n - \alpha_m)] - \cos[x(k_n + k_m) + (\alpha_n + \alpha_m)]\}\\ &= 0 \qquad\qquad \text{for } m \neq n\\ &= L/2 \qquad\quad \text{for } m = n > 0\\ &= L(\sin\alpha_0)^2 \quad \text{for } m = n = 0\end{aligned}$$

2.6.5 Example 6.5

The clamped beam shown in Fig. 2.13 is excited by a force $F = F_0 \cdot e^{i\omega t}$ at $x = x_1$. The forced response of the beam is

$$\xi(x,t) = h(x) \cdot e^{i\omega t} \tag{2.6.5.1}$$

Fig. 2.13 A clamped beam excited by a force F



The displacement $\xi(x, t)$ should satisfy the wave equation

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{\rho}{E} \cdot \frac{\partial^2 \xi}{\partial t^2} = -\frac{F \cdot \delta(x - x_1)}{SE} \tag{2.6.5.2}$$

Consequently,

$$\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} + k^2 \cdot h = -\frac{F_0 \cdot \delta(x - x_1)}{SE}; \quad k^2 = \frac{\omega^2 \rho}{E} \tag{2.6.5.3}$$

The solutions to Eq. (2.6.5.3) are

$$h_- = A_1 \sin(k_l x) + B_1 \cos(k_l x) \quad 0 \leqslant x \leqslant x_1 \tag{2.6.5.4}$$

$$h_+ = A_2 \sin k_l (L - x) + B_2 \cos k_l (L - x) \quad x_1 \leqslant x \leqslant L \tag{2.6.5.5}$$

The boundary conditions are

$$h_-(0) = 0 \tag{2.6.5.6}$$

$$h_+(L) = 0 \tag{2.6.5.7}$$

$$h_-(x_1) = h_+(x_1) \tag{2.6.5.8}$$

$$\left[\frac{\mathrm{d}h_+}{\mathrm{d}x}\right]_{x=x_1} - \left[\frac{\mathrm{d}h_-}{\mathrm{d}x}\right]_{x=x_1} = -\frac{F_0}{SE} \tag{2.6.5.9}$$

The Eqs. (2.6.5.6)–(2.6.5.9) give

$$B_1 = 0 \tag{2.6.5.10}$$

$$B_2 = 0 \tag{2.6.5.11}$$

$$A_1 \sin(k_l x_1) = A_2 \sin k_l (L - x_1) \tag{2.6.5.12}$$

$$k_l \left[A_2 \cos k_l (L - x_1) + A_1 \cos(k_l x_1)\right] = \frac{F}{SE} \tag{2.6.5.13}$$

The solutions are

$$A_2 = \frac{F_0 \cdot \sin(k_l x_1)}{SEk_l \sin(k_l L)} \tag{2.6.5.14}$$

$$A_1 = \frac{F_0 \cdot \sin k_l (L - x_1)}{SEk_l \sin(k_l L)} \tag{2.6.5.15}$$

The response is

$$\xi(x,t) = F_0(x) \cdot e^{i\omega t} \cdot G(x\,|x_1\,)$$

$$G(x\,|x_1\,) = G_1(x\,|x_1\,) = \frac{\sin(k_l x_1) \cdot \sin k_l(L-x)}{SEk_l \sin(k_l L)} \quad 0 \leqslant x_1 \leqslant x \leqslant L$$

$$G(x\,|x_1\,) = G_2(x\,|x_1\,)\frac{\sin k_l(L-x_1) \cdot \sin(k_l x)}{SEk \sin(k_l L)} \quad 0 \leqslant x \leqslant x_1 \leqslant L$$

2.6.6 Example 6.6

The response ξ of the beam excited by a force $F'$is according to Eq. (6.51) given by

$$\xi(x,t) = \int_0^L F'(\varsigma,t)G(x\,|\varsigma)\mathrm{d}\varsigma = \int_0^x F'(\varsigma,t)G_1(x\,|\varsigma)\mathrm{d}\varsigma + \int_x^L F'(\varsigma,t)G_2(x\,|\varsigma)\mathrm{d}\varsigma \tag{2.6.6.1}$$

The Green's function for a beam with clamped ends is given in Eq. (6.53). Thus, the response is given by

$$\begin{aligned}\xi(x,t) &= -\frac{F_0}{L} \cdot e^{i\omega t} \cdot \frac{1}{SEk_l \sin(k_l L)} \cdot \\ &\qquad \left\{\int_0^x \mathrm{d}\varsigma \cdot \sin(k_l\varsigma) \cdot \sin k_l(x-L) + \int_{\mathrm{x}}^L \mathrm{d}\varsigma \cdot \sin k_l(\varsigma - L) \cdot \sin(k_l x)\right\} \\ &= -\frac{F_0}{L} \cdot e^{i\omega t} \cdot \frac{1}{SEk_l \sin(k_l L)} \cdot \Omega(x)\end{aligned} \tag{2.6.6.2}$$

The function $\Omega(x)$is

$$\begin{aligned}\Omega(x) &= \frac{1}{k} \cdot \{\sin k_l(x-L) \cdot [\,1 - \cos(k_l x)] + \sin(k_l x) \cdot [\,\cos k_l(x-L) - 1]\} \\ &= \frac{1}{k} \cdot \{\sin(k_l L) + \sin k_l(x-L) - \sin(k_l x)\}\end{aligned} \tag{2.6.6.3}$$

The Eqs. (2.6.6.2) and (2.6.6.3) give the response of the beam as

$$\xi(x,t) = -\frac{F_0}{L} \cdot e^{i\omega t} \cdot \frac{1}{SEk^2 \sin(k_l L)} \cdot \{\sin(k_l L) + \sin k_l(x-L) - \sin(k_l x)\} \tag{2.6.6.4}$$

2.6.7 Example 6.7

The differential equation governing the response of the beam is

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{\rho}{E} \cdot \frac{\partial^2 \xi}{\partial t^2} = -\frac{F'(x,t)}{SE} \tag{2.6.7.1}$$

The external force per unit length is

$$F'(x,t) = \frac{F}{L} \cdot e^{i\omega t} \cdot \sin\left(\frac{\pi x}{L}\right) \tag{2.6.7.2}$$

The response is written as

$$\xi(x,t) = e^{i\omega t} \cdot \sum C_n \cdot \varphi_n(x) \tag{2.6.7.3}$$

where the eigenfunction satisfying the boundary conditions for a clamped beam is

$$\varphi_n(x) = \sin(n\pi x/L) \tag{2.6.7.4}$$

The eigenfunction satisfies

$$\varphi_n''(x) = -k_n^2 \varphi_n; \quad k_n = n\pi/L \tag{2.6.7.5}$$

ξ and φ_n satisfy the same boundary conditions. The Eqs. (2.6.7.1)–(2.6.7.3) give

$$\begin{aligned} &\sum C_n \cdot \varphi_n'' + k_l^2 \sum C_n \cdot \varphi_n = -\frac{F}{SEL} \cdot \sin\left(\frac{\pi x}{L}\right) \\ &\sum C_n \cdot \varphi_n \cdot \left[k_l^2 - k_n^2\right] = -\frac{F}{SEL} \cdot \sin\left(\frac{\pi x}{L}\right) \end{aligned} \tag{2.6.7.6}$$

Multiply Eq. (2.6.7.6) by φ_n and integrate over x. The result is

$$\begin{aligned} &C_n \langle \varphi_n | \varphi_n \rangle \cdot \left[k_l^2 - k_n^2\right] = -\frac{F}{SEL} \cdot \int_0^L \mathrm{d}x \cdot \sin\left(\frac{\mathrm{n}\pi x}{L}\right) \cdot \sin\left(\frac{\pi x}{L}\right) \\ &\frac{C_n \cdot L}{2} \cdot \left[k_l^2 - k_n^2\right] = -\frac{F}{SEL} \cdot \frac{L}{2} \quad \text{for } n = 1, \text{ otherwise zero. Thus} \\ &C_1 = \frac{-F}{SEL \cdot [k_l^2 - k_1^2]}; \quad C_n = 0 \quad \text{for } n \neq 1 \end{aligned} \tag{2.6.7.7}$$

The Eqs. (2.6.7.3) and (2.6.7.7) give the response as

$$\xi(x,t) = \frac{-F \cdot e^{i\omega t} \cdot \sin(\pi x/L)}{SEL \cdot \left[k_l^2 - (\pi/L)^2\right]} \tag{2.6.7.8}$$

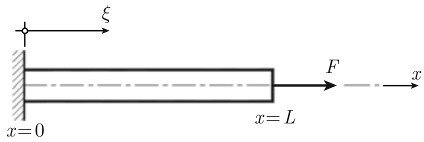


Fig. 2.14 A beam clamped at one end and excited by a force at the other end

2.6.8 Example 6.8

The displacement ξ of the beam is governed by the differential equation (Fig. 2.14)

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{\rho}{E} \cdot \frac{\partial^2 \xi}{\partial t^2} = -\frac{F(t) \cdot \delta(x-L)}{SE} \tag{2.6.8.1}$$

For $t \leqslant 0$ the beam is at rest and $\dot{\xi} = \ddot{\xi} = 0$. For $t \leqslant 0$ the general solution to Eq. (2.6.8.1) is $\xi = Ax + B$. The beam is clamped at $x = 0$. Thus $\xi(0) = 0$ yields $B = 0$. For $x = L$ the boundary condition is

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)_{x=L} = \frac{F}{SE} \quad \Rightarrow \quad \xi(x, 0) = \frac{Fx}{SE} \tag{2.6.8.2}$$

According to Eq. (6.13) the eigenfunction for a clamped-free beam is

$$\varphi_n(x) = \sin(k_n x); \quad k_n = (\pi/L)(n + 1/2) \tag{2.6.8.3}$$

For $t > 0$ the external force $F(t)$ is zero. The displacement of the beam is for $t > 0$ written as

$$\xi(x, t) = \sum g_n(t) \cdot \varphi_n(x) \tag{2.6.8.4}$$

The function $g_n(t)$ satisfies the differential equation (6.18), i.e.

$$k_n^2 g_n + (\rho/E)\ddot{g}_n = 0; \quad E = E_0(1 + i\eta) \tag{2.6.8.5}$$

Introduce ω_n as

$$\begin{aligned} \omega_n &= (E/\rho)^{1/2} \cdot k_n = (E_0/\rho)^{1/2} \cdot (1 + i\eta/2)\, k_n \; = \omega_{n0} \cdot (1 + i\eta/2) \\ \omega_{n0} &= (E_0/\rho) k_n \end{aligned} \tag{2.6.8.6}$$

The solution to Eq. (2.6.8.5) can according to Eq. (6.20) be written as

$$g_n(t) = [A_n \cos(\omega_{0n} t) + B_n \sin(\omega_{n0} t)] \cdot e^{-\omega_{n0}\eta t/2} \tag{2.6.8.7}$$

The initial condition is $\dot{\xi}(x, 0) = 0$. Equation (2.6.8.4) gives $\dot{g}_n(0) = 0$. From Eq. (2.6.8.7) $B_n = 0$ and $g_n(0) = A_n$. The Eqs. (2.6.8.2), (2.6.8.4), and (2.6.8.7) give

$$\xi(x, 0) = \sum A_n \cdot \varphi_n(x) = \frac{Fx}{SE} \tag{2.6.8.8}$$

The eigenfunctions are orthogonal thus

$$\begin{aligned} A_n \langle \varphi_n \mid \varphi_n \rangle &= \frac{F}{SE} \cdot \int_0^L \mathrm{d}x \cdot x \cdot \sin(k_n x) \\ A_n \cdot (L/2) &= \frac{F}{SEk_n^2} \cdot \sin[\pi(n + 1/2)] \end{aligned} \tag{2.6.8.9}$$

The response for $t > 0$

$$\xi(x, t) = \sum A_n \cdot \varphi_n(x) \cdot \cos(\omega_{0n} t) \cdot e^{-\omega_0 \eta t/2} \tag{2.6.8.10}$$

with

$$A_n = \frac{2FL}{SE\pi^2(n + 1/2)^2} \cdot (-1)^n;$$

$$\varphi_n(x) = \sin[(x/L)(n + 1/2)]; \quad \omega_{0n} = (\pi/L)(n + 1/2)\sqrt{E_0/\rho}$$

2.6.9 *Example 6.9*

The differential equation governing the displacement of a beam which is excited by a force at mid-point is

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{\rho}{E}^2 \cdot \frac{\partial^2 \xi}{\partial t^2} = -\frac{F_0 \cdot \delta(x - L/2)}{SE} \cdot \exp(i\omega_0 t) \tag{2.6.9.1}$$

Assume a solution

$$\xi(x, t) = \exp(i\omega_0 t) \cdot \sum C_n \cdot \varphi_n(x) \tag{2.6.9.2}$$

where the eigenfunctions for the clamped been are

$$\varphi_n(x) = \sin(k_n x); \quad k_n = \frac{n\pi}{L} \tag{2.6.9.3}$$

The eigenfunction satisfies the equation

$$\varphi_n''(x) = -k_n^2 \varphi_n \tag{2.6.9.4}$$

The eigenfunction $\varphi_n(x)$ and $\xi(x)$ satisfy the same boundary conditions. Thus Eq. (2.6.9.2) can be inserted in Eq. (2.6.9.1) giving the result

$$\sum C_n \cdot \varphi_n \left[k_l^2 - k_n^2\right] = -\frac{F}{SE} \cdot \delta(x - L/2) \tag{2.6.9.5}$$

where $k^2 = \omega_0^2 \rho / E$. Equation (2.6.9.5) is multiplied by φ_n and integrated over x.

$$(C_n L/2) \cdot \left[k_l^2 - k_n^2\right] = -[F/(SE)] \cdot \sin(n\pi/2) \tag{2.6.9.6}$$

The Eqs. (2.6.9.2) and (2.6.9.6) give

$$\xi(x, t) = \frac{2F \cdot \exp(i\omega_0 t)}{SEL} \cdot \sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)\sin(n\pi/2)}{(n\pi/L)^2 - k_l^2} \tag{2.6.9.7}$$

For frequencies well below the first natural frequency $k_l \ll \pi/L$, Eq. (2.6.9.7) is written

$$\xi(x, t) = \frac{2F \cdot \exp(i\omega_0 t)}{SEL} \left\{ \frac{\sin(\pi x/L)}{(\pi/L)^2 - k_l^2} - \frac{\sin(3\pi x/L)}{(3\pi/L)^2 - k_l^2} \right.$$
$$\left. + \frac{\sin(5\pi x/L)}{(5\pi/L)^2 - k_l^2} \cdots \right\}$$

Thus for $k_l \ll \pi/L$ the response is approximately given by

$$\xi(x, t) \approx \frac{2F \cdot \exp(i\omega_0 t)}{SEL} \left\{ \frac{\sin(\pi x/L)}{(\pi/L)^2} \right\} \tag{2.6.9.8}$$

2.6.10 Example 6.10

The velocities and forces at the two ends of the coupled beams are related as (Fig. 2.15)

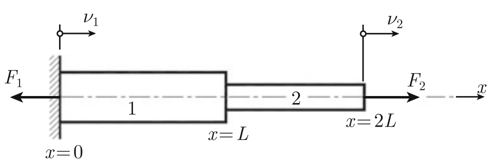


Fig. 2.15 Two coupled beams. Beam 1 is mounted to an infinitely stiff structure

$$\begin{Bmatrix} v_2 \\ F_2 \end{Bmatrix} = [A]_2 \cdot [A]_1 \cdot \begin{Bmatrix} v_1 \\ F_1 \end{Bmatrix} = [B] \cdot \begin{Bmatrix} v_1 \\ F_1 \end{Bmatrix} \tag{2.6.10.1}$$

Since $S_1 = 4S_2$ the matrices $[A]_1$ and $[A]_2$ are written

$$[A]_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad [A]_2 = \begin{bmatrix} a_{11} & a_{12} \\ 4a_{21} & a_{22} \end{bmatrix} \tag{2.6.10.2}$$

The elements a_{ij} are defined in Eq. (6.110) as

$$\begin{gathered} a_{11} = a_{22} = \cos(k_l L); \quad a_{12} = -i\omega \sin(k_l L)/(SEk_l); \\ a_{21} = -iSEk_l \sin(k_l L)/\omega \end{gathered} \tag{2.6.10.3}$$

Equation (2.6.10.1)

$$\Rightarrow v_2 = B_{11} \cdot v_1 + B_{12} \cdot F_1 \tag{2.6.10.4}$$

$$F_2 = B_{21} \cdot v_1 + B_{22} \cdot F_1 \tag{2.6.10.5}$$

The boundary conditions are $F_2 = v_1 = 0$. Equation (2.6.10.5) gives

$$B_{22} \cdot F_1 = 0 \quad \text{and} \quad B_{22} = 0$$

According to the Eqs. (2.6.10.1) and (2.6.10.2), B_{22} is

$$B_{22} = a_{12} \cdot a_{21}/4 + a_{22}^2 \tag{2.6.10.6}$$

Equations (2.6.10.3) and (2.6.10.6) give for $B_{22} = 0$.

$$-\left[\sin(k_l L)\right]^2/4 + \left[\cos(k_l L)\right]^2 = 0 \tag{2.6.10.7}$$

The solution to Eq. (2.6.10.7) is

$$\tan(k_l L) = \pm 2; \quad k_{l1} L = \arctan(2) \approx 1.1 \tag{2.6.10.8}$$

The first eigenfrequency f_1 is

$$f_1 = \frac{k_{l1}}{(2\pi)} \cdot \sqrt{\frac{E_0}{\rho}} \approx \frac{1.1}{(2\pi L)} \cdot \sqrt{\frac{E_0}{\rho}} \tag{2.6.10.9}$$

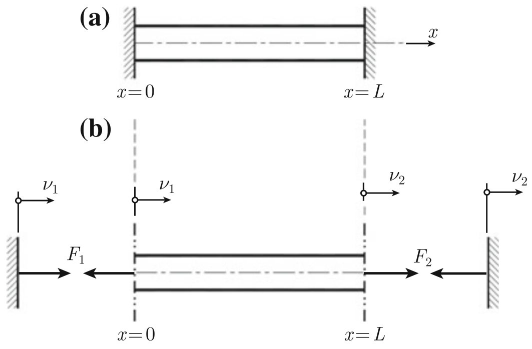


Fig. 2.16 A beam mounted between two structures each having a point mobility Y

2.6.11 Example 6.11

The forces acting on the beam and the resulting velocities are shown in Fig. 2.16b. The velocities at the two ends are according to Eqs. (6.95) and (6.96) given by

$$\hat{v}_1 = -\hat{F}_1 \cdot Y_{11} + \hat{F}_2 \cdot Y_{21} \tag{2.6.11.1}$$

$$\hat{v}_2 = \hat{F}_2 \cdot Y_{22} - \hat{F}_1 \cdot Y_{12} \tag{2.6.11.2}$$

where

$$Y_{11} = Y_{22} = -\frac{i\omega}{SEk_l \tan(k_l L)} \text{ and } Y_{12} = Y_{21} = -\frac{i\omega}{SEk_l \sin(k_l L)} \tag{2.6.11.3}$$

The mobility of the adjoining structure is Y. According to Fig. 2.16b

$$\hat{v}_1 = \hat{F}_1 \cdot Y \quad \text{and } \hat{v}_2 = -\hat{F}_2 \cdot Y \tag{2.6.11.4}$$

Equations (2.6.11.1), (2.6.11.2), and (2.6.11.4) give

$$\hat{F}_1 \cdot Y = -\hat{F}_1 \cdot Y_{11} + \hat{F}_2 \cdot Y_{21} \tag{2.6.11.5}$$

$$-\hat{F}_2 \cdot Y = \hat{F}_2 \cdot Y_{22} - \hat{F}_1 \cdot Y_{12} \tag{2.6.11.6}$$

By eliminating $\hat{F}_1$ and $\hat{F}_2$ from Eqs. (2.6.11.4) and (2.6.11.5) the result is

$$\frac{Y + Y_{11}}{Y_{12}} = \frac{Y_{21}}{Y + Y_{22}} \tag{2.6.11.7}$$

If Y is known the frequency or rather the natural frequency satisfying Eq. (2.6.11.7) can be calculated numerically. For the special case that the adjoining structure is a rigid mass M then $\hat{F} = i\omega M\hat{v}$ and $Y = \hat{v}/\hat{F} = 1/(i\omega M)$. For this particular case Eq. (2.6.11.7) gives:

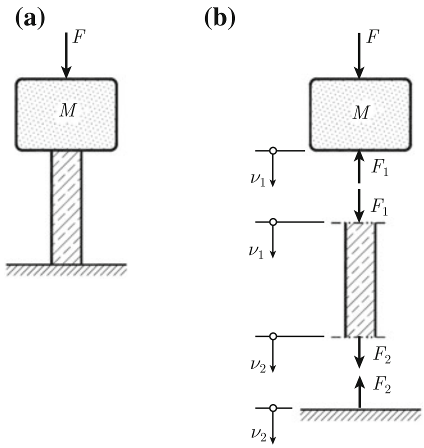


Fig. 2.17 Stiff mass mounted on a rod

$$\left[SEk_l \sin(k_l L) + \omega^2 M \cos(k_l L)\right] = \pm\omega^2 M \tag{2.6.11.8}$$

As $M \to \infty$, Eq. (2.6.11.8) approaches $\cos(k_l L) = \pm 1$. The solution to this limiting case is $k_l L = n\pi$ and the resulting eigenfrequencies are $f_n = [n/(2L)]\sqrt{E_0/\rho}$. These are also the eigenfrequencies for a clamped beam. For $M = 0$ Eq. (2.6.11.8) reads $\sin(k_l L) = 0$ giving the eigenfrequencies for a free beam. The solution is again $k_l L = n\pi$. The corresponding eigenfrequencies are $f_n = [n/(2L)]\sqrt{E_0/\rho}$.

2.6.12 Example 6.12

According to Fig. 2.17.

$$F - F_1 = i\omega M \cdot \nu_1 \tag{2.6.12.1}$$

$$v_1 = F_1 \cdot Y_{12} + F_2 \cdot Y_{22} \tag{2.6.12.2}$$

$$v_2 = F_1 \cdot Y_{12} + F_2 \cdot Y_{22} \tag{2.6.12.3}$$

$$v_2 = -F_2 \cdot Y \tag{2.6.12.4}$$

Y is the mobility of the foundation. The transfer mobilities Y_{ij} are given by Eq. (6.96) as

$$Y_{11} = Y_{22} = -\frac{i\omega}{SEk_l \cdot \tan(k_l L)} \tag{2.6.12.5}$$

$$Y_{21} = Y_{12} = -\frac{i\omega}{SEK_l \cdot \sin(k_l L)} \tag{2.6.12.6}$$

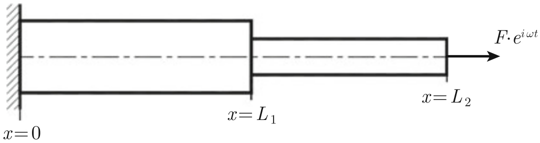


Fig. 2.18 Two coupled beams. One end clamped and the other excited by a force

The Eqs. (2.6.12.1) through (2.6.12.4) give

$$F_2 = F\left\{i\omega M \cdot \frac{[Y_{12}\cdot Y_{21} - Y_{22}\cdot Y_{11}]}{Y_{12}} - \frac{i\omega M\cdot Y_{11}\cdot Y}{Y_{12}} - \frac{[Y_{22}+Y]}{Y_{12}}\right\}^{-1} \tag{2.6.12.7}$$

The Eqs. (2.6.12.5)–(2.6.12.7) give

$$\begin{aligned} F_2 =&F\cdot\left\{\frac{\omega^2 M\cdot\sin(k_l L)}{SEk_l} - \cos(k_l L)\right.\\ &\left.+ i\cdot\left[\frac{YSEk_l\cdot\sin(k_l L)}{\omega} - \omega M\mathrm{Y}\cdot\cos(k_l L)\right]\right\}^{-1} \end{aligned} \tag{2.6.12.8}$$

The power induced in the plate is

$$\bar{\Pi}_{\text{in}} = \frac{1}{2}\cdot\mathrm{Re}\cdot\left\{F_2\cdot v_2^*\right\} = \frac{|F_2|^2}{2}\cdot\mathrm{Re}Y$$

2.6.13 Example 6.13

The displacements in the beams are (Fig. 2.18)

Beam 1

$$\xi_1 = A_1\sin(k_1 x) + B_1\cos(k_1 x);\quad 0\leqslant x\leqslant L_1 \tag{2.6.13.1}$$

Beam 2

$$\xi_2 = A_2\sin[k_2(x-L_1)] + B_2\cos[k_2(x-L_1)];\quad L_1\leqslant x\leqslant L_1+L_2 \tag{2.6.13.2}$$

The boundary conditions are:

$$\xi_1 = 0\quad\text{for}\quad x=0 \tag{2.6.13.3}$$

$$\xi_1 = \xi_2\quad\text{for}\quad x=L_1 \tag{2.6.13.4}$$

$$S_1E_1\frac{\partial\xi_1}{\partial x} = S_2E_2\frac{\partial\xi_2}{\partial x}\quad\text{for}\quad x=L_1 \tag{2.6.13.5}$$

$$F = S_2E_2\frac{\partial\xi_2}{\partial x}\quad\text{for}\quad x=L_1+L_2 \tag{2.6.13.6}$$

The boundary conditions (2.6.13.3)–(2.6.13.6) give when introducing $\alpha = k_1 L_1$ and $\beta = k_2 L_2$

$$B_1 = 0 \tag{2.6.13.7}$$

$$A_1 \sin\alpha = B_2 \tag{2.6.13.8}$$

$$S_1 E_1 k_1 A_1 \cos\alpha = S_2 E_2 k_2 A_2 \tag{2.6.13.9}$$

$$S_2 E_2 k_2 \left[A_2 \cos\beta - B_2 \sin\beta\right] = F \tag{2.6.13.10}$$

The Eqs. (2.6.13.8)–(2.6.13.10) give

$$A_1 = F/\left[S_1 E_1 k_1 \cos\alpha \cos\beta - S_2 E_2 k_2 \sin\alpha \sin\beta\right] \tag{2.6.13.11}$$

The velocity v_1 at the junction between the two beams is $v_1(L_1) = i\omega\xi_1(L_1)$. Thus from Eqs. (2.6.13.1), (2.6.13.7) and (2.6.13.11)

$$v_1(L_1) = i\omega F \sin\alpha/\left[S_1 E_1 k_1 \cos\alpha \cos\beta - S_2 E_2 k_2 \sin\alpha \sin\beta\right] \tag{2.6.13.12}$$

2.6.14 Example 6.14

The external force exciting the beam at $x = L/2$ is $F(t) = F_0 \exp(i\omega t)$. As defined by Eq. (6.110) a transfer matrix $[A]$ for a straight and homogeneous beam of length $L/2$ is given by

$$[A] = \begin{bmatrix} \cos\alpha & -i\omega \sin\alpha/\Lambda \\ -i\Lambda \sin\alpha/\omega & \cos\alpha \end{bmatrix} \tag{2.6.14.1}$$

where $\alpha = kL/2$ and $\Lambda = SEk$ and k the wavenumber for L-waves. The velocities v_1 and v_2 are equal to zero for the beam being clamped. The displacement of beam 1 is

$$\xi(x,t) = \frac{1}{i\omega} \cdot \frac{v_0(t)\sin(kx)}{\sin\alpha} \tag{2.6.14.2}$$

where $v_0(t)$ is the velocity at the excitation point. According to Eq. (6.116) velocities and forces shown in Fig. 2.19 are related as

$$\begin{Bmatrix} v_2 \\ F_2 \end{Bmatrix} = [A]\cdot[A] \begin{Bmatrix} v_1 \\ F_1 \end{Bmatrix} + [A] \begin{Bmatrix} 0 \\ F \end{Bmatrix} \tag{2.6.14.3}$$

where F is the external force. However the beam is clamped. Thus $v_1 = v_2 = 0$. Equation (2.6.14.3) gives

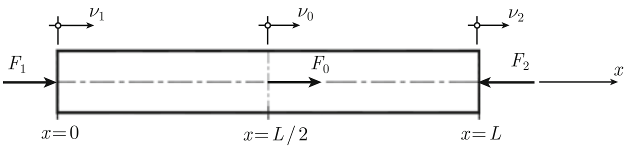


Fig. 2.19 A clamped beam excited at midpoint

$$F_1 = -F \cdot \sin\alpha / \sin(2\alpha) \tag{2.6.14.4}$$

$$\begin{aligned} F_2 &= F_1 \cdot \cos(2\alpha) + F \cdot \cos\alpha = F\,[\cos\alpha - \sin\alpha \cdot \cos(2\alpha)/\sin(2\alpha)] \\ &= F \cdot \sin\alpha / \sin(2\alpha) \end{aligned} \tag{2.6.14.5}$$

For beam 1

$$\begin{Bmatrix} v_0 \\ F_x \end{Bmatrix} = [A] \cdot \begin{Bmatrix} v_1 \\ F_1 \end{Bmatrix} \tag{2.6.14.6}$$

where F_x is the force acting on the beam at the excitation point. For v_1 Eq. (2.6.14.6) gives

$$v_0 = \cos\alpha \cdot v_1 - i\omega \sin\alpha F_1/\Lambda = i\omega(\sin\alpha)^2 F/[\sin(2\alpha)] \tag{2.6.14.7}$$

The displacement for beam element 1 is obtained from Eqs. (2.6.14.2) and (2.6.14.7) as

$$\xi(x,t) = \frac{1}{i\omega} \cdot \frac{v_0}{\sin\alpha} \cdot \sin(kx) = \frac{F(t)\sin(kx)}{2\Lambda\cos\alpha} = \frac{F_0 \exp(i\omega t)\sin(kx)}{2SEk\cos\alpha} \tag{2.6.14.8}$$

2.6.15 Example 6.15

According to Eq. (6.17), the eigenfunctions are orthogonal if the function I is equal to zero for $m \neq n$ where I is defined as

$$I = \left[\varphi_m \frac{\mathrm{d}\varphi_n}{\mathrm{d}x} - \varphi_n \frac{\mathrm{d}\varphi_m}{\mathrm{d}x}\right]_0^L \tag{2.6.15.1}$$

The boundary conditions are

$$\frac{\mathrm{d}\varphi_m}{\mathrm{d}x} = q\varphi_m \quad \text{for } x = 0; \qquad \frac{\mathrm{d}\varphi_m}{\mathrm{d}x} = -q\varphi_m \quad \text{for } x = L \tag{2.6.15.2}$$

The boundary conditions (2.6.15.2) inserted in (2.6.15.1) result in $I = 0$, i.e. $\int_0^L \varphi_m\varphi_n \mathrm{d}x = 0$.

For $m = n$ and $\varphi_m = \sin(k_n x + \alpha_n)$

$$\int_0^L \varphi_m \varphi_m \mathrm{d}x = \int_0^L \sin^2(k_n x + \alpha_n)\mathrm{d}x = \frac{L}{2} - \frac{1}{4k_n}\left[\sin(4\alpha_n) - \sin(2\alpha_n)\right]$$

where $\alpha_n = k_n L$ and according to Problem 6.3

$$\tan(\alpha_n) = \frac{2 \cdot \dfrac{SE\alpha_n}{Lk_s}}{\left[\left(\dfrac{SE\alpha_n}{Lk_s}\right)^2 - 1\right]}$$

Thus $\int_0^L \varphi_m \varphi_n \mathrm{d}x = 0$ for $m \neq n$ and $\int_0^L \varphi_m \varphi_n \mathrm{d}x \neq 0$ for $m = n$. Consequently, the eigenfunctions are orthogonal.

2.6.16 Example 6.16

The cross-sectional area of beam I is S and for beam II $2S$. The length, Young's modulus and density of each beam are denoted L, E and ρ respectively. The wavenumber k for longitudinal waves is defined as $k = \left(\frac{\omega^2 \rho}{E}\right)^{1/2}$. The quantity kL is defined as α.

Beam I
The point and transfer mobilities for beam I are defined as

$$Y_{11}^I = Y_{22}^I = -\frac{i\omega}{SEk \tan\alpha} = Y_A \tag{2.6.16.1}$$

$$Y_{12}^I = Y_{21}^I = -\frac{i\omega}{SEk \sin\alpha} = Y_B \tag{2.6.16.2}$$

Beam II
The point and transfer mobilities for beam II are defined as

$$Y_{11}^{II} = Y_{22}^{II} = -\frac{i\omega}{2SEk \tan\alpha} = Y_A/2 \tag{2.6.16.3}$$

$$Y_{12}^{II} = Y_{21}^{II} = -\frac{i\omega}{2SEk \sin\alpha} = Y_B/2 \tag{2.6.16.4}$$

The velocities and forces at the ends of the beams are related as

Beam I

$$v_1 = F_1 Y_{11}^I + F_x Y_{21}^I = F_1 Y_A + F_x Y_B \tag{2.6.16.5}$$

$$v_x = F_1 Y_{12}^I + F_x Y_{22}^I = F_1 Y_B + F_x Y_A \tag{2.6.16.6}$$

Beam II

$$v_x = -F_x Y_{11}^{II} + F_2 Y_{21}^{II} = -F_x Y_A/2 + F_2 Y_B/2 \tag{2.6.16.7}$$

$$v_2 = F_2 Y_{22}^{II} - F_x Y_{12}^{II} = -F_x Y_B/2 + F_2 Y_A/2 \tag{2.6.16.8}$$

Equations (2.6.16.6) and (2.6.16.7) give

$$F_x = \frac{F_2 Y_B - 2F_1 Y_B}{3Y_A} \tag{2.6.16.9}$$

Equations (2.6.16.9) and (2.6.16.5) give

$$v_1 = F_1 \left[\frac{3Y_A^2 - 2Y_B^2}{3Y_A} \right] + F_2 \frac{Y_B^2}{3Y_A} \tag{2.6.16.10}$$

Equations (2.6.16.9) and (2.6.16.8) give

$$v_2 = F_2 \left[\frac{3Y_A^2 - Y_B^2}{6Y_A} \right] + F_1 \frac{Y_B^2}{3Y_A} \tag{2.6.16.11}$$

According to definition, the velocities at the two ends of the total structure can also be written as

$$v_1 = F_1 Y_{11}^{\text{tot}} + F_2 Y_{21}^{\text{tot}} \quad \text{and } v_2 = F_1 Y_{12}^{\text{tot}} + F_2 Y_{22}^{\text{tot}} \tag{2.6.16.12}$$

Equations (2.6.16.10), (2.6.16.11), (2.6.16.12) and (2.6.16.1) and (2.6.16.2) give

$$Y_{11}^{\text{tot}} = \left[\frac{3Y_A^2 - 2Y_B^2}{3Y_A} \right] = -\frac{i\omega(3\cos^2\alpha - 2)}{3SEk \sin\alpha\cos\alpha}$$

$$Y_{21}^{\text{tot}} = Y_{12}^{\text{tot}} = \frac{Y_B^2}{3Y_A} = -\frac{i\omega}{3SEK \sin\alpha\cos\alpha}$$

$$Y_{22}^{\text{tot}} = \left[\frac{3Y_A^2 - Y_B^2}{6Y_A} \right] = \left[\frac{3Y_A^2 - 2Y_B^2}{3Y_A} \right] = -\frac{i\omega(3\cos^2\alpha - 1)}{6SEk \sin\alpha\cos\alpha}$$

2.7 Chapter 7

2.7.1 Example 7.1

The eigenfunction $\varphi_n(x)$ should satisfy the differential Eq. (7.3) or

$$\frac{\mathrm{d}^4\varphi_n}{\mathrm{d}x^4} + \kappa_n^4\varphi_n = 0 \tag{2.7.1.1}$$

The general solution to Eq. (2.7.1.1) is

$$\varphi_n(x) = A_1 \sin(\kappa_n x) + A_2 \cos(\kappa_n x) + A_3 \sinh(\kappa_n x) + A_4 \cosh(\kappa_n x) \tag{2.7.1.2}$$

The boundary conditions for a sliding edge are

$$\varphi_n'(x) = \varphi_n''' = 0 \text{ for } x = 0 \text{ and } x = L \tag{2.7.1.3}$$

Introduce $\beta = kL$. The boundary conditions in combination with Eq. (2.7.1.1) give for x=0

$$A_1 + A_3 = 0 \tag{2.7.1.4}$$

$$-A_1 + A_3 = 0 \tag{2.7.1.5}$$

Equations (2.7.1.4) and (2.7.1.5) give $A_1 = A_3 = 0$. The boundary conditions at $x = L$ give

$$-A_2 \sin\beta + A_4 \sinh\beta = 0 \tag{2.7.1.6}$$

$$A_2 \sin\beta + A_4 \sinh\beta = 0 \tag{2.7.1.7}$$

Equations (2.7.1.6) and (2.7.1.7) give $A_4 \sinh\beta = 0$. This is only satified if $A_4 = 0$. For Eqs. (2.7.1.6) and (2.7.1.7) to equal zero $\sin\beta$ must also equal zero. This condition is satisfied for $\beta = n\pi$. Thus the eigenfunctions are $\varphi_n = \cos(k_n x)$; $k_n = n\pi/L$ for $n = 0, 1, 2, \ldots$.

The eigen function φ_n is orthogonal since

$$\int \varphi_m(x) \cdot \varphi_n(x)\mathrm{d}x = \int_0^L \cos\left(\frac{m\pi x}{L}\right) \cdot \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = 0 \quad \text{for } m \neq n$$

For $m = n > 0$

$$\int \varphi_m(x) \cdot \varphi_n(x)\mathrm{d}x = L/2$$

For $m = n = 0$

$$\int \varphi_m(x) \cdot \varphi_n(x)\mathrm{d}x = L$$

2.7.2 Example 7.2

The centre frequencies f_c for the octave bands are 63, 125, 250, 500, 1000, 2000, 4000, and 8000 Hz. The upper frequency limit for an octave band is $f_c \cdot \sqrt{2}$. The lower limit is $f_c/\sqrt{2}$. The natural frequencies are obtained from Eq. (7.14) as

$$f_n = \omega_n/(2\pi) = \kappa_n^2 \cdot L^2 \cdot C \tag{2.7.2.1}$$

where C is a constant. According to Table 7.2

$$L\kappa_1 = 4.73004 \quad \text{for } f_1 = 52\,\text{Hz} \quad \text{Eq. (2.7.2.1)} \Rightarrow C = 2.3242$$

The consecutive natural frequencies are obtained from Eq. (2.7.2.1) and Table 7.2.

$$L\kappa_2 = 7.853 \quad \Rightarrow \quad f_2 = 143\,\text{Hz}$$

$$L\kappa_3 = 10.996 \quad \Rightarrow \quad f_3 = 271\,\text{Hz}$$

$$L\kappa_n = (2n+1)\cdot\pi/2 \quad \Rightarrow \quad f_n = C \cdot \frac{(2n+1)^2 \cdot \pi^2}{4} \quad \Rightarrow$$

$$(2n+1) = \frac{2}{\pi}\sqrt{\frac{f}{C}} \quad \Rightarrow \quad n = \frac{1}{\pi}\sqrt{\frac{f}{C}} - \frac{1}{2}$$

The number of modes for frequencies below f is

$$n = \text{Int} \cdot \left(\frac{1}{\pi}\sqrt{\frac{f}{C}} - \frac{1}{2}\right)$$

The number of modes in an octave band with centre frequency f_c is (Table 2.1)

$$n_{\Delta f} = \text{Int} \cdot \left[\frac{1}{\pi}\sqrt{\frac{f_c \cdot \sqrt{2}}{C}} - \frac{1}{2}\right] - \text{Int} \cdot \left[\frac{1}{\pi}\sqrt{\frac{f_c}{C \cdot \sqrt{2}}} - \frac{1}{2}\right]$$

Table 2.1 Number of modes per OB

f_c/Hz	63	125	250	500	1000	2000	4000	8000
$n_{\Delta\mathrm{f}}$	1	1	1	2	2	3	5	6

2.7.3 Example 7.3

The beam is excited by a point force at $x = x_1$. The one-sided power spectral density of the force is G_{FF}. For white noise excitation G_{FF} is constant. The time average of the total energy $\bar{\mathcal{E}}_n$ of mode n is according to Eq. (7.66) given by

$$\bar{\mathcal{E}}_n = G_{\mathrm{FF}} \cdot \varphi_n^2(x_1) \cdot \frac{1}{4M_n \cdot \omega_{0n}\eta} \tag{2.7.3.1}$$

The modal mass M_n is $M_n = m'L/2$. The frequency band Δf includes N natural frequencies where

$$N = \Delta f \cdot \mathcal{N}_f = \Delta f \cdot \left\{\frac{m'}{D'_0}\right\}^{\frac{1}{4}} \cdot \frac{L}{\sqrt{2\pi f}} \tag{2.7.3.2}$$

$\mathcal{N}_f$ is the modal density defined in Eq. (7.20) and f the centre frequency of the frequency band.

The average of $\varphi_n^2(x_1) = \sin^2\left(\frac{n\pi x_1}{L}\right)$ is $\frac{1}{2}$. The total energy within the band is

$$\bar{\mathcal{E}}_{\Delta f} = N \cdot \bar{\mathcal{E}}_n = \Delta f \cdot \left(\frac{m'}{D'_0}\right)^{1/4} \cdot \frac{G_F}{\sqrt{2\pi f} \cdot 8\pi \cdot m' f \eta} \tag{2.7.3.3}$$

The power input to mode n is according to Eq. (7.65)

$$\bar{\Pi}_n = G_F \varphi_n^2(x_1)/(2m'L) \tag{2.7.3.4}$$

The power input within the frequency band Δf with the centre frequency f is

$$\bar{\Pi}_{\Delta f} = N\bar{\Pi}_n \tag{2.7.3.5}$$

Again it has been assumed that the average of $\varphi_n^2(x_1) = \sin^2(n\pi x_1/L)$ is 1/2.

The Eqs. (2.7.3.2), (2.7.3.4), and (2.7.3.5) give

$$\bar{\Pi}_{\Delta f} = N\frac{G_{FF}}{4m'L} = \Delta f \cdot \left(\frac{m'}{D'_0}\right)^{1/4} \frac{G_{FF}}{\sqrt{2\pi f} \cdot 4m'} \tag{2.7.3.6}$$

The Eqs. (2.7.3.3) and (2.7.3.6) give

$$\bar{\Pi}_{\Delta f} = \omega_{0n}\eta\bar{E}_{\Delta f} \tag{2.7.3.7}$$

2.7.4 Example 7.4

The one-dimensional equation governing L-waves is

$$SE \cdot \frac{\partial^2 \xi}{\partial x^2} - \rho S \cdot \frac{\partial^2 \xi}{\partial t^2} = -F' \tag{2.7.4.1}$$

A beam is clamped at both ends. The eigenfunction φ_n is $\varphi_n(x) = \sin(n\pi x/L)$. The eigenvalue k_n of the eigenfunction is $k_n = n\pi/L$. Thus $\varphi''_n = -k_n^2\varphi_n$. The displacement ξ is expanded along the eigenfunctions as

$$\xi(x,t) = \sum \varphi_n(x) \cdot g_n(t) \tag{2.7.4.2}$$

The displacement ξ and the eigenfunction φ_n satisfy the same boundary conditions. Equation (2.7.4.2) can therefore be inserted in Eq. (2.7.4.1). The result is

$$\sum_n (-SE \cdot g_n \cdot k_n^2 \cdot \varphi_n - \rho S \cdot \varphi_n \cdot \ddot{g}_n) = -F' \tag{2.7.4.3}$$

Multiply by φ_n and integrate over x

$$g_n \langle \varphi_n \mid \varphi_n \rangle SE \cdot g_n \cdot k_n^2 + \rho S \langle \varphi_n \mid \varphi_n \rangle \ddot{g}_n = \langle F' \mid \varphi_n \rangle \tag{2.7.4.4}$$

The norm of the eigenvector is $\langle \varphi_n | \varphi_n \rangle = L/2$. Together with Eq. (2.7.4.4) this gives

$$g_n \cdot \frac{SELk_n^2}{2} + \rho\frac{SL}{2} \cdot \ddot{g}_n = \langle F' \mid \varphi_n \rangle \tag{2.7.4.5}$$

Introducing the modal stiffness K_n, modal mass M_n and modal force F_n Eq. (2.7.4.5) is written

$$g_n \cdot K_n + \ddot{g}_n \cdot M_n = F_n \tag{2.7.4.6}$$

By identifying the modal parameters using the Eqs. (2.7.4.5) and (2.7.4.6) the result is

$$K_n = SELk_n^2/2 = SE(n\pi)^2/(2L)\,; \quad M_n = \rho SL/2 = M/2 \tag{2.7.4.7}$$

$$F_n = \langle F' \mid \varphi_n \rangle = \int_0^L F'(x)\cdot\varphi_n(x)\cdot \mathrm{d}\,x \tag{2.7.4.8}$$

where M is the total mass of the beam.

2.7.5 Example 7.5

The forces are random and uncorrelated. The system—the vibrating beam is linear. The power spectral density of the velocity is therefore equal to the sum of the power spectral densities induced by each force. The response $w_1(x,t)$ of the beam caused by the force $F_1 e^{i\omega t}$ is written $w_1(x,t) = y_1(x)e^{i\omega t}$. If F_1 is the FT of the force function then y_1 is the FT of the displacement. The function $y_1(x)$ should satisfy the differential Eq. (7.30). Thus

$$\frac{\mathrm{d}^4 y_1}{\mathrm{d}x^4} - \kappa^4\cdot y_1 = \frac{F_1}{D'}\delta\left(x - \frac{L}{4}\right) \tag{2.7.5.1}$$

The eigenfunction for a simply supported beam is

$$\varphi_n(x) = \sin(\kappa_n x); \quad \kappa_n = n\pi/L \tag{2.7.5.2}$$

The eigenfunctions are orthogonal. The eigenfunction satisfies the differential equation

$$\mathrm{d}^4\varphi_n/\mathrm{d}x^4 - \kappa_n^4\varphi_n = 0 \tag{2.7.5.3}$$

The function $y_1(x)$ is written

$$y_1(x) = \sum C_n\cdot\varphi_n(x) \tag{2.7.5.4}$$

The displacement and the eigenfunction satisfy the same boundary conditions, i.e. Eq. (2.7.5.4) can be inserted in Eq. (2.7.5.1). The result using Eq. (2.7.5.3) is

$$\sum C_n\varphi_n(\kappa_n^4 - \kappa^4) = [F_1\delta(x - L/4)]/D' \tag{2.7.5.5}$$

Equation (2.7.5.5) is multiplied by φ_n and integrated over x. The norm of the eigenvector is $L/2$. Thus

$$(C_n L/2) \cdot (\kappa_n^4 - \kappa^4) = F_1 \varphi_n(L/4)/D' \quad \Rightarrow$$
$$C_n = \frac{2F_1 \varphi_n(L/4)}{D' L(\kappa_n^4 - \kappa^4)} \tag{2.7.5.6}$$

The velocity $v_1(x, t)$ of the beam due to F_1 is $v_1 = i\omega y_1 e^{i\omega t}$. The FT of the velocity of the beam is

$$\hat{v}_1(x, \omega) = i\omega y(x) = \sum i\omega C_n \cdot \varphi_n$$

The space average of the square of FT of velocity is $\langle \hat{v}^2 \rangle = \dfrac{1}{L} \int_0^L \hat{v}^2 \mathrm{d}x$ which gives

$$\langle \hat{v}_1^2 \rangle = \frac{1}{2} \sum \omega^2 \left|C_n\right|^2 = \frac{2F_1^2 \omega^2}{(D'L)^2} \sum \frac{\varphi_n^2(L/4)}{\left|\kappa_n^4 - \kappa^4\right|}$$

The one-sided power spectral density of the velocity is thus

$$G_{vv1} = G_{FF_1} \frac{2\omega^2}{(D'L)^2} \sum \frac{\varphi_n^2(L/4)}{\left|\kappa_n^4 - \kappa^4\right|^2} \tag{2.7.5.7}$$

By writing $\kappa^4 = m'(2\pi f)^2/D'$ and $\kappa_n^4 = m'(2\pi f_n)^2/D_0'$ where f_n are the natural frequencies of the simply—supported beam Eq. (2.7.5.7) is rewritten as

$$G_{vv1} = G_{FF_1} \frac{2f^2}{M^2(2\pi)^2} \sum \frac{\varphi_n^2(L/4)}{\left|(f_n^2 - f^2)^2 - (f_n^2 \eta)^2\right|} \tag{2.7.5.8}$$

where $M = m'L$.

The power spectral density G_{vv2} due to the force F_2 is obtained in a similar way as

$$G_{vv2} = G_{FF2} \frac{2f^2}{M^2(2\pi)^2} \sum \frac{\varphi_n^2(3L/4)}{\left|(f_n^2 - f^2)^2 - (f_n^2 \eta)^2\right|} \tag{2.7.5.9}$$

However, $G_{FF1} = G_{FF2} = G_{FF}$ and $\varphi_n^2(L/4) = \varphi_n^2(3L/4)$. The total power spectral density G_v is thus

$$G_{vv} = G_{vv1} + G_{vv2} = 4G_{FF} \frac{f^2}{M^2(2\pi)^2} \sum \frac{\varphi_n^2(3L/4)}{\left|(f_n^2 - f^2)^2 - (f_n^2 \eta)^2\right|} \tag{2.7.5.10}$$

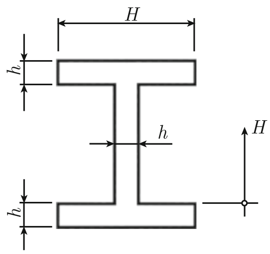


Fig. 2.20 Cross section of beam

where

$$\begin{aligned}\varphi_n^2(3L/4) &= \sin^2(3n\pi/4) = 0 && \text{for } n = 0, 4, 8, \ldots \\ \varphi_n^2(3L/4) &= \sin^2(3n\pi/4) = 1/2 && \text{for } n = 1, 3, 5, \ldots \\ \varphi_n^2(3L/4) &= \sin^2(3n\pi/4) = 1 && \text{for } n = 2, 6, 10, \ldots\end{aligned} \tag{2.7.5.11}$$

2.7.6 Example 7.6

The mass per unit length is

$$m' = 3H \cdot h \cdot \rho. \tag{2.7.6.1}$$

The neutral axis is at the symmetry plane of the beam. See Fig. 2.20. The bending stiffness D' of the beam is

$$D' = E \cdot \left\{ \int_{-H/2}^{H/2} \mathrm{d}y \cdot h \cdot y^2 + H \cdot h \cdot \frac{H^2}{4} \right\} = \frac{EhH^3}{3} \tag{2.7.6.2}$$

The first natural frequency for a clamped beam is obtained when $\kappa L = 4.73$—Table 7.2
The wavenumber κ is $\kappa = \left[m'\omega^2/D'\right]^{1/4}$. The first natural frequency is

$$f_1 = \frac{1}{2\pi}\left(\frac{4.73}{L}\right)^2 \cdot \left\{\frac{D'}{m'}\right\}^{1/2} \tag{2.7.6.3}$$

For a steel beam $E = 2.1 \times 10^{11}$ N/m^2 and $\rho = 7600$ kg/m^3. The length of the beam is 5 m. The first natural frequency of the beam is obtained from Eq. (2.7.6.3) as

$$f_1 = \frac{1}{2\pi} \cdot \left(\frac{4.73}{5}\right)^2 \cdot \left\{\frac{EH^2}{9\rho}\right\}^{1/2} = 12\,\text{Hz}$$

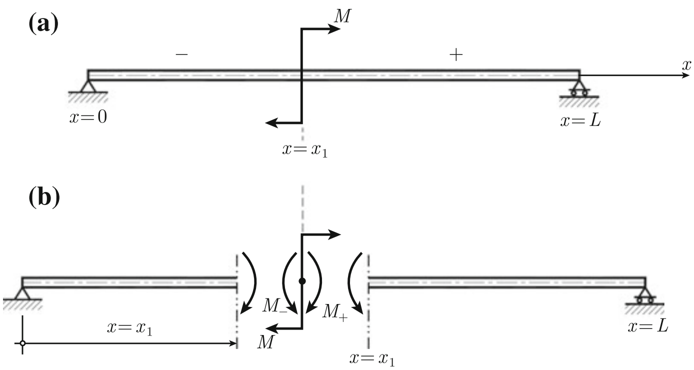


Fig. 2.21 A simply supported beam excited by a bending moment

2.7.7 Example 7.7

The displacement of the beam is given by

$$w(x,t) = y(x) \cdot e^{i\omega t} \tag{2.7.7.1}$$

The displacement to the left of the bending moment exciting the beam is given $y_-(x)$. See Fig. 2.21. The solution to the right of the excitation point is $y_+(x)$. The solutions can according to Eqs. (7.31) and (7.32) be expressed as

$$y_- = A_1 \sin(\kappa x) + A_2 \cos(\kappa x) + A_3 \sinh(\kappa x) + A_4 \cosh(\kappa x) \tag{2.7.7.2}$$

$$\begin{aligned} y_+ = {} & B_1 \sin[\kappa(L-x)] + B_2 \cos[\kappa(L-x)] + B_3 \sinh[\kappa(L-x)] \\ & + B_4 \cosh[\kappa(L-x)] \end{aligned} \tag{2.7.7.3}$$

The beam is simply supported at each end. The boundary conditions at $x = 0$ and $x = L$ are

$$y_-(0) = y_-''(0) = 0 \tag{2.7.7.4}$$

$$y_+(L) = y_+''(L) = 0 \tag{2.7.7.5}$$

The Eqs. (2.7.7.2) through (2.7.7.5) give

$$A_2 = A_4 = B_2 = B_4 = 0 \tag{2.7.7.6}$$

The boundary conditions at $x = x_1$ are

$$y_- = y_+ \tag{2.7.7.7}$$

$$y_-' = y_+' \tag{2.7.7.8}$$

The sum of the bending moments should equal zero or $M = M_{-} - M_{+}$ resulting in the boundary condition

$$M = D' \left[\frac{\mathrm{d}^2 y_+}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 y_-}{\mathrm{d}x^2} \right]_{x=x_1} \tag{2.7.7.9}$$

There is no external force. Thus,

$$\frac{\mathrm{d}^3 y_+}{\mathrm{d}x^3} = \frac{\mathrm{d}^3 y_-}{\mathrm{d}x^3} \quad \text{for } x = x_1 \tag{2.7.7.10}$$

Let $\alpha = \kappa x_1$ and $\beta = \kappa(L - x_1)$. The boundary conditions (2.7.7.7)–(2.7.7.10) give:

$$A_1 \sin\alpha + A_3 \sinh\alpha = B_1 \sin\beta + B_3 \sinh\beta \tag{2.7.7.11}$$

$$A_1 \cos\alpha + A_3 \cosh\alpha = -B_1 \cos\beta - B_3 \cosh\beta \tag{2.7.7.12}$$

$$M = D'\kappa^2 \{A_1 \sin\alpha - A_3 \sinh\alpha - B_1 \sin\beta + B_3 \sinh\beta\} \tag{2.7.7.13}$$

$$-A_1 \cos\alpha + A_3 \cosh\alpha = B_1 \cos\beta - B_3 \cosh\beta \tag{2.7.7.14}$$

The solutions to Eqs. (2.7.7.11)–(2.7.7.12) are

$$A_1 = \frac{M \cdot \cos\beta}{2D'\kappa^2 \cdot \sin(\kappa L)}; \quad B_1 = -\frac{M \cdot \cos\alpha}{2D'\kappa^2 \cdot \sin(\kappa L)} \tag{2.7.7.15}$$

$$A_3 = -\frac{M \cdot \cosh\beta}{2D'\kappa^2 \cdot \sinh(\kappa L)}; \quad B_3 = \frac{M \cdot \cosh\alpha}{2D'\kappa^2 \cdot \sinh(\kappa L)} \tag{2.7.7.16}$$

The results (2.7.7.2), (2.7.7.6), (2.7.7.15), and (2.7.7.16) give the displacement of the beam.

2.7.8 Example 7.8

The displacement of the beam can by using Green's function and Eq. (7.42) be written as:

$$\begin{aligned} w(x,t) = \int_0^L F'(\zeta,t) \cdot G(\zeta\,|x)\,\mathrm{d}\zeta = \int_0^x F'(\zeta,t) \cdot G_1(\zeta\,|x)\,\mathrm{d}\zeta \\ + \int_x^L F'(\zeta,t) \cdot G_2(\zeta\,|x)\,\mathrm{d}\zeta \end{aligned} \tag{2.7.8.1}$$

For a beam with simply supported ends Green's function is defined in Eq. (7.40). The force F' per unit length exciting the beam is

$$F'(x,t) = (F/L)\sin(\pi x/L)\cdot\exp(i\omega t) \tag{2.7.8.2}$$

Equations (2.7.8.1) and (2.7.8.2) in combination with Eq. (7.40) give

$$\begin{aligned}
w(x,t) =& \frac{e^{i\omega t}}{2D'\kappa^3}\cdot\frac{F}{L}\left\{\frac{\sin\kappa(L-x)}{\sin(\kappa L)}\cdot\int_0^x \sin\left(\frac{\pi\varsigma}{L}\right)\cdot\sin\kappa\varsigma\cdot d\varsigma\right.\\
&-\frac{\sinh\kappa(L-x)}{\sinh(\kappa L)}\cdot\int_0^x \sin\left(\frac{\pi\varsigma}{L}\right)\cdot\sinh(\kappa\varsigma)\cdot d\varsigma\\
&+\frac{\sin(\kappa x)}{\sin(\kappa L)}\cdot\int_x^L \sin\left(\frac{\pi\varsigma}{L}\right)\cdot\sin\kappa(L-\varsigma)\cdot d\varsigma\\
&\left.-\frac{\sinh(\kappa x)}{\sin(\kappa L)}\cdot\int_x^L \sin\left(\frac{\pi\varsigma}{L}\right)\cdot\sinh\kappa(L-\varsigma)\cdot d\varsigma\right\}\\
=& \frac{F\cdot e^{i\omega t}\sin(\kappa x)}{LD'\cdot[(\pi/L)^4-\kappa^4]}
\end{aligned}$$

2.7.9 Example 7.9

The force per unit length exciting the beam is

$$F'(x,t) = (F/L)\sin(\pi x/L)\exp(i\omega t) = f(x)\exp(i\omega t) \tag{2.7.9.1}$$

The displacement is

$$w(x,t) = w(x)\cdot e^{i\omega t} \tag{2.7.9.2}$$

The function $w(x)$ should satisfy the equation

$$\frac{d^4 w}{dx^4} - \kappa^4 w = \frac{f}{D'};\quad f(x) = (F/L)\sin(\pi x/L) \tag{2.7.9.3}$$

The beam is simply supported. The displacement can be expanded in a series by means of the eiegenfunctions $\varphi_n(x)$ or

$$w = \sum C_n\cdot\varphi_n(x);\quad \varphi_n(x) = \sin(n\pi x/L) \tag{2.7.9.4}$$

The eigenfunctions must satisfy the differential equation

$$\frac{d^4\varphi_n}{dx^4} = \kappa_n^4\varphi_n;\quad \kappa_n = (n\pi/L) \tag{2.7.9.5}$$

The displacement $w(x)$ and the eigenfunction $\varphi_n(x)$ satisfy the same boundary conditions. Equation (2.7.9.4) can therefore be inserted in Eq. (2.7.9.3). The result, considering Eq. (2.7.9.5), is

$$\sum C_n\varphi_n \cdot (\kappa_n^4 - \kappa^4) = f/D' \tag{2.7.9.6}$$

The expression is multiplied by φ_n and integrated over the length of the beam

$$(C_n L/2) \cdot (\kappa_n^4 - \kappa^4) = \langle f \,|\varphi_n \rangle /D' \tag{2.7.9.7}$$

where

$$\begin{aligned} \langle f \mid \varphi_n\rangle &= \frac{F}{L} \cdot \int_0^L \sin\left(\frac{\pi x}{L}\right) \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot \mathrm{d}x = \frac{F}{2} \quad \text{for } n = 1 \\ \langle f \mid \varphi_n\rangle &= \frac{F}{L} \cdot \int_0^L \sin\left(\frac{\pi x}{L}\right) \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot \mathrm{d}x = 0 \quad \text{for } n \neq 1 \end{aligned} \tag{2.7.9.8}$$

For $n = 1$

$$C_n = C_1 = \frac{F}{LD' \cdot (\kappa_1^4 - \kappa^4)} \quad \Rightarrow \quad w(x,t) = \frac{F \cdot e^{i\omega t} \cdot \sin(\pi x/L)}{LD' \cdot (\kappa_1^4 - \kappa^4)}; \quad \kappa_1 = \pi/L$$

2.7.10 Example 7.10

The modal energy for the beam is according to Eq. (7.66)

$$\bar{\mathcal{E}}_n = \frac{G_{FF}\varphi_n^2(x_1)}{2m' L\omega_{0n}\eta} \tag{2.7.10.1}$$

The total energy within a frequency band, width Δf and centre frequency f is

$$\bar{\mathcal{E}} = \mathcal{N}_f \cdot \Delta f \cdot \langle \bar{\mathcal{E}}_n \rangle \tag{2.7.10.2}$$

where $\mathcal{N}_f$ is the modal density and $\langle \bar{\mathcal{E}}_n \rangle$ the average of $\bar{\mathcal{E}}_n$ with respect to the coordinates of the excitation point. The average of $\varphi_n^2(x_1)$ is 1/2. The modal density is given by Eq. (7.20) as

$$N_f = \left\{\frac{m'}{D'}\right\}^{1/4} \cdot \frac{L}{\sqrt{2\pi f}} \tag{2.7.10.3}$$

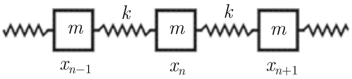


Fig. 2.22 An infinite number of coupled mass-spring systems

The Eqs. (2.7.10.1)–(2.7.10.3) give

$$\bar{\mathcal{E}} = \Delta f \cdot \left\{\frac{m'}{D'}\right\}^{1/4} \cdot \frac{G_{FF}}{\sqrt{2\pi f} \cdot 4m' \cdot \eta(2\pi f)} = \frac{C}{(m')^{3/4} \cdot (D')^{1/4} \cdot \eta} \tag{2.7.10.4}$$

where C is a constant. The ratio between energies before $\bar{\mathcal{E}}_1$ and after the changes $\bar{\mathcal{E}}_2$ is

$$\frac{\bar{\mathcal{E}}_1}{\bar{\mathcal{E}}_2} = \left[\frac{m'_2}{m'_1}\right]^{3/4} \left[\frac{D'_2}{D'_1}\right]^{1/4} \left[\frac{\eta_2}{\eta_1}\right] = [1.2]^{3/4}\,[1.4]^{1/4}\,10 \approx 12.5 \tag{2.7.10.5}$$

or $10 \cdot \log(\bar{\mathcal{E}}_1/\bar{\mathcal{E}}_2) = 11\,\text{dB}$.

2.7.11 Example 7.11

The equation of motion for mass n is (Fig. 2.22)

$$m\ddot{x}_n + k(x_n - x_{n+1}) + k(x_n - x_{n-1}) = 0 \tag{2.7.11.1}$$

Assume:

$$x_{n+1} = e^{i\varphi} \cdot x_n = z \cdot x_n \tag{2.7.11.2}$$

and

$$x_{n-1} = x_n/z; \quad |z| \leq 1 \tag{2.7.11.3}$$

Let the time dependence be $\exp(i\omega t)$. This gives $\ddot{x}_n = -\omega^2 x_n$. Considering this the basic equation (2.7.11.1) is written

$$-m\omega^2 + k(1 - z) + k(1 - 1/z) = 0 \tag{2.7.11.4}$$

The parameter ω_0 is defined as $\omega_0 = \sqrt{k/m}$. Equation (2.7.11.4) now reads

$$z^2 - z\left[2 - (\omega/\omega_0)^2\right] + 1 = 0 \tag{2.7.11.5}$$

There is no attenuation of the wave motion as long as $|z| = 1$

The solution to Eq. (2.7.11.5) is

$$z = 1 - (1/2)(\omega/\omega_0)^2 \pm \left[(1/4)(\omega/\omega_0)^4 - (\omega/\omega_0)^2\right]^{1/2} \tag{2.7.11.6}$$

The minus sign in front of the bracket must be neglected since $|z| \leqslant 1$ and $|\ x_{n+1}\ | \leqslant |\ x_n\ |$.

For $\omega > 2\omega_0$ $\ z$ is real and less than unity resulting in an attenuated wave.
For $\omega = 2\omega_0$ $\ z = 1$
For $0 < \omega < 2\omega_0$ $\ z$ is complex and equal to

$$z = 1 - (1/2)(\omega/\omega_0)^2 \pm i\left[(\omega/\omega_0)^2 - (1/4)(\omega/\omega_0)^4\right]^{1/2} \tag{2.7.11.7}$$

Consequently, $|z| = 1$. For $\omega = 0$, $\quad z = 1$.

Thus there is attenuation of the wave motion as long as $0 \leqslant \omega \leqslant 2\omega_0$.

If an infinite homogeneous beam exposed to L-waves is modelled as an infinite number of mass spring systems, each mass Δm and each stiffness Δk representing a section Δx of the beam would be

$$\Delta m = \rho S \Delta x; \quad \Delta k = ES/\Delta x \tag{2.7.11.8}$$

The cross-sectional area of the beam is S, the E-modulus E, and density ρ. The natural frequency ω_0 is obtained as $\omega_0^2 = \dfrac{\Delta k}{\Delta m} = \dfrac{E}{\rho(\Delta x)^2} \to \infty$ as $\Delta x \to 0$. The parameter z is obtained from (2.7.11.7) when excluding higher order terms in $1/\Delta x$ as

$$z = 1 \pm i\omega/\omega_0 = 1 \pm i\omega\Delta x\sqrt{\rho/E} \tag{2.7.11.9}$$

However, $z = e^{i\varphi} = e^{i\lambda\Delta x}$ where λ is a wavenumber for longitudinal waves propagating in an infinite beam. For $\Delta x \ll 1$

$$z = e^{i\varphi} = e^{i\lambda\Delta x} = 1 + i\lambda\Delta x \tag{2.7.11.10}$$

Equations (2.7.11.9) and (2.7.11.10) give $\lambda = \pm\omega\sqrt{\rho/E}$ which is equal to the wavenumber for L-waves propagating along the positive or negative axis of a slender beam.

2.7.12 *Example 7.12*

Let the time dependence be $e^{i\omega t}$. The equation of motion for mass n is (Fig. 2.23)

$$m\ddot{x}_n + k(2x_n - x_{n+1} - x_{n-1}) = F \cdot e^{i\omega t} \tag{2.7.12.1}$$

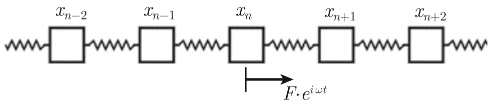


Fig. 2.23 An infinite number of coupled mass-spring systems. One mass excited by a force

The disturbances in the infinite chain are propagating away from the mass n.

On the right-hand side of the excitation point the displacements can, according to Floquet's theorem (7.99), be written

$$x_{n+1} = z \cdot x_n; \quad x_{n+2} = z \cdot x_{n+1} = z^2 \cdot x_n \tag{2.7.12.2}$$

It is required that $0 \leqslant |z| \leqslant 1$. On the left-hand side the disturbances are also propagating away from the excitation point. Thus again

$$x_{n-1} = z \cdot x_n; \quad x_{n-2} = z \cdot x_{n-1} = z^2 \cdot x_n \tag{2.7.12.3}$$

The Eqs. (2.7.12.1) through (2.7.12.3) give

$$F \cdot e^{i\omega t} = x_n \cdot \left\{ -m\omega^2 + 2k - 2kz \right\} \tag{2.7.12.4}$$

By introducing $\omega_0^2 = k/m$ the displacement x_n of the mass being excited is obtained from Eq. (2.7.12.4) as

$$x_n = \frac{F \cdot e^{i\omega t}}{2k\left[-\frac{1}{2}\left(\frac{\omega}{\omega_0}\right)^2 + 1 - z\right]} \tag{2.7.12.5}$$

The equation of motion for mass $n+1$ is

$$m\ddot{x}_{n+1} + k\left[2x_{n+1} - x_n - x_{n+2}\right] = 0$$

This equation in combination with Eq. (2.7.12.2) gives

$$x_{n+1} \cdot \left\{ -m\omega^2 + 2k - k/z - kz \right\} = 0 \tag{2.7.12.6}$$

For Eq. (2.7.12.6) to be satisfied it follows that

$$z^2 - 2z\left[1 - (\omega/\omega_0)^2/2\right] + 1 = 0 \tag{2.7.12.7}$$

Fig. 2.24 Beam simply supported at one end and free at the other

as already shown in Example 7.11. The solution to Eq. (2.7.12.7) is considering that $0 \leqslant |z| \leqslant 1$

$$z = 1 - [\omega/\omega_0]^2/2 + \sqrt{[\omega/\omega_0]^4/4 - [\omega/\omega_0]^2} \quad \text{for } \omega \geqslant 2\omega_0 \tag{2.7.12.8}$$

$$z = 1 - [\omega/\omega_0]^2/2 - i\sqrt{[\omega/\omega_0]^2 - [\omega/\omega_0]^4/4} \quad \text{for } 0 \leqslant \omega \leqslant 2\omega_0 \tag{2.7.12.9}$$

The displacement of the mass n is given by the results (2.7.12.5), (2.7.12.8) and (2.7.12.9) as

$$x_n = -\frac{F \cdot e^{i\omega t}}{2k\sqrt{[\omega/\omega_0]^4/4 - [\omega/\omega_0]^2}} \quad \text{for } \omega \geqslant 2\omega_0 \tag{2.7.12.10}$$

$$x_n = -\frac{iF \cdot e^{i\omega t}}{2k\sqrt{[\omega/\omega_0]^2 - [\omega/\omega_0]^4/4}} \quad \text{for } 0 \leqslant \omega \leqslant 2\omega_0 \tag{2.7.12.11}$$

When including losses k and ω_0 are complex, the displacement is therefore finite.

2.7.13 Example 7.13

The displacement of the beam is $w(x,t) = \varphi(x)g(t)$ where according to Eq. (7.5) φ is (Fig. 2.24)

$$\varphi = A_1 \sin(\kappa x) + A_2 \cos(\kappa x) + A_3 \sinh(\kappa x) + A_4 \cosh(\kappa x) \tag{2.7.13.1}$$

The first few derivatives of φ are

$$\varphi' = \kappa \cdot \{A_1 \cos(\kappa x) - A_2 \sin(\kappa x) + A_3 \cosh(\kappa x) + A_4 \sinh(\kappa x)\} \tag{2.7.13.2}$$
$$\varphi'' = k^2 \cdot \{-A_1 \sin(\kappa x) - A_2 \cos(\kappa x) + A_3 \sinh(\kappa x) + A_4 \cosh(\kappa x)\} \tag{2.7.13.3}$$
$$\varphi''' = \kappa^3 \cdot \{-A_1 \cos(\kappa x) + A_2 \sin(\kappa x) + A_3 \cosh(\kappa x) + A_4 \sinh(\kappa x)\} \tag{2.7.13.4}$$

The boundary conditions for the beam are

$$\varphi(0) = \varphi''(0) = 0 \tag{2.7.13.5}$$

$$\varphi''(L) = \varphi'''(L) = 0 \tag{2.7.13.6}$$

The Eqs. (2.7.13.1), (2.7.13.3), and (2.7.13.5) give

$$A_2 = A_4 = 0 \tag{2.7.13.7}$$

The Eqs. (2.7.13.1), (2.7.13.3), and (2.7.13.4) give with $\beta = \kappa L$

$$-A_1 \cdot \sin\beta + A_3 \cdot \sinh\beta = 0 \tag{2.7.13.8}$$

$$-A_1 \cdot \cos\beta + A_3 \cdot \cosh\beta = 0 \tag{2.7.13.9}$$

The Eqs. (2.7.13.8) and (2.7.13.9) are only satisfied when $\tan\beta = \tanh\beta$, i.e., there is an infinite number of solutions to this equation corresponding to the eigenvalues κ_n. The eigenvalues are the solutions to

$$\tan(\kappa_n L) = \tanh(\kappa_n L) \tag{2.7.13.10}$$

For $A_1 = 1$ the eigenfunctions are

$$\varphi_n = \sin(\kappa_n x) + \frac{\sinh(\kappa_n x) \cdot \sin(\kappa_n L)}{\sinh(\kappa_n L)} \tag{2.7.13.11}$$

2.7.14 Example 7.14

G and w satisfy the equations

$$D' \cdot \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - m'\omega^2 \cdot w = F'(x) \tag{2.7.14.1}$$

$$D' \cdot \frac{\mathrm{d}^4 G}{\mathrm{d}x^4} - m'\omega^2 \cdot G = \delta(x - x_0) \tag{2.7.14.2}$$

Equation (2.7.14.1) is multiplied by G and Eq. (2.7.14.2) by w. The result is

$$D' \cdot G \cdot \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - m'\omega^2 \cdot w \cdot G = F'(x) \cdot G(x\,|x_0) \tag{2.7.14.3}$$

$$D' \cdot w \cdot \frac{\mathrm{d}^4 G}{\mathrm{d}x^4} - m'\omega^2 \cdot w \cdot G = w \cdot \delta(x - x_0) \tag{2.7.14.4}$$

Equation (2.7.14.3) is subtracted from Eq. (2.7.14.4) resulting in

$$D'\left[w \cdot \frac{\mathrm{d}^4 G}{\mathrm{d}x^4} - G \cdot \frac{\mathrm{d}^4 w}{\mathrm{d}x^4}\right] = -F' \cdot G + w \cdot \delta(x - x_0) \tag{2.7.14.5}$$

Equation (2.7.14.5) is integrated with respect to x giving

$$I = D' \left\{ \int_0^L \mathrm{d}x \left[w \cdot \frac{\mathrm{d}^4 G}{\mathrm{d}x^4} - G \cdot \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} \right] \right\} = \int_0^L \mathrm{d}x \cdot w(x) \cdot \delta(x - x_0) - \int_0^L \mathrm{d}x F' G \tag{2.7.14.6}$$

Integrate the left-hand side by parts. This gives

$$\begin{aligned} I =& D \left[w \cdot G''' - G \cdot w''' - w' \cdot G'' + G' \cdot w'' \right]_0^L + \int \mathrm{d}x \cdot (w'' \cdot G'' - G'' \cdot w'') \\ =& \left[w \cdot G''' - G \cdot w''' - w' \cdot G'' + G' \cdot w'' \right]_0^L = 0 \end{aligned}$$

for any of the natural boundary conditions.
For $I = 0$ Eq. (2.7.14.6) gives

$$\int \mathrm{d}x \cdot w(x) \cdot \delta(x - x_0) = \int F'(x) \cdot G(x\,|x_0\,) \cdot \mathrm{d}x \text{ or}$$

$$w(x_0) = \int_0^L F'(x) \cdot G(x\,|x_0\,) \cdot \mathrm{d}x \tag{2.7.14.7}$$

Since $G(x\,|x_0\,) = G(x_0\,|x\,)$ it follows that

$$w(x) = \int_0^L \mathrm{d}\varsigma \cdot F'(\varsigma) \cdot G(x\,|\varsigma\,)$$

For $F' = F' \cdot e^{i\omega\, t} \quad \Rightarrow$

$$w(x,t) = e^{i\omega t} \int_0^L \mathrm{d}\varsigma \cdot F'(\varsigma) \cdot G(x\,|\varsigma\,) \tag{2.7.14.8}$$

2.7.15 Example 7.15

The transfer matrix for beam I is $[A]_1$. The elements a_{ij} are given in Eq. (7.95) with $\beta = \kappa\xi$. For beam II the matrix is $[A]_2$ with $\beta = \kappa(L - \xi)$. The wavenumber is κ. The field parameters at the support 2 are according to Eq. (7.98) (Fig. 2.25).

$$\begin{Bmatrix} w_2 \\ \Theta_2 \\ M_2 \\ F_2 \end{Bmatrix}_2 = [A]_2 \cdot [A]_1 \cdot \begin{Bmatrix} w_1 \\ \Theta_1 \\ M_1 \\ F_1 \end{Bmatrix}_1 + [A]_2 \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ F \end{Bmatrix} \tag{2.7.15.1}$$

However, $[A]_2 \cdot [A]_1 = [A]$ where $[A]$ is the transfer matrix for the entire beam. The elements a_{ij} are given in Eq. (7.95) by letting $\beta = \kappa L$. For a simply supported

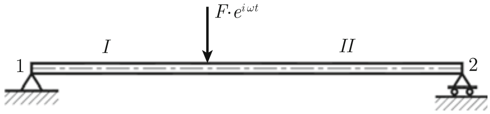


Fig. 2.25 A simply supported beam excited by a force

beam $w_1 = w_2 = M_1 = M_2 = 0$. These boundary conditions in combination with Eq. (2.7.15.1) give

$$\begin{Bmatrix} 0 \\ \Theta_2 \\ 0 \\ F_2 \end{Bmatrix}_2 = [A] \cdot \begin{Bmatrix} 0 \\ \Theta_1 \\ 0 \\ F_1 \end{Bmatrix}_1 + [A]_2 \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ F \end{Bmatrix} \tag{2.7.15.2}$$

This matrix equation can also be expressed as

$$0 = a_{12} \cdot \Theta_1 + a_{14} \cdot F_1 + (a_{14})_2 \cdot F \tag{2.7.15.3}$$
$$\Theta_2 = a_{22} \cdot \Theta_1 + a_{24} \cdot F_1 + (a_{24})_2 \cdot F \tag{2.7.15.4}$$
$$0 = a_{32} \cdot \Theta_1 + a_{34} \cdot F_1 + (a_{34})_2 \cdot F \tag{2.7.15.5}$$
$$F_2 = a_{42} \cdot \Theta_1 + a_{44} \cdot F_1 + (a_{44})_2 \cdot F \tag{2.7.15.6}$$

Equations (2.7.15.3) and (2.7.15.5) give

$$\begin{bmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{bmatrix} \cdot \begin{Bmatrix} \Theta_1 \\ F_1 \end{Bmatrix} = -F \cdot \begin{Bmatrix} (a_{14})_2 \\ (a_{34})_2 \end{Bmatrix} \Rightarrow$$
$$\begin{Bmatrix} \Theta_1 \\ F_1 \end{Bmatrix} = -F \cdot \begin{bmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{bmatrix}^{-1} \cdot \begin{Bmatrix} (a_{14})_2 \\ (a_{34})_2 \end{Bmatrix} \tag{2.7.15.7}$$

The Eqs. (2.7.15.4) and (2.7.15.6) give

$$\begin{Bmatrix} \Theta_2 \\ F_2 \end{Bmatrix} = \begin{bmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ F_1 \end{Bmatrix} + F \cdots \begin{Bmatrix} (a_{24})_2 \\ (a_{44})_2 \end{Bmatrix} \tag{2.7.15.8}$$

For $0 \leqslant x \leqslant \xi$ the displacement w of the beam is obtained from (7.91) and (7.93) as

$$w = A_1 \sin(\kappa x) + A_2 \cos(\kappa x) + A_3 \sinh(\kappa x) + A_4 \cosh(\kappa x)$$

For a simply supported beam $A_2 = A_4 = 0$ and from (7.93)

$$A_1 = \frac{1}{2}\left\{\frac{\Theta_1}{\kappa} + \frac{F_1}{D'\kappa^3}\right\}; \quad A_3 = \frac{1}{2}\left\{\frac{\Theta_1}{\kappa} - \frac{F_1}{D'\kappa^3}\right\} \quad \text{or}$$
$$\begin{Bmatrix} A_1 \\ A_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1/\kappa & 1/(D'\kappa^3) \\ 1/\kappa & -1/(D'\kappa^3) \end{bmatrix} \cdot \begin{Bmatrix} \Theta_1 \\ F_1 \end{Bmatrix} \tag{2.7.15.9}$$

For $\xi \leqslant x \leqslant L$ the displacement is

$$w = C_1 \sin \kappa(L-x) + C_2 \cos \kappa(L-x) + C_3 \sinh \kappa(L-x) + C_4 \cosh \kappa(L-x)$$

For a simply supported beam $C_2 = C_4 = 0$

$$\begin{Bmatrix} C_1 \\ C_3 \end{Bmatrix} = -\frac{1}{2} \begin{bmatrix} 1/\kappa & 1/(D'\kappa^3) \\ 1/\kappa & -1/(D'\kappa^3) \end{bmatrix} \cdot \begin{Bmatrix} \Theta_2 \\ F_2 \end{Bmatrix} \tag{2.7.15.10}$$

The parameters Θ_1, Θ_2, F_1 and F_1 are obtained from Eqs. (2.7.15.7) and (2.7.15.8).

2.7.16 Example 7.16

For a clamped-clamped beam

$$\varphi_n = \cosh(\kappa_n x) - \cos(\kappa_n x) - \frac{\cosh(\kappa_n L) - \cos(\kappa_n L)}{\sinh(\kappa_n L) - \sin(\kappa_n L)} \cdot [\sinh(\kappa_n x) - \sin(\kappa_n x)] \tag{2.7.16.1}$$

The eigenvalues κ_n are the solutions to

$$\cos(\kappa_n L) \cdot \cosh(\kappa_n L) = 1 \tag{2.7.16.2}$$

When $\kappa_n L \rightarrow \infty$ then $\cosh(\kappa_n L) \rightarrow \infty$. Thus, Eq. (2.7.16.2) is only satisfied if $\cos(\kappa_n L) \approx 0$ or when $\kappa_n L = \pi/2 + n\pi$ for n large.

The last part of Eq. (2.7.16.1) is written

$$\begin{aligned} I &= \frac{\cosh(\kappa_n L) - \cos(\kappa_n L)}{\sinh(\kappa_n L) - \sin(\kappa_n L)} = \frac{1 + e^{-2\beta_n} - 2 \cdot e^{-\beta_n} \cdot \cos(\kappa_n L)}{1 - e^{-2\beta_n} - 2 \cdot e^{-\beta_n} \cdot \sin(\kappa_n L)} \\ \beta_n &= \kappa_n L \approx \pi/2 + n\pi \quad \text{for } n \text{ large} \\ I &= \frac{1}{1 - 2 \cdot e^{-\beta_n} \cdot \sin \beta_n} \approx 1 + 2 \cdot e^{-\beta_n} \cdot \sin \beta_n \end{aligned} \tag{2.7.16.3}$$

Equations (2.7.16.1) and (2.7.16.3) give

$$\begin{aligned} \varphi_n =& \frac{e^{\kappa_n x}}{2} + \frac{e^{-\kappa_n x}}{2} - \cos(\kappa_n x) \\ & - \left[1 + 2 \cdot e^{-\beta_n} \cdot \sin \beta_n\right] \cdot \left[\frac{e^{\kappa_n x}}{2} - \frac{e^{-\kappa_n x}}{2} - \sin(\kappa_n x)\right] \\ & \approx \sin(\kappa_n x) - \cos(\kappa_n x) + e^{-\kappa_n x} - \sin \beta_n \cdot e^{\kappa_n (x-L)} \end{aligned}$$

For $\kappa_n x \gg 1$ and $\kappa_n(L-x) \gg 1$ it follows that $e^{-\kappa_n x} \ll 1$ and $e^{\kappa_n(x-L)} \ll 1$. For these cases, the eigenfunctions and eigenvalues are for n large

$$\varphi_n = \sin(\kappa_n x) - \cos(\kappa_n x) = \sqrt{2}\cdot \sin(\kappa_n x - \pi/4) \tag{2.7.16.4}$$

$$\kappa_n = \pi(n+1/2)/L \tag{2.7.16.5}$$

For a beam with free ends and for $n > 4$

$$\varphi_n = \sqrt{2}\cdot \sin(\kappa_n x - \pi/4) \text{ and } \kappa_n = \pi(n+1/2)/L \tag{2.7.16.6}$$

For a beam clamped at $x = 0$ and simply supported at $x = L$ and for $n > 4$

$$\varphi_n = \sqrt{2}\cdot \sin(\kappa_n x - \pi/4) \text{ and } \kappa_n = \pi(n+1/4)/L \tag{2.7.16.7}$$

For a beam clamped at $x = 0$ and free at $x = L$ and for $n > 4$

$$\varphi_n = \sqrt{2}\cdot \sin(\kappa_n x - \pi/4) \text{ and } \kappa_n = \pi(n-1/2)/L \tag{2.7.16.8}$$

2.7.17 Example 7.17

The equation of motion for mass n is

$$m\ddot{x}_n + k(x_n - x_{n+1}) + k(x_n - x_{n-1}) = 0 \tag{2.7.17.1}$$

By setting $x_{n+1} = x_n \cdot e^{i\varphi}$and $m\ddot{x}_n = -m\omega^2 x_n$ and inserting these expressions in (2.7.17.1) the result is

$$\cos\varphi = 1 - \frac{\omega^2}{2\omega_0^2}; \quad \omega_0^2 = k/m \tag{2.7.17.2}$$

The equation of motion of mass 1 is

$$m\ddot{x}_1 + k(x_1 - x_2) + kx_1 = 0 \tag{2.7.17.3}$$

The equation of motion for mass N is

$$m\ddot{x}_N + k(x_N - x_{N-1}) = 0 \tag{2.7.17.4}$$

The general solution for x_n is

$$x_n = Z\cdot e^{in\varphi} = A\cos(n\varphi) + B\sin(n\varphi) \tag{2.7.17.5}$$

This expression inserted in Eq. (2.7.17.3) gives

$$2\left(1-\frac{\omega^2}{2\omega_0^2}\right)x_1 = x_2$$

By inserting (2.7.17.2) and (2.7.17.5) the result is

$$A(2\cos^2\varphi - \cos 2\varphi) = B(\sin 2\varphi - 2\sin\varphi\cos\varphi) = 0 \tag{2.7.17.6}$$

Consequently, the displacement is written $x_n = B\sin(n\varphi)$. This expression inserted in Eq. (2.7.17.4) gives

$$(1-\omega^2/\omega_0^2)\cdot\sin(N\varphi) = \sin[(N-1)\varphi] \tag{2.7.17.7}$$

Equation (2.7.17.2) gives $\omega^2/\omega_0^2 = 2(1-\cos\varphi)$. This expression in combination with (2.7.17.6) gives

$$\begin{aligned}&\sin(N\varphi)(2\cos\varphi - 1) = \sin[(N-1)\varphi] \quad \text{or}\\ &2\cos[\varphi(N+1/2)]\sin(\varphi/2) = 0\end{aligned} \tag{2.7.17.8}$$

The solutions are $\varphi_n = 2n\pi$ and $\varphi_n = \dfrac{\pi(1+2n)}{2N+1}$. The corresponding natural frequencies are

$$\omega_n = 2\sqrt{\frac{k}{m}}\cdot\sin(\varphi_n/2) = 2\sqrt{\frac{k}{m}}\cdot\sin\left[\frac{\pi(1+2n)}{4N+2}\right] \tag{2.7.17.9}$$

2.7.18 Example 7.18

For $x \leqslant L/2$, $w(x) = w_-(x)$; for $x \geqslant L/2$, $w(x) = w_+(x) = w_-(L-x)$. The solution is symmetric with respect to $x = L/2$. General solution,

$$w_- = A_1\sin\kappa x + A_2\cos\kappa x + A_3\sinh\kappa x + A_4\cosh\kappa x \tag{2.7.18.1}$$

Boundary conditions

$$w = w' = 0 \quad \text{for } x = 0 \tag{2.7.18.2}$$

$$w' = 0 \tag{2.7.18.3}$$

$$w''' = -F/(2D') \tag{2.7.18.4}$$

$$\text{for } x = L/2$$

Let $\alpha = \kappa L/2$.

Equations (2.7.18.1) and (2.7.18.2) give

$$A_2 + A_4 = 0; \quad A_1 + A_3 = 0 \tag{2.7.18.5}$$

Equations (2.7.18.1), (2.7.18.3) and (2.7.18.5) give

$$A_1(\cos\alpha - \cosh\alpha) - A_2(\sin\alpha + \sinh\alpha) = 0 \tag{2.7.18.6}$$

Equations (2.7.18.1), (2.7.18.4) and (2.7.18.5) give

$$-\frac{F}{2D'\kappa^3} = -A_1(\cos\alpha + \cosh\alpha) + A_2(\sin\alpha - \sinh\alpha) \tag{2.7.18.7}$$

Equations (2.7.18.6) and (2.7.18.7) give

$$A_1 = \frac{F}{4D'\kappa^3}\left(\frac{\sin\alpha + \sinh\alpha}{\sin\alpha\cosh\alpha + \sinh\alpha\cos\alpha}\right) \tag{2.7.18.8}$$

The force at the support at $x = 0$ is

$$\begin{aligned} F_-(0) &= -\,D'w'''(0) = -D'\kappa^3(-A_1 + A_3) = 2A_1 D'\kappa^3 \\ &= \frac{F}{2}\left(\frac{\sin\alpha + \sinh\alpha}{\sin\alpha\cosh\alpha + \sinh\alpha\cos\alpha}\right) \end{aligned}$$

The bending moment at $x = 0$ is

$$M_-(0) = -D'w_-''(0) = \frac{F}{2\kappa}\left(\frac{\cos\alpha - \cosh\alpha}{\sin\alpha\cosh\alpha + \sinh\alpha\cos\alpha}\right)$$

2.8 Chapter 8

2.8.1 Example 8.1

The natural frequencies f_{mn} for a simply supported rectangular plate are in Eq. (8.16) given as

$$f_{mn} = \frac{\pi}{2}\sqrt{\frac{D_0}{\rho h}} \cdot \left[\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2\right] \tag{2.8.1.1}$$

This expression is rewritten

$$f_{mn} = \left(C/L_x^2\right)\left[m^2 + \left(nL_x/L_y\right)^2\right] \tag{2.8.1.2}$$

where C is a constant.
Case 1

$$L_x = L_y; \quad S = L_x L_y = L_x^2 \tag{2.8.1.3}$$

Equations (2.8.1.2) and (2.8.1.3) give

$$f_{mn} = (C/S)\,(m^2 + n^2) \tag{2.8.1.4}$$

For $m = n = 1, \quad f_{11} = 20\,\text{Hz} \Rightarrow C/S = 10$, thus

$$f_{mn} = 10 \cdot (m^2 + n^2) \tag{2.8.1.5}$$

Case 2

$$L_x = 2L_y; \quad S = L_x L_y = L_x^2/2 \tag{2.8.1.6}$$

Equations (2.8.1.2) and (2.8.1.6) give

$$f_{mn} = [C/(2S)](m^2 + 4n^2) \tag{2.8.1.7}$$

However, $C/S = 10$, thus

$$f_{mn} = 5(m^2 + 4n^2) \tag{2.8.1.8}$$

The results are

Case 1			Case 2		
m	n	f_{mn}/Hz	m	n	f_{mn}/Hz
1	1	20	1	1	25
2	1	50	2	1	40
1	2	50	3	1	65
3	1	100	1	2	85
1	3	100	4	1	100
3	2	130	2	2	100
2	3	130	3	2	125
3	3	180	5	1	145

2.8.2 Example 8.2

Let the corners of the plate be at (0, 0), $(L_x, 0)$, $(L_x, L_y)$and $(0, L_y)$. The differential equation governing the forced response of a plate is given by Eq. (8.20). For the static case $\ddot{w} = 0$. For a static force F at $(L_x/2, L_y/2)$ Eq. (8.20) reads

$$\nabla^2(\nabla^2 w) = (F/D) \cdot \delta(x - L_x/2) \cdot \delta(y - L_y/2) \tag{2.8.2.1}$$

For a simply supported plate the response w can be expanded by means of the orthogonal eigenfunctions

$$\varphi_{mn}(x, y) = \sin(m\pi x/L_x) \cdot \sin(n\pi y/L_y) \tag{2.8.2.2}$$

The eigenfunctions satisfy the equation

$$\nabla^2(\nabla^2 \varphi_{mn}) = \kappa_{mn}^4 \varphi_{mn}; \quad \kappa_{mn}^2 = (m\pi/L_x)^2 + (n\pi/L_y)^2 \tag{2.8.2.3}$$

The solution w is written

$$w = \sum_{m,n} C_{mn}\varphi_{mn} \tag{2.8.2.4}$$

The Eqs. (2.8.2.1) through (2.8.2.4) give

$$\sum C_{mn} \nabla^2(\nabla^2 \varphi_{mn}) = \sum C_{mn} \kappa_{mn}^4 \varphi_{mn} = (F/D) \cdot \delta(x - L_x/2) \cdot \delta(y - L_y/2) \tag{2.8.2.5}$$

This operation is allowed since w and φ_{mn} satisfy the same boundary conditions. According to standard procedure Eq. (2.8.2.5) is multiplied by φ_{mn} and integrated over the entire plate area to determine the parameters C_{mn}. Since the norm of φ_{mn} is $\langle \varphi_{mn} \mid \varphi_{mn} \rangle = L_x L_y/4$ it follows that

$$C_{mn}\kappa_{mn}^4 L_x L_y/4 = (F/D)\varphi_{mn}(L_x/2, L_y/2) \tag{2.8.2.6}$$

The response w is obtained from Eqs. (2.8.2.4) and (2.8.2.6) as

$$w(x, y) = \sum_{m,n} \frac{4F\varphi_{mn}(L_x/2, L_y/2) \cdot \varphi_{mn}(x, y)}{DL_x L_y \kappa_{mn}^4} \tag{2.8.2.7}$$

At $t = 0$ the static force F is removed. The response of the plate is for $t \geqslant 0$ written as

$$u(x, y, t) = \sum_{m,n} \varphi_{mn}(x, y) \left[A_{mn} \cos(\omega_{mn} t) + B_{mn} \sin(\omega_{mn} t)\right] \cdot \exp\left[-\omega_{mn}\eta t/2\right] \tag{2.8.2.8}$$

The result is obtained following the procedure discussed in Sect. 7.2. Consequently,

$$\omega_{mn} = \kappa_{mn}^2 \sqrt{D_0/\mu} \tag{2.8.2.9}$$

where D_0 is the real part of the bending stiffness of the plate, μ its mass per unit area and η the lossfactor. At $t = 0$ the plate is at rest having the displacement $w(x, y)$ given by Eq. (2.8.2.7). The initial conditions are

$$u(x, y, 0) = w(x, y) = \sum_{m,n} C_{mn}\varphi_{mn}; \quad \dot{u}(x, y, 0) = 0 \tag{2.8.2.10}$$

Equations (2.8.2.8) and (2.8.2.10) give $A_{mn} = C_{mn}$ and $B_{mn} = 0$. The response of the plate for $t \geqslant 0$ is consequently

$$u(x, y, t) = \sum_{m,n} \frac{4F\varphi_{mn}(L_x/2, L_y/2) \cdot \varphi_{mn}(x, y) \cdot \cos(\omega_{mn}t) \cdot \exp(-\omega_{mn}\eta t/2)}{L_x L_y D\kappa_{mn}^4} \tag{2.8.2.11}$$

2.8.3 Example 8.3

The angular frequency ω for the first mode is according to Eqs. (8.86) and (8.87) given by

$$\omega^2 = \frac{D}{\rho h} \cdot \frac{\iint \mathrm{d}x\mathrm{d}y[(\partial^2 F/\partial x^2)^2 + 2(\partial^2 F/\partial x \partial y)^2 + (\partial^2 F/\partial y^2)^2]}{\iint \mathrm{d}x\mathrm{d}y F^2} \tag{2.8.3.1}$$

The displacement F for the first vibrational mode is approximated by

$$F = A\left(x^4 - 2x^3 L_x + xL_x^3\right)\left(y^4 - 2y^3 L_y + yL_y^3\right) = A \cdot g(x) \cdot h(y) \tag{2.8.3.2}$$

Introduce the notations

$$\begin{aligned}
I_1 &= \int_0^{L_x} \mathrm{d}x\,(g(x))^2; \quad I_2 = \int_0^{L_x} \mathrm{d}x\,\left(g'(x)\right)^2; \quad I_3 = \int_0^{L_x} \mathrm{d}x\,\left(g''(x)\right)^2 \\
H_1 &= \int_0^{Ly} \mathrm{d}y\,(h(y))^2; \quad H_2 = \int_0^{Ly} \mathrm{d}y\,\left(h'(y)\right)^2; \quad H_3 = \int_0^{Ly} \mathrm{d}y\,\left(h''(y)\right)^2
\end{aligned} \tag{2.8.3.3}$$

Equations (2.8.3.1) through (2.8.3.3) give

$$\omega^2 = \frac{D}{\rho h} \cdot \frac{I_3 H_1 + 2I_2 H_2 + I_1 H_3}{I_1 H_1} \tag{2.8.3.4}$$

$$\begin{aligned} I_1 &= L_x^9 \cdot (31/630)\,; \quad H_1 = L_y^9 \cdot (31/630) \\ I_2 &= L_x^7 \cdot (17/35)\,; \quad H_2 = L_y^7 \cdot (17/35) \\ I_3 &= L_x^5 \cdot (144/30)\,; \quad H_3 = L_y^5 \cdot (144/30) \end{aligned} \tag{2.8.3.5}$$

Since $L_x = 3L_y$, Eqs. (2.8.3.4) and (2.8.3.5) give

$$\omega^2 = \frac{D}{\rho h} \cdot \frac{1}{L_y^4} \cdot \frac{\left[82 \cdot 31 \cdot 144 \cdot 21 + 18 \cdot (17 \cdot 630/35)^2\right]}{3^4 \cdot 31^2} = \frac{D}{\rho h} \cdot \frac{1}{L_y^4} \cdot 120.405 \tag{2.8.3.6}$$

The first natural frequency, assuming a displacement given by Eq. (2.8.3.2) and estimated by means of the Ritz technique, is obtained from Eq. (2.8.3.6) as

$$(f_{11})_{\text{estimated}} = \omega/(2\pi) = \sqrt{\frac{D}{\rho h}} \cdot \frac{1}{2\pi L_y^2} \cdot \sqrt{120.405} = \sqrt{\frac{D}{\rho h}} \cdot \frac{1}{2\pi L_y^2} \cdot 10.973 \tag{2.8.3.7}$$

The correct value is according to Eq. (8.16)

$$(f_{11})_{\text{correct}} = \frac{\pi}{2}\sqrt{\frac{D}{\rho h}} \cdot \left[\left(\frac{1}{L_x}\right)^2 + \left(\frac{1}{L_y}\right)^2\right] = \sqrt{\frac{D}{\rho h}} \cdot \frac{1}{2\pi L_y^2} \cdot 10.966 \tag{2.8.3.8}$$

The relative error is 6×10^{-4}.

2.8.4 Example 8.4

According to Eqs. (8.86) and (8.87), the first natural frequency f_{11} of the clamped plate can be written as

$$\begin{aligned} f_{11}^2 =& \frac{D}{(2\pi)^2} \cdot \frac{\iint \mathrm{d}x\mathrm{d}y[(\partial^2 F/\partial x^2)^2 + 2(\partial^2 F/\partial x \partial y)^2 + (\partial^2 F/\partial y^2)^2]}{\iint \mathrm{d}x\mathrm{d}y \mu F^2} \\ =& \frac{D}{\mu(2\pi)^2} \cdot \frac{I}{J} \end{aligned} \tag{2.8.4.1}$$

The bending stiffness of the plate is D and its mass per unit area is μ. The function F is defined as

$$F(x, y) = g(x)h(y); \quad g(x) = 1 - \cos(2\pi x/L_x); \quad h(y) = 1 - \cos(2\pi y/L_y) \tag{2.8.4.2}$$

The function I is according to Eqs. (2.8.4.1) and (2.8.4.2)

$$\begin{aligned} I = \iint \mathrm{d}x\mathrm{d}y\Big[h^2(y) \cdot (2\pi/L_x)^4 \cdot (\cos(2\pi x/L_x))^2 \\ + g^2(x) \cdot \left(2\pi/L_y\right)^4 \cdot \left(\cos(2\pi y/L_y)\right)^2 \Big] \\ + 2\iint \mathrm{d}x\mathrm{d}y\, (2\pi/L_x)^2 \left(2\pi/L_y\right)^2 \left(\sin(2\pi y/L_y)\right)^2 (\sin(2\pi x/L_x))^2 \end{aligned} \tag{2.8.4.3}$$

With $\int_0^{L_x} \mathrm{d}x\, (\cos(2\pi x/L_x))^2 = L_x/2$ and $\int_0^{L_x} \mathrm{d}x g^2(x) = 3L_x/2$ it follows that

$$I = \frac{L_x L_y}{4} \cdot (2\pi)^4 \left[\frac{3}{L_x^4} + \frac{3}{L_y^4} + \frac{2}{L_x^2 L_y^2}\right] \tag{2.8.4.4}$$

$$J = \frac{9L_x L_y}{4} \tag{2.8.4.5}$$

With $L_x = L \cdot \xi$ and $L_y = L/\xi$ Eqs. (2.8.4.1) through (2.8.4.5) give

$$f_{11}^2 = \frac{D}{\mu} \cdot \frac{(2\pi)^2}{9L^4} \left[3\xi^4 + 3/\xi^4 + 2\right] \tag{2.8.4.6}$$

The ratio $f_{11}(\xi)/f_{11}(1)$ is thus

$$\frac{f_{11}(\xi)}{f_{11}(1)} = \left[\frac{3\xi^4 + 3/\xi^4 + 2}{8}\right]^{1/2} \tag{2.8.4.7}$$

The first natural frequency of a rectangular plate with a constant area has a minimum when the sides have the same length, i.e. when $\xi = 1$.

2.8.5 Example 8.5

The corners of the simply supported plate are at $(0, 0)$, $(L_x, 0)$, (L_x, L_y) and at $(0, L_y)$. A point force $F_0 \cdot e^{i\omega t}$ excites the plate at (x_0, y_0). The displacement of the plate is $w(x, y) \cdot e^{i\omega t}$. The differential equation governing w is

$$\nabla^2(\nabla^2 w) - \kappa^4 w = (F_0/D) \cdot \delta(x - x_0) \cdot \delta(y - y_0) \tag{2.8.5.1}$$

where $\kappa^4 = \mu\omega^2/D$, μ being the mass per unit area of the plate and D its bending stiffness. The orthogonal eigenfunctions φ_{mn} for a simply supported plate are

$$\varphi_{mn} = \sin(m\pi x/L_x)\sin(n\pi y/L_y) \tag{2.8.5.2}$$

The eigenfunctions satisfy the equation

$$\nabla^2(\nabla^2\varphi_{mn}) = \kappa_{mn}^4\varphi_{mn}; \quad \kappa_{mn}^2 = (m\pi/L_x)^2 + (n\pi/L_y)^2 \tag{2.8.5.3}$$

The displacement is written

$$w(x, y) = \sum_{m,n} C_{mn}\cdot\varphi_{mn} \tag{2.8.5.4}$$

The displacement w and the eigenfunction φ_{mn} satisfy the same boundary conditions. Consequently, Eq. (2.8.5.4) can be inserted in the basic differential equation (2.8.5.1) resulting in

$$\sum C_{mn}\nabla^2(\nabla^2\varphi_{mn}) - \sum C_{mn}\kappa^4\varphi_{mn} = (F_0/D)\cdot\delta(x-x_0)\cdot\delta(y-y_0) \tag{2.8.5.5}$$

The Eqs. (2.8.5.3) and (2.8.5.5) give

$$\sum C_{mn}(\kappa_{mn}^4 - \kappa^4)\varphi_{mn} = (F_0/D)\cdot\delta(x-x_0)\cdot\delta(y-y_0) \tag{2.8.5.6}$$

Equation (2.8.5.6) is multiplied by φ_{mn} and integrated over the plate area. The eigenfunctions being orthogonal with the norm $\langle\varphi_{mn} \mid \varphi_{mn}\rangle = L_xL_y/4$ give

$$C_{mn}(\kappa_{mn}^4 - \kappa^4)L_xL_y/4 = (F_0/D)\cdot\varphi_{mn}(x_0, y_0) \tag{2.8.5.7}$$

Using Eq. (8.16) and the definition of the wavenumber κ it follows that $\kappa_{mn}^4 = (2\pi)^2(\mu/D_0)f_{mn}^2$ and $\kappa^4 = (2\pi)^2(\mu/D)f^2$ where f_{mn} is the natural frequency for mode (m, n) and $D = D_0(1+i\eta)$, η being the loss factor of the structure. The parameters C_{mn} are thus obtained from Eq. (2.8.5.7) as

$$C_{mn} = \frac{4F_0\varphi_{mn}(x_0, y_0)}{(2\pi)^2M[f_{mn}^2(1+i\eta) - f^2]}; \quad M = \mu L_xL_y \tag{2.8.5.8}$$

The total mass of the plate is M. The space and time averages of the velocity squared of the plate is

$$\langle\bar{v}^2\rangle = \frac{1}{L_xL_y}\iint \mathrm{d}x\mathrm{d}y\frac{1}{2}\left|i\omega w\right|^2 = \frac{1}{8}\sum\omega^2\left|C_{mn}\right|^2 \tag{2.8.5.9}$$

since $\iint \mathrm{d}x\mathrm{d}y\varphi_{mn}\varphi_{kl} = L_x L_y/4$ if $m = k$ and $n = l$, otherwise zero. If only the first mode is considered Eq. (2.8.5.9) gives

$$\langle \bar{v}^2 \rangle \approx \frac{1}{8}\omega^2 \left| C_{11} \right|^2 = \frac{16F_0^2\left(\varphi_{mn}(x_0, y_0)\right)^2}{(2\pi)^4 M^2[(f_{11}^2 - f^2)^2 + (\eta f_{11}^2)^2]} = \frac{Q}{M^2[(f_{11}^2 - f^2)^2 + (\eta f_{11}^2)^2]} \tag{2.8.5.10}$$

where Q is a constant.

Two cases are considered.

Case 1

$M = M_1$; $\eta = 0.01$; $D_0 = D_1$; $f = 0.9 \cdot f_{11}$. These parameters inserted in Eq. (2.8.5.10) give

$$\langle \bar{v}_1^2 \rangle \approx \frac{Q}{M_1^2 f^4 \left[(1/0.81 - 1)^2 + (0.01/0.81)^2\right]} = \frac{Q \cdot 100}{M_1^2 f^4 \cdot 5.52} \tag{2.8.5.11}$$

Case 2

$M = 1.35 \cdot M_1$; $\eta = 0.1$; $D_0 = 1.2 \cdot D_1$. The first natural frequency f_{11} of the plate is changed since both the mass and the stiffness of the plate are changed. The first natural frequency is

$$f_{11} = (\kappa_{11}^2/2\pi)\sqrt{D_0/\mu} = (\kappa_{11}^2/2\pi)\sqrt{1.2D_1/(1.35M)} = (f/0.9)\sqrt{1.2/1.35} = 1.048 \cdot f$$

These parameters inserted in Eq. (2.8.5.10) give

$$\langle \bar{v}_2^2 \rangle \approx \frac{Q}{(1.35)^2 M_1^2 f^4 \left[(1.048^2 - 1)^2 + 1.048^4/100^2\right]} = \frac{Q \cdot 100}{M_1^2 f^4 \cdot 3.96} \tag{2.8.5.12}$$

The velocity level of the plate is increased due to the changes. The main reason being that after the change the new natural frequency of the plate is closer to the frequency of the force exciting the plate. The velocity level increase is

$$\Delta L_v = 10 \cdot \log\left[\frac{\langle \bar{v}_2^2 \rangle}{\langle \bar{v}_1^2 \rangle}\right] = 10 \cdot \log(5.52/3.96) = 1.4\,\mathrm{dB}$$

2.8.6 Example 8.6

The corners of the simply supported plate are at $(0, 0)$, $(L_x, 0)$, (L_x, L_y) and at $(0, L_y)$. Two forces excite the plate one $F_0 \cdot e^{i\omega t}$ at $(L_x/4; L_y/2)$ and another one

$-F_0 \cdot e^{i\omega t}$ at $(3L_x/4; L_y/2)$. The displacement of the plate is $w(x, y) \cdot e^{i\omega t}$. The differential equation governing w is

$$\nabla^2(\nabla^2 w) - \kappa^4 w = (F_0/D) \cdot [\delta(x - L_x/4)\delta(y - L_y/2) - \delta(x - 3L_x/4)\delta(y - L_y/2)] \quad (2.8.6.1)$$

where $\kappa^4 = \mu\omega^2/D$, μ being the mass per unit area of the plate and D its bending stiffness. The orthogonal eigenfunctions φ_{mn} for a simply supported plate are

$$\varphi_{mn} = \sin(m\pi x/L_x)\sin(n\pi y/L_y) \quad (2.8.6.2)$$

The eigenfunctions satisfy the equation

$$\nabla^2(\nabla^2 \varphi_{mn}) = \kappa_{mn}^4 \varphi_{mn}; \quad \kappa_{mn}^2 = (m\pi/L_x)^2 + (n\pi/L_y)^2 \quad (2.8.6.3)$$

The displacement is written

$$w(x, y) = \sum_{m,n} C_{mn} \cdot \varphi_{mn} \quad (2.8.6.4)$$

The displacement w and the eigenfunction φ_{mn} satisfy the same boundary conditions. Consequently, Eq. (2.8.6.4) can be inserted in the basic differential equation (2.8.6.1). The parameters C_{mn} using the procedure outlined in Example 8.5 are obtained as

$$\begin{aligned} C_{mn} &= \frac{4F_0 \sin(n\pi/2)\,[\sin(m\pi/4) - \sin(3m\pi/4)]}{DL_xL_y(\kappa_{mn}^4 - \kappa^4)} \\ &= \frac{-8F_0 \sin(n\pi/2)\cos(m\pi/2)\sin(m\pi/4)}{DL_xL_y(\kappa_{mn}^4 - \kappa^4)} \end{aligned} \quad (2.8.6.5)$$

$C_{mn} \neq 0$ for $m = 2, 6, 10$ etc and $n = 1, 3, 5$ etc.

The time average of the kinetic energy is

$$\bar{T} = \iint \mathrm{d}x\mathrm{d}y \frac{\mu}{4}\,|i\omega w|^2 = \frac{L_xL_y\mu}{16}\sum \omega^2\,|C_{mn}|^2 \quad (2.8.6.6)$$

since $\iint \mathrm{d}x\mathrm{d}y \varphi_{mn}\varphi_{kl} = L_xL_y/4$ if $m = k$ and $n = l$, otherwise zero. Using Eq. (8.16) and the definition of the wavenumber κ it follows that $\kappa_{mn}^4 = (2\pi)^2(\mu/D_0)$ f_{mn}^2 and $\kappa^4 = (2\pi)^2(\mu/D)f^2$ where f_{mn} is the natural frequency for mode (m, n) and $D = D_0(1 + i\eta)$, η being the loss factor of the structure. The parameters $|C_{mn}|^2$ are thus obtained from Eq. (2.8.6.5) as

$$|C_{mn}|^2 = \frac{16\,|F_0|^2}{M^2(2\pi)^4[(f_{mn}^2 - f^2)^2 + (\eta f_{mn}^2)^2]}; \quad M = \mu L_xL_y \quad (2.8.6.7)$$

for $m = 2, 6, 10$ etc and $n = 1, 3, 5$ etc. The total mass of the plate is M. The time average of the kinetic energy is obtained from Eqs. (2.8.6.6) and (2.8.6.7) as

$$\bar{\mathcal{T}} = \sum_{m,n} \frac{|F_0|^2 f^2}{M(2\pi)\left[(f_{mn}^2 - f^2)^2 + (\eta f_{mn}^2)^2\right]} \tag{2.8.6.8}$$

for $m = 2, 6, 10$ etc and $n = 1, 3, 5$ etc.

2.8.7 Example 8.7

The corners of the simply supported plate are at $(0, 0)$, $(L_x, 0)$, (L_x, L_y) and at $(0, L_y)$. A force $f(x, y)e^{i\omega t}$ per unit area excites the plate. The function $f(x, y)$ is

$$f(x, y) = F_0 \cdot xy(L_x - x)(L_y - y)/(L_x L_y)^3 \tag{2.8.7.1}$$

The displacement of the plate is $w(x, y) \cdot e^{i\omega t}$. The differential equation governing w is

$$\nabla^2(\nabla^2 w) - \kappa^4 w = f(x, y)/D \tag{2.8.7.2}$$

where $\kappa^4 = \mu\omega^2/D$, μ being the mass per unit area of the plate and D its bending stiffness. The orthogonal eigenfunctions φ_{mn} for a simply supported plate are

$$\varphi_{mn} = \sin(m\pi x/L_x)\sin(n\pi y/L_y) \tag{2.8.7.3}$$

The eigenfunctions satisfy the equation

$$\nabla^2(\nabla^2 \varphi_{mn}) = \kappa_{mn}^4 \varphi_{mn}; \quad \kappa_{mn}^2 = (m\pi/L_x)^2 + (n\pi/L_y)^2 \tag{2.8.7.4}$$

The displacement is written

$$w(x, y) = \sum_{m,n} C_{mn} \cdot \varphi_{mn} \tag{2.8.7.5}$$

The displacement w and the eigenfunction φ_{mn} satisfy the same boundary conditions. Consequently, Eq. (2.8.7.4) can be inserted in the basic differential equation (2.8.7.1). The parameters C_{mn} using the procedure outlined in Example 8.5 and are obtained as

$$C_{mn} = \frac{4\langle f \mid \varphi_{mn}\rangle}{DL_x L_y(\kappa_{mn}^4 - \kappa^4)} \tag{2.8.7.6}$$

where, using Eqs. (2.8.7.1) and (2.8.7.3)

$$\langle f \mid \varphi_{mn} \rangle = \iint \mathrm{d}x\mathrm{d}y f(x, y)\varphi_{mn}(x, y)$$
$$= F_0 \left[\frac{2 \cdot \cos(m\pi)}{(m\pi)^3} + \frac{\sin(m\pi)}{(m\pi)^2} \right] \cdot \left[\frac{2 \cdot \cos(n\pi)}{(n\pi)^3} + \frac{\sin(n\pi)}{(n\pi)^2} \right] \tag{2.8.7.7}$$

The displacement w is given by Eqs. (2.8.7.5), (2.8.7.6) and (2.8.7.7).

2.8.8 *Example 8.8*

The corners of the simply supported plate are at $(0, 0)$, $(L_x, 0)$, (L_x, L_y) and at $(0, L_y)$. The corners of the limp material are at $(L_x/4, L_y/4)$, $(L_x/4, 3L_y/4)$, $(3L_x/4, 3L_y/4)$ and at $(3L_x/4, L_y/4)$. The total area of the plate is S and the area of the added material is S_0. The stiffness of the plate and thus the potential energy of the plate is not changed by the addition of the limp mass. The added mass influences the kinetic energy. The angular frequency ω of the system can according to Eq. (8.87) be written as

$$\omega^2 = \frac{C}{\iint \mathrm{d}x\mathrm{d}y \mu(x, y) F^2(x, y)} \tag{2.8.8.1}$$

where C is a constant proportional to the potential energy of the plate. The mass per unit area of the plate is $\mu(x, y)$. For the first mode of vibration $F(x, y) = \sin(\pi x/L_x) \cdot \sin(\pi y/L_y)$.

Case 1

$\mu(x, y) = \mu_0 = M/(L_x L_y)$ where M is the total mass of the plate. Equation (2.8.8.1) gives

$$\omega^2 = \frac{C}{\mu_0 \iint dxdy F^2(x, y)} = \frac{4C}{M} \tag{2.8.8.2}$$

Case 2

$$\mu(x, y) = \mu_0 \quad \text{for } 0 \leqslant x \leqslant L_x/4 \text{ and } 3L_x/4 \leqslant x \leqslant L_x$$
$$\text{and } 0 \leqslant y \leqslant L_y/4 \text{ and } 3L_y/4 \leqslant y \leqslant L_y$$
$$\mu(x, y) = 2\mu_0 \quad \text{otherwise}$$

The integral in the denominator of Eq. (2.8.8.2) is

$$\begin{aligned}
&\iint \mathrm{d}x\mathrm{d}y\mu(x, y)F^2(x, y) = \iint_S \mathrm{d}x\mathrm{d}y\mu_0 F^2 + \iint_{S_0} \mathrm{d}x\mathrm{d}y\mu_0 F^2 \\
&= M/4 + \mu_0 \int_{L_x/4}^{3L_x/4} \mathrm{d}x\,(\sin(\pi x/L_x))^2 \int_{L_y/4}^{3L_y/4} \mathrm{d}y\,(\sin(\pi y/L_x))^2 \\
&= M/4 \cdot \left[1 + (1/2 + 1/\pi)^2\right] \qquad (2.8.8.3)
\end{aligned}$$

The ratio between the natural frequencies before and after the change is

$$\frac{(f_{11})_{\text{case1}}}{(f_{11})_{\text{case2}}} = \left[1 + (1/2 + 1/\pi)^2\right]^{1/2} \qquad (2.8.8.4)$$

2.8.9 Example 8.9

The plate is simply supported along the edges $y = 0$ and $y = L_y$ and free along the edges $x = 0$ and $x = L_x$. The displacement w of the plate is $w(x, y) \cdot \exp(i\omega t)$. For a plate with two opposite sides simply supported the function w can, according to Eq. (8.49), be written as

$$w(x, y) = \sum_{m,n} X_n(x)\varphi_n(y); \quad \varphi_n(y) = \sin(n\pi y/L_y) \qquad (2.8.9.1)$$

For free vibrations the functions X_n should satisfy Eq. (8.51) with $F = 0$. Thus

$$\frac{\mathrm{d}^4 X_n}{\mathrm{d}x^4} - 2k_n^2 \frac{\mathrm{d}^2 X_n}{\mathrm{d}x^2} + k_n^4 X_n - \kappa^4 X_n = 0 \qquad (2.8.9.2)$$

$$\kappa^4 = \mu\omega^2/D; \quad k_n^2 = (n\pi/L_y)^2 \qquad (2.8.9.3)$$

By introducing

$$\kappa_1 = \sqrt{\kappa^2 - k_n^2} \quad \text{and} \quad \kappa_2 = \sqrt{\kappa^2 + k_n^2} \qquad (2.8.9.4)$$

the general solution to Eq. (2.8.9.2) is written

$$X_n = A_1 \sin(\kappa_1 x) + A_2 \cos(\kappa_1 x) + A_3 \sinh(\kappa_2 x) + A_4 \cosh(\kappa_2 x) \qquad (2.8.9.5)$$

The displacement of mode (1,1) is $w_{11}(x, y) = X_1(x)\varphi_1(y)$. The boundary conditions corresponding to simply supported edges along the lines $y = 0$ and $y = L_y$ are satisfied by the function φ_1. For free edges along the two other boundaries the bending and force should equal zero. Thus according to Table 3.3, (p. 104), w_{11} should satisfy

$$M_y' = -D\left[\frac{\partial^2 w_{11}}{\partial x^2} + \nu\frac{\partial^2 w_{11}}{\partial x^2}\right] = 0 \quad \text{for } x = 0 \text{ and } x = L_x \tag{2.8.9.6}$$

$$F_x' = -D\left[\frac{\partial^3 w_{11}}{\partial x^3} + (2-\nu)\frac{\partial^3 w_{11}}{\partial x \partial y^2}\right] = 0 \quad \text{for } x = 0 \text{ and } x = L_x \tag{2.8.9.7}$$

The expression $w_{11}(x, y) = X_1(x)\varphi_1(y)$ and the Eqs. (2.8.9.6) and (2.8.9.7) give for $x = 0$ and $x = L_x$

$$\left[\frac{\mathrm{d}^2 X_1}{\mathrm{d}x^2} - \nu k_1^2 X_1\right] = 0; \quad \left[\frac{\mathrm{d}^3 X_1}{\mathrm{d}x^3} - (2-\nu)k_1^2\frac{\mathrm{d}X_1}{\mathrm{d}x}\right] = 0 \tag{2.8.9.8}$$

The Eqs. (2.8.9.5) and (2.8.9.8) give a system of equations which in matrix form is

$$[B]\cdot\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = 0;$$

$$[B] = \begin{bmatrix} 0 & -(\kappa_1^2 + \nu k_1^2) & 0 & (\kappa_2^2 - \nu k_1^2) \\ -\kappa_1\left[\kappa_1^2 + (2-\nu)k_1^2\right] & 0 & \kappa_2\left[\kappa_2^2 - (2-\nu)k_1^2\right] & 0 \\ -S(\kappa_1^2 + \nu k_1^2) & -C(\kappa_1^2 + \nu k_1^2) & SH(\kappa_2^2 - \nu k_1^2) & CH(\kappa_2^2 - \nu k_1^2) \\ -\kappa_1 C\left[\kappa_1^2 + (2-\nu)k_1^2\right] & \kappa_1 S\left[\kappa_1^2 + (2-\nu)k_1^2\right] & \kappa_2 CH\left[\kappa_2^2 - (2-\nu)k_1^2\right] & \kappa_2 SH\left[\kappa_2^2 - (2-\nu)k_1^2\right] \end{bmatrix} \tag{2.8.9.9}$$

where $\alpha = \kappa_1 L_x$ and $\beta = \kappa_2 L_x$ and $S = \sin\alpha$, $C = \cos\alpha$, $CH = \cosh\beta$ and $SH = \sinh\beta$. The first natural frequency for the plate with two opposite simply supported edges and with the two remaining sides free is the solution to $\mathrm{Det}[B] = 0$. There are two simple limiting cases. The first when $L_y \to \infty$. In the limit $k_1 = 0$ and $\kappa_1 = \kappa_2 = \kappa$ and $\alpha = \beta$. For this particular case the matrix $[B]$ is

$$[B] = \begin{bmatrix} 0 & -\kappa^2 & 0 & \kappa^2 \\ -\kappa^3 & 0 & \kappa^3 & 0 \\ -\kappa^2\sin\alpha & -\kappa^2\cos\alpha & \kappa^2\sinh\alpha & \kappa^2\cosh\alpha \\ -\kappa^3\cos\alpha & \kappa^3\sin\alpha & \kappa^3\cosh\alpha & \kappa^3\sinh\alpha \end{bmatrix} \tag{2.8.9.10}$$

The determinat of $[B]$ is zero when

$$\cos\alpha\cdot\cosh\alpha = 1 \tag{2.8.9.11}$$

which according to Table 7.3 gives the natural frequency for a beam with both ends free.

The other simple case is when $L_x \to \infty$ leading to $\sinh\beta \approx \cosh\beta \to \infty$. The determinant of $[B]$ is for this case zero when $\alpha = 0$. The first natural frequency is consequently the solution to $\kappa = \pi/L_y$ which is equivalent to the natural frequency of a beam simply supported at both ends.

2.8.10 Example 8.10

The total potential energy of the system is stored in the plate and in the two springs. The potential energy stored in one spring with the spring constant s is $s/2 \cdot (\Delta x)^2$ where Δx is the compression of the spring. Assuming a deflection $F$of the first vibrational mode of the plate where

$$F(x, y) = \sin(\pi x/L_x)\sin(\pi y/L_y) \tag{2.8.10.1}$$

it follows that the maximum potential energy stored in the plate and the springs is

$$\begin{aligned} U_{\max} =& \frac{D}{2}\iint \mathrm{d}x\mathrm{d}y \left[\left(\frac{\partial^2 F}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 F}{\partial y^2}\right)^2\right] \\ &+ \frac{s}{2}\left[F^2(x_1, y_1) + F^2(x_2, y_2)\right] \end{aligned} \tag{2.8.10.2}$$

where (x_1, y_1) and (x_2, y_2) are the coordinates for the mounting of the springs. The Eqs. (2.8.10.1) and (2.8.10.2) give

$$\begin{aligned} \mathcal{U}_{\max} &= (1/2)\cdot\left[DL_xL_y\kappa_{11}^4/4 + s\cdot\sin^2(\pi/2)\cdot\left(\sin^2(\pi/4) + \sin^2(3\pi/4)\right)\right] \\ &= (1/2)\cdot\left[DL_xL_y\kappa_{11}^4/4 + s\right] \end{aligned} \tag{2.8.10.3}$$

with $\kappa_{11}^2 = (\pi/L_x)^2 + \left(\pi/L_y\right)^2$. According to Eqs. (8.22) and (8.23), the first natural frequency f_{11} is the solution to the expression

$$(2\pi f_{11})^2 = 2\mathcal{U}_{\max}\Big/\left[\iint \mathrm{d}x\mathrm{d}y\mu F^2\right] = \frac{DL_xL_y\kappa_{11}^4/4 + s}{\mu L_xL_y/4} \tag{2.8.10.4}$$

The mass per unit area of the plate is μ and the total mass M of the plate is $M = \mu L_xL_y$. The first natural frequency is thus

$$f_{11} = \frac{1}{2\pi}\cdot\left(\frac{D\kappa_{11}^4}{\mu} + \frac{4s}{M}\right)^{1/2} \tag{2.8.10.5}$$

The expression is only valid for small pertubations, i.e. $\dfrac{4s}{D\kappa_{11}^4}\cdot\dfrac{\mu}{M} \ll 1$.

2.8.11 Example 8.11

According to Eq. (8.39), the time average of the modal energy of a plate excited by white noise with a one-sided power spectral density G_{FF} is

$$\bar{\mathcal{E}}_{mn} = G_{FF}/(4\omega_{mn}\eta M) \tag{2.8.11.1}$$

where M is the total mass of the plate and η its loss factor. For viscous losses and white noise excitation $\bar{\mathcal{E}}_{mn}$ is constant and idependent of the mode numbers m and n. This is equipartition of energy. The total energy within a frequency band Δf is thus

$$\bar{\mathcal{E}}_{\Delta f} = \mathcal{N}_f \cdot \Delta f \cdot \bar{\mathcal{E}}_{mn} \tag{2.8.11.2}$$

where $\mathcal{N}_f$ is the modal density for the plate or

$$\mathcal{N}_f = \frac{L_x L_y}{2}\sqrt{\frac{\rho h}{D_0}} = \frac{S}{2}\sqrt{\frac{\rho h}{D_0}} \tag{2.8.11.3}$$

S being the area of the plate. Equations (2.8.11.1)–(2.8.11.3) give

$$\bar{\mathcal{E}}_{\Delta f} = \Delta f \cdot \frac{S}{2}\sqrt{\frac{\rho h}{D_0}} \cdot \frac{G_{FF}}{4\omega_{mn}\eta M} \tag{2.8.11.4}$$

The total energy is also given by

$$\bar{\mathcal{E}}_{\Delta f} = M\langle\left|\bar{v}^2\right|\rangle_{\Delta f} \tag{2.8.11.5}$$

Equations (2.8.11.4) and (2.8.11.5) give

$$\langle\left|\bar{v}^2\right|\rangle_{\Delta f} = \Delta f \cdot \frac{S}{2}\sqrt{\frac{\rho h}{D_0}} \cdot \frac{G_{FF}}{4\omega_{mn}\eta M^2} \tag{2.8.11.6}$$

$D_0 = E_0 h^3/\left[12(1-\nu^2)\right]$ and $M = \rho h S$ inserted in (2.8.11.6) give

$$\langle\left|\bar{v}^2\right|\rangle_{\Delta f} = \Delta f \cdot \frac{S}{2}\sqrt{\frac{\rho 12(1-\nu^2)}{E_0 h^2}} \cdot \frac{G_{FF}}{4\omega_{mn}\eta\rho^2 h^2 S^2} = \frac{C}{\eta h^3} \tag{2.8.11.7}$$

where C is a constant.

2.8.12 Example 8.12

An eigenvalue problem is given by

$$L\varphi_m = \lambda_m \varphi_m \tag{2.8.12.1}$$

The operator L is

$$L = L^{(0)} + Q \tag{2.8.12.2}$$

The operator Q is small as compared to $L^{(0)}$ which corresponds to an undisturbed case for which

$$L^{(0)}\varphi_m^{(0)} = \lambda_m^{(0)}\varphi_m^{(0)} \tag{2.8.12.3}$$

Define eigenvalues and eigenfunctions as

$$\lambda_m = \lambda_m^{(0)} + \lambda_m^{(1)} + \lambda_m^{(2)} + \cdots \tag{2.8.12.4}$$

$$\varphi_m = \varphi_m^{(0)} + \sum_n a_{mn}\varphi_n^{(0)} + \sum_r b_{mr}\varphi_r^{(0)} + \cdots \tag{2.8.12.5}$$

Equations (2.8.12.2), (2.8.12.4), and (2.8.12.5) inserted in (2.8.12.1) yield when neglecting higher order terms

$$\begin{aligned}
&(L^{(0)} + Q)\left[\varphi_m^{(0)} + \sum_n a_{mn}\varphi_n^{(0)} + \sum_r b_{mr}\varphi_r^{(0)}\right] \\
&= \left[\lambda_{\mathrm{m}}^{(0)} + \lambda_m^{(2)} + \lambda_m^{(2)}\right]\left[\varphi_m^{(0)} + \sum_n a_{mn}\varphi_n^{(0)} + \sum_r b_{mr}\varphi_r^{(0)}\right]
\end{aligned} \tag{2.8.12.6}$$

Solving this expression and neglecting third- and fourth-order terms the result is

$$\begin{aligned}
&L^{(0)}\varphi_m^{(0)} + \sum_n a_{mn}L^{(0)}\varphi_n^{(0)} + \sum_r b_{mr}L^{(0)}\varphi_r^{(0)} \\
&+ Q\varphi_m^{(0)} + \sum_n a_{mn}Q\varphi_n^{(0)} + \sum_r b_{mr}Q\varphi_r^{(0)} \\
&\quad = \lambda_m^{(0)}\varphi_m^{(0)} + \lambda_m^{(0)}\sum_n a_{mn}\varphi_n^{(0)} \\
&\qquad + \lambda_m^{(0)}\sum_r b_{mr}\varphi_r^{(0)} + \lambda_m^{(1)}\varphi_m^{(0)} + \lambda_m^{(1)}\sum_n a_{mn}\varphi_n^{(0)} + \lambda_m^{(1)}\sum_r b_{mr}\varphi_r^{(0)} \\
&\qquad + \lambda_m^{(2)}\varphi_m^{(0)} + \lambda_m^{(2)}\sum_n a_{mn}\varphi_n^{(0)} + \lambda_m^{(2)}\sum_r b_{mr}\varphi_r^{(0)}
\end{aligned} \tag{2.8.12.7}$$

The unperturbed solution should, as given by Eq. (2.8.12.1), satisfy

$$L^{(0)}\varphi_m^{(0)} = \lambda_m^{(0)}\varphi_m^{(0)}$$

Terms of the first order should satisfy Eq. (8.114) or

$$\sum_n a_{mn}L^{(0)}\varphi_n^{(0)} + Q\varphi_m^{(0)} = \lambda_m^{(0)}\sum_n a_{mn}\varphi_n^{(0)} + \lambda_m^{(1)}\varphi_m^{(0)} \tag{2.8.12.8}$$

Terms of third or fourth order are $\sum_r b_{mr}Q\varphi_r^{(0)}$, $\lambda_m^{(1)}\sum_r b_{mr}\varphi_r^{(0)}$, $\lambda_m^{(2)}\sum_n a_{mn}\varphi_n^{(0)}$ and $\lambda_m^{(2)}\sum_r b_{mr}\varphi_r^{(0)}$.

Neglecting terms of third and higher orders in Eq. (2.8.12.7) and using Eqs. (2.8.12.3) and (2.8.12.8), Eq. (2.8.12.7) is reduced to

$$\begin{aligned}
&\sum_{r\neq m} b_{mr}L^{(0)}\varphi_r^{(0)} + \sum_{n\neq m} a_{mn}Q\varphi_n^{(0)} \\
&\quad = \lambda_m^{(0)}\sum_{r\neq m} b_{mr}\varphi_r^{(0)} + \lambda_m^{(1)}\sum_{n\neq m} a_{mn}\varphi_n^{(0)} + \lambda_m^{(2)}\varphi_m^{(0)}
\end{aligned} \tag{2.8.12.9}$$

where according to Eq. (2.8.12.1) $L^{(0)}\varphi_r^{(0)} = \lambda_r^{(0)}\varphi_r^{(0)}$. Equation (2.8.12.9) is multiplied by $\varphi_m^{(0)}$ and integrated. The result is

$$\begin{aligned}
&b_{mm}\lambda_m^{(0)}\left\langle\varphi_m^{(0)}\,\middle|\,\varphi_m^{(0)}\right\rangle + \sum_n a_{mn}\left\langle\varphi_m^{(0)}\,\middle|\,Q\varphi_n^{(0)}\right\rangle \\
&\quad = \lambda_m^{(0)}b_{mm}\left\langle\varphi_m^{(0)}\,\middle|\,\varphi_m^{(0)}\right\rangle + \lambda_m^{(1)}a_{mm}\left\langle\varphi_m^{(0)}\,\middle|\,\varphi_m^{(0)}\right\rangle + \lambda_m^{(2)}\left\langle\varphi_m^{(0)}\,\middle|\,\varphi_m^{(0)}\right\rangle
\end{aligned} \tag{2.8.12.10}$$

However, $a_{mm} = b_{mm} = 0$. Thus Eq. (2.8.12.10) gives

$$\begin{aligned}
&\lambda_m^{(2)} = \sum_{n\neq m} a_{mn}Q_{mn}; \quad a_{mn} = \frac{Q_{nm}}{\lambda_m^{(0)} - \lambda_n^{(0)}}; \\
&Q_{mn} = \left\langle\varphi_m^{(0)}\,\middle|\,Q\varphi_n^{(0)}\right\rangle / \left\langle\varphi_n^{(0)}\,\middle|\,\varphi_n^{(0)}\right\rangle
\end{aligned} \tag{2.8.12.11}$$

The parameters b_{mr} are obtained by multiplying Eq. (2.8.12.9) by $\varphi_r^{(0)}$ and integrating and using $L^{(0)}\varphi_r^{(0)} = \lambda_r^{(0)}\varphi_r^{(0)}$. The result is

$$\begin{aligned}
&b_{mr}\lambda_r^{(0)}\left\langle\varphi_r^{(0)}\,\middle|\,\varphi_r^{(0)}\right\rangle + \sum_{n\neq m} a_{mn}\left\langle\varphi_r^{(0)}\,\middle|\,Q\varphi_n^{(0)}\right\rangle \\
&\quad = b_{mr}\lambda_m^{(0)}\left\langle\varphi_r^{(0)}\,\middle|\,\varphi_r^{(0)}\right\rangle + a_{mr}\lambda_m^{(1)}\left\langle\varphi_r^{(0)}\,\middle|\,\varphi_r^{(0)}\right\rangle
\end{aligned}$$

Thus

$$b_{mr} = \frac{\sum_{n \neq m} a_{mn} Q_{rn} - a_{mr} \lambda_m^{(1)}}{\lambda_m^{(0)} - \lambda_r^{(0)}} \tag{2.8.12.12}$$

However, $\lambda_m^{(1)} = Q_{mm}$ reducing Eq. (2.8.12.12) to

$$b_{mr} = \frac{\sum_{n \neq m} a_{mn} Q_{rn} - a_{mr} Q_{mm}}{\lambda_m^{(0)} - \lambda_r^{(0)}} \tag{2.8.12.13}$$

2.8.13 Example 8.13

Following the discussion in Sect. 8.10, Eq. (8.121), the operator L is

$$L = L^{(0)} + Q; \quad L^{(0)} = \frac{D}{\mu} \nabla^2 \nabla^2; \quad Q = -\frac{D \Delta \mu}{\mu^2} \nabla^2 \nabla^2 \tag{2.8.13.1}$$

The eigenfunction for the unperturbed case is $\varphi_m^{(0)}$. The eigenvalue, including second-order terms, is

$$\lambda_m = \lambda_m^{(0)} + \lambda_m^{(1)} + \lambda_m^{(2)} \tag{2.8.13.2}$$

where from Eqs. (8.116) and (8.119)

$$\lambda_m^{(0)} = \left\langle \varphi_m^{(0)} \,\middle|\, L^{(0)} \varphi_m^{(0)} \right\rangle; \quad \lambda_m^{(1)} = \left\langle \varphi_m^{(0)} \,\middle|\, Q \varphi_m^{(0)} \right\rangle \tag{2.8.13.3}$$

Equation (2.8.12.11) in Example 8.12 reads

$$\begin{gathered} \lambda_m^{(2)} = \sum_n a_{mn} Q_{mn}; \quad a_{mn} = \frac{Q_{nm}}{\lambda_m^{(0)} - \lambda_n^{(0)}}; \\ Q_{mn} = \left\langle \varphi_m^{(0)} \,\middle|\, Q \varphi_n^{(0)} \right\rangle / \left\langle \varphi_n^{(0)} \,\middle|\, \varphi_n^{(0)} \right\rangle \end{gathered} \tag{2.8.13.4}$$

The eigenfunctions for a simply supported plate are

$$\varphi_{mn}^{(0)} = \sin(m\pi x/L_x) \sin(n\pi y/L_y) \tag{2.8.13.5}$$

Equations (2.8.13.3) and (2.8.13.5) give as presented in Eq. (8.122)

$$\lambda_m^{(0)} = \frac{D}{\mu}\kappa_{mn}^4; \quad \kappa_{mn}^4 = \left[\left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2\right]^2 \tag{2.8.13.6}$$

$$\lambda_m^{(1)} = -\frac{4D}{\mu} \cdot \frac{\Delta M}{M} \cdot \kappa_{mn}^4 \left[\varphi_{mn}^{(0)}(\boldsymbol{r}_0)\right]^2 \tag{2.8.13.7}$$

where ΔM is the added mass and M the mass of the plate and $\boldsymbol{r}_0$ the coordinates for the added mass. The second-order term is given by Eq. (2.8.13.4) as

$$\lambda_m^{(2)} = \sum_{n \neq m} \frac{Q_{nm} Q_{mn}}{\lambda_m^{(0)} - \lambda_n^{(0)}} \tag{2.8.13.8}$$

In 2D

$$Q_{mn} \to Q_{mnrs} = \frac{\left\langle \varphi_{mn}^{(0)} \middle| Q\varphi_{rs}^{(0)} \right\rangle}{\left\langle \varphi_{rs}^{(0)} \middle| \varphi_{rs}^{(0)} \right\rangle} = -\frac{4D}{\mu} \cdot \frac{\Delta M}{M} \cdot \kappa_{rs}^4 \left[\varphi_{rs}^{(0)}(\boldsymbol{r}_0)\right]\left[\varphi_{mn}^{(0)}(\boldsymbol{r}_0)\right] \tag{2.8.13.9}$$

$$Q_{nm} \to Q_{rsmn} = \frac{\left\langle \varphi_{rs}^{(0)} \middle| Q\varphi_{mn}^{(0)} \right\rangle}{\left\langle \varphi_{mn}^{(0)} \middle| \varphi_{mn}^{(0)} \right\rangle} = -\frac{4D}{\mu} \cdot \frac{\Delta M}{M} \cdot \kappa_{mn}^4 \left[\varphi_{rs}^{(0)}(\boldsymbol{r}_0)\right]\left[\varphi_{mn}^{(0)}(\boldsymbol{r}_0)\right] \tag{2.8.13.10}$$

Equations (2.8.13.8), (2.8.13.9), and (2.8.13.10) give

$$\lambda_m^{(2)} = \frac{D}{\mu}\left[\frac{4\Delta M}{M}\right]^2 \kappa_{mn}^4 \sum_r \sum_s \frac{\left[\varphi_{mn}^{(0)}(\boldsymbol{r}_0)\right]^2 \left[\varphi_{rs}^{(0)}(\boldsymbol{r}_0)\right]^2 \kappa_{rs}^4}{\kappa_{mn}^4 - \kappa_{rs}^4} \tag{2.8.13.11}$$

Including the second-order terms, the eigenvalue is obtained from (2.8.13.6), (2.8.13.7), and (2.8.13.11) as

$$\lambda_m = \lambda_m^{(0)} + \lambda_m^{(1)} + \lambda_m^{(2)} = \frac{D}{\mu}\kappa_{mn}^4$$
$$\left\{1 - \frac{4\Delta M}{M}\left[\varphi_{mn}^{(0)}(\boldsymbol{r}_0)\right]^2 + \left[\frac{4\Delta M}{M}\right]^2 \sum_r \sum_s \frac{\kappa_{rs}^4 \left[\varphi_{mn}^{(0)}(\boldsymbol{r}_0)\varphi_{rs}^{(0)}(\boldsymbol{r}_0)\right]^2}{\left[\kappa_{mn}^4 - \kappa_{rs}^4\right]}\right\}$$

The summations do not include the case for which $r = m$ and $s = r$ simultaneously. The eigenfrequencies are

$$f_{mn} = \frac{\sqrt{\lambda_m}}{2\pi} \tag{2.8.13.12}$$

2.8.14 Example 8.14

The critical frequency is according to Eq. (8.134) higher for the top plate than for the bottom plate or $f_{c2} > f_{c1}$. The velocity-level difference between the bottom and top plates is thus given by Eqs. (8.132) and (8.136) as

$$\Delta L_v = C_2 + 25 \log f; \quad C_2 = 10 \log \left[\frac{\pi\delta(f_{c1}^2 - f_{c2}^2)^2(L_x + L_y)}{2cf_2^2 f_{c1}^{3/2} f_{c2}^2} \right] \quad (2.8.14.1)$$

The parameters changed are the thickness of the top plate, the stiffness of the resilient layer, and the loss factor. Further $f_{c2} \gg f_{c1}$. The parameter C_2 is, using the Eqs. (8.134) and (8.135), reduced to

$$C_2 = 10 \log \left[\frac{\delta}{h_2^2 E_w} \right] + K \quad (2.8.14.2)$$

where K is a constant. The material parameters for the two floors are using Table 8.7, p. 352:
Floor 1 $\delta = 0.2$; $E_w = 4.2 \times 10^5$ N/m^2; $h_2 = 4 \times 10^{-3}$ m
Inserting these values in Eq. (2.8.14.2) gives $C_2 = K - 15$ dB.
Floor 2 $\delta = 0.28$; $E_w = 1.3 \times 10^5$ N/m^2; $h_2 = 2 \times 10^{-3}$ m
Inserting these values in Eq. (2.8.14.2) gives $C_2 = K - 3$ dB.
In the high-frequency region the improvement is

$$(C_2)_{\text{floor2}} - (C_2)_{\text{floor1}} = 12\,\text{dB}$$

2.8.15 Example 8.15

The eigenfrequencies for an isotropic rectangular plate with free edges are given by Eq. (8.90) as

$$f_{mn} = \frac{\pi}{2}\sqrt{\frac{D_0}{\rho h}} \cdot \left[\left(\frac{G_m}{L_x}\right)^4 + \left(\frac{G_n}{L_y}\right)^4 + \frac{2J_m J_n + 2\nu(H_m H_n - J_m J_n)}{(L_x L_y)^2} \right]^{1/2} \quad (2.8.15.1)$$

Let an orthotropic plate have the bending stiffness D_x in the x-direction and D_y in the y-direction. The torsional rigidity B of the plate is approximated by $B = \sqrt{D_x D_y}$ as discussed in Sect. 3.10. For describing the displacement and eigenfrequencies of this type of orthotropic plate a coordinate transformation can be made in such a way that the orthotropic plate with the dimensions L_x and L_y is replaced by an isotropic plate with the bending stiffnes$D_0 = \sqrt{D_x D_y}$ and the dimensions $L_x \left(D_y/D_x\right)^{1/8}$ and

$L_y\left(D_x/D_y\right)^{1/8}$ in the x- and y-directions of the plate. By inserting these transforms in Eq. (2.8.15.1) the result is

$$f_{mn} = \frac{\pi}{2}\sqrt{\frac{1}{\rho h}}\cdot\left[D_x\left(\frac{G_m}{L_x}\right)^4 + D_y\left(\frac{G_n}{L_y}\right)^4 + \sqrt{D_x D_y}\frac{2J_m J_n + 2\nu(H_m H_n - J_m J_n)}{(L_x L_y)^2}\right]^{1/2} \tag{2.8.15.2}$$

where ρ is the density of the plate h its thickness and L_x and L_y the lengths of the two sides.

2.8.16 Example 8.16

The eigenfrequencies for a simply supported rectangular plate are

$$f_{mn} = \frac{\pi}{2}\sqrt{\frac{D_0}{\rho h}}\cdot\left[\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2\right] \tag{2.8.16.1}$$

According to the discussion in Sect. 8.6, the eigenfrequencies for the same plate but with clamped edges are

$$f_{mn} = \frac{\pi}{2}\sqrt{\frac{D_0}{\rho h}}\cdot\left[\left(\frac{G_m}{L_x}\right)^4 + \left(\frac{G_n}{L_y}\right)^4 + \frac{2H_m H_n}{(L_x L_y)^2}\right]^{1/2} \tag{2.8.16.2}$$

For $m > 2$ the parameters G_m and H_m are approximated by

$$G_m = m + 1/2; \quad H_m = G_m^2\left(1 - \frac{2}{\pi G_m}\right) \tag{2.8.16.3}$$

Equations (2.8.16.2) and (2.8.16.3) give as m and n approach infinity

$$\begin{aligned} f_{mn} &\to \frac{\pi}{2}\sqrt{\frac{D_0}{\rho h}}\cdot\left[\left(\frac{G_m}{L_x}\right)^4 + \left(\frac{G_n}{L_y}\right)^4 + \frac{2G_m^2 G_n^2}{(L_x L_y)^2}\right]^{1/2} \\ &= \frac{\pi}{2}\sqrt{\frac{D_0}{\rho h}}\cdot\left[\left(\frac{G_m}{L_x}\right)^2 + \left(\frac{G_n}{L_y}\right)^2\right] \\ &\to \frac{\pi}{2}\sqrt{\frac{D_0}{\rho h}}\cdot\left[\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2\right] \end{aligned} \tag{2.8.16.4}$$

Consequently, for high mode numbers the natural frequencies for clamped edges, Eq. (2.8.16.4), are the same as for a simply supported plate, Eq. (2.8.16.1).

However, the limiting eigenfrequencies for the same plate but with free edges are

$$f_{mn} \to \frac{\pi}{2}\sqrt{\frac{D_0}{\rho h}} \cdot \left[\left(\frac{G_m}{L_x}\right)^4 + \left(\frac{G_n}{L_y}\right)^4 + \frac{2(1-\nu)G_m^2 G_n^2}{(L_x L_y)^2}\right]^{1/2}$$
$$\to \frac{\pi}{2}\sqrt{\frac{D_0}{\rho h}} \cdot \left[\left(\frac{m}{L_x}\right)^4 + \left(\frac{n}{L_y}\right)^4 + 2(1-\nu)\left(\frac{m}{L_x}\right)^2\left(\frac{n}{L_y}\right)^2\right]^{1/2}$$

2.8.17 Example 8.17

Consider the eigenvalue problem

$$Lw = \lambda w; \quad L = L^0 + Q \tag{2.8.17.1}$$

Equation (2.8.17.1) is rewritten

$$L^0 w + Qw = \lambda w \text{ and again as } L^0 w - \lambda w = -Qw = f \tag{2.8.17.2}$$

The function f is introduced as a source function. The orthogonal eigenfunctions $\varphi_n^{(0)}$ satisfy

$$L^0 \varphi_n^{(0)} = \lambda_n^{(0)} \varphi_n^{(0)} \tag{2.8.17.3}$$

The solution to Eq. (2.8.17.2) is written

$$w = \sum_n C_n \varphi_n^{(0)} \tag{2.8.17.4}$$

Equations (2.8.17.2), (2.8.17.3) and (2.8.17.4) give

$$C_n = \frac{\left\langle \varphi_n^{(0)} \,\middle|\, f \right\rangle}{\left\langle \varphi_n^{(0)} \,\middle|\, \varphi_n^{(0)} \right\rangle \left(\lambda_n^{(0)} - \lambda\right)} = -\frac{\left\langle \varphi_n^{(0)} \,\middle|\, Qw \right\rangle}{\left\langle \varphi_n^{(0)} \,\middle|\, \varphi_n^{(0)} \right\rangle \left(\lambda_n^{(0)} - \lambda\right)} \tag{2.8.17.5}$$

$$w = \sum_n \frac{\varphi_n^{(0)} \left\langle \varphi_n^{(0)} \,\middle|\, Qw \right\rangle}{\left\langle \varphi_n^{(0)} \,\middle|\, \varphi_n^{(0)} \right\rangle \left(\lambda - \lambda_n^{(0)}\right)} \tag{2.8.17.6}$$

The unknown function w appears on both the left- and right-hand sides of the equation. A solution can be obtaned by using an interative method. The first step is to assume that in Eq. (2.8.17.6) $w \to \varphi_n^{(0)}$ and $\lambda \to \lambda_n^{(0)}$ as $Q \to 0$. The resulting solution is denoted φ_n and obtained from Eq. (2.8.17.6) as

$$\varphi_n = \sum_m \frac{\varphi_m^{(0)} \left\langle \varphi_m^{(0)} \,\middle|\, Q\varphi_n \right\rangle}{\left\langle \varphi_m^{(0)} \,\middle|\, \varphi_m^{(0)} \right\rangle \left(\lambda - \lambda_m^{(0)}\right)} = K \cdot \varphi_n^{(0)} + \sum_{m \neq n} \frac{\varphi_m^{(0)} \left\langle \varphi_m^{(0)} \,\middle|\, Q\varphi_n \right\rangle}{\left\langle \varphi_m^{(0)} \,\middle|\, \varphi_m^{(0)} \right\rangle \left(\lambda - \lambda_m^{(0)}\right)} \tag{2.8.17.7}$$

where

$$K = \frac{\left\langle \varphi_n^{(0)} \,\middle|\, Q\varphi_n \right\rangle}{\left\langle \varphi_n^{(0)} \,\middle|\, \varphi_n^{(0)} \right\rangle \left(\lambda - \lambda_n^{(0)}\right)} \tag{2.8.17.8}$$

The result (2.8.17.7) i.e $\varphi_n = K\varphi_n^{(0)} + \sum_{m \neq n} \cdots$ is inserted on the right-hand side of Eq. (2.8.17.7). The process is repeated in an iterative way

$$\begin{aligned} \varphi_n = K\varphi_n^{(0)} &+ \sum_{m \neq n} \frac{\varphi_m^{(0)} \left\langle \varphi_m^{(0)} \,\middle|\, QK\varphi_n^{(0)} \right\rangle}{\left\langle \varphi_m^{(0)} \,\middle|\, \varphi_m^{(0)} \right\rangle \left(\lambda - \lambda_m^{(0)}\right)} \\ &+ \sum_{m \neq n} \sum_{p \neq n} \frac{\varphi_m^{(0)} \left\langle \varphi_m^{(0)} \,\middle|\, Q\varphi_p^{(0)} \right\rangle}{\left\langle \varphi_m^{(0)} \,\middle|\, \varphi_m^{(0)} \right\rangle \left(\lambda - \lambda_m^{(0)}\right)} \frac{\left\langle \varphi_p^{(0)} \,\middle|\, QK\varphi_n^{(0)} \right\rangle}{\left\langle \varphi_p^{(0)} \,\middle|\, \varphi_p^{(0)} \right\rangle \left(\lambda - \lambda_p^{(0)}\right)} + \cdots \end{aligned} \tag{2.8.17.9}$$

which gives

$$\varphi_n = K \left[\varphi_n^{(0)} + \sum_{m \neq n} \frac{\varphi_m^{(0)} Q_{mn}}{\left(\lambda_n - \lambda_m^{(0)}\right)} + \sum_{m \neq n} \sum_{p \neq n} \frac{\varphi_m^{(0)} Q_{mp} Q_{pn}}{\left(\lambda_n - \lambda_m^{(0)}\right)\left(\lambda_n - \lambda_p^{(0)}\right)} + \cdots \right] \tag{2.8.17.10}$$

where

$$Q_{mn} = \left\langle \varphi_m^{(0)} \,\middle|\, Q\varphi_n^{(0)} \right\rangle / \left\langle \varphi_n^{(0)} \,\middle|\, \varphi_n^{(0)} \right\rangle \tag{2.8.17.11}$$

However, according to Eq. (2.8.17.8)

$$K\left(\lambda_n - \lambda_n^{(0)}\right) = \frac{\left\langle \varphi_n^{(0)} \,\middle|\, Q\varphi_n \right\rangle}{\left\langle \varphi_n^{(0)} \,\middle|\, \varphi_n^{(0)} \right\rangle} \tag{2.8.17.12}$$

By inserting φ_n given by Eq. (2.8.17.10) in (2.8.17.12) the result is

$$K\left(\lambda_n - \lambda_n^{(0)}\right) = K\Bigg[Q_{nn} + \sum_{m \neq n} \frac{Q_{nm} Q_{mn}}{\lambda_n - \lambda_m^{(0)}} + \sum_{m \neq n} \sum_{p \neq n} \frac{Q_{nm} Q_{mp} Q_{pn}}{\left(\lambda_n - \lambda_m^{(0)}\right)\left(\lambda_n - \lambda_p^{(0)}\right)} + \cdots\Bigg] \tag{2.8.17.13}$$

which gives λ_n as

$$\lambda_n = \lambda_n^{(0)} + Q_{nn} + \sum_{m \neq n} \frac{Q_{nm} Q_{mn}}{\lambda_n - \lambda_m^{(0)}} + \sum_{m \neq n} \sum_{p \neq n} \frac{Q_{nm} Q_{mp} Q_{pn}}{\left(\lambda_n - \lambda_m^{(0)}\right)\left(\lambda_n - \lambda_p^{(0)}\right)} \tag{2.8.17.14}$$

Again λ_n is found on both the left and right-hand sides of the equation. However, $\lambda_n$can be solved by iteration in a rather convenient way.

2.9 Chapter 9

2.9.1 Example 9.1

The kinetic energy is $\mathcal{T} = \dfrac{m\dot{x}^2}{2}$, the potential energy $\mathcal{U} = 0$ and the potential energy due to the external force is $\mathcal{A} = -F_x x$. Hamilton's principle (9.4) yields

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{m\dot{x}^2}{2} + F_x x\right) = \int \mathrm{d}t \,(m\dot{x}\delta\dot{x} + F_x \delta x) = 0 \tag{2.9.1.1}$$

Partial integration of the last integral gives

$$\int \mathrm{d}t \,(m\dot{x}\delta\dot{x} + F_x\delta x) = [\delta x \cdot m\dot{x}]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathrm{d}t \,[\delta x \,(m\ddot{x} - F_x)] = 0 \tag{2.9.1.2}$$

For $x(t_1) = x(t_2) = 0$ it follows that the parenthesis of the last integral of Eq. (2.9.1.2) must be zero. Thus $F_x = m\ddot{x}$.

2.9.2 Example 9.2

The kinetic and potential energies are $\mathcal{T} = m\dot{z}^2/2$; $\mathcal{V} = mgz$. Hamilton's principle (9.2) gives

$$\delta \int \mathrm{d}t \left[m\dot{z}^2/2 - mgz \right] = \int \mathrm{d}t \, [m\dot{z}\delta\dot{z} - mg\delta z] = [m\dot{z}\delta z]_{t_1}^{t_2} - \int \mathrm{d}t\delta z \, (m\ddot{z} + mg) = 0 \qquad (2.9.2.1)$$

The equation of motion is $\ddot{z} + g = 0$. The initial conditions are

$$\dot{z}(0) = 0 \text{ and } z(0) = z_0 \qquad (2.9.2.2)$$

Thus, Eqs. (2.9.1.1) and (2.9.1.2) give

$$z = z_0 - gt^2/2 \text{ for } t \geqslant 0 \qquad (2.9.2.3)$$

and until impact.

2.9.3 Example 9.3

Let the beam be oriented along the x-axis of a coordinate system. The length of the beam is L, cross section S, density ρ, and Young's modulus E. The displacement along the x-axis is ξ. The kinetic energy of the beam is $\mathcal{T} = \int_0^L \mathrm{d}x m'\dot{\xi}^2/2$, where $m' = \rho S$.

The potential energy is $\mathcal{U} = \dfrac{SE}{2}\left(\dfrac{\partial\xi}{\partial x}\right)^2$

The potential energy induced by the external forces is

$$\mathcal{A} = +F_1 \cdot \xi(0) - F_2 \cdot \xi(L) - \int_0^L \mathrm{d}x F'(x)\xi(x) \qquad (2.9.3.1)$$

Hamilton's principle gives

$$\delta \int_{t_1}^{t_2} \mathrm{d}t(\mathcal{T} - \mathcal{U} - \mathcal{A}) = \int_{t_1}^{t_2} \mathrm{d}t \int_0^L \mathrm{d}x \left[m'\dot{\xi}\delta\dot{\xi} - SE\frac{\partial\xi}{\partial x}\frac{\partial\delta\xi}{\partial x} + F'\delta\xi \right] + \int_{t_1}^{t_2} \mathrm{d}t \, [F\xi]_0^L = 0 \qquad (2.9.3.2)$$

Partial integration of the first part of Eq. (2.9.3.2) gives

$$\int_{t_1}^{t_2} \mathrm{d}t \int_0^L \mathrm{d}x \left[\delta\xi \left(SE\frac{\partial^2\xi}{\partial x^2} - m'\ddot{\xi} + F' \right) \right] + \int_{t_1}^{t_2} \mathrm{d}t\delta\xi \left[F - SE\left(\frac{\partial\xi}{\partial x}\right) \right]_0^L = 0 \qquad (2.9.3.3)$$

For Eq. (2.9.3.3) to be zero, it follows that

$$SE\frac{\partial^2 \xi}{\partial x^2} - m'\ddot{\xi} = -F' \tag{2.9.3.4}$$

The boundary conditions are obtained from the last integral of Eq. (2.9.3.3) as

$$F = ES\frac{\partial \xi}{\partial x} \text{ or } \xi = 0 \quad \text{for } x = 0 \text{ and } x = L \tag{2.9.3.5}$$

2.9.4 Example 9.4

Equations (9.35) and (9.36) give

$$-G_e S\left\{\frac{\partial^2 w}{\partial x^2} - \frac{\partial \beta}{\partial x}\right\} + 2D'_2\left\{\frac{\partial^4 w}{\partial x^4} - \frac{\partial^3 \beta}{\partial x^3}\right\} + m'\frac{\partial^2 w}{\partial t^2} - F' = 0 \tag{2.9.4.1}$$

$$-G_e S\left\{\frac{\partial w}{\partial x} - \beta\right\} - D'_1\frac{\partial^2 \beta}{\partial x^2} + 2D'_2\left\{\frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 \beta}{\partial x^2}\right\} + I_\omega\frac{\partial^2 \beta}{\partial t^2} = 0 \tag{2.9.4.2}$$

Set $w = W\exp[i(\omega t - kx)]$, $\beta = B\exp[i(\omega t - kx)]$. Equations (2.9.4.1) and (2.9.4.2) give

$$W(G_e S k^2 - m'\omega^2) + B(2ik^3 D_2 - ikG_e S) = F'_0 \tag{2.9.4.3}$$

$$B[G_e S + (D_1 + 2D_2)k^2 - I_\omega\omega^2] + W[2ik^3 D_2 - ikG_e S] = 0 \tag{2.9.4.4}$$

Eliminating B from (2.9.4.3) and (2.9.4.4) gives

$$\begin{aligned}
&2D'_1 D'_2 k^6 W - 2k^4\omega^2 D'_2 I_\omega W + k^4 G_e S D'_1 W\\
&-\left[(D'_1 + 2D'_2)m' + G_e S I_\omega\right]k^2\omega^2 W - \omega^2 G_e S m' W + m' I_\omega \omega^4 W\\
&\quad = G_e S F'_0 + k^2\left(D'_1 + 2D'_2\right)F'_0 - \omega^2 I_\omega F'_0
\end{aligned} \tag{2.9.4.5}$$

Interpreting $(-ik)^n W$ as $\partial^n w/\partial x^n$ and $(i\omega)^n W$ as $\partial^n w/\partial t^n$ etc give together with Eq. (2.9.4.5)

$$\begin{aligned}
&-2D'_1 D'_2\frac{\partial^6 w}{\partial x^6} + 2D'_2 I_\omega\frac{\partial^6 w}{\partial x^4 \partial t^2} + G_e S D'_1\frac{\partial^4 w}{\partial x^4}\\
&-\left[(D'_1 + 2D'_2)m' + G_e S I_\omega\right]\frac{\partial^4 w}{\partial x^2 \partial t^2} + G_e S m'\frac{\partial^2 w}{\partial t^2} + m' I_\omega\frac{\partial^4 w}{\partial t^4}\\
&\quad = G_e S F' - \left(D'_1 + 2D'_2\right)\frac{\partial^2 F'}{\partial x^2} + I_\omega\frac{\partial^2 F'}{\partial t^2}
\end{aligned} \tag{2.9.4.6}$$

2.9.5 Example 9.5

From Eqs. (9.50)–(9.52), the following expressions are obtained

$$(D_1 + D_3)\frac{\partial^4 w}{\partial x^4} + \mu_0 \ddot{w} - G\frac{h_e^2}{h_2}\frac{\partial^2 w}{\partial x^2} - G\frac{h_e}{h_2}\left(\frac{\partial \xi_3}{\partial x} - \frac{\partial \xi_1}{\partial x}\right) = p \tag{2.9.5.1}$$

$$E_1 h_1 \frac{\partial^2 \xi_1}{\partial x^2} - \mu_1 \ddot{\xi}_1 + G\frac{h_e}{h_2}\frac{\partial w}{\partial x} + G\frac{1}{h_2}(\xi_3 - \xi_1) = 0 \tag{2.9.5.2}$$

$$E_3 h_3 \frac{\partial^2 \xi_3}{\partial x^2} - \mu_3 \ddot{\xi}_3 - G\frac{h_e}{h_2}\frac{\partial w}{\partial x} - G\frac{1}{h_2}(\xi_3 - \xi_1) = 0 \tag{2.9.5.3}$$

Neglecting $\ddot{\xi}_i$ the last two equations (2.9.5.2) and (2.9.5.3), are written

$$X_1 \frac{\partial^2 \xi_1}{\partial x^2} + A\frac{\partial w}{\partial x} + B(\xi_3 - \xi_1) = 0 \tag{2.9.5.4}$$

$$X_3 \frac{\partial^2 \xi_3}{\partial x^2} - A\frac{\partial w}{\partial x} - B(\xi_3 - \xi_1) = 0 \tag{2.9.5.5}$$

with

$$X_i = E_i h_i; \quad A = G\frac{h_e}{h_2}; \quad B = G\frac{1}{h_2} \tag{2.9.5.6}$$

Set $\dfrac{\partial^n \xi}{\partial x^n} = (-ik)^n\, \xi$. Solving ξ_1 and ξ_3 from Eqs. (2.9.5.4) and (2.9.5.5) gives

$$\xi_1 = -\frac{ikwAX_3}{k^2 X_1 X_3 + BX_1 + BX_3} \tag{2.9.5.7}$$

$$\xi_3 = -\frac{ikwAX_1}{k^2 X_1 X_3 + BX_1 + BX_3} \tag{2.9.5.8}$$

Thus

$$\left(\frac{\partial \xi_3}{\partial x} - \frac{\partial \xi_1}{\partial x}\right) = \frac{k^2 wA(X_1 + X_3)}{k^2 X_1 X_3 + BX_1 + BX_3} \tag{2.9.5.9}$$

Writing $\dfrac{\partial^n w}{\partial x^n} = (-ik)^n\, w$, $\dfrac{\partial^n w}{\partial t^n} = (i\omega)^n\, w$ and inserting these expressions plus (2.9.5.9) and $p = 0$ in (2.9.5.1) yields

$$(D_1 + D_3)k^6 w X_1 X_3 + B(X_1 + X_3)(D_1 + D_3)k^4 w + \mu_0 k^2 \ddot{w} X_1 X_3$$
$$+ B(X_1 + X_3)\mu_0 \ddot{w} + G\frac{h_e^2}{h_2} w k^4 X_1 X_3$$
$$+ wk^2 B(X_1 + X_3)Gh_2 - k^2 Aw(X_1 + X_3) = 0 \tag{2.9.5.10}$$

Considering that $\dfrac{\partial^n w}{\partial x^n} = (-ik)^n\, w$ the Eq. (2.9.5.10) is rewritten

$$\frac{\partial^6 w}{\partial x^6} + \frac{\partial^4 w}{\partial x^4}\left[\frac{B(X_1+X_3)}{X_1X_3} + \frac{Gh_4^2}{h_2(D_1+D_3)}\right] + \frac{\mu_0}{D_1+D_3}\left[\frac{\ddot{w}B(X_1+X_3)}{X_1X_3} - \frac{\partial^2 \ddot{w}}{\partial x^2}\right]$$
$$+ \frac{\partial^2 w}{\partial x^2}\left[\frac{(X_1+X_3)Gh_e(Bh_e - A)}{h_2 X_1 X_3 (D_1+D_3)}\right] = 0$$

However, $Bh_e - A = 0$. The resulting differential equation is thus

$$\frac{\partial^6 w}{\partial x^6} - Z(1+\mathcal{Y})\frac{\partial^4 w}{\partial x^4} + \frac{\mu_0}{D_1+D_3}\left[\frac{\partial^2 \ddot{w}}{\partial x^2} - \ddot{w}\mathcal{Z}\right] = 0 \tag{2.9.5.11}$$

where

$$\mathcal{Z} = \frac{G}{h_2}\left[\frac{E_1h_1 + E_3h_3}{E_1h_1E_3h_3}\right]; \quad \mathcal{Y} = \frac{[h_2 + (h_1+h_3)/2]^2}{(D_1+D_3)}\left[\frac{E_1h_1E_3h_3}{E_1h_1 + E_3h_3}\right]$$

2.9.6 Example 9.6

The requirement is $\Delta \leqslant \dfrac{1}{10}$ where Δ is defined in Eq. (9.78) as

$$\Delta = \frac{k_L^2}{4\kappa^2}\left[1 + \left(\frac{k_T}{k_L T_p}\right)^2\right] \text{ for } \nu = 0.3, \left(\frac{k_T}{k_L T_p}\right)^2 = \frac{2(1+\nu)}{T_p^2} \approx \frac{2.6}{0.93^2} \approx 3 \tag{2.9.6.1}$$

Thus

$$\Delta \approx \frac{k_L^2}{\kappa^2} = \frac{\omega^2 \rho}{E} \cdot \left[\frac{Eh^2}{\rho\omega^2 12}\right]^{1/2} = \frac{(h\kappa)^2}{12} \tag{2.9.6.2}$$

Consequently, $(h\kappa)^2 < \dfrac{12}{10}$ or approximately $h\kappa < 1$.

2.9.7 Example 9.7

According to definitions in Sect. 8.6.
Rayleigh-Ritz
The max potential energy is given by

$$\mathcal{U}_{\max} = \frac{D}{2}\int \mathrm{d}x\mathrm{d}y\Bigg\{\left(C_1\frac{\partial^2\phi_1}{\partial x^2} + C_2\frac{\partial^2\phi_2}{\partial x^2}\right)^2 + \left(C_1\frac{\partial^2\phi_1}{\partial y^2} + C_2\frac{\partial^2\phi_2}{\partial y^2}\right)^2 + 2\left(C_1\frac{\partial^2\phi_1}{\partial x\partial y} + C_2\frac{\partial^2\phi_2}{\partial x\partial y}\right)^2\Bigg\}$$

It follows that

$$\frac{\partial\mathcal{U}_{\max}}{\partial C_1} = D\int \mathrm{d}S\Bigg\{C_1\left[\left(\frac{\partial^2\phi_1}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi_1}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2\phi_1}{\partial x\partial y}\right)^2\right] + C_2\Bigg[\left(\frac{\partial^2\phi_1}{\partial x^2}\right)\left(\frac{\partial^2\phi_2}{\partial x^2}\right) + \left(\frac{\partial^2\phi_1}{\partial y^2}\right)\left(\frac{\partial^2\phi_2}{\partial y^2}\right) + 2\left(\frac{\partial^2\phi_1}{\partial x\partial y}\right)\left(\frac{\partial^2\phi_2}{\partial x\partial y}\right)\Bigg]\Bigg\}$$

etc.
The max kinetic energy is

$$\mathcal{T}_{\max} = \omega^2\frac{\mu}{2}\int \mathrm{d}x\mathrm{d}y\,[C_1\phi_1 + C_2\phi_2]^2$$

Thus

$$\frac{\partial\mathcal{T}_{\max}}{\partial C_1} = \mu\omega^2\int \mathrm{d}x\mathrm{d}y\left[C_1\phi_1^2 + C_2\phi_1\phi_2\right]$$

The elements in the matrix (8.94) are

$$a_{11} = D\int \mathrm{d}S\left[\left(\frac{\partial^2\phi_1}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi_1}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2\phi_1}{\partial x\partial y}\right)^2\right];$$

$$a_{22} = D\int \mathrm{d}S\left[\left(\frac{\partial^2\phi_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi_2}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2\phi_2}{\partial x\partial y}\right)^2\right]$$

$$a_{12} = a_{21} = D\int \mathrm{d}S\Bigg[\left(\frac{\partial^2\phi_1}{\partial x^2}\right)\left(\frac{\partial^2\phi_2}{\partial x^2}\right) + \left(\frac{\partial^2\phi_1}{\partial y^2}\right)\left(\frac{\partial^2\phi_2}{\partial y^2}\right) + 2\left(\frac{\partial^2\phi_1}{\partial x\partial y}\right)\left(\frac{\partial^2\phi_2}{\partial x\partial y}\right)\Bigg]$$

$$b_{11} = \mu \int \mathrm{d}S\phi_1^2; \quad b_{12} = b_{21} = \mu \int \mathrm{d}S\phi_1\phi_2; \quad b_{22} = \mu \int \mathrm{d}S\phi_2^2$$

The first few natural frequencies are obtained by setting the determinant of the matrix $[A]$ equal to zero, where

$$[A] = \begin{bmatrix} a_{11} - \omega^2 b_{11} & a_{12} - \omega^2 b_{12} \\ a_{21} - \omega^2 b_{21} & a_{22} - \omega^2 b_{22} \end{bmatrix}$$

Garlekin

The method is discussed in Sect. 9.9. For no external forces, the parameters B_i are equal to zero. The elements in the matrix $[A]$ defined in Eq. (9.106) are

$$A_{ij} = \int \mathrm{d}S \left[\phi_i \nabla^2(\nabla^2\phi_j) - \mu\omega^2\phi_i\phi_j\right]$$

Partial integration gives

$$A_{11} = D \int \mathrm{d}S \left[\left(\frac{\partial^2\phi_1}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi_1}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2\phi_1}{\partial x\partial y}\right)^2 - \mu\omega^2\phi_1^2 \right]$$

$$A_{12} = A_{21} = D \int \mathrm{d}S \Bigg[\left(\frac{\partial^2\phi_1}{\partial x^2}\right)\left(\frac{\partial^2\phi_2}{\partial x^2}\right) + \left(\frac{\partial^2\phi_1}{\partial y^2}\right)\left(\frac{\partial^2\phi_2}{\partial y^2}\right) + 2\left(\frac{\partial^2\phi_1}{\partial x\partial y}\right)\left(\frac{\partial^2\phi_1}{\partial x\partial y}\right) - \mu\omega^2\phi_1\phi_2 \Bigg].$$

$$A_{22} = D \int \mathrm{d}S \left[\left(\frac{\partial^2\phi_2}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi_2}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2\phi_2}{\partial x\partial y}\right)^2 - \mu\omega^2\phi_2^2 \right]$$

The natural frequencies are obtaind by setting the determinant of $[A]$ equal to zero.

The results show that $A_{ij} = a_{ij} - \omega^2 b_{ij}$. Thus, the Rayleigh–Ritz and Garlekin methods give the same natural frequencies.

2.9.8 Example 9.8

The first two expressions of (9.93) are:

$$\frac{\partial^2\xi}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2\xi}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2\eta}{\partial x\partial y} - \frac{1-\nu^2}{Eh}\mu\ddot{\xi} = 0 \tag{2.9.8.1}$$

$$\frac{1+\nu}{2}\frac{\partial^2\xi}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} - \frac{1-\nu^2}{Eh}\mu\ddot{\eta} = 0 \tag{2.9.8.2}$$

Using a scalar and a vector potential, the displacements ξ and η are are according to Eq. (4.5) given by (no z dependence)

$$\xi = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}; \quad \eta = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \tag{2.9.8.3}$$

Equation (2.9.8.3) inserted in (2.9.8.1) yields

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 \phi}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 \phi}{\partial y^2} + \frac{1-\nu^2}{Eh}\mu\omega^2\phi\right) \\ &+ \frac{\partial}{\partial y}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 \psi}{\partial y^2} - \frac{1+\nu}{2}\frac{\partial^2 \psi}{\partial x^2} + \frac{1-\nu^2}{Eh}\mu\omega^2\psi\right) = 0 \end{aligned} \tag{2.9.8.4}$$

or

$$\frac{\partial}{\partial x}\left(\nabla^2\phi + \frac{1-\nu^2}{Eh}\mu\omega^2\phi\right) + \frac{\partial}{\partial y}\left(\frac{1-\nu}{2}\nabla^2\Psi + \frac{1-\nu^2}{Eh}\mu\omega^2\psi\right) = 0 \tag{2.9.8.5}$$

Or since $\mu = \rho h$

$$\frac{\partial}{\partial x}\left(\nabla^2\phi + \frac{1-\nu^2}{Eh}\mu\omega^2\phi\right) + \left(\frac{1-\nu}{2}\right)\frac{\partial}{\partial y}\left(\nabla^2\Psi + \frac{2(1+\nu)\rho}{E}\omega^2\psi\right) = 0 \tag{2.9.8.6}$$

The expression inside the first parenthesis of Eq. (2.9.8.6) is the differential equation for longitudinal waves and the expression inside the second represents the differential equation for transverse waves, compare Eq. (4.8).

2.9.9 Example 9.9

The equation governing the vibration of a Timoshenko beam is given by Eq. (9.60) as

$$G_e S D_1' \frac{\partial^4 w}{\partial x^4} - \left[D_1' m' + G_e S I_\omega'\right] \frac{\partial^4 w}{\partial x^2 \partial t^2} + G_e S m' \frac{\partial^2 w}{\partial t^2} + m' I_\omega' \frac{\partial^4 w}{\partial t^4} = 0 \tag{2.9.9.1}$$

The boundary conditions for a simply supported beam are according to Table 9.2 given by

$$w = 0; \quad \partial\beta/\partial x = 0 \tag{2.9.9.2}$$

For free vibrations, no external forces, the displacements w and β satisfy the same differential equation (2.9.9.1). Let

$$w = A_1 \sin\kappa_1 x + A_2 \cos\kappa_1 x + A_3 \sinh\kappa_2 x + A_4 \cosh\kappa_2 x \tag{2.9.9.3}$$

$$\beta = B_1 \sin \kappa_1 x + B_2 \cos \kappa_1 x + B_3 \sinh \kappa_2 x + B_4 \cosh \kappa_2 x \tag{2.9.9.4}$$

By setting $w = Q \cdot \exp[i(\omega t - \kappa x)]$ in Eq. (2.9.9.1) the wavenumber κ is the solution to the equation

$$G_e S D_1' \kappa^4 - \left[D_1' m' + G_e S I_\omega'\right] \omega^2 \kappa^2 - \omega^2 G_e S m' + \omega^4 m' I_\omega' = 0 \tag{2.9.9.5}$$

The four solutions are written $\kappa = \pm\kappa_1$ and $\kappa = \pm i\kappa_2$.

The displacements w and β should satisfy Eq. (9.36) with $D_2' = 0$. Thus

$$-G_e S \left\{ \frac{\partial w}{\partial x} - \beta \right\} - D_1' \frac{\partial^2 \beta}{\partial x^2} + I_\omega' \frac{\partial^2 \beta}{\partial t^2} = 0 \tag{2.9.9.6}$$

This gives

$$\begin{aligned} B_2 &= A_1 \frac{G_e S \kappa_1}{G_e S + D_1' \kappa_1^2 - \omega^2 I_\omega'}; \quad B_1 = -A_2 \frac{G_e S \kappa_1}{G_e S + D_1' \kappa_1^2 - \omega^2 I_\omega'} \\ B_4 &= A_3 \frac{G_e S \kappa_2}{G_e S - D_1' \kappa_2^2 - \omega^2 I_\omega'}; \quad B_3 = A_4 \frac{G_e S \kappa_2}{G_e S - D_1' \kappa_2^2 - \omega^2 I_\omega'} \end{aligned} \tag{2.9.9.7}$$

The boundary conditions are satisfied for $A_2 = A_3 = A_4 = B_1 = B_3 = B_4 = 0$ and $\kappa_1 = n\pi/L$. The natural angular frequencies ω_n are the solutions to Eq. (2.9.9.5) with $\kappa = n\pi/L$ or

$$G_e S D_1' (n\pi/L)^4 - \left[D_1' m' + G_e S I_\omega'\right] \omega_n^2 (n\pi/L)^2 - \omega_n^2 G_e S m' + \omega_n^4 m' I_\omega' = 0 \tag{2.9.9.8}$$

2.9.10 Example 9.10

According to Eq. (4.32), the wavenumbers describing bending of a Timoshenko beam are

$$k_1 = \pm \left\{ \frac{1}{2} \left[k_l^2 + k_t^2/T_b\right] + \left[4\kappa^4 + (k_l^2 - k_t^2/T_b)^2\right]^{1/2} \right\}^{1/2} \tag{2.9.10.1}$$

$$k_2 = \pm \left\{ \frac{1}{2} \left[k_l^2 + k_t^2/T_b\right] - \left[4\kappa^4 + (k_l^2 - k_t^2/T_b)^2\right]^{1/2} \right\}^{1/2} \tag{2.9.10.2}$$

where T_b is the Timoshenko constant and

$$k_l = \omega\sqrt{\frac{\rho}{E}};\quad k_t = \omega\sqrt{\frac{\rho}{G}};\quad \kappa = \left(\frac{m'\omega^2}{D'}\right)^{1/4};\quad m' = S\rho;\quad D' = \frac{bh^3 E}{12} \tag{2.9.10.3}$$

The height of the beam is h, its width b, and cross-sectional area $S = bh$. In the low-frequency region, the solutions are $k_1 = \pm\kappa$ and $k_2 = \pm i\kappa$. For high frequencies, $k_1 \to \pm k_l$ and $k_2 \to k_t/\sqrt{T_b}$.

Let the force excite the beam at $x = 0$. For $x > 0$ and a time dependence $e^{i\omega t}$ assume the displacements w and β to be

$$w = A_1 e^{-ik_1 x} + A_2 e^{-ik_2 x};\quad \beta = iB_1 e^{-ik_1 x} + iB_2 e^{-ik_2 x} \tag{2.9.10.4}$$

The displacements w and β should for $D_2' = 0$ satisfy Eq. (9.36) or

$$T_b GS\left(\frac{\partial w}{\partial x} - \beta\right) + D'\frac{\partial^2\beta}{\partial x^2} - I_\omega'\frac{\partial^2\beta}{\partial t^2} = 0 \tag{2.9.10.5}$$

Equations (2.9.10.4) and (2.9.10.5) give

$$A_1 = B_1\frac{\omega^2 I_\omega' - D'k_1^2 - T_b GS}{k_1 T_b GS};\quad A_2 = B_2\frac{\omega^2 I_\omega' - D'k_2^2 - T_b GS}{k_2 T_b GS} \tag{2.9.10.6}$$

The boundary conditions at the excitation point, $x = 0$, are given by (9.61)

$$\frac{\partial w}{\partial x} = 0 \tag{2.9.10.7}$$

$$\frac{F}{2} = T_b GS\left(\frac{\partial w}{\partial x} - \beta\right) \tag{2.9.10.8}$$

Equations (2.9.10.4) and (2.9.10.7) give

$$k_1 A_1 = -k_2 A_2 \tag{2.9.10.9}$$

Equations (2.9.10.7) and (2.9.10.4) inserted in (2.9.10.8) result in

$$B_1 + B_2 = -\frac{F}{2iT_b GS} \tag{2.9.10.10}$$

Equations (2.9.10.9) and (2.9.10.6) give

$$B_2 = -B_1\frac{(\omega^2 I_\omega' - D'k_1^2 - T_b GS)}{(\omega^2 I_\omega' - D'k_2^2 - T_b GS)} \tag{2.9.10.11}$$

Equations (2.9.10.10) and (2.9.10.11) give

$$B_1 = \frac{iF}{2T_b GS} \cdot \frac{(\omega^2 I'_\omega - D'k_2^2 - T_b GS)}{D'(k_1^2 - k_2^2)} \tag{2.9.10.12}$$

The displacement at the excitation point is

$$w(0) = A_1 + A_2 = A_1\left(\frac{k_2 - k_1}{k_1}\right) = B_1\left(\frac{k_2 - k_1}{k_1^2}\right)\left(\frac{\omega^2 I'_\omega - D'k_1^2 - T_b GS}{T_b GS}\right) \tag{2.9.10.13}$$

Equations (2.9.10.12) and (2.9.10.13) give

$$w(0) = \frac{iF(k_2 - k_1)(\omega^2 I'_\omega - D'k_2^2 - T_b GS)(\omega^2 I'_\omega - D'k_1^2 - T_b GS)}{2(k_1 T_b GS)^2 D'(k_1^2 - k_2^2)} \tag{2.9.10.14}$$

The point mobility is

$$Y = \frac{i\omega w(0)}{F} = \frac{\omega(\omega^2 I'_\omega - D'k_2^2 - T_b GS)(\omega^2 I'_\omega - D'k_1^2 - T_b GS)}{2(k_1 T_b GS)^2 D'(k_1 + k_2)} \tag{2.9.10.15}$$

For an Euler beam $G \to \infty$ and $k_1 \to \kappa$ and $k_2 \to i\kappa$ resulting in Eq. (2.9.10.15) being reduced to

$$Y = \frac{(1 - i)\kappa}{4m'\omega} \tag{2.9.10.16}$$

This is the point mobility of an Euler beam as derived in Eq. (5.39).

2.9.11 Example 9.11

The corners of a rectangular and homogeneous plate with constant thickness are located at $(0, 0)$, $(L_x, 0)$, (L_x, L_y) and $(0, L_y)$. The mass per unit area of the plate is μ and its bending stiffness is D. The plate is excited by a pressure $p(x, y)$. The displacement is w. The pressure and displacement are positive in the same direction. Resulting forces and moments are indicated in Fig. 2.26.

The potential energy per unit area of the plate is given by Eq. (3.124) as

$$\mathcal{U}_S = \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + 2(1 - \nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] \tag{2.9.11.1}$$

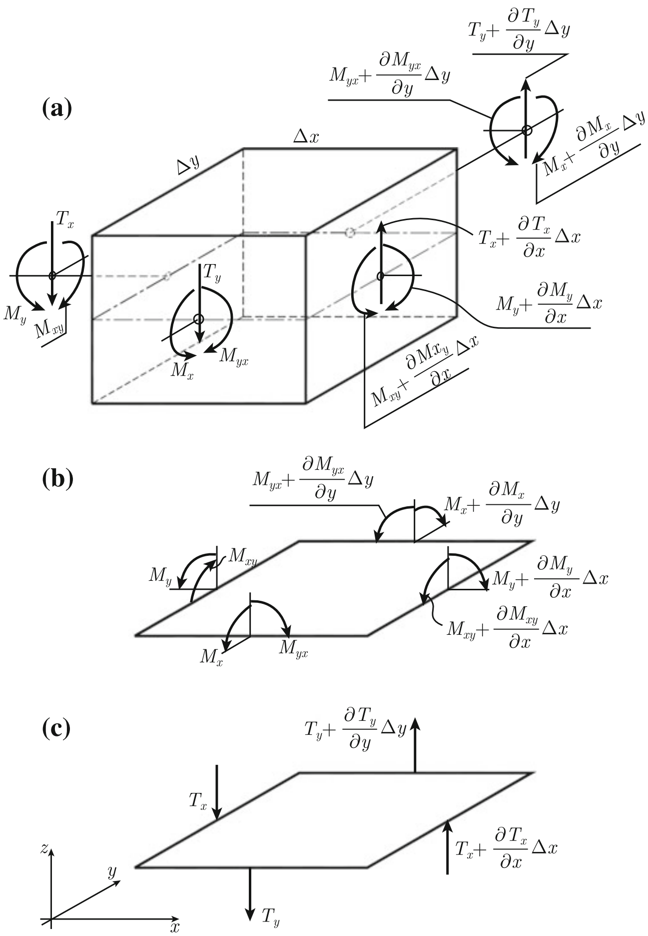


Fig. 2.26 Forces and moments acting on a plate element

The kinetic energy per unit area is

$$\mathcal{T}_S = \frac{\mu}{2}\left(\frac{\partial w}{\partial t}\right)^2 \tag{2.9.11.2}$$

The potential energy induced by external forces is

$$A = -\iint_S wp\mathrm{d}x\mathrm{d}y - \int_0^{L_y}\left[T_x w - M_y'\frac{\partial w}{\partial x} - M_{xy}'\frac{\partial w}{\partial y}\right]_0^{L_x}\mathrm{d}y$$
$$-\int_0^{L_x}\left[T_y w - M_x'\frac{\partial w}{\partial y} - M_{xy}'\frac{\partial w}{\partial x}\right]_0^{L_y}\mathrm{d}x \tag{2.9.11.3}$$

where T_x is the the force per unit length of the plate along $x = 0$ or $x = L_x$ and M'_x is the bending moment per unit length around a line parallel to the x-axis. T_y is the the force per unit length of the plate along $y = 0$ or $y = L_y$ and M'_y is the bending moment per unit length around a line parallel to the y-axis. M'_{xy} is the bending moment per unit length due to shear.

According to Hamilton's principle, Eq. (9.4)

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y(\mathcal{T}_S - \mathcal{U}_S) - \delta \int_{t_1}^{t_2} \mathrm{d}t A = 0 \tag{2.9.11.4}$$

The first part of Eq. (2.9.11.4) is obtained as

$$\begin{aligned}\delta \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y\mathcal{T}_S &= -\mu \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y \frac{\partial^2 w}{\partial t^2}\delta w + \mu \iint_S \mathrm{d}x\mathrm{d}y \left[\frac{\partial w}{\partial t}\delta w\right]_{t_1}^{t_2} \\ &= -\mu \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y \frac{\partial^2 w}{\partial t^2}\delta w \end{aligned} \tag{2.9.11.5}$$

The displacement is zero for $t = t_1$ and t_2.

The second part of Eq. (2.9.11.4) is

$$\begin{aligned}&-\delta \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y\mathcal{U}_S = -D \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y \left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 \delta w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^2 \delta w}{\partial y^2}\right] \\ &-D \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y \left[\nu\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 \delta w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^2 \delta w}{\partial x^2} + 2(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial^2 \delta w}{\partial x\partial y}\right]\end{aligned} \tag{2.9.11.6}$$

The last part of Eq. (2.9.11.4) gives

$$\begin{aligned}-\delta y \int_{t_1}^{t_2} \mathrm{d}t\mathcal{A} = \int_{t_1}^{t_2} \mathrm{d}t \Bigg\{ \iint_s \mathrm{d}x\mathrm{d}y p\delta w + \int_0^{L_y} \mathrm{d}y \left[T_x\delta w - M'_y\frac{\partial\delta w}{\partial x} - M'_{xy}\frac{\partial\delta w}{\delta y}\right]_0^{L_x} \\ + \int_0^{L_x} \mathrm{d}x \left[T_y\delta w - M'_x\frac{\partial\delta w}{\partial y} - M'_{xy}\frac{\partial\delta w}{\delta x}\right]_0^{L_y}\Bigg\}\end{aligned} \tag{2.9.11.7}$$

Partial integration of the various expressions of Eq. (2.9.11.6) gives

$$\begin{aligned}&-D \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y \left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 \delta w}{\partial x^2}\right] = -\int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y\delta w D\frac{\partial^4 w}{\partial x^4} \\ &-\int_{t_1}^{t_2} \mathrm{d}t \int_0^{L_y} \mathrm{d}y D\left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial\delta w}{\partial x} - \frac{\partial^3 w}{\partial x^3}\cdot\delta w\right]_0^{L_x} \\ &-D \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y \left[\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^2 \delta w}{\partial y^2}\right] = -\int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x\mathrm{d}y\delta w D\frac{\partial^4 w}{\partial y^4}\end{aligned} \tag{2.9.11.8}$$

$$-\int_{t_1}^{t_2} \mathrm{d}t \int_0^{L_x} \mathrm{d}x D \left[\frac{\partial^2 w}{\partial y^2} \cdot \frac{\partial \delta w}{\partial y} - \frac{\partial^3 w}{\partial y^3} \cdot \delta w \right]_0^{L_y} \tag{2.9.11.9}$$

$$- D \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x \mathrm{d}y \left[\nu \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 \delta w}{\partial y^2} \right] = - \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x \mathrm{d}y \delta w \nu D \frac{\partial^4 w}{\partial x^2 \partial y^2}$$
$$+ \int_{t_1}^{t_2} \mathrm{d}t \int_0^{L_x} \mathrm{d}x \nu D \left[\frac{\partial^3 w}{\partial x^2 \partial y} \cdot \delta w \right]_0^{L_y} - \int_{t_1}^{t_2} \mathrm{d}t \int_0^{L_y} \mathrm{d}y \nu D \left[\frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial \delta w}{\partial y} \right]_0^{L_y} \tag{2.9.11.10}$$

$$- D \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x \mathrm{d}y \left[\nu \frac{\partial^2 w}{\partial y^2} \cdot \frac{\partial^2 \delta w}{\partial x^2} \right] = - \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x \mathrm{d}y \delta w \nu D \frac{\partial^4 w}{\partial x^2 \partial y^2}$$
$$+ \int_{t_1}^{t_2} \mathrm{d}t \int_0^{L_y} \mathrm{d}y \nu D \left[\frac{\partial^3 w}{\partial x \partial y^2} \cdot \delta w \right]_0^{L_x} - \int_{t_1}^{t_2} \mathrm{d}t \int_0^{L_x} \mathrm{d}x \nu D \left[\frac{\partial^2 w}{\partial y^2} \cdot \frac{\partial \delta w}{\partial x} \right]_0^{L_x} \tag{2.9.11.11}$$

$$- D \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x \mathrm{d}y \left[2(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial^2 \delta w}{\partial x \partial y} \right]$$
$$= -2(1-\nu) D \int_{t_1}^{t_2} \mathrm{d}t \left\{ \iint_S \mathrm{d}x \mathrm{d}y \frac{\partial^4 w}{\partial x^2 \partial y^2} \delta w - \int_0^{L_x} \mathrm{d}x \left[\frac{\partial^3 w}{\partial x^2 \partial y} \cdot \delta w \right]_0^{L_y} \right.$$
$$\left. + \int_0^{L_y} \mathrm{d}y \left[\frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial \delta w}{\partial y} \right]_0^{L_x} \right\} \tag{2.9.11.12}$$

If the order of integration is changed the result of Eq. (2.9.11.12) is

$$-D \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x \mathrm{d}y \left[2(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial^2 \delta w}{\partial x \partial y} \right]$$
$$= -2(1-\nu) D \int_{t_1}^{t_2} \mathrm{d}t \left\{ \iint_S \mathrm{d}x \mathrm{d}y \frac{\partial^4 w}{\partial x^2 \partial y^2} \delta w - \int_0^{L_y} \mathrm{d}y \left[\frac{\partial^3 w}{\partial x \partial y^2} \cdot \delta w \right]_0^{L_x} \right.$$
$$\left. + \int_0^{L_y} \mathrm{d}y \left[\frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial \delta w}{\partial x} \right]_0^{L_y} \right\} \tag{2.9.11.13}$$

By adding the two solutions (2.9.11.12) and (2.9.11.13) and by dividing by a factor 2 the result is

$$-D \int_{t_1}^{t_2} \mathrm{d}t \iint_S \mathrm{d}x \mathrm{d}y \left[2(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial^2 \delta w}{\partial x \partial y} \right]$$
$$= - 2(1-\nu) D \int_{t_1}^{t_2} \mathrm{d}t \left\{ \iint_S \mathrm{d}x \mathrm{d}y \frac{\partial^4 w}{\partial x^2 \partial y^2} \delta w \right\}$$
$$- (1-\nu) D \int_{t_1}^{t_2} \mathrm{d}t \left\{ - \int_0^{L_y} \mathrm{d}y \left[\frac{\partial^3 w}{\partial x \partial y^2} \cdot \delta w \right]_0^{L_x} \right.$$

$$
\begin{aligned}
&+\int_0^{L_y} \mathrm{d}y \left[\frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial \delta w}{\partial x}\right]_0^{L_y} - \int_0^{L_x} \mathrm{d}x \left[\frac{\partial^3 w}{\partial x^2 \partial y} \cdot \delta w\right]_0^{L_y} \\
&+\int_0^{L_y} \mathrm{d}y \left[\frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial \delta w}{\partial y}\right]_0^{L_x} \Bigg\}
\end{aligned} \tag{2.9.11.14}
$$

A summation of all contributions, Eqs. (2.9.11.5), (2.9.11.7), (2.9.11.8), (2.9.11.9), (2.9.11.10), (2.9.11.11), and (2.9.11.14) gives

$$
\begin{aligned}
&\int_{t_1}^{t_2} \mathrm{d}t \Bigg\{ \iint_s \mathrm{d}x \mathrm{d}y G_1 \delta w + \int_0^{L_y} \mathrm{d}y \left[G_2 \frac{\partial \delta w}{\partial x}\right]_0^{L_x} + \int_0^{L_x} \mathrm{d}x \left[G_3 \frac{\partial \delta w}{\partial y}\right]_0^{L_y} \\
&+ \int_0^{L_y} \mathrm{d}y \left[G_4 \delta w\right]_0^{L_x} + \int_0^{L_x} \mathrm{d}x \left[G_5 \delta w\right]_0^{L_y} \Bigg\} \\
&+ \int_{t_1}^{t_2} \mathrm{d}t \left\{ \int_0^{L_y} \mathrm{d}y \left[G_6 \frac{\partial \delta w}{\partial y}\right]_0^{L_x} + \int_0^{L_y} \mathrm{d}y \left[G_7 \frac{\partial \delta w}{\partial x}\right]_0^{L_y} \right\} = 0
\end{aligned} \tag{2.9.11.15}
$$

where

$$
\begin{aligned}
G_1 =& -\mu \frac{\partial^2 w}{\partial t^2} + p - D \frac{\partial^4 w}{\partial x^4} - D \frac{\partial^4 w}{\partial y^4} - \nu D \frac{\partial^4 w}{\partial x^2 \partial y^2} \\
& - \nu D \frac{\partial^4 w}{\partial x^2 \partial y^2} - 2D(1-\nu) \frac{\partial^4 w}{\partial x^2 \partial y^2} \\
=& p - \nabla^2(\nabla^2 w) - \mu \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{2.9.11.16}
$$

$$
G_2 = - M_y' - D \frac{\partial^2 w}{\partial x^2} - \nu D \frac{\partial^2 w}{\partial y^2} \tag{2.9.11.17}
$$

$$
G_3 = - M_x' - D \frac{\partial^2 w}{\partial y^2} - \nu D \frac{\partial^2 w}{\partial x^2} \tag{2.9.11.18}
$$

$$
\begin{aligned}
G_4 =& T_x + D \frac{\partial^3 w}{\partial x^3} + \nu D \frac{\partial^3 w}{\partial x \partial y^2} + 2(1-\nu) D \frac{\partial^3 w}{\partial x \partial y^2} \\
=& T_x + D \frac{\partial^3 w}{\partial x^3} + (2-\nu) D \frac{\partial^3 w}{\partial x \partial y^2}
\end{aligned} \tag{2.9.11.19}
$$

$$
\begin{aligned}
G_5 =& T_y + D \frac{\partial^3 w}{\partial y^3} + \nu D \frac{\partial^3 w}{\partial x^2 \partial y} + 2(1-\nu) D \frac{\partial^3 w}{\partial x^2 \partial y} \\
=& T_x + D \frac{\partial^3 w}{\partial y^3} + (2-\nu) D \frac{\partial^3 w}{\partial x^2 \partial y}
\end{aligned} \tag{2.9.11.20}
$$

$$
G_6 = - M_{xy}' - (1-\nu) D \frac{\partial^2 w}{\partial x \partial y} \tag{2.9.11.21}
$$

$$
G_7 = - M_{xy}' - (1-\nu) D \frac{\partial^2 w}{\partial x \partial y} \tag{2.9.11.22}
$$

For the expression (2.9.11.15) to equal zero all integrals must also equal zero.

For G_1 equal zero it follows that either $w = 0$ or

$$\nabla^2(\nabla^2 w) + \mu \frac{\partial^2 w}{\partial t^2} = p \tag{2.9.11.23}$$

This is the governing equation for a plate in flexure.

For the second integral to be zero either $\partial w/\partial x$ or G_2 is zero along the sides $x = L_x$ and $x = 0$ of the plate. Thus, along these sides the requirements are

$$M'_y = -D\frac{\partial^2 w}{\partial x^2} - \nu D\frac{\partial^2 w}{\partial y^2} \quad \text{or } \partial w/\partial x = 0 \text{ for } x = 0 \text{ and } x = L_x \tag{2.9.11.24}$$

For the fourth integral to be zero the requirements are

$$T_x = -D\frac{\partial^3 w}{\partial x^3} - (2-\nu)D\frac{\partial^3 w}{\partial x \partial y^2} \text{ or } w = 0 \text{ for } x = 0 \text{ and } x = L_x \tag{2.9.11.25}$$

For the sixth integral to be zero the requirements are

$$M'_{xy} = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} \text{ or } \partial w/\partial y = 0 \text{ for } x = 0 \text{ and } x = L_x \tag{2.9.11.26}$$

The corresponding expressions along the y-axis are

$$M'_x = -D\frac{\partial^2 w}{\partial y^2} - \nu D\frac{\partial^2 w}{\partial x^2} \text{ or } \partial w/\partial y = 0 \text{ for } y = 0 \text{ and } y = L_y \tag{2.9.11.27}$$

$$T_y = -D\frac{\partial^3 w}{\partial y^3} - (2-\nu)D\frac{\partial^3 w}{\partial x^2 \partial y} \text{ or } w = 0 \text{ for } y = 0 \text{ and } y = L_y \tag{2.9.11.28}$$

$$M'_{xy} = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} \text{ or } \partial w/\partial x = 0 \text{ for } y = 0 \text{ and } y = L_y \tag{2.9.11.29}$$

For the side $x = L_x$ to be simply supported the requirements are $w = 0$ and $M'_x = 0$. For w to be zero along the side it follows that also $\partial w/\partial y = 0$. Thus, the boundary conditions are

$$w = 0; \quad \frac{\partial^2 w}{\partial x^2} = 0 \tag{2.9.11.30}$$

For the same side to be clamped the requirements are

$$w = 0; \quad \frac{\partial w}{\partial x} = 0 \tag{2.9.11.31}$$

For a free edge $M'_y = 0$ and $T_x = 0$ or

$$\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} = 0; \quad \frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2} = 0 \tag{2.9.11.32}$$

The moment M'_{xy} is in general different from zero.

In principle for a free edge there are three boundary conditions. Along the edge $x = L_x$ the three apparent boundary conditions are M'_y, T_x and M'_{xy} equal to zero. However, only two conditions can be satisfied since the displacement of the plate is governed by a fourth-order differential equation. This anomaly was in the 19th century considered by Cauchy, Navier, Kirchhoff, and Lord Kelvin amongst others. It was suggested by Kirchhoff that the effect of the twisting moment M'_{xy} can be included as a force, i.e.

$$T_x = -D\frac{\partial}{\partial x}\left(\nabla^2 w\right) + \frac{\partial M'_{xy}}{\partial y} = -D\frac{\partial^3 w}{\partial x^3} - (2-\nu)D\frac{\partial^3 w}{\partial x \partial y^2}$$

This expression was obtained directly in Eq. (2.9.11.24) using Hamilton's principle. Lord Kelvin concluded that for a thin plate the detailed description of the stresses in the plate can not be given within a distance equal to the plate thickness to the edge. This is in accordance with the principle of Saint-Venant. Consequently, for a free edge there are two boundary conditions to be satisfied. For the edge $x = L_x$ these are

$$\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} = 0; \quad \frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2} = 0$$

The same result—two boundary conditions—is obtained from Hamilton's principle if instead of Eq. (2.9.11.14) the Eq. (2.9.11.12) is used to derive the function G_6.

2.10 Chapter 10

2.10.1 Example 10.1

The kinetic and potential energies of the system are

$$\mathcal{T} = m(\dot{x}_1^2 + \dot{x}_2^2)/2; \quad \mathcal{U} = k[(x_1 - x_2)^2 + x_1^2 + x_2^2]/2 - F_1 x_1 - F_2 x_2 + \mathcal{U}_0 \tag{2.10.1.1}$$

Based on the Eqs. (10.6) and (10.23), the resulting equations governing the displacements of the masses are

$$m\ddot{x}_1 + k(2x_1 - x_2) = F_1; \quad m\ddot{x}_2 + k(2x_2 - x_1) = F_2 \tag{2.10.1.2}$$

In matrix form this is equivalent to

$$\boldsymbol{M}\cdot\ddot{\boldsymbol{X}}+\boldsymbol{K}\cdot\boldsymbol{X}=\boldsymbol{F};\quad \boldsymbol{M}=m\begin{bmatrix}1&0\\0&1\end{bmatrix};$$
$$\boldsymbol{K}=k\begin{bmatrix}2&-1\\-1&2\end{bmatrix};\quad \boldsymbol{F}=\begin{Bmatrix}F_1\\F_2\end{Bmatrix}e^{i\omega t}\tag{2.10.1.3}$$

The eigenvalues for the undamped system are obtained from Eq. (10.12) as

$$\det\begin{bmatrix}2k-m\lambda&-k\\-k&2k-m\lambda\end{bmatrix}=0\tag{2.10.1.4}$$

By introducing $\omega_0^2=k/m$ the result is

$$\lambda_1=3\omega_0^2 \text{ and } \lambda_2=\omega_0^2\tag{2.10.1.5}$$

The eigenvectors $\boldsymbol{X}_r$ corresponding to λ_r are obtained from Eq. (10.13) as

$$[\boldsymbol{K}-\lambda_r\boldsymbol{M}]\,\boldsymbol{X}_r=[\boldsymbol{K}-\lambda_r\boldsymbol{M}]\begin{Bmatrix}x_1\\x_2\end{Bmatrix}_r=0\tag{2.10.1.6}$$

$$\boldsymbol{X}_1=\begin{Bmatrix}x_1\\x_2\end{Bmatrix}_1=\begin{Bmatrix}-1\\1\end{Bmatrix};\quad \boldsymbol{X}_2=\begin{Bmatrix}x_1\\x_2\end{Bmatrix}_2=\begin{Bmatrix}1\\1\end{Bmatrix}\tag{2.10.1.7}$$

According to (10.18) and (10.20), the displacement of the masses are given by

$$\boldsymbol{X}=\frac{-F_1+F_2}{6k-2m\omega^2}\begin{Bmatrix}-1\\1\end{Bmatrix}e^{i\omega t}+\frac{F_1+F_2}{2k-2m\omega^2}\begin{Bmatrix}1\\1\end{Bmatrix}e^{i\omega t}\tag{2.10.1.8}$$

2.10.2 Example 10.2

The ratio between the mobilities is

$$\frac{Y_{12}^s}{Y_{11}^s}=\cos(k_lL);\quad k_l=\omega\sqrt{\frac{\rho}{E}}$$

The mass of the mount is $m=\rho SL$. Thus $m\to 0\Rightarrow\rho\to 0\Rightarrow k_l\to 0\Rightarrow\cos(k_lL)\to 1$. Consequently, the ratio Y_{12}^s/Y_{11}^s=1 for a mass-less spring.

The equivalent mobility for a mount is defined by Eq. (10.54) as

$$Y_{eq}^s=\frac{Y_{11}^sY_{22}^s-Y_{12}^sY_{21}^s}{Y_{12}^s}\tag{2.10.2.1}$$

For $Y_{11}^s = Y_{22}^s = -i\omega/[SEk_l \tan(k_l L)]$ and $Y_{12}^s = Y_{21}^s = -i\omega/[SEk_l \sin(k_l L)]$ the result is

$$Y_{eq}^s = \frac{\left(-\dfrac{i\omega}{SEk_l}\right)^2 \left(\dfrac{\cos^2(k_l L)}{\sin^2(k_l L)} - \dfrac{1}{\sin^2(k_l L)}\right)}{\left(-\dfrac{i\omega}{SEk_l}\right) \dfrac{1}{\sin(k_l L)}} = \frac{i\omega}{SEk_l} \sin(k_l L) \tag{2.10.2.2}$$

For $m \to 0$ then $k_l \to 0$ and $\sin(k_l L) \to k_l L$. Thus $Y_{eq}^s \to \dfrac{i\omega}{SE/L}$ as $m \to 0$. However, SE/L is according to Eq. (3.4) equal to k_{eq} of a mass-less spring. Consequently,

$$Y_{eq}^s \to \frac{i\omega}{k_{eq}} \text{ as } m \to 0 \tag{2.10.2.3}$$

2.10.3 Example 10.3

The beam is oriented along the x-axis. The bending moment is exciting the beam at $x = 0$. The displacement w, time dependence $\exp(i\omega t)$, of the beam is given by

$$w_+ = A_1 e^{-i\kappa x} + B_1 e^{-\kappa x} \text{ for } x \geqslant 0; \quad w_- = A_2 e^{-i\kappa x} + B_2 e^{-\kappa x} \text{ for } x \leqslant 0 \tag{2.10.3.1}$$

Boundary conditions at the excitation point are

$$w_+(0) = w_-(0);\ w'_+(0) = w'_-(0);\ D[w_+(0) - w_-(0)] = M;\ w'''_+(0) = w'''_-(0) \tag{2.10.3.2}$$

Equations (2.10.3.1) and (2.10.3.2) give

$$A_1 = \frac{M}{4\kappa^2 D} = -A_2 = -B_1 = B_2 \tag{2.10.3.3}$$

The velocity of rotation around the y-axis is

$$\dot{\omega}_y = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial w}{\partial x}\right)_{x=0} = i\omega\kappa A_1(1-i) = \frac{\omega M}{4\kappa D'}(1+i) \tag{2.10.3.4}$$

According to definition $\hat{\dot{\omega}}_y = Y_{M\dot{\omega}} \hat{M}$. Thus from Eqs. (2.10.3.3) and (2.10.3.4)

$$Y_{M\dot{\omega}} = \frac{\omega}{4\kappa D'}(1+i) \tag{2.10.3.5}$$

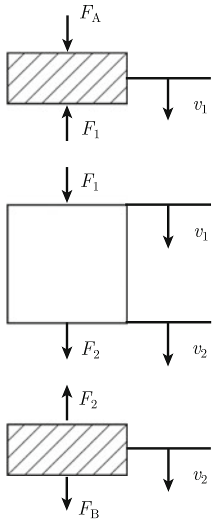


Fig. 2.27 A resilient mount consisting of a rubber cylinder coupled at both ends to stiff masses

2.10.4 Example 10.4

The mobility for each mass is $Y_M = 1/(i\omega M)$. According to Fig. 2.27, the following equations are obtained

$$\hat{F}_A - \hat{F}_1 = \frac{\hat{v}_1}{Y^m}; \quad \hat{F}_B - \hat{F}_2 = \frac{\hat{v}_2}{Y^m} \tag{2.10.4.1}$$

$$\hat{v}_1 = \hat{F}_1 Y_{11} + \hat{F}_2 Y_{21} = \hat{F}_A Y_{11}^t + \hat{F}_B Y_{21}^t \tag{2.10.4.2}$$

$$\hat{v}_2 = \hat{F}_2 Y_{11} + \hat{F}_1 Y_{21} = \hat{F}_B Y_{11}^t + \hat{F}_A Y_{21}^t \tag{2.10.4.3}$$

The mobilities for the entire construction are denoted Y_{ij}^t.

The equalities $Y_{11} = Y_{22}$, $Y_{12} = Y_{21}$, $Y_{11}^t = Y_{22}^t$ and $Y_{12}^t = Y_{21}^t$ have been used. Equation (2.10.4.1) inserted in (2.10.4.3) yields

$$\hat{F}_1(Y_{11} - Y_{11}^t) + \hat{F}_2(Y_{21} - Y_{21}^t) = \frac{\hat{v}_1 Y_{11}^t + \hat{v}_2 Y_{21}^t}{Y^m} \tag{2.10.4.4}$$

Equation (2.10.4.4) in combination with $\hat{v}_2 = \hat{F}_2 Y_{11} + \hat{F}_1 Y_{21}$ results in

$$\hat{v}_1 = \frac{\hat{F}_1}{Y_{11}^t}[Y^m(Y_{11} - Y_{11}^t) - Y_{21}Y_{21}^t] + \frac{\hat{F}_2}{Y_{11}^t}[Y^m(Y_{21} - Y_{21}^t) - Y_{11}Y_{21}^t] \tag{2.10.4.5}$$

However, $\hat{v}_1$ is also given by Eq. (2.10.4.3). An identification of parameters gives

$$Y_{11} = \frac{1}{Y_{11}^t}[Y^m(Y_{11} - Y_{11}^t) - Y_{21}Y_{21}^t] \tag{2.10.4.6}$$

$$Y_{21} = \frac{1}{Y_{11}^t}[Y^m(Y_{21} - Y_{21}^t) - Y_{11}Y_{21}^t] \tag{2.10.4.7}$$

The Eqs. (2.10.4.6) and (2.10.4.7) give

$$Y_{11} = \frac{Y_{11}^t(Y_M)^2 + Y^m(Y_{21}^t)^2 - Y_M(Y_{11}^t)^2}{(Y_M - Y_{11}^t)^2 - (Y_{21}^t)^2} \tag{2.10.4.8}$$

$$Y_{21} = \frac{Y_{21}^t(Y_M)^2 - Y^m Y_{21}^t Y_{11}^t + Y_M Y_{11}^t Y_{21}^t}{(Y_M - Y_{11}^t)^2 - (Y_{21}^t)^2} \tag{2.10.4.9}$$

2.10.5 Example 10.5

According to Sect. 10.10 and using the notations of Fig. 10.21, Vol. 2

$$\hat{v}_0 = \hat{F}_{ext}Y_{12}^m; \quad Y_{12}^m = Y_{22}^m = 1/(i\omega M) \tag{2.10.5.1}$$

The point mobility of the foundation Y^f is

$$Y^f = \frac{1}{8\sqrt{\mu D}} = \frac{1}{8h^2}\sqrt{\frac{12(1-\nu^2)}{\rho E}} \tag{2.10.5.2}$$

Equation (10.116) gives

$$F_f = \frac{\hat{v}_0}{Y^m Y_{22}^s/Y_{12}^s + Y^m Y^f/Y_{12}^s + Y^f Y_{11}^s/Y_{12}^s + Y_{eq}^s Y_{12}^s/Y_{12}^s} \tag{2.10.5.3}$$

For an infinitely stiff mount Eq. (10.117) gives

$$\hat{F}_f^0 = \frac{\hat{v}_0}{(Y^m + Y^f)} \tag{2.10.5.4}$$

According to Eq. (6.96),

$$Y_{11}^s = Y_{22}^s = -\frac{i\omega}{SEk_l\tan(k_l L)}; \quad Y_{12}^s = Y_{21}^s = -\frac{i\omega}{SEk_l\sin(k_l L)} \tag{2.10.5.5}$$

According to Eqs. (10.53) and (10.54),

$$Y_{eq}^s = \frac{i\omega \sin(k_l L)}{SEk_l} \tag{2.10.5.6}$$

Equations (2.10.5.5) and (2.10.5.6) inserted in (2.10.5.3) gives

$$\hat{F}_f = \frac{\hat{v}_0}{(Y^m + Y^f)\cos(k_l L) + i\sin(k_l L)[Y^m Y^f SEk_l/\omega + \omega/(SEk_l)]} \tag{2.10.5.7}$$

The insertion loss is thus

$$IL = 20\log\left|\frac{\hat{F}_f^0}{\hat{F}_f}\right| = 20\log \left|\frac{(Y^m + Y^f)\cos(k_l L) + i\sin(k_l L)[Y^m Y^f SEk_l/\omega + \omega/(SEk_l)]}{(Y^m + Y^f)}\right| \tag{2.10.5.8}$$

where $k_l = k_{l0}(1 - i\eta/2)$, $Y^f = \dfrac{1}{8h^2}\sqrt{\dfrac{12(1-\nu^2)}{\rho E}}$, $Y^m = \dfrac{1}{i\omega M}$.

2.10.6 Example 10.6

When the source is turned off the external force F_{ext} is zero. Thus $\hat{v}_0 = 0$. The source is excited by a force F_0 parallel to F_2 in Fig. 2.28.

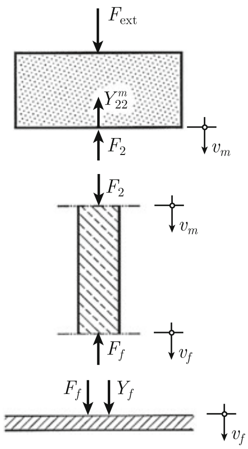


Fig. 2.28 A stiff mass mounted on a rod which in turn is mounted to a foundation having the point mobility Y_f

The measured point mobility between the mount and the source is

$$(Y^m)_{\text{measured}} = \hat{v}_m/\hat{F}_0 \tag{2.10.6.1}$$

The velocity v_m is, for $v_0 = 0$ obtained from (10.107) as

$$\hat{v}_m = -Y^m(\hat{F}_2 + \hat{F}_0) \tag{2.10.6.2}$$

The Eqs. (10.114) and (10.115) give

$$\hat{v}_m = \hat{F}_2 Y^s_{11} - \hat{F}_f Y^s_{21}; \quad \hat{v}_f = \hat{F}_2 Y^s_{12} - \hat{F}_f Y^s_{22}; \quad \hat{v}_f = \hat{F}_f Y^f \tag{2.10.6.3}$$

By eliminating $\hat{F}_2$, $\hat{F}_f$ and $\hat{v}_f$ it is found that

$$\begin{aligned}(Y^m)_{\text{measured}} = \frac{\hat{v}_m}{\hat{F}_0} &= \frac{Y^m(Y^s_{11}Y^f + Y^s_{12}Y_{eq})}{Y^s_{11}Y^f + Y^s_{12}Y_{eq} + Y^mY^f + Y^mY^s_{22}} \\ &= \frac{Y^m(Y^f + Y_{eq}Y^s_{12}/Y^s_{11})}{Y^f + Y_{eq}Y^s_{12}/Y^s_{11} + Y^mY^f/Y^s_{11} + Y^mY^s_{22}/Y^s_{11}}\end{aligned} \tag{2.10.6.4}$$

For a mass less spring $Y^s_{12}/Y^s_{11} = Y^s_{22}/Y^s_{11} = 1$ and $Y^f \ll Y^s_{11}$. Thus

$$(Y^m)_{\text{measured}} = \frac{\hat{v}_m}{\hat{F}_0} = \frac{Y^m(Y^f + Y_{eq})}{Y^f + Y_{eq} + Y^m} \tag{2.10.6.5}$$

In a similar way, the measured point mobility of the foundation is obtained as

$$(Y^f)_{\text{measured}} = \frac{\hat{v}_f}{\hat{F}_0} = \frac{Y^f(Y^m + Y_{eq})}{Y^f + Y_{eq} + Y^m} \tag{2.10.6.6}$$

2.10.7 Example 10.7

From Problem 10.6, Eq. (2.10.6.5)

$$(Y^m)_{\text{measured}} = \frac{Y^m(Y^f + Y_{eq})}{Y^f + Y_{eq} + Y^m} = Q_1 \tag{2.10.7.1}$$

The measured point mobility of the foundation is obtained from Problem 10.6, Eq. (2.10.6.6) as

$$(Y^f)_{\text{measured}} = \frac{Y^f(Y^m + Y_{eq})}{Y^f + Y_{eq} + Y^m} = Q_2 \tag{2.10.7.2}$$

From Eqs. (10.133) and (10.134)

$$\hat{v}_0 = \hat{v}_m \left(\frac{Y^f + Y_{eq} + Y^m}{Y^f + Y_{eq}} \right) \tag{2.10.7.3}$$

$$\hat{v}_0 = \hat{v}_f \left(\frac{Y^f + Y_{eq} + Y^m}{Y^f} \right) \tag{2.10.7.4}$$

The four unknown parameters Y^m, Y^f, Y_{eq} and v_0 are solved from the Eqs. (2.10.7.1) through (2.10.7.4). Thus,

$$\hat{v}_0 = \frac{\hat{v}_2(\hat{v}_1^2 Q_2 - \hat{v}_2^2 Q_1)}{\hat{v}_1(\hat{v}_1 Q_2 - \hat{v}_2 Q_1)} \tag{2.10.7.5}$$

where $v_m = v_2$ and $v_f = v_1$.

2.10.8 *Example 10.8*

Set $Y^m = U_m + iV_m$ and $Y^f = U_f + iV_f$. According to (10.111), the power input to the foundation is

$$ReG_\Pi = G_{v_0v_0} \cdot \frac{ReY^f}{\left|Y^f + Y^m\right|^2} = G_{v_0v_0} \cdot \frac{U_f}{(U_v + U_f)^2 + (V_v + V_f)^2} \tag{2.10.8.1}$$

Differentiating with respect to U_f and V_f gives

$$\begin{aligned} dG_\Pi =& \frac{(U_v + U_f)^2 + (V_v + V_f)^2 - 2U_f(U_v + U_f)}{\left[(U_m + U_f)^2 + (V_m + V_f)^2\right]^2} \cdot \mathrm{d}U_f \\ & - \frac{2U_f(V_v + V_f)}{\left[(U_m + U_f)^2 + (V_m + V_f)^2\right]^2} \cdot \mathrm{d}U_f \end{aligned} \tag{2.10.8.2}$$

For $dG_\Pi = 0$ the result is

$$2U_f(V_v + V_f) = 0 \tag{2.10.8.3}$$

$$(U_v + U_f)^2 + (V_v + V_f)^2 - 2U_f(U_v + U_f) = 0 \tag{2.10.8.4}$$

For a nontrivial solution to Eqs. (2.10.8.3) and (2.10.8.4), it follows that

$$U_v = U_f \text{ and } V_v = -V_f \Rightarrow Y^f = (Y^m)^* \tag{2.10.8.5}$$

2.10.9 Example 10.9

For any solid the displacement can be described by a combination of longitudinal and transverse waves. The L-waves are governed by the differential equation, Eq. (4.8) as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\omega^2}{c_l^2}\phi = 0 \tag{2.10.9.1}$$

Let the ratio between full and model scale dimensions be Z. Let the coordinates for the full scale structure be x, y and z and for the model scale $x_1 = x/Z$, $y_1 = y/Z$ and $z_1 = z/Z$. Since $\partial/\partial x = (1/Z)\partial/\partial x_1$ it follows that the wave equation can be written as

$$\frac{1}{Z^2}\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial y_1^2} + \frac{\partial^2 \phi}{\partial z_1^2}\right) + \frac{\omega^2}{c_l^2}\phi = 0$$

or

$$\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial y_1^2} + \frac{\partial^2 \phi}{\partial z_1^2}\right) + \frac{Z^2\omega^2}{c_l^2}\phi = 0 \tag{2.10.9.2}$$

Thus by introducing $\omega_1 = Z\omega$ the initial differential equation is reduced to

$$\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial y_1^2} + \frac{\partial^2 \phi}{\partial z_1^2}\right) + \frac{\omega_1^2}{c_l^2}\phi = 0 \tag{2.10.9.3}$$

which is on the form as the initial governing equation. By reducing the length scale by a factor Z, the frequency must be increased by the same factor Z.

The same discussion can be carried out for transverse waves. Every quantity like Helmholtz numbers type κL or $k_l L$ or mobilities etc derived using the basic equation for longitudinal and transverse waves must satisfy the conditions. So for $x_1 = x/Z$ etc it follows that $f_1 = Zf$.

2.10.10 Example 10.10

According to Eq. (10.47), the Bishop model for a circular mount, radius a, is governed by the equation

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{\rho}{E}\cdot\frac{\partial^2 \xi}{\partial t^2} + \frac{\rho\nu^2 a^2}{2E}\cdot\frac{\partial^4 \xi}{\partial x^2 \partial t^2} - \left(\frac{\rho}{E}\right)^2 \frac{\nu^2 a^4}{6}(2\nu^2 + \nu - 1)\frac{\partial^6 \xi}{\partial x^2 \partial t^4} = 0 \tag{2.10.10.1}$$

Assuming $\xi = A \cdot \exp[i(\omega t - k_b x)]$ the wavenumber k_b is obtained as

$$k_b = \omega\sqrt{\frac{\rho}{E}} \cdot \left[1 - \frac{\omega^2 \rho \nu^2 a^2}{2E} - \omega^4 \left(\frac{\rho}{E}\right)^2 \frac{\nu^2 a^4}{6}(2\nu^2 + \nu - 1)\right]^{-1/2} \quad (2.10.10.2)$$

The normal stress σ_x should according to Eq. (10.48) satisfy the expression

$$\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial^2 \xi}{\partial t^2} \quad (2.10.10.3)$$

Equations (2.10.10.1) and (2.10.10.3) give

$$\frac{\partial \sigma_x}{\partial x} = E\frac{\partial^2 \xi}{\partial x^2} + \frac{\rho \nu^2 a^2}{2} \cdot \frac{\partial^4 \xi}{\partial x^2 \partial t^2} - \frac{\rho^2}{E}\frac{\nu^2 a^4}{6}(2\nu^2 + \nu - 1)\frac{\partial^6 \xi}{\partial x^2 \partial t^4} \quad (2.10.10.4)$$

An integration with respect to x gives

$$\sigma_x = E\frac{\partial \xi}{\partial x} + \frac{\rho \nu^2 a^2}{2} \cdot \frac{\partial^3 \xi}{\partial x \partial t^2} - \frac{\rho^2}{E}\frac{\nu^2 a^4}{6}(2\nu^2 + \nu - 1)\frac{\partial^5 \xi}{\partial x \partial t^4} \quad (2.10.10.5)$$

2.10.11 Example 10.11

Assume that the displacement of the mass is x and the displacement of the foundation is y. The external force exciting the mass is $F(t) = F_0 \cdot \sin(\omega_1 t)$. The equation of motion for the mass is

$$\begin{aligned} m\ddot{x} + k(x - y) &= F(t) \text{ or} \\ m\ddot{x} + kx &= F(t) + H(t) \end{aligned} \quad (2.10.11.1)$$

where $H(t) = ky$ is the force on the mass caused by the motion of the foundation. The FT of x is obtained by substituting x by $\hat{x}e^{i\omega t}$, F and H by $\hat{F}e^{i\omega t}$ and $\hat{H}e^{i\omega t}$ respectively. The FT of x is obtained from Eq. (2.10.11.1) as

$$\hat{x} = \frac{\hat{F} + \hat{H}}{k - m\omega^2} = \frac{\hat{F} + k\hat{y}}{m(\omega_0^2 - \omega^2 + i\omega_0^2 \delta)} \quad (2.10.11.2)$$

where $k = k_0(1 + i\delta)$ and $\omega_0 = \sqrt{k_0/m}$.

The FT of the velocity is

$$\hat{v} = i\omega\hat{x} = \frac{i\omega(\hat{F} + \hat{H})}{k - m\omega^2} = \frac{i\omega(\hat{F} + k\hat{y})}{m(\omega_0^2 - \omega^2 + i\omega_0^2 \delta)} \quad (2.10.11.3)$$

The force F and the displacement y are completely uncorrelated since y is random. Consequently, F and H are also uncorrelated. The two-sided power spectral density of the force $F(t)$

$$S_{FF} = F_0^2/4 \cdot [\delta(f - f_1) + \delta(f + f_1)] \tag{2.10.11.4}$$

The two-sided power spectral density for H is defined as

$$S_{HH} = \lim_{T\to\infty} \frac{\left|\hat{H}\right|^2}{T} \tag{2.10.11.5}$$

Since F and H are uncorrelated the power spectral density of the total force acting on the mass is $S_{FF} + S_{HH}$.

The two-sided power spectral density of the velocity is obtained as

$$\begin{aligned} S_{vv} = \lim_{T\to\infty} \frac{\left|\hat{v}\right|^2}{T} &= \frac{\omega^2}{m^2\left[(\omega_0^2 - \omega^2)^2 + (\omega_0^2\delta)^2\right]} \lim_{T\to\infty} \left[\frac{\left|\hat{F}\right|^2}{T} + \frac{k_0^2\left|\hat{y}\right|^2}{T}\right] \\ &= \frac{\omega^2(S_F + k_0^2 S_y)}{m^2\left[(\omega_0^2 - \omega^2)^2 + (\omega_0^2\delta)^2\right]} \end{aligned} \tag{2.10.11.6}$$

The time average of the velocity squared is defined as

$$\bar{v}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{vv}\mathrm{d}\omega$$

Thus

$$\begin{aligned} \bar{v}^2 =& \frac{F_0^2}{4}\int_{-\infty}^{\infty} \mathrm{d}f \cdot \frac{\omega^2[\delta(f - f_1) + \delta(f + f_1)]}{m^2[(\omega^2 - \omega_0^2)^2 + (\omega_0^2\delta)^2]} \\ &+ \frac{k_0^2 G_{yy}}{2}\int_{-\infty}^{\infty} \mathrm{d}f \cdot \frac{\omega^2}{m^2[(\omega^2 - \omega_0^2)^2 + (\omega_0^2\delta)^2]} \\ =& \frac{F_0^2}{2}\left\{\frac{\omega_1^2}{m^2[(\omega_1^2 - \omega_0^2)^2 + (\omega_0^2\delta)^2]}\right\} \\ &+ \frac{k_0^2 G_{yy}}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \cdot \frac{\omega^2}{m^2[(\omega^2 - \omega_0^2)^2 + (\omega_0^2\delta)^2]} \\ =& \frac{F_0^2}{2}\left\{\frac{\omega_1^2}{m^2[(\omega_1^2 - \omega_0^2)^2 + (\omega_0^2\delta)^2]}\right\} + \frac{G_{yy}\omega_0^3}{4\delta} \end{aligned} \tag{2.10.11.7}$$

The identity $\frac{k_0}{m} = \omega_0^2$ has been used.

The second integral is solved as described in Sect. 2.5 using

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \cdot \frac{g(\omega)}{(\omega^2-\omega_0^2)^2+(\delta\,\omega_0^2)^2} = \frac{\pi g(\omega_0)}{\omega_0^3\delta}$$

2.10.12 Example 10.12

The equation of motion for the mass is

$$m\ddot{x} + kx = F_0 \sin\omega_1 t \tag{2.10.12.1}$$

The solution is

$$x = X_0 \sin\omega_1 t; \quad X_0 = \frac{F_0}{m(\omega_0^2-\omega_1^2+i\omega_0^2\delta)} \tag{2.10.12.2}$$

The time average of the velocity squared is

$$|\bar{v}|^2 = \frac{\omega^2\,|X_0|^2}{2} = \frac{f_1^2\,|F_0|^2}{(2\pi)^2 m^2[(f_0^2-f_1^2)^2+(f_0^2\delta)^2]} \tag{2.10.12.3}$$

By increasing the losses by a factor Q the resulting velocity is v_Q where from (2.10.12.3)

$$\frac{|\bar{v}|^2}{\left|\bar{v}_Q\right|^2} = \frac{[(f_0^2-f_1^2)^2+(f_0^2 Q\delta)^2]}{[(f_0^2-f_1^2)^2+(f_0^2\delta)^2]} \tag{2.10.12.4}$$

For $f_0 = f_1$ the effect of increasing the losses is significant. However, for $f_1 \gg f_0$ the effect is insignificant since typically $Q\delta < 1$.

2.10.13 Example 10.13

The shape function for a circular mount is according to Eqs. (10.35) and (10.61) given by

$$S = \frac{R}{2(L-d)} \tag{2.10.13.1}$$

The apparent E-modulus is for $\mathcal{B} = 2$ according to Eq. (10.59) equal to

$$E_a = E \cdot (1 + 2S^2) \tag{2.10.13.2}$$

The compression d of the mount due to the static load is

$$d = \frac{FL}{\pi R^2 E_a} = \frac{FL}{\pi R^2 E\left[1 + \dfrac{R^2}{2(L-d)^2}\right]} \tag{2.10.13.3}$$

For small deflections, $d \ll L$, Eq. (2.10.13.3) is reduced to

$$d \approx \frac{FL}{\pi R^2 E\left[1 + \dfrac{R^2}{2L^2}\right]} \tag{2.10.13.4}$$

In the same way and for $d \ll L$,

$$E_a \approx E\left(1 + \frac{R^2}{2L^2}\right)$$

According to the Love model, Eq. (10.44) the wavenumber for effective longtudinal waves is

$$k_l = \omega\sqrt{\frac{\rho}{E_a}} \cdot \left[1 - \frac{\omega^2 \nu^2 \rho R^2}{2E_a}\right]^{-1/2} \tag{2.10.13.5}$$

The equivalent stiffness of a mount is according to Eq. (10.53)

$$k_{eq} = \frac{\pi R^2 E_a k_l}{\sin[k_l(L-d)]} \tag{2.10.13.6}$$

In the low-frequency region as $\omega \to 0$ and $k_l \to 0$

$$k_{eq} = \frac{\pi R^2 E_a}{L-d} = \frac{\pi R^2 E_a}{L[1 - F/(\pi R^2 E_a)]} \tag{2.10.13.7}$$

The equivalent stiffness is increasing in the low-frequency region as the force F is increased.

The first maximum of k_{eq} is obtained when $Re[k_l(L-d)] = \pi/2$, i.e. at the frequency f where f is the solution to

$$f = \frac{1}{4}\sqrt{\frac{E_a}{\rho}} \cdot \frac{\left(1 - \dfrac{4\pi^2 f^2 \nu^2 \rho R^2}{2E_a}\right)}{L\left(1 - \dfrac{F}{\pi R^2 E_a}\right)} \tag{2.10.13.8}$$

As long as the Love correction, $4\pi^2 f^2 \nu^2 \rho R^2/(2E_a)$, is small the frequency for the first maximum is increasing for an increasing static load F. The main reason is that the length of the mount is decreased as the static force is increased. Compare Figs. 10.10 and 10.18.

2.11 Chapter 11

2.11.1 Example 11.1

The wave equation giving the pressure p in a fluid moving with the vector velocity $\boldsymbol{u}$ is according to Sect. 11.1 given as

$$\nabla^2 p - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot \text{grad}\right)^2 p = 0 \tag{2.11.1.1}$$

Assume $p(x,t) = p_0 \exp[i(\omega t - kx)]$. Thus,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot \text{grad}\right) p = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) p = i\,(\omega - uk_+)\,p \tag{2.11.1.2}$$

Inserting Eq. (2.11.1.2) in Eq. (2.11.1.1) gives

$$\frac{1}{c^2}(\omega - uk)^2 = k^2 \tag{2.11.1.3}$$

For $u/c = M$ the solutions are

$$k_+ = \frac{\omega}{c}/\,(1+M) = k_0/\,(1+M)\,; \quad k_- = -\frac{\omega}{c}/\,(1-M) = -k_0/\,(1-M) \tag{2.11.1.4}$$

where k_0 is the wavenumber in a fluid at rest and k_+ the wavenumber for a wave propagating in the direction of the flow or in this case along the positive x-axis. k_- is the wavenumber for a wave propagating in the opposite direction.

For $M \ll 1$,

$$k_+ = k_0/\,(1+M) \approx k_0\,(1-u/c) \text{ and } k_- = -k_0/\,(1-M) \approx -k_0\,(1+u/c) \tag{2.11.1.5}$$

2.11.2 Example 11.2

Assume the velocity potential to be

$$\Phi(x,t) = \Phi_0(x)\exp(i\omega t) \tag{2.11.2.1}$$

The velocity potential should satisfy Eq. (11.19). Thus,

$$\Phi_0 = A\sin kx + B\cos kx \tag{2.11.2.2}$$

where k is the wavenumber in the fluid. Since $p = -\rho_0 \partial\phi/\partial t$ and $v_x = \partial\phi/\partial x$ it follows that

$$p = -i\omega\rho_0(A\sin kx + B\cos kx); \quad v_x = k(A\cos kx - B\sin kx) \tag{2.11.2.3}$$

The boundary conditions are

$$v_x = u_0 \text{ for } x = 0 \text{ and } p/v_x = Z \text{ for } x = L \tag{2.11.2.4}$$

The boundary conditions give

$$A = \frac{u_0 c}{\omega}; \quad B = \frac{u_0 c}{\omega}\left(\frac{iZ\cos kL - \rho_0 c\sin kL}{iZ\sin kL + \rho_0 c\cos kL}\right) \tag{2.11.2.5}$$

2.11.3 Example 11.3

Conservation of momentum

$$\frac{\partial}{\partial t}(\rho_t \boldsymbol{v}) + \boldsymbol{v}\cdot\mathrm{div}(\rho_t\boldsymbol{v}) + \mathbf{grad}\,p = 0 \tag{2.11.3.1}$$

Conservation of mass

$$\frac{\partial\rho_t}{\partial t} + \mathrm{div}(\rho_t\boldsymbol{v}) = 0 \tag{2.11.3.2}$$

The divergence of Eq. (2.11.3.1) gives

$$\mathrm{div}(\rho_t\dot{\boldsymbol{v}}) + \mathrm{div}(\dot{\rho}_t\boldsymbol{v}) + \mathrm{div}[\boldsymbol{v}\cdot\mathrm{div}(\rho_t\boldsymbol{v})] + \nabla^2 p = 0 \tag{2.11.3.3}$$

The time derivative of (2.11.3.2) gives

$$\frac{\partial^2\rho_t}{\partial t^2} + \mathrm{div}(\rho_t\dot{\boldsymbol{v}}) + \mathrm{div}(\dot{\rho}_t\boldsymbol{v}) = 0 \tag{2.11.3.4}$$

Equation (2.11.3.3) minus Eq. (2.11.3.4) yields

$$\nabla^2 p - \frac{\partial^2 \rho}{\partial t^2} = -\mathrm{div}[\boldsymbol{v} \cdot \mathrm{div}(\rho_t \boldsymbol{v})] \tag{2.11.3.5}$$

Equations (11.15), (11.16) and (11.18) give $\partial\rho/\partial p = 1/c^2$. Thus, Eq. (2.11.3.5) is rewritten as

$$c^2 \nabla^2 \rho - \frac{\partial^2 \rho}{\partial t^2} = -\mathrm{div}[\boldsymbol{v} \cdot \mathrm{div}(\rho_t \boldsymbol{v})] \tag{2.11.3.6}$$

If the term on the right-hand side is neglected the basic wave equation is obtained if also the relationship between ρ and p is considered. The source term, which has a quadruple character is non-negligible in a region of violent fluid motion. Equation (2.11.3.6) is usually impossible to solve exactly.

2.11.4 Example 11.4

For a tyre rotating at a constant speed U, the gas inside the tyre also rotates. The resulting velocity potential inside the cavity should satisfy the equation

$$\nabla^2 \Phi - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + u_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi}\right)^2 \Phi = 0 \tag{2.11.4.1}$$

The pressure and particle velocity inside the fluid are

$$p = -\rho_0 \left(\frac{\partial}{\partial t} + u_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi}\right)\Phi; \quad \boldsymbol{v} = \mathbf{grad}\Phi + (0, u_\varphi, 0) \tag{2.11.4.2}$$

Using cylindrical coordinates the velocity inside the tyre is $u_\varphi = \dfrac{Ur}{R_0}$ for $r_0 \leqslant r \leqslant R_0$.

In cylindrical coordinates, the governing differential equation reads

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} - \frac{1}{c^2}\left[\frac{\partial^2 \Phi}{\partial t^2} + 2\frac{U_0}{R_0}\frac{\partial^2 \Phi}{\partial \varphi \partial t} + \left(\frac{U_0}{R_0}\right)^2 \frac{\partial^2 \Phi}{\partial \varphi^2}\right] = 0 \tag{2.11.4.3}$$

A factored solution is assumed. Thus,

$$\Phi(r, \varphi, z, t) = g(r)h(\varphi)Z(z)e^{i\omega t} \tag{2.11.4.4}$$

The solution is written

$$\Phi(r,\varphi,z,t)=\sum_{m=-\infty}^{\infty} g_{mn}(k_{mn}r)e^{im\varphi}\cos(n\pi z/z_0)e^{i\omega t} \tag{2.11.4.5}$$

The width of the tyre is z_0. The function $g_{mn}(k_{mn}r)$ should satisfy

$$\frac{\partial^2 g_{mn}}{\partial r^2}+\frac{1}{r}\frac{\partial g_{mn}}{\partial r}-\frac{m^2}{r^2}g_{mn}+\left[\left(\frac{\omega}{c}+\frac{mU_0}{cR_0}\right)^2-\left(\frac{n\pi}{z_0}\right)^2\right]=0 \tag{2.11.4.6}$$

For $k_{mn}r>0$ a solution is

$$g_{mn}(k_{mn}r)=A_{mn}J_m(k_{mn}r)+B_{mn}Y_m(k_{mn}r)$$

$$k_{mn}=\left[\left(\frac{\omega}{c}+\frac{mU_0}{cR_0}\right)^2-\left(\frac{n\pi}{z_0}\right)^2\right]^{1/2} \tag{2.11.4.7}$$

The boundary conditions are $v_r=\dfrac{\partial\phi}{\partial r}=0$ for $r=r_0$ and $v_z=\dfrac{\partial\phi}{\partial z}=0$ for $z=0$ and $z=z_0$

The solution is

$$\Phi=e^{i\omega t}\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{V_{mn}[J_m(k_{mn}r)Y'_m(k_{mn}r_0)-J'_m(k_{mn}r_0)Y_m(k_{mn}r)]}{\pi k_{mn}z_0\varepsilon_n[J'_m(k_{mn}R_0)Y'_m(k_{mn}r_0)-J'_m(k_{mn}r_0)Y'_m(k_{mn}R_0)]} \tag{2.11.4.8}$$

where $\varepsilon_n=1$ for $n>0$ and $\varepsilon_n=2$ for $n=0$. The parameter V_{mn} is determined by the vibration of the tyre. The velocity potential has maxima whenever

$$Q_{mn}=[J'_m(k_{mn}R_0)Y'_m(k_{mn}r_0)-J'_m(k_{mn}r_0)Y'_m(k_{mn}R_0)]=0 \tag{2.11.4.9}$$

The first few natural frequencies in the air cavity are obtained for $n=0$. $Q_{mn}=0$ for $k_{mn}=\lambda_{mn}$. The resulting natural frequencies for $n=0$ are

$$f_{m0}=\frac{c}{2\pi}\left(\lambda_{m0}-\frac{mU_0}{cR_0}\right)\quad m=\pm1;\pm2,\ldots \tag{2.11.4.10}$$

The dimensions of a standard tyre are $r_0=0.21$ m, $R_0=0.275$ m, $z_0=0.205$ m. The parameter λ_{m0} is equal to 4.1, 8.3, 12.4 $\mathrm{m}^{-1}$for $m=1,2,3$ respectively. For $U=0$ the first few natural frequencies are 221, 449 and 671 Hz. For $U\neq0$, the natural frequencies inside the tyre are split due to the velocity of the tyre. The frequency split is proportional to the velocity of the tyre.

2.11.5 Example 11.5

The velocity potential induced by a dipole is according to Eq. (11.66) given as

$$\begin{aligned}
\phi(x,t) &= \Phi_0(x)\exp(i\omega t)\\
\Phi_0 &= -\frac{ike^{-ikr}}{4\pi r^2} rD\left(1+\frac{1}{ikr}\right)\\
&= -\frac{ike^{-ikr}xD}{4\pi r^2}\left(1+\frac{1}{ikr}\right)\\
&= -\frac{ike^{-ikr}D\cos\varphi}{4\pi r}\left(1+\frac{1}{ikr}\right)
\end{aligned} \tag{2.11.5.1}$$

The pressure in the fluid is

$$p = -\rho_0\frac{\partial\Phi}{\partial t} = -i\omega\rho_0\Phi_0\exp(i\omega t) \tag{2.11.5.2}$$

The particle velocity v_r is

$$v_r = \exp(i\omega t)\frac{\partial\Phi_0}{\partial r} = \exp[(i\omega t - kr)]\cdot\frac{k^2D\cos\varphi}{4\pi r}\left(-1+\frac{2i}{kr}+\frac{2}{(kr)^2}\right) \tag{2.11.5.3}$$

The time average of the intensity is obtained from Eqs. (2.11.5.2) and (2.11.5.3) as

$$\bar{I}_r = \frac{1}{2}\mathrm{Re}(p\boldsymbol{v}_r^*) = \frac{\rho_0 ck^4D^2\cos^2\varphi}{2(4\pi r)^2} \tag{2.11.5.4}$$

The total power radiated is

$$\bar{\Pi} = 2\pi r^2\int_0^{\pi}\bar{I}_r\sin\varphi\,\mathrm{d}\varphi = \frac{\rho_0 cD^2k^4}{24\pi} \tag{2.11.5.5}$$

2.11.6 Example 11.6

The velocity potential induced by the vibrating sector on the sphere is written

$$\Phi(r,\theta,t) = \Phi_0(r,\theta)\exp(i\omega t) \tag{2.11.6.1}$$

According to Eq. (11.73)

$$\Phi_0(r,\theta) = \sum_m B_m P_m(\cos\theta)h_m^{(2)}(kr) \tag{2.11.6.2}$$

The particle velocity on the sphere is

$$v_r(r_0,\theta) = [\partial\Phi_0/\partial r]_{r=r_0} = \sum_m B_m P_m(\cos\theta)k[h_m^{(2)}(kr_0)]'$$
$$= \sum_m W_m P_m(\cos\theta) \tag{2.11.6.3}$$

The parameters W_m are obtained as

$$W_m = \frac{2m+1}{2}\int_{-1}^{1} P_m(z)f(z)\mathrm{d}z = -\frac{2m+1}{2}\int_{\theta_0}^{0} P_m(\cos\theta)u_0\sin\theta\mathrm{d}\theta \tag{2.11.6.4}$$

For θ_0 small $\cos\theta_0 = 1-\theta_0^2/2$ and $P_m(\cos\theta)\approx 1$. Consequently, Eq. (2.11.6.4) gives

$$W_m = \frac{2m+1}{2}u_0(1-\cos\theta_0) \approx \frac{2m+1}{4}u_0\theta_0^2 \tag{2.11.6.5}$$

Equations (2.11.6.3) and (2.11.6.5) give

$$\Phi_0 = u_0\sum_m \frac{2m+1}{4}\cdot\frac{\theta_0^2}{k}\cdot\frac{h_m^{(2)}(kr)}{[h_m^{(2)}(kr_0)]'}P_m(\cos\theta) \tag{2.11.6.6}$$

2.11.7 Example 11.7

The velocity potential is written

$$\Phi(r,\varphi,t) = \Phi_0(r,\varphi)\cdot\exp(i\omega t) \tag{2.11.7.1}$$

According to Eq. (11.80), the function Φ_0 is

$$\Phi_0(r,\varphi) = \sum_m A_m\cdot H_m^{(2)}(kr)\cdot\cos(m\varphi) \tag{2.11.7.2}$$

The particle velocity on the surface of the cylinder is

$$v_r(r_0,\varphi) = [\partial\phi_0/\partial r]_{r=r_0} = \sum_m A_m\cdot k\left[H_m^{(2)}(kr)\right]'_{r=r_0}\cdot\cos(m\varphi) \tag{2.11.7.3}$$

The velocity of the cylinder is $u(\varphi) = u_0$ for $-\varphi_0 \leqslant \varphi \leqslant \varphi_0$ otherwise zero.

The boundary condition is also written

$$u(\varphi) = \sum_m W_m \cos(m\varphi)$$

$$W_m = \frac{u_0}{\pi}\int_{-\varphi_0}^{\varphi_0} \cos(m\theta)\mathrm{d}\theta = \frac{2u_0}{m\pi}\sin(m\varphi_0) \quad \text{for } m > 0$$

$$W_0 = 2u_0\varphi_0 \quad \text{for } m = 0 \tag{2.11.7.4}$$

Equations (2.11.7.1) through (2.11.7.4) give

$$\Phi(r,\varphi,t) = \frac{u_0\cdot\exp(i\omega t)}{\pi k}\left[\frac{\varphi_0 H_0^{(2)}(kr)}{[H_0^{(2)}(kr_0)]'} + \sum_{m=1}^{\infty}\frac{H_m^{(2)}(kr)}{[H_m^{(2)}(kr_0)]'}\frac{2\varphi_0}{m\pi}\sin(m\varphi_0)\cos(m\varphi)\right] \tag{2.11.7.5}$$

2.11.8 *Example 11.8*

The time average of pressure squared at the observation point is according to Eq. (11.119) given as

$$|\bar{p}|^2 = |\bar{p}_0|^2\, 4\cos^2(k\Delta r) \tag{2.11.8.1}$$

where $|\bar{p}_0|^2$ is the pressure at the same point under free field conditions.

The frequency average of the measured pressure is

$$|\bar{p}|^2_{\Delta f} = \frac{1}{\Delta f}\int_{f-\Delta f/2}^{f+\Delta f/2} \mathrm{d}f\; |\bar{p}_0|^2\, 4\cos^2(2\pi f\Delta r/c)$$
$$= \frac{2}{\Delta f}|\bar{p}_0|^2\left[f + \frac{\sin(4\pi f\Delta r/c)}{4\pi\Delta r/c}\right]_{f-\Delta f/2}^{f+\Delta f/2} \tag{2.11.8.2}$$

$$|\bar{p}|^2_{\Delta f} = \frac{1}{\Delta f}|\bar{p}_0|^2\left[\Delta f + \frac{\cos(4\pi f\Delta r/c)\sin(2\pi\Delta f\Delta r/c)}{2\pi\Delta r/c}\right] \tag{2.11.8.3}$$

For $f\Delta r/c \ll 1$, $\cos(4\pi f\Delta r/c) \approx 1$ and $\sin(2\pi f\Delta r/c) \approx 2\pi f\Delta r/c$. Thus, $|\bar{p}|^2_{\Delta f} \approx 2|\bar{p}_0|^2$ and Thus L_p(measured) $\approx L_p$(freefield) + 3 dB. However, right on the reflecting surface $|\bar{p}|^2_{\Delta f} = 4|\bar{p}_0|^2$ and L_p(measured) $\approx L_p$(freefield) +6 dB.

2.11.9 Example 11.9

The velocity potential induced in the fluid is written

$$\Phi(r, \varphi, t) = \Phi_0(r, \varphi) \cdot \exp(i\omega t)$$

$$\Phi_0(r, \varphi) = \sum_m A_m \cdot H_m^{(2)}(kr) \cdot \sin(m\varphi) \tag{2.11.9.1}$$

Only sine terms are considered since the velocity is negative in the upper half plane and positive in the lower. Due to the reflection in the water surface, assume that under free field conditions the velocity of the cylinder is $u(\varphi, t) = u(\varphi) \cdot \exp(i\omega t)$ where

$u(\varphi) = -u_0$ for $0 \leqslant \varphi \leqslant \pi$; $u(\varphi) = u_0$ for $\pi < \varphi < 2\pi$

The velocity is written

$$u(\varphi) = \sum_m W_m \sin(m\varphi)$$

$$\pi W_m = -u_0 \int_0^{\pi} \sin(m\theta)\mathrm{d}\theta + u_0 \int_{\pi}^{2\pi} \sin(m\theta)\mathrm{d}\theta$$

$$= -\frac{2u_0}{m}(1 - \cos m\pi) \tag{2.11.9.2}$$

The resulting velocity potential is

$$\Phi(r, \varphi, t) = -\frac{2u_0 \cdot \exp(i\omega t)}{\pi k}\left[\sum_{m=1}^{\infty} \frac{H_m^{(2)}(kr)}{m[H_m^{(2)}(kr_0)]'}[1 - \cos(m\pi)]\sin(m\varphi)\right] \tag{2.11.9.3}$$

2.11.10 Example 11.10

Considering the image effects the pressure in the water can be calculated as if the cylinder is in an unbounded medium. The velocity $u(\varphi, t) = u(\varphi)\exp(i\omega t)$ of the surface of the cylinder as seen from the water is

$u(\varphi) = -u_0$ for $0 \leqslant \varphi \leqslant \varphi_0$ and $u(\varphi) = u_0$ for $-\varphi_0 < \varphi < 0$

otherwise zero.

Thus

$$u(\varphi) = \sum_m W_m \sin(m\varphi)$$

$$W_m = -\frac{u_0}{\pi}\left[\int_0^{\varphi_0} \sin(m\varphi)\mathrm{d}\varphi - \int_{-\varphi_0}^{0} \sin(m\varphi)\mathrm{d}\varphi\right] = -\frac{2u_0}{m\pi}\left[1 - \cos(m\varphi_0)\right] \tag{2.11.10.1}$$

The resulting velocity potential is

$$\phi(r, \varphi, t) = -\frac{2u_0 \cdot \exp(i\omega t)}{\pi k} \cdot \left[\sum_{m=1}^{\infty} \frac{H_m^{(2)}(kr)}{m[H_m^{(2)}(kr_0)]'}[1 - \cos(m\varphi_0)]\sin(m\varphi)\right] \tag{2.11.10.2}$$

2.11.11 Example 11.11

The equation for an ellipsoid is

$$\left(\frac{x}{A_x}\right)^2 + \left(\frac{y}{A_y}\right)^2 + \left(\frac{z}{A_z}\right)^2 = 1 \tag{2.11.11.1}$$

The volume of the ellipsoid is

$$V = \frac{4\pi}{3} A_x A_y A_z \tag{2.11.11.2}$$

The natural frequencies in a room are

$$f_{lmn}^2 = \frac{c^2}{4}\left[\left(\frac{l}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2 + \left(\frac{n}{L_z}\right)^2\right] \tag{2.11.11.3}$$

Thus $\left(\frac{l}{A_x}\right)^2 + \left(\frac{m}{A_y}\right)^2 + \left(\frac{n}{A_z}\right)^2 = 1$; where $A_x = \frac{2fL_x}{c}$; $A_y = \frac{2fL_y}{c}$; $A_z = \frac{2fL_z}{c}$.

The number of modes for which l, m, and n all are positive or 1/8 of the total

$$N = \int\int\int \mathrm{d}l\mathrm{d}m\mathrm{d}n = \frac{1}{8}\frac{4\pi}{3} A_x A_y A_z = \frac{4\pi V f^3}{3c^3} \tag{2.11.11.4}$$

The modal density is

$$\mathcal{N}_f = \frac{\Delta N}{\Delta f} = \frac{4\pi f^2 V}{c^3} \tag{2.11.11.5}$$

2.11.12 Example 11.12

At time t_1 the source emits a signal which reaches the observer at time t'_1 or $t'_1 = t_1 + r_1/c$ where r_1 is the distance between source and observer. At a later time $t_1 + \Delta t$ the source emits another signal reaching the observer at t'_2 where $t'_2 = t_1 + \Delta t + r_2/c$. The observed time interval is

$$\Delta t' = t'_2 - t'_1 = \Delta t + (r_2 - r_1)/c \tag{2.11.12.1}$$

The vector from the observer to the source at $t = t_1$ is defined as $\boldsymbol{r}_1$. At $t = t_2$ the vector is $\boldsymbol{r}_2$ where

$$\boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{v} \cdot \Delta t \tag{2.11.12.2}$$

Thus

$$r_2^2 = r_1^2 + u^2(\Delta t)^2 + 2\boldsymbol{r}_1 \boldsymbol{v} \Delta t \tag{2.11.12.3}$$

For $\Delta t \ll 1$, $r_2 = r_1 + \boldsymbol{r}_1 \boldsymbol{v} \Delta t / r_1$. This result inserted in Eq. (2.11.12.1) gives

$$\Delta t' = \Delta t \left(1 + \frac{\boldsymbol{r}_1 \mathbf{v}}{r_1 c}\right)$$

$$\boldsymbol{r}_1 \boldsymbol{v} = r_1 v \cos\varphi$$

where φ is the angle between the two vectors as shown in Fig. 2.29. The frequency f of the source as compared to the observed frequency f' are related as

$$\frac{f'}{f} = \frac{\Delta t}{\Delta t'} = \left[\frac{1}{1 + \boldsymbol{r}_1 \boldsymbol{v}/(r_1 c)}\right]; \quad \boldsymbol{r}_1 \boldsymbol{v} = r_1 v \cos\varphi \tag{2.11.12.4}$$

Thus

$$\frac{f'}{f} = \frac{\Delta t}{\Delta t'} = \left(\frac{1}{1 + v\cos\varphi/c}\right) \tag{2.11.12.5}$$

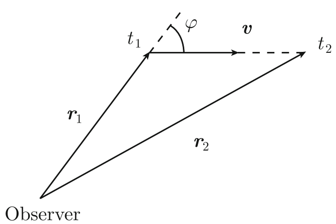


Fig. 2.29 A noise source having the vector velocity $\boldsymbol{v}$ travelling past an observer

The frequency of the signal experienced by the observer is decreased as the source is moving away from the observer and increased as the source is approaching as compared to the actual frequency of the source.

2.11.13 Example 11.13

Case 1
Let the velocity potential be

$$\Phi = \Phi_0 \exp(i\omega t) \tag{2.11.13.1}$$

The function Φ_0 should satisfy the wave equation

$$\partial^2\Phi_0/\partial x^2 + k^2\Phi_0 = Q_0\delta(x - x_0) \tag{2.11.13.2}$$

The particle velocity is zero at both ends of the duct. The boundary conditions are

$$\partial\Phi_0/\partial x = 0 \text{ for } x = 0 \text{ and } x = L \tag{2.11.13.3}$$

The resulting eigenfunction are

$$\varphi_n = \cos(n\pi x/L); \quad k_n = n\pi/L \tag{2.11.13.4}$$

The resulting field inside the duct is $\Phi = \Phi_0 \exp(i\omega t)$ where

$$\Phi_0 = \sum_n \frac{2Q_0\varphi_n(x)\varphi_n(x_0)}{L(k^2 - k_n^2)} \tag{2.11.13.5}$$

The FT of the pressure is $\hat{p} = -i\omega\rho_0\Phi_0$. The space average of $\hat{p}^2$ is $\langle|\hat{p}|^2\rangle = \frac{1}{L}\int_0^L \mathrm{d}x\,|\hat{p}|^2$.

The result is

$$\langle|\hat{p}|^2\rangle = (\omega\rho_0)^2 \sum_n \frac{2\,|Q_0|^2\,\varphi_n^2(x_0)}{L^2\,|k^2 - k_n^2|^2} \text{ for } L > x_0 > 0 \tag{2.11.13.6}$$

Case 2
The velocity potential is again given by $\Phi = \Phi_0 \exp(i\omega t)$. The function Φ_0 should satisfy the wave equation

$$\partial^2\Phi_0/\partial x^2 + k^2\Phi_0 = Q_0[\delta(x - x_0) + \delta(x + x_0)] \tag{2.11.13.7}$$

The boundary conditions are

$$\partial \Phi_0/\partial x = 0 \text{ for } x = -L \text{ and } x = L \tag{2.11.13.8}$$

The resulting eigenfunction is

$$\varphi_n = \cos(n\pi x/L); \quad k_n = n\pi/L \tag{2.11.13.9}$$

The resulting field in the duct is

$$\Phi_0 = \sum_n \frac{2Q\varphi_n(x)\varphi_n(x_0)}{L(k^2 - k_n^2)} \tag{2.11.13.10}$$

The FT of the pressure is $\hat{p} = -i\omega\rho_0\Phi_0$. The space average of $\hat{p}^2$ is $\langle|\hat{p}|^2\rangle = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} \mathrm{d}x\, |\hat{p}|^2$.

The result is $\langle|\hat{p}|^2\rangle = (\omega\rho_0)^2 \displaystyle\sum_n \frac{2\,|Q_0|^2\,\varphi_n^2(x_0)}{L^2\,|k^2 - k_n^2|^2}$ for $-L < x_0 < L$ (2.11.13.11)

For $0 < x_0 < L$ the results (2.11.13.6) and (2.11.13.11) are identical. As $x_0 \to 0$ both Eqs. (2.11.13.6) and (2.11.13.11) approach the same result

$$\langle|\hat{p}|^2\rangle = (\omega\rho_0)^2 \sum_n \frac{2\,|Q_0|^2}{L^2\,|k^2 - k_n^2|^2} \tag{2.11.13.12}$$

However, if the source is assumed to be located at $x_0 = 0$ Eq. (2.11.13.2) can not be solved directly. The solution would involve an intergral of the type

$$I = \int_0^L \delta(x)\varphi_m(x)\mathrm{d}x \tag{2.11.13.13}$$

The Dirac function is not defined for $x = 0$. Junger and Feit, Ref. [165] in vol II, argue that the solution to Eq. (2.11.13.13) can be written

$$I = \int_0^L \delta(x)\varphi_m(x)\mathrm{d}x = \varphi_m(0)/2 \tag{2.11.13.14}$$

When the source is mounted right on the end section of the duct the source strength should be doubled. Thus, according to [165], Eq. (2.11.3.2) should be written

$$\partial^2\Phi_0/\partial x^2 + k^2\Phi_0 = 2Q_0\delta(x)$$

The procedure outlined as Case 2 always give a correct answer without any undue mathematical manipulations. Also the other procedure, Case 1, gives the correct result if in the final solution the position of the source is allowed to approach the duct wall.

2.11.14 Example 11.14

The space average of the pressure squared in the room is given by $\langle \left|\hat{p}\right|^2 \rangle = \sum_{l,m,n} \langle \left|\hat{p}_{lmn}\right|^2 \rangle$ where

$$\langle \left|\hat{p}_{lmn}\right|^2 \rangle = \frac{2c^4 \rho_0^2 f^2 \left|Q_0\right|^2 \varphi_{lmn}^2(\boldsymbol{r}_0)}{\varepsilon_l \varepsilon_m \varepsilon_n \pi^2 V^2 [(f^2 - f_{lmn}^2)^2 + (\delta f_{lmn}^2)^2]} \tag{2.11.14.1}$$

For sufficiently high frequencies $\varepsilon_l = \varepsilon_m = \varepsilon_n = 1$. Thus, following the results of Sect. 2.7, the average is given as

$$\langle \left|\bar{\hat{p}}_{lmn}\right|^2 \rangle = \frac{1}{\Delta f} \int \mathrm{d}f \langle \left|\hat{p}_{lmn}\right|^2 \rangle = \frac{c^4 \rho_0 \left|Q_0\right|^2 \varphi_{lmn}^2(\boldsymbol{r}_0)}{\Delta f \cdot \pi \delta f_{lmn} V^2}; \quad \Delta f = \frac{c^3}{4\pi f^2 V} \tag{2.11.14.2}$$

Setting $f_{lmn} = f$ the result is

$$\langle \left|\bar{\hat{p}}_{lmn}\right|^2 \rangle = \frac{4c\rho_0^2 f \left|Q_0\right|^2 \varphi_{lmn}^2(\boldsymbol{r}_0)}{\delta \cdot V} \tag{2.11.14.3}$$

2.11.15 Example 11.15

Equation (11.175) gives $R = 25.5$ dB with air gap and 35 dB without the gap.

2.11.16 Example 11.16

The energy $\mathcal{E}_\mathcal{V}$ per unit volume in the room should according to Eq. (11.152) satisfy the equation

$$V \frac{\partial \mathcal{E}_\mathcal{V}}{\partial t} + c\mathcal{E}_\mathcal{V} A/4 + \omega \delta_a \mathcal{E}_\mathcal{V} V = \Pi \tag{2.11.16.1}$$

The general solution is

$$\mathcal{E}_{\mathcal{V}} = \frac{1}{V} e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda\tau} \Pi(\tau) \mathrm{d}\tau; \quad \lambda = \frac{cA}{4V} + \omega\delta_a \tag{2.11.16.2}$$

For $\Pi(t) = \Pi_0$ for $0 \leqslant t \leqslant t_0$ otherwise zero the result is

$$\mathcal{E}_{\mathcal{V}}(t) = \frac{\Pi_0}{\lambda V}(1 - e^{-\lambda t}) \text{ for } 0 \leqslant t \leqslant t_0$$

$$\mathcal{E}_{\mathcal{V}}(t) = \frac{\Pi_0 e^{-\lambda(t-t_0)}}{\lambda V}(1 - e^{-\lambda t_0}) = \mathcal{E}_{\mathcal{V}}(t_0) e^{-\lambda(t-t_0)} \text{ for } t > t_0 \tag{2.11.16.3}$$

The pressure in the room is obtained from $|p|^2 = \rho_0 c^2 \mathcal{E}_V$.

2.11.17 Example 11.17

The transforms are given as $x_1 = \gamma(x - ut)$, $y_1 = y$, $z_1 = z$, $t_1 = \gamma(t - ux/c^2)$ and $\gamma = c/\sqrt{c^2 - u^2}$. Compare Sect. 11.8. Thus,

$$x = \gamma(x_1 + ut_1); \quad t = \frac{\gamma u}{c^2} x_1 + \gamma t_1 \tag{2.11.17.1}$$

$$\frac{\partial}{\partial x_1} = \frac{\mathrm{d}x}{\mathrm{d}x_1}\frac{\partial}{\partial x} + \frac{\mathrm{d}t}{\mathrm{d}x_1}\frac{\partial}{\partial t} \tag{2.11.17.2}$$

From (2.11.17.1) $\mathrm{d}x/\mathrm{d}x_1 = \gamma$ and $\mathrm{d}t/\mathrm{d}x_1 = \gamma u/c^2$. These expressions inserted in (2.11.17.2) give

$$\frac{\partial}{\partial x_1} = \gamma\frac{\partial}{\partial x} + \frac{\gamma u}{c^2}\frac{\partial}{\partial t} \tag{2.11.17.3}$$

and

$$\frac{\partial^2}{\partial x_1^2} = \gamma^2\frac{\partial^2}{\partial x^2} + \left(\frac{\gamma u}{c^2}\right)^2\frac{\partial^2}{\partial t^2} + 2\frac{\gamma^2 u}{c^2}\frac{\partial^2}{\partial x \partial t} \tag{2.11.17.4}$$

In a similar way

$$\frac{\partial^2}{\partial t_1^2} = (\gamma u)^2\frac{\partial^2}{\partial x^2} + \gamma^2\frac{\partial^2}{\partial t^2} + 2\gamma^2 u\frac{\partial^2}{\partial x \partial t} \tag{2.11.17.5}$$

Thus,

$$
\begin{aligned}
&\nabla_1^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t_1^2}\\
&= \gamma^2\frac{\partial^2\Phi}{\partial x^2} + \left(\frac{\gamma u}{c^2}\right)^2\frac{\partial^2\Phi}{\partial t^2} + 2\frac{\gamma^2 u}{c^2}\frac{\partial^2\Phi}{\partial x\partial t} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}\\
&\quad - \left(\frac{\gamma u}{c}\right)^2\frac{\partial^2\Phi}{\partial x^2} - \left(\frac{\gamma}{c}\right)^2\frac{\partial^2\Phi}{\partial t^2} - 2\frac{\gamma^2 u}{c^2}\frac{\partial^2\Phi}{\partial x\partial t}\\
&= \gamma^2\left(1-\frac{u^2}{c^2}\right)\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} - \left(\frac{\gamma}{c}\right)^2\frac{\partial^2\Phi}{\partial t^2}\left(1-\frac{u^2}{c^2}\right)
\end{aligned}
\tag{2.11.17.6}
$$

However, $\gamma^2 = \left(1-\frac{u^2}{c^2}\right)^{-1}$. Thus,

$$
\nabla_1^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t_1^2} = \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \tag{2.11.17.7}
$$

2.12 Chapter 12

2.12.1 Example 12.1

The wavenumber κ_x for flexural waves propagating along a fluid loaded plate (fluid load on one side) is the solution to Eq. (12.12),

$$
\kappa_x^4 = \kappa^4 + \frac{\omega^2\rho_0}{D\sqrt{\kappa_x^2 - k^2}} \tag{2.12.1.1}
$$

For finding a solution as $\kappa_x \to k$ set $\kappa_x = k(1+\xi)$where $\xi \ll 1$. Insert this expression in the basic equation (2.12.1.1). The result is

$$
k^4(1+\xi)^4 = \kappa^4 + \frac{\omega^2\rho_0}{Dk\sqrt{2\xi+\xi^4}} = \kappa^4 + \frac{\kappa^4\rho_0}{\mu k\sqrt{2\xi+\xi^2}} \tag{2.12.1.2}
$$

For $\xi \ll 1$ and $k = \kappa$ the equation is approximated by

$$
4\xi = \frac{\rho_0}{\mu k\sqrt{2\xi}} \tag{2.12.1.3}
$$

Thus,

$$\xi \approx \frac{1}{4}\left(\frac{\rho_0\sqrt{2}}{\mu k}\right)^{2/3} \tag{2.12.1.4}$$

For a 4 mm steel plate with a fluid load of water on one side $\xi \approx 9/100$ for $\kappa = k$.

For $k \gg \kappa$ the parameter ξ is always positive. Consequently, $\kappa_x > k$. This type of wave does not radiate sound.

2.12.2 Example 12.2

Omitting the time dependence the Eqs. (12.33) and (12.45) give the sound pressure in the fluid as

$$p(x, y, z) = \iint_{S_0} i\omega\rho_0 v(x_0, y_0) G(x, y, z|x_0, y_0, 0)\mathrm{d}x_0\mathrm{d}y_0 \tag{2.12.2.1}$$

$$G(x, y, z|x_0, y_0, 0)$$
$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}k_x\mathrm{d}k_y \frac{\exp[ik_x(x-x_0)+ik_y(y-y_0)-z\sqrt{k_x^2+k_y^2-k^2}]}{\sqrt{k_x^2+k_y^2-k^2}} \tag{2.12.2.2}$$

$$v(x_0, y_0) = v_0\exp(-i\kappa x_0) \tag{2.12.2.3}$$

Equations (2.12.2.2) and (2.12.2.3) inserted in (2.12.2.1) give

$$p(x, y, z) = \frac{i\omega\rho_0 v_0}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}k_x\mathrm{d}k_y \frac{\exp[ik_x x + ik_y y - z\sqrt{k_x^2+k_y^2-k^2}]}{\sqrt{k_x^2+k_y^2-k^2}} \cdot I \tag{2.12.2.4}$$

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ix_0(k_x+\kappa)-iy_0k_y}\mathrm{d}x_0\mathrm{d}y_0 \tag{2.12.2.5}$$

According to definition $\delta(\xi + k_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix_0(k_x+\xi)}\mathrm{d}x_0$. Thus, the solution to Eq. (2.12.2.5) is

$$I = (2\pi)^2\delta(k_x+\kappa)\delta(k_y) \tag{2.12.2.6}$$

Consequently,

$$p(x, y, z) = i\omega\rho_0 v_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}k_x \mathrm{d}k_y \frac{\exp[ik_x x + ik_y y - z\sqrt{k_x^2 + k_y^2 - k^2}]}{\sqrt{k_x^2 + k_y^2 - k^2}} \delta(k_x + \kappa)\delta(k_y)$$

Thus,

$$p(x, y, z) = \frac{i\omega\rho_0 v_0}{\sqrt{\kappa^2 - k^2}} \exp[-i\kappa x - z\sqrt{\kappa^2 - k^2}] \quad \text{for } \kappa > k \tag{2.12.2.7}$$

The pressure is decaying exponentially away from the plate. No acoustic intensity is radiated away from the plate.

$$p(x, y, z) = \frac{\omega\rho_0 v_0}{\sqrt{k^2 - \kappa^2}} \exp[-i\kappa x - iz\sqrt{k^2 - \kappa^2}] \quad \text{for } \kappa < k \tag{2.12.2.8}$$

A pressure wave is propagating away from the plate.

2.12.3 Example 12.3

According to Eq. (12.98), the radiation ratio for $f < f_c$ is

$$\bar{\sigma}_r = \frac{L_x + L_y}{L_x L_y} h(f/f_c) \tag{2.12.3.1}$$

where $h(f/f_c)$ is a function independent on the dimensions of the plate. For $L_x/L_y = \xi$ and $L_x L_y = S_0$ the sides can be written as $L_x = \sqrt{S_0\xi}$ and $L_y = \sqrt{S_0/\xi}$. Consequently, the radiation ratio is written

$$\bar{\sigma}_r = \frac{1/\sqrt{\xi} + \sqrt{\xi}}{\sqrt{S_0}} h(f/f_c) \tag{2.12.3.2}$$

The radiation ratio has a minimum for a quadratic shape, i.e. for $\xi = 1$.

2.12.4 Example 12.4

For a plate with a fluid load on one side, the added mass at the first natural frequency is

$$\Delta\mu = \frac{\rho_0}{\kappa_{11}}; \quad \kappa_{11} = \left[\left(\frac{\pi}{L_x}\right)^2 + \left(\frac{\pi}{L_y}\right)^2\right]^{1/2} \tag{2.12.4.1}$$

Thus, the added mass is

$$\Delta\mu = \frac{\rho_0 L_x L_y}{\pi\sqrt{L_x^2 + L_y^2}} \tag{2.12.4.2}$$

2.12.5 Example 12.5

The radiation ratio is given by Eq. (12.116) as

$$\sigma_m = \frac{2}{\pi k r_0 \left|\left[H_m^{(2)}(z)\right]'_{z=kr_0}\right|^2} \tag{2.12.5.1}$$

For $z \to 0$ the Hankel function is approximated by

$$H_m^{(2)}(z) = 1 + \frac{2i}{\pi}\left[\ln\left(\frac{z}{2}\right) + \gamma\right] \quad \text{for } m = 0 \tag{2.12.5.2}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[H_m^{(2)}(z)\right] \approx \frac{2i}{\pi z} \quad \text{for } m = 0 \tag{2.12.5.3}$$

For $m > 0$ and $z \to 0$,

$$H_m^{(2)}(z) = \frac{1}{m!}\left(\frac{z}{2}\right)^m + \frac{i(m-1)!}{\pi}\left(\frac{2}{z}\right)^m \tag{2.12.5.4}$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}z}H_m^{(2)}(z) = \left[H_m^{(2)}(z)\right]' = \frac{m}{m!}\left(\frac{z}{2}\right)^{m-1} - \frac{i2^m m!}{\pi z^{m+1}} \tag{2.12.5.5}$$

Equations (2.12.5.3) and (2.12.5.5) inserted in Eq. (2.12.5.1) give for $kr_0 \ll 1$

$$\sigma_0 = \frac{\pi}{2}(kr_0) \quad \text{for } m = 0 \tag{2.12.5.6}$$

$$\sigma_m = \frac{4\pi}{(m!)^2}\left(\frac{kr_0}{2}\right)^{2m+1} \quad \text{for } m > 0 \tag{2.12.5.7}$$

2.12.6 Example 12.6

The eigenfunction for φ_n for a clamped beam is from Table 7.2 given as

$$\sqrt{2}\cdot\varphi_n = \cosh(\kappa_n x) - \cos(\kappa_n x) - \frac{\cosh(\kappa_n L) - \cos(\kappa_n L)}{\sinh(\kappa_n L) - \sin(\kappa_n L)} \cdot [\sinh(\kappa_n x) - \sin(\kappa_n x)] \tag{2.12.6.1}$$

The eigenvalues κ_n are the solutions to $\cos(\kappa_n L) \cdot \cosh(\kappa_n L) = 1$. For $n \geqslant 4$

$$\kappa_n L = \pi/2 + n\pi \tag{2.12.6.2}$$

For n large the eigenfunctions can according to Problem 7.16 be approximated by

$$\varphi_n = \sin(\kappa_n x - \pi/4) + \left[e^{-\kappa_n x} - \sin(\kappa_n L)\cdot e^{\kappa_n(x-L)}\right]\Big/\sqrt{2} \tag{2.12.6.3}$$

The radiation area of the cross mode not being cancelled is

$$\int_0^L \varphi_n(x)\mathrm{d}x = \frac{4}{\kappa_n\sqrt{2}} = \frac{4L}{\pi\sqrt{2}(n+1/2)} = S_1 \text{ for } n \text{ odd} \tag{2.12.6.4}$$

For simply supported edges $\varphi_n = \sin(\kappa_n x)$, $\kappa_n = n\pi/L$. Thus,

$$\int_0^L \varphi_n(x)\mathrm{d}x = \frac{2}{\kappa_n} = \frac{2L}{\pi n} = S_2 \tag{2.12.6.5}$$

The ratio $\left(\dfrac{S_1}{S_2}\right)^2 = 2\left(\dfrac{n}{n+\dfrac{1}{2}}\right)^2 \to 2$ as $n \to \infty$.

The edgemode for a clamped plate would therefore radiate twice as much as the edge mode for a simply supported plate.

2.12.7 Example 12.7

For a fluid loaded infinite plate the real part of the point mobility is

$$ReY_{f\infty} = \frac{1}{10}\left(\frac{\omega}{D^3\rho_0^2}\right)^{1/5} = \frac{1}{8\sqrt{D}}\frac{8}{10}\left(\frac{\omega}{D^{1/2}\rho_0^2}\right)^{1/5} \tag{2.12.7.1}$$

The wavenumber for flexural waves on the plate is

$$\kappa_x^4 = \frac{\mu_{\text{apparent}}\omega^2}{D} \Rightarrow \frac{\omega}{\sqrt{D}} = \frac{\kappa_x^2}{\sqrt{\mu_{\text{apparent}}}} \tag{2.12.7.2}$$

Equations (2.12.7.1) and (2.12.7.2) give

$$ReY_{f\infty} = \frac{1}{8\sqrt{D}}\frac{8}{10}\left(\frac{\kappa_x^2}{\mu_{\text{apparent}}^{1/2}\rho_0^2}\right)^{1/5} \tag{2.12.7.3}$$

The real part of the point mobility is also written

$$ReY_{f\infty} = \frac{1}{8\sqrt{D\mu_{\text{apparent}}}} \tag{2.12.7.4}$$

where μ_{apparent} is the apparent mass of the plate at the excitation point. Equations (2.12.7.4) and (2.12.7.3) give

$$\mu_{\text{apparent}} = \left(\frac{5}{4}\right)^2 \frac{\mu_{\text{apparent}}^{1/5}\rho_0^{4/5}}{\kappa_x^{4/5}}$$

or

$$\mu_{\text{apparent}}^{4/5} = \left(\frac{5}{4}\right)^2 \frac{\rho_0^{4/5}}{\kappa_x^{4/5}} \tag{2.12.7.5}$$

The added weight to a reverberant fluid loaded plate (one side) is from (12.15)

$$\mu_{\text{add}} \approx \rho_0/\kappa_x \tag{2.12.7.6}$$

Equations (2.12.7.5) and (2.12.7.6) give

$$\frac{\mu_{\text{apparent}}^{4/5}}{\mu_{\text{add}}^{4/5}} = \left(\frac{5}{4}\right)^2 \Rightarrow \frac{\mu_{\text{apparent}}}{\mu_{\text{add}}} = \left(\frac{5}{4}\right)^{2.5} \approx 1.7 \tag{2.12.7.7}$$

2.12.8 Example 12.8

The plate is assumed to be completely submerged in water. The possible effects of reflections in the water surface are neglected. The total mass μ_{tot} per unit area of the water loaded plate for $f \ll f_c$ but for frequencies above the first natural frequency of the plate is according to Eq. (12.15) approximately given as

$$\mu_{\text{tot}} = \mu_0 + 2\rho/\kappa_0 \tag{2.12.8.1}$$

where μ_0 is the mass per area of the plate itself and $\kappa_0 = \left(\frac{\mu(2\pi f)^2}{D}\right)^{1/4}$ the wavenumber for the plate in vacuo. The radiation ratio $\bar{\sigma}_r$ is given by Eq. (12.98) as

$$\bar{\sigma}_r = \frac{L_x + L_y}{\pi q k L_x L_y \sqrt{q^2 - 1}} \left[\ln\left(\frac{q+1}{q-1}\right) + \frac{2q}{q^2 - 1}\right] \tag{2.12.8.2}$$

where L_x and L_y are the lengths of the sides of the rectangular panel. The parameter q is according to Sect. 12.9 defined as

$$q = \left(\frac{c^2 \mu_{\text{tot}}^{1/2}}{2\pi D^{1/2} f}\right)^{1/2} \tag{2.12.8.3}$$

For a fluid loaded plate the total mass is frequency dependent as given by Eq. (2.12.8.1).

The loss factor due to radiation from the plate is given by

$$\eta = \frac{2\rho_0 c \bar{\sigma}_r}{\omega \mu_{\text{tot}}} \tag{2.12.8.4}$$

The wave impedance $\rho_0 c$ is approximately equal to 1.5×10^6 kg/(m^2 s).

2.13 Chapter 13

2.13.1 Example 13.1

From Eq. (13.39) it follows

f(Hz)	1	5	6.3	8	10	12.5	16	20	25	31.5
σ_a	1.5	1.6	1.6	1.7	1.7	1.8	1.9	2.0	2.1	2.3

f(Hz)	40	50	63	80	100	125	160	200	250	315
σ_a	2.4	2.6	2.7	2.9	3.1	3.3	3.5	3.7	3.9	4.1

f(Hz)	400	500
σ_a	4.3	4.5

2.13.2 Example 13.2

The sound transmission loss for an infinite plate is given by Eqs. (13.22) and (13.31). The mass per unit area is $\mu = \rho h$ and the critical frequency $f_c = \dfrac{c^2}{2\pi}\sqrt{\dfrac{\mu}{D_0}} = \dfrac{c^2}{2\pi}\sqrt{\dfrac{12(1-\nu^2)\rho}{E_0 h^2}} \propto \dfrac{1}{h}$.

By changing the thickness from h_1 to h_2 the sound transmission loss is changed by ΔR. Thus $\Delta R = 20\log(h_2/h_1)$ for $f \ll f_c$ and $\Delta R = 30\log(h_2/h_1)$ for $f \gg f_c$

2.13.3 Example 13.3

The function $\cos[\lambda_{mn}(x-d)]$ is continuous in the interval $0 \leqslant x \leqslant a$ and can be expanded in a cosine series in this interval. Thus

$$\cos[\lambda_{mn}(x-d)] = \sum_{l=0}^{\infty} A_l \varepsilon_l \cos\left(\frac{l\pi x}{d}\right) \tag{2.13.3.1}$$

The parameters A_l are

$$A_l = \frac{2}{d}\int_0^a \varepsilon_l \cos[\lambda_{mn}(x-d)]\cos\left(\frac{l\pi x}{d}\right) = \frac{2}{d}\varepsilon_l \frac{\lambda_{mn}\sin(\lambda_{mn}d)}{\lambda_{mn}^2 - (l\pi/d)^2} \tag{2.13.3.2}$$

$$\lambda_{mn}^2 = k^2 - k_{mn}^2 - (l\pi/d)^2 = k^2 - k_{lmn}^2 \tag{2.13.3.3}$$

Equations (2.13.3.1) to (2.13.3.3) give

$$\cos[\lambda_{mn}(x-d)] = \sum_{l=0}^{\infty} \varepsilon_l^2 \frac{\lambda_{mn}\sin(\lambda_{mn}d)}{k^2 - k_{lmn}^2}\cos\left(\frac{l\pi x}{d}\right) \tag{2.13.3.4}$$

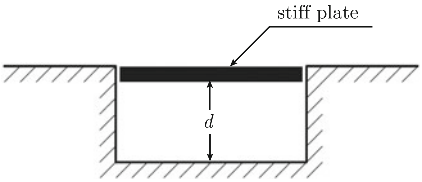


Fig. 2.30 Stiff plate coupled to a cavity

For $x = 0$

$$\sum_{l=0}^{\infty} \frac{\varepsilon_l^2}{k^2 - k_{lmn}^2} = \frac{d}{2\lambda_{mn} \tan(\lambda_{mn} d)}$$

2.13.4 Example 13.4

The plate is located in the y-z-plane at $x = 0$. The dimensions of the plate are L_y and L_z. The depth of the cavity is d. The width and breadth of the cavity is the same as the plate. An external force $\hat{F} \exp(i\omega t)$ is acting on the infinitely stiff plate. The velocity v of the plate is (Fig. 2.30)

$$i\omega\hat{v}\mu = \hat{F}/S - \hat{p} \tag{2.13.4.1}$$

where p is the pressure in the enclosed fluid. The FT of the velocity potential in the fluid is

$$\hat{\Phi} = \frac{\hat{v} \cdot \cos[k(x+d)]}{k \sin(kd)} \tag{2.13.4.2}$$

The FT of the particle velocity $\hat{\nu}_x$ is $\partial\hat{\Phi}/\partial x$ satisfying the boundary conditions $\hat{\nu}_x = 0$ for $x = -d$ and $\hat{\nu}_x = -\hat{\nu}$ for $x = 0$. The FT of the pressure p on the plate is

$$\hat{p} = i\omega\rho_0 \frac{\hat{v}}{k \tan(kd)} \tag{2.13.4.3}$$

This expression inserted in Eq. (2.13.4.1) yields

$$Y = \frac{\hat{v}}{\hat{F}} = \frac{1}{i\omega[\mu + \rho_0/(k \tan kd)]} \tag{2.13.4.4}$$

The mobility of the plate is consequently very low whenever $\mathrm{Re}(kd) = n\pi$ corresponding to the natural frequencies in the cavity. In the high-frequency range or rather when $kd \to \infty$, $\tan(kd) \to -i$ and the mobility tends to $1/(i\omega\mu)$ if there are losses in the fluid or when k is complex.

2.13.5 Example 13.5

The function $Y_{mn}(a)$ is defined as

$$Y_{mn}(a) = \frac{1}{[\lambda_{mn}\sin(\lambda_{mn}a)]^2}\left[\frac{\sin(2\lambda_{mn}a)}{2\lambda_{mn}a} + 1\right] \tag{2.13.5.1}$$

The parameter λ_{mn} is given by

$$\lambda_{mn} = \sqrt{k_0^2 - k_{mn}^2 - i\delta k_0^2} \tag{2.13.5.2}$$

The function $Y_{mn}(a)$ has maxima for $k_0 = k_{0N} = \sqrt{k_{mn}^2 + (N\pi/a)^2}$. Let the wavenumber k_0 around this value be described by $k_0 = k_{0N}(1+\xi)$. Thus, for $|\xi| \ll 1$ and $\delta \ll 1$

$$\begin{aligned}\lambda_{mn} &\approx \left[(N\pi/a)^2 + (2\xi - i\delta)k_0^2\right]^{1/2} \approx (N\pi/a)\left[1 + (2\xi - i\delta)k_0^2/(N\pi/a)^2\right]^{1/2} \\ &\approx (N\pi/a) + (\xi - i\delta/2)k_0^2/(N\pi/a)\end{aligned} \tag{2.13.5.3}$$

An expansion of $\sin(\lambda_{mn}a)$ in a Taylor series gives

$$\sin(\lambda_{mn}a) = \sin(N\pi) + a^2k_0^2(\xi - i\delta/2)\cos(N\pi)/(N\pi) \tag{2.13.5.4}$$

Thus to the first order of smallness in ξ and δ

$$|Y_{mn}(a)| = \frac{1}{(ak_0^2)^2(\xi^2 + \delta^2/4)} \tag{2.13.5.5}$$

2.13.6 Example 13.6

The expression has two poles in the upper half plane, $\xi_1 = i\delta_1/2$ and $\xi_2 = i\delta_2/2$. According to Eq. (2.61), an integration along a path shown in Fig. 2.5 gives

$$J = \oint \frac{\mathrm{d}\xi}{h(\xi)} = 2\pi i \sum_n \frac{1}{h'(\xi_n)} \tag{2.13.6.1}$$

With $h(\xi) = (\xi^2 + \delta_1^2/4)(\xi^2 + \delta_2^2/4)$ the function $h'(\xi)$ is

$$h'(\xi) = 2\xi(\xi^2 + \delta_2^2/4) + 2\xi(\xi^2 + \delta_1^2/4) \tag{2.13.6.2}$$

Thus,

$$J = 2\pi i \left[\frac{1}{i\delta_1(-\delta_1^2/4 + \delta_2^2/4)} + \frac{1}{i\delta_2(\delta_1^2/4 - \delta_2^2/4)} \right]$$
$$= \frac{8\pi(\delta_1 - \delta_2)}{(\delta_1^2 - \delta_2^2)\delta_1\delta_2} = \frac{8\pi}{(\delta_1 + \delta_2)\delta_1\delta_2} \tag{2.13.6.3}$$

2.13.7 Example 13.7

According to Eq. (13.97), the velocity v of a panel separating two rooms is the solution to

$$\nabla^2(\nabla^2 v) - \kappa^4 v = \sum_{mn} Q_{mn}\varphi_{mn} \tag{2.13.7.1}$$

where is φ_{mn} are the eigenfunction for the cross section of the room. The velocity is written

$$v = \sum_{qr} B_{qr} g_{qr}(y, z); \quad \nabla^2(\nabla^2 g_{qr}) = k_{qr}^4 g_{qr} \tag{2.13.7.2}$$

The eigenfunction g_{qr} satisfies the boundary conditions of the panel. The eigenfunctions are orthogonal for any of the simple boundary conditions. The amplitudes B_{qr} are obtained from (2.13.7.1) and (2.13.7.2) as

$$B_{qr} = \frac{1}{(k_{qr}^4 - \kappa^4)\langle g_{qr}|g_{qr}\rangle} \sum_{mn} Q_{mn}\langle\varphi_{mn}|g_{qr}\rangle \tag{2.13.7.3}$$

The frequency and space average of the pressure squared in the receiving room is proportional the space and frequency average of the square of the plate velocity. Thus,

$$\langle\bar{v}^2\rangle \propto \sum_{qr} \left|\bar{B}_{qr}\right|^2 \tag{2.13.7.4}$$

Assuming as before, Sect. 13.6, that the frequency average of the product $Q_{mn} \cdot Q_{rs}$ is equal to $\left|\bar{Q}\right|^2 \delta_{mr}\delta_{ns}$ it follows that

$$\langle\bar{v}^2\rangle \propto \sum_{qr} \frac{1}{\Delta\kappa} \int \frac{d\kappa}{\left|k_{qr}^4 - \kappa^4\right|^2} \sum_{mn} \frac{(\langle\varphi_{mn}|g_{qr}\rangle)^2}{(\langle g_{qr}|g_{qr}\rangle)^2} \tag{2.13.7.5}$$

The dominating contribution to the velocity squared is given by resonant modes for $k_{qr} \approx \kappa$ when at the same time the coupling between the modes in the room and on the plate is maximum. The coupling is maximum when the wavenumber on the plate coincides or is close to the wavenumber in the y-z-plane of the acoustic field. These effects can only occur simultaneously if $k \geqslant k_{qr} \approx \kappa$ or for $f > f_c$. For $k_{mn} \gg k_{qr}$ the product $\left|\langle\varphi_{mn}|g_{qr}\rangle\right|^2$ is small and decreases rapidly for increasing k_{mn} as k_{mn}^{-4}. Hence the error is small if the summation in (2.13.7.5) is extended to include all m and n instead of those for which $k_{mn} < k$. The sum over m and n in Eq. (2.13.7.5) is rearranged as

$$\sum_{mn} \frac{(\langle\varphi_{mn}|g_{qr}\rangle)^2}{(\langle g_{qr}|g_{qr}\rangle)^2} = \frac{1}{(\langle g_{qr}|g_{qr}\rangle)^2} \int \mathrm{d}S \int \mathrm{d}S_0 g_{qr}(r) g_{qr}(r_0) \sum_{m,n=0}^{\infty} \varphi_{mn}(r)\varphi_{mn}(r_0) \tag{2.13.7.6}$$

The completeness relation, as given in for example on p. 254 of Ref. [69] of vol. II, states

$$\sum_{m,n=0}^{\infty} \varphi_{mn}(r)\varphi_{mn}(r_0) = \delta(r - r_0) \tag{2.13.7.7}$$

Consequently,

$$\sum_{mn} \frac{(\langle\varphi_{mn}|g_{qr}\rangle)^2}{(\langle g_{qr}|g_{qr}\rangle)^2} = \frac{1}{(\langle g_{qr}|g_{qr}\rangle)^2} \int \mathrm{d}S \int \mathrm{d}S_0 g_{qr}(r) g_{qr}(r_0)\delta(r - r_0) = 1 \tag{2.13.7.8}$$

for any orthogonal eigenfunction, i.e. independent of free, clamped or simply supported boundary conditions for the plate.

Returning to Eq. (2.13.7.5) the velocity squared of the panel is at a certain frequency for which $k_{qr} \approx \kappa$ given by

$$\langle \bar{v}^2 \rangle \propto \frac{1}{\Delta\kappa} \int_{k_{qr}-\Delta\kappa/2}^{k_{qr}+\Delta\kappa/2} \frac{\mathrm{d}\kappa}{\left|k_{qr}^4 - \kappa^4\right|^2} \tag{2.13.7.9}$$

for any boundary condition having orthogonal eigenfunctions.

2.13.8 Example 13.8

Notations etc as in Sect. 13.6. For a one-dimensional simply supported structure the boundary conditions are $v = \partial^2 v/\partial y^2 = 0$ for $y = 0 = L_y$. The velocity is $v = v_1 + v_2$ where, from (13.99) and (13.100)

$$v_1 = \sum_m \frac{Q_m \varphi_m(y)}{k_m^4 - \kappa^4} \approx -\sum_m \frac{Q_m \varphi_m(y)}{\kappa^4} \text{ for } f \ll f_c \tag{2.13.8.1}$$

The complementary and symmeteric function v_2 is

$$v_2 = B_1 \cos \kappa(y - L_y/2) + B_2 \cosh \kappa(y - L_y/2) = \sum_m v_{2m} \varphi_m(y) \tag{2.13.8.2}$$

The boundary conditions yield

$$B_1 = \frac{1}{2\cos(\kappa L_y/2)} \sum_m \frac{Q_m}{\kappa^4} \left(1 + \frac{k_m^2}{\kappa^2}\right) \approx \frac{1}{2\cos(\kappa L_y/2)} \sum_m \frac{Q_m}{\kappa^4} \tag{2.13.8.3}$$

$$B_2 = \frac{1}{2\cosh(\kappa L_y/2)} \sum_m \frac{Q_m}{\kappa^4} \left(1 - \frac{k_m^2}{\kappa^2}\right) \approx \frac{1}{2\cosh(\kappa L_y/2)} \sum_m \frac{Q_m}{\kappa^4} \tag{2.13.8.4}$$

$$v_2 = \frac{1}{2} \left(\sum_m \frac{Q_m}{\kappa^4}\right) \left[\frac{\cos \kappa(y - L_y/2)}{\cos(\kappa L_y/2)} + \frac{\cosh \kappa(y - L_y/2)}{\cosh(\kappa L_y/2)}\right] \tag{2.13.8.5}$$

By expanding the complementary solution along the eigenfunctions the result is

$$v_2 = \sum_m v_{2m} \varphi_m(y) \tag{2.13.8.6}$$

where the parameters v_{2m} are

$$v_{2m} = \frac{4}{\kappa L_y} \left[\frac{1}{\cot(\kappa L_y/2)} + 1\right] \sum_m \frac{Q_m}{\kappa^4}$$

The parameter v_{2m} is large when $\text{Re}[\cot(\kappa L_y/2)] = 0$. Therefore, close to a maximum

$$v_{2m} \approx \frac{4}{\kappa L_y} \left[\frac{1}{\cot(\kappa L_y/2)}\right] \sum_m \frac{Q_m}{\kappa^4} \tag{2.13.8.7}$$

The absolute value of this maximum amplitude is 1/2 of the the same amplitude for a clamped plate. The correction term for a simply suported beam is therefore $1/4$ of the correction term for clamped edge since the transmitted power is proportional to the square of the velocity.

2.13.9 Example 13.9

Assume that the transmission losses of the two panels are R_1 and R_2. The sound pressure level in the source room is L_1 and in the receiving room L_2 and in the cavity L_0. The equivalent absorption areas is A_0 in the cavity and A_2 in the receiving room. Thus,

$$L_1 - L_0 = R_1 + 10\log(A_0/S) \tag{2.13.9.1}$$

$$L_0 - L_2 = R_2 + 10\log(A_2/S) \tag{2.13.9.2}$$

These equations give

$$L_1 - L_2 = R_1 + R_2 + 10\log(A_0/S) + 10\log(A_2/S) \tag{2.13.9.3}$$

The result indicates that the total transmission loss increases with added sound absorption in the cavity. In practice, there is a limit to the correction. Often $10\log(A_0/S)$ is set to equal 6 dB as an upper limit. Compare Figs. 13.18 and 13.19 in Vol. 2.

2.13.10 Example 13.10

Returning to Problem 13.4 it was found that the pressure p in a fluid filled cavity was given by $p = i\omega\rho_0 v/\tan(kd)$ where v was the velocity of an infinitely stiff structure enclosing the fluid. The velocity is, for harmonic motion, also written $v = i\omega x$ with x being the displacement of the plate. Thus,

$$p = -\omega^2\rho_0\frac{x}{k\cdot\tan(kd)} \approx -\frac{\omega^2\rho_0 x}{k^2 d} \text{ for } kd \ll 1 \tag{2.13.10.1}$$

The "spring constant per unit area" s is obtained as

$$s = -\frac{\partial p}{\partial x} = \frac{\rho_0 c^2}{d} \tag{2.13.10.2}$$

If the plates of the double wall structure has the mass per unit area of μ_1 and μ_2 the double wall resonace is

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{s(\mu_1+\mu_2)}{\mu_1\mu_2}} = \frac{c}{2\pi}\sqrt{\frac{\rho_0(\mu_1+\mu_2)}{d\mu_1\mu_2}} \tag{2.13.10.3}$$

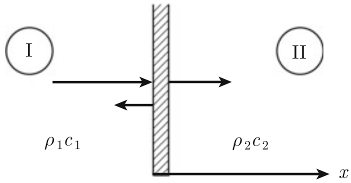


Fig. 2.31 Infinite structure in between two different fluids. A plane wave is incident on the structure in fluid I

2.13.11 Example 13.11

The configuration is shown in Fig. 2.31. The velocity potentials in space I and space II are

$$\Phi_{\mathrm{I}} = \exp[i(\omega t - k_1 x)] + R \cdot \exp[i(\omega t + k_1 x)] \quad x < 0 \tag{2.13.11.1}$$

$$\Phi_{\mathrm{II}} = T \cdot \exp[i(\omega t - k_2 x)] \tag{2.13.11.2}$$

The velocity of the plate is v. Thus,

$$i\omega\mu v = p_{\mathrm{I}} - p_{\mathrm{II}} = -i\omega\rho_1(1 + R) + i\omega\rho_2 T \tag{2.13.11.3}$$

Continuity of velocity gives

$$(\partial\Phi_{\mathrm{I}}/\partial x)_{x=0} = (\partial\Phi_{\mathrm{II}}/\partial x)_{x=0} = v \tag{2.13.11.4}$$

The Eqs. (2.13.11.1) through (2.13.11.4) give

$$T = \frac{2i\rho_1}{k_2\mu - i\rho_1 k_2/k_1 - i\rho_2} \tag{2.13.11.5}$$

The incident intensity is $\left(\bar{I}_x\right)_{in} = \omega k_1 \rho_1/2$ and the transmitted intensity $\left(\bar{I}_x\right)_{\mathrm{trans}} = \omega k_2 \rho_2 \left|T\right|^2/2$

The transmission coefficient τ is

$$\tau = \frac{\left(\bar{I}_x\right)_{\mathrm{trans}}}{\left(\bar{I}_x\right)_{\mathrm{in}}} = \frac{\rho_2 c_1}{\rho_1 c_2} \cdot |T|^2 = \frac{4(\rho_1 c_1)(\rho_2 c_2)}{(\omega\mu)^2 + (\rho_1 c_1 + \rho_2 c_2)^2} \tag{2.13.11.6}$$

The result shows that reciprocity holds. If fluid II is water $(\rho_2 c_2) \approx 1.5 \times 10^6\,\mathrm{kg/(s\,m^2)}$, fluid I air $(\rho_1 c_1) \approx 415\,\mathrm{kg/(s\,m^2)}$ the transmission coefficient is almost independent of the mass μ of the plate giving

$$\tau \approx \frac{4(\rho_1 c_1)}{(\rho_2 c_2)} \approx 10^{-3} \tag{2.13.11.7}$$

2.13.12 Example 13.12

Following the discussion in Sect. 13.3, Eq. (13.45), the velocity of a finite panel is written

$$v(y, z, t) = \sum_{mn} v_{mn}\varphi_{mn}(y, z) \cdot \exp(i\omega t) \tag{2.13.12.1}$$

According to Eq. (13.57), the modal amplitude v_{mn} of the panel is

$$v_{mn}\left[k_{mn}^4 - \kappa^4 + \frac{\rho_0\omega^2}{D}\left(\frac{1}{\lambda_{mn}\tan(\lambda_{mn}d)} + \frac{1}{\lambda_{mn}\tan(\lambda_{mn}a)}\right)\right] = \frac{\rho_0\omega^2 dP}{2D\lambda_{mn}\tan(\lambda_{mn}d)} \tag{2.13.12.2}$$

The wavenumber in a fluid with losses is according to Eq. (11.26) given by $k = k_0(1 - i\delta/2)$. The parameter λ_{mn} is

$$\lambda_{mn} = \sqrt{k_0^2 - k_{mn}^2 - i\delta k_0^2} = \lambda_{mn0}(1 - i\gamma) \tag{2.13.12.3}$$

where λ_{mn0} and γ are positive and real quantities. Thus,

$$\tan(\lambda_{mn}a) = \frac{e^{i\lambda_{mn}a} - e^{-i\lambda_{mn}a}}{i(e^{i\lambda_{mn}a} + e^{-i\lambda_{mn}a})} \tag{2.13.12.4}$$

By inserting Eq. (2.13.12.3) in Eq. (2.13.12.4) it follows that

$$\lim_{a\to\infty}\tan(\lambda_{mn}a) = 1/i \quad \lim_{d\to\infty}\tan(\lambda_{mn}d) = 1/i \tag{2.13.12.5}$$

In addition $\lambda_{mn} = k\cos\varphi$ and $k_{mn} = k\sin\varphi$. Considering this Eq. (2.13.12.2) is written

$$v_{mn} = \frac{\mathrm{d}P}{4\left[1 + \frac{iDk\cos\varphi}{2\rho_0\omega^2 c}\left(k^4\sin^4\varphi - \kappa^4\right)\right]} \tag{2.13.12.6}$$

According to Eqs. (13.73), (13.79), and (13.80) the space and frequency average of p_2^2 is

$$\langle|\bar{p}_2|^2\rangle = \frac{S^2}{\pi A_2}\int_0^{\pi/2}(\omega\rho_0)^2\,|v_{mn}|^2\sin\varphi\cos\varphi\mathrm{d}\varphi \tag{2.13.12.7}$$

According to (13.62) and since $\delta k = A/(4V)$

$$\langle \left| \bar{p}_1^2 \right| \rangle = \frac{|\rho_0 \omega P|^2 \, V_1}{32\pi\delta_1 k} = \frac{|\rho_0 \omega P|^2 \, V_1^2}{8\pi A_1} \tag{2.13.12.8}$$

$$\langle |\bar{p}_2|^2 \rangle = \langle |\bar{p}_1|^2 \rangle \frac{A_1}{2A_2} \int_0^{\pi/2} \frac{\sin\varphi \cos\varphi \mathrm{d}\varphi}{1 + \left[\dfrac{Dk\cos\varphi}{2\omega^2 \rho_0 c} (k^4 \sin^4\varphi - \kappa^4) \right]^2} \tag{2.13.12.9}$$

For $p_{\text{in}} = p_1/2$ and $A_1 = A_2$ it follows that

$$\tau_d = 2 \int_0^{\pi/2} \frac{\sin\varphi \cos\varphi \mathrm{d}\varphi}{1 + \left[\dfrac{Dk\cos\varphi}{2\omega^2 \rho_0 c} (k^4 \sin^4\varphi - \kappa^4) \right]^2} \tag{2.13.12.10}$$

This expression is the same as that given for an infinite panel. Compare Eq. (13.9) and Eq. (13.16).

2.14 Chapter 14

2.14.1 Example 14.1

The wave equation for waves in a fluid and assuming a time dependence $\exp(i\omega t)$ is in cylindrical coordinates given by

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \varphi^2} + k^2 \Phi = 0 \tag{2.14.1.1}$$

where Φ is the velocity potential. The general solution to Eq. (2.14.1.1) is

$$\Phi = \sum_{m=0}^{\infty} A_m J_m(kr) \cos(m\varphi) \tag{2.14.1.2}$$

The particle velocity normal to the wall is zero. Thus $J_m'(kr_0) = 0$, where r_0 is the radius of the duct. For each m there is an infinite number of zeros. The first few zeros are according to Ref. [43] in volume II given as

$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
0	1.841	3.054	4.201	5.318	6.416
3.832	5.331	6.706	8.015	9.282	10.52
7.016	8.536	9.969	11.346	12.682	13.99

The cut-on frequencies are the solutions to $kr_0 = \alpha$ where α is listed in the table above. The first five cut-on frequencies are obtained for $f_n = \alpha_n \cdot c/(2\pi r_0)$where c is the speed of sound in the fluid and α_n equal to 1.841, 3.054, 3.832, 4.201 and 5.318 for $n = 1$ to 5.

2.14.2 Example 14.2

The bending moment exciting the waveguide at $x = 0$ can be written as

$$M \exp(i\omega t) = \exp(i\omega t) \sum_n M_n \varphi_n(y); \quad \varphi_n(y) = \sin(n\pi y/L_y) \tag{2.14.2.1}$$

The amplitudes M_n are

$$M_n = \frac{2M}{L_y} \int_0^{L_y} \sin(n\pi y/L_y)\mathrm{d}y = \frac{4M}{n\pi} \text{for } n \text{ odd, otherwise zero} \tag{2.14.2.2}$$

The displacement w of the plate is according to Eq. (14.7) given by

$$w(x, y, t) = e^{i\omega t} \sum_{n=1}^{\infty} \varphi_n(y) \cdot \left(A_n \cdot e^{-i\kappa_1 x} + B_n \cdot e^{-\kappa_2 x} \right) \tag{2.14.2.3}$$

The boundary conditions are $w(0) = 0$ and $-Dw''(0) = M' = M/L_y$. Thus

$$A_n = \frac{2M}{n\pi D L_y \kappa^2} \text{ for } n \text{ odd, otherwise zero, } B_n = -A_n \tag{2.14.2.4}$$

The energy flow in the waveguide is obtained from Eq. (14.10) as

$$\bar{\Pi}_x = \frac{1}{2} Re \left\{ \omega \sum_{n\text{odd}} \frac{4M^2 \left[\kappa^2 - (n\pi/L_y)^2 \right]^{3/2}}{n^2 \pi^2 D \kappa^4 L_y} \right\} \tag{2.14.2.5}$$

2.14.3 Example 14.3

Assume the displacement of a beam to be

$$w = A_1 \sin(\kappa x) + A_2 \cos(\kappa x) + A_3 \sinh(\kappa x) + A_4 \cosh(\kappa x) \tag{2.14.3.1}$$

The boundary conditions are $w(0) = w(L) = 0$; $w'(0) = \gamma_m$; $w'(L) = \gamma_n$. The boundary conditions give the parameters A_i. The bending moments at each side of the beam are

$$M_{mn}(0) = -D\left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right]_{x=0} = D\kappa^2(A_2 - A_4) \tag{2.14.3.2}$$

$$M_{mn}(L) = -D\left[\frac{d^2 w}{dx^2}\right]_{x=L} = D\kappa^2(A_1 \sin\alpha + A_2\cos\alpha - A_3\sinh\alpha - A_4\cosh\alpha) \tag{2.14.3.3}$$

where $\alpha = \kappa L$. The bending moment $M_{mn}(L)$ is for $\alpha \gg 1$ written

$$M_{mn}(L) = Y_{mn}\gamma_m - X_{mn}\gamma_n \text{where}$$

$$X_{mn} = D_{mn}\frac{\kappa_{mn}\cos\alpha_{mn} - \kappa_{mn}\sin\alpha_{mn}}{\cos\alpha_{mn}}; \quad Y_{mn} = D_{mn}\frac{\kappa_{mn}}{\cos\alpha_{mn}} \tag{2.14.3.4}$$

The bending stiffness of an element between m and n is given by D_{mn}. The parameter α_{mn} is equal to

$$\alpha_{mn} = \kappa_{mn} L_{mn} \tag{2.14.3.5}$$

where κ_{mn} is the wavenumber for flexural waves propagating along the element between m and n. The length of the element is L_{mn}. The bending at the other end of the element is

$$M_{mn}(0) = X_{mn}\gamma_m - Y_{mn}\gamma_n \tag{2.14.3.6}$$

By insering Eq. (2.14.3.4) in Eq. (14.25) the elements in the matrix $[A]$ are obtained.

2.14.4 Example 14.4

Equation (14.34) reads

$$\begin{aligned} &2D_1' D_2' k_x^6 - 2D_2' I_\omega k_x^4 \omega^2 - [m'(D_1' + 2D_2') + I_\omega' G_e S]k_x^2\omega^2 \\ &\quad + G_e S[D_1' k_x^4 - m'\omega^2] + m' I_\omega' \omega^4 = 0 \end{aligned} \tag{2.14.4.1}$$

In the low-frequency region and for $k_x \propto \sqrt{\omega}$ and neglecting terms of ω^3 and higher orders the equation is reduced to $G_e H[D_1' k_x^4 - m'\omega^2] = 0$. Thus,

$$k_x = \pm\left(\frac{m'\omega^2}{D_1'}\right)^{1/4} \text{ and } k_x = \pm i\left(\frac{m'\omega^2}{D_1'}\right)^{1/4} \quad \text{as } f \to 0 \tag{2.14.4.2}$$

For k_x constant and independent of ω the equation is reduced to

$$2D_1' D_2' k_x^6 + G_e S D_1' k_x^4 = 0 \Rightarrow k_x = \pm i \left(\frac{G_e S}{2D_2'} \right)^{1/2} \text{ as } f \to 0 \tag{2.14.4.3}$$

For large ω and considering only the highest order of ω and assuming $k_x \propto \omega$, Eq. (2.14.4.1) gives

$$2D_1' D_2' k_x^6 - 2D_2' I_\omega k_x^4 \omega^2 = 0 \Rightarrow k_x = \pm \left(\frac{I_\omega' \omega^2}{D_1'} \right)^{1/2} \text{ as } f \to \infty \tag{2.14.4.4}$$

Assuming $k_x \propto \sqrt{\omega}$ gives

$$\begin{aligned} & -2D_2' I_\omega k_x^4 \omega^2 + m' I_\omega' \omega^4 = 0 \\ \Rightarrow & k_x = \pm \left(\frac{m' \omega^2}{2D_2'} \right)^{1/4} \text{ and } k_x = \pm i \left(\frac{m' \omega^2}{2D_2'} \right)^{1/4} \text{ as } f \to \infty \end{aligned} \tag{2.14.4.5}$$

2.14.5 Example 14.5

The displacement w and angular displacement β are given by Eqs. (14.37), (14.38) and (14.39). The boundary conditions for a free-free beam are given in Table 14.1. Neglecting I_ω' the boundary conditions are

$$\partial\beta/\partial x = 0;\ \partial^2 w/\partial x^2 = 0;\ \partial^2 \beta/\partial x^2 = 0 \text{ for } x = 0 \text{ and } x = L \tag{2.14.5.1}$$

The six boundary conditions give a system of equations which in matrix form is written

$$\begin{bmatrix} 0 & X_2\kappa_1 & -X_3\kappa_2 & X_4\kappa_2 e^{-\kappa_2 L} & -X_5\kappa_3 & X_6\kappa_3 e^{-\kappa_3 L} \\ -X_1\kappa_1 \sin\kappa_1 L & X_2\kappa_1 \cos\kappa_1 L & -X_3\kappa_2 e^{-\kappa_2 L} & X_4\kappa_2 & -X_5\kappa_3 e^{-\kappa_3 L} & X_6\kappa_6 \\ 0 & -\kappa_1^2 & \kappa_2^2 & \kappa_2^2 e^{-\kappa_2 L} & \kappa_3^2 & \kappa_3^2 e^{-\kappa_3 L} \\ -\kappa_1^2 \sin\kappa_1 L & -\kappa_1^2 \cos\kappa_1 L & \kappa_2^2 e^{-\kappa_2 L} & \kappa_2^2 & \kappa_3^2 e^{-\kappa_3 L} & \kappa_3^2 \\ -X_1\kappa_1^2 & 0 & X_3\kappa_2^2 & X_4\kappa_2^2 e^{-\kappa_2 L} & X_5\kappa_3^2 & X_6\kappa_3^2 e^{-\kappa_3 L} \\ -X_1\kappa_1^2 \cos\kappa_1 L & -X_2\kappa_1^2 \sin\kappa_1 L & X_3\kappa_2^2 e^{-\kappa_2 L} & X_4\kappa_2^2 & X_5\kappa_3^2 e^{-\kappa_3 L} & X_6\kappa_3^2 \end{bmatrix}$$

$$\cdot \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix} = 0$$

The first line is obtained for $\partial\beta/\partial x = 0$ at $x = 0$ and the second at $x = L$. The third and fourth are for $\partial^2 w/\partial x^2 = 0$ first at $x = 0$ and then at $x = L$. The last two lines are obtained when $\partial^2\beta/\partial x^2 = 0$ for $x = 0$ and $x = L$ respectively. The eigenfrequencies are obtained as solutions to the determinant of the matrix being zero.

2.14.6 Example 14.6

Assume

$$w = \sum_m A_m \sin k_m x; \quad \beta = \sum_m B_m \cos k_m x; \quad k_m = m\pi/L \tag{2.14.6.1}$$

Inserting (2.14.6.1) in Eqs. (14.28) and (14.29) and neglecting I'_ω gives

$$\begin{aligned}\sum_m \sin k_m x[G_e S(A_m k_m^2 - B_m k_m) + 2D'_2(A_m k_m^4 - B_m k_m^3) - A_m m'\omega^2] \\ = F\delta(x - x_1)\end{aligned} \tag{2.14.6.2}$$

$$\sum_m \cos k_m x[-G_e S(A_m k_m - B_m) + B_m D'_1 k_m^2 - 2D'_2(A_m k_m^3 - B_m k_m^2)] = 0 \tag{2.14.6.3}$$

By multiplying Eq. (2.14.6.2) by $\sin k_m x$ and (2.14.6.3) by $\cos k_m x$ and integrating over the length of the beam the parameters A_m and B_m are solved. The response is obtained as

$$w(x,t) = \frac{2F \cdot e^{i\omega t}}{Lm'} \sum_{m=1}^{\infty} \frac{\sin(m\pi x/L)\sin(m\pi x_1/L)}{(2\pi)^2[f_m^2(1+i\eta) - f^2]} \tag{2.14.6.4}$$

$$f_m = \frac{m^2\pi}{2L^2}\left\{\frac{D'_1[2D'_2\pi^2 m^2 + G_e S L^2]}{m'[(D'_1 + 2D'_2)\pi^2 m^2 + G_e S L^2]}\right\}^{1/2} \tag{2.14.6.5}$$

2.14.7 Example 14.7

For an infinite beam oriented along the x-axis of a coordinate system and being excited by a force at $x = 0$ the displacement w and the angular displacement β are for $x \geqslant 0$

$$w_1 = A_1 e^{-i\kappa_1 x} + A_2 e^{-\kappa_2 x} + A_3 e^{-\kappa_3 x}; \quad \beta_1 = B_1 e^{-i\kappa_1 x} + B_2 e^{-\kappa_2 x} + B_3 e^{-\kappa_3 x} \tag{2.14.7.1}$$

The time dependence $\exp(i\omega t)$ has been omitted. The expressions w_1 and β_1 of Eq. (2.14.7.1) should satisfy Eq. (14.29). Neglecting I'_ω the parameters B_i are obtained as

$$B_1 = -i\kappa_1 Y_1 A_1; \quad B_2 = -\kappa_2 Y_2 A_2; \quad B_3 = -\kappa_3 Y_3 A_3 \tag{2.14.7.2}$$

where

$$\begin{aligned} Y_1 &= \frac{2D'_2\kappa_1^2 + G_e S}{(D'_1 + 2D'_2)\kappa_1^2 + G_e S} \\ Y_2 &= \frac{2D'_2\kappa_2^2 - G_e S}{(D'_1 + 2D'_2)\kappa_2^2 - G_e S} \\ Y_3 &= \frac{2D'_2\kappa_3^2 - G_e S}{(D'_1 + 2D'_2)\kappa_3^2 - G_e S} \end{aligned} \tag{2.14.7.3}$$

The boundary conditions at $x = 0$ are

$$\frac{\partial w}{\partial x} = 0; \quad \beta = 0; \quad F = 2D'_1 \frac{\partial^2 \beta}{\partial x^2} \tag{2.14.7.4}$$

The resulting point mobility is obtained as

$$\begin{aligned} Y_{\text{sandbeam}} &= \frac{i\omega \hat{w}_1(0)}{\hat{F}} \\ &= \frac{\omega}{2D'_1\kappa_1\kappa_2\kappa_3}\left[\frac{\kappa_2\kappa_3(Y_3 - Y_2) + i\kappa_1\kappa_3(Y_1 - Y_3) - i\kappa_1\kappa_2(Y_1 - Y_2)}{\kappa_1^2 Y_1(Y_3 - Y_2) - \kappa_2^2 Y_2(Y_1 - Y_3) + \kappa_3^2 Y_3(Y_1 - Y_2)}\right] \end{aligned} \tag{2.14.7.5}$$

2.14.8 Example 14.8

The differential equation governing the displacement of a cylinder is given by Eq. (14.76) as

$$\Omega^6 - K_2\Omega^4 + K_1\Omega^2 - K_0 = 0 \tag{2.14.8.1}$$

$$K_2 = 1 + \frac{(3-\nu)}{2}[n^2 + \lambda_n^2 R^2] + \frac{h^2}{12R^2}[n^2 + \lambda_n^2 R^2]^2 \tag{2.14.8.2}$$

$$K_1 = \frac{1-\nu}{2}\left[(3+2\nu)\lambda_n^2 R^2 + n^2 + \left(n^2 + \lambda_n^2 R^2\right)^2 + \frac{(3-\nu)}{(1-\nu)}\frac{h^2}{12R^2}\left(n^2+\lambda_n^2 R^2\right)^3\right] \tag{2.14.8.3}$$

$$K_0 = \frac{(1-\nu)}{2}\left[(1-\nu^2)\lambda_n^4 R^4 + \frac{h^2}{12R^2}\left(n^2+\lambda_n^2 R^2\right)^4\right] \tag{2.14.8.4}$$

The parameter Ω is defined as

$$\Omega = R\omega\sqrt{\frac{\rho(1-\nu^2)}{E}} = \frac{R\omega}{c_l} \tag{2.14.8.5}$$

First assume that the wavenumber λ_n is $\lambda_n \propto f$. In the high-frequency range as $f \to \infty$

$$K_2 \to \frac{h^2\lambda_n^4 R^4}{12R^2} \tag{2.14.8.6}$$

$$K_1 \to \frac{(3-\nu)}{2}\frac{h^2}{12R^2}\left(\lambda_n^2 R^2\right)^3 \tag{2.14.8.7}$$

$$K_0 \to \frac{(1-\nu)}{2}\frac{h^2}{12R^2}\left(\lambda_n^2 R^2\right)^4 \tag{2.14.8.8}$$

For high frequencies including only the highest order in f Eq. (2.14.8.1) is reduced to

$$-\lambda_m^4 R^4 \Omega^4 + \frac{3-\nu}{2}\lambda_m^6 R^6 \Omega^2 - \frac{1-\nu}{2}\lambda_m^8 R^8 = 0; \quad \Omega = \frac{\omega R}{c_l} \tag{2.14.8.9}$$

Thus

$$\lambda_m^4 - \frac{3-\nu}{1-\nu}\cdot\frac{\lambda_m^2\omega^2}{c_l^2} + \frac{\omega^4}{c_l^4}\cdot\frac{2}{1-\nu} = 0 \tag{2.14.8.10}$$

The solutions are

$$\lambda_m = \pm\frac{\omega}{c_l}; \quad \lambda_m = \pm\frac{\omega}{c_l}\cdot\frac{2}{1-\nu} = \pm\frac{\omega}{c_t} \tag{2.14.8.11}$$

The wavenumbers represent L- and T-waves respectively.

In the second case, assume $\lambda_m \propto \sqrt{f}$ as for flexural waves. Only including the highest order of f Eq. (2.14.8.1) is reduced to

$$\Omega^6 - \Omega^4\frac{h^2\lambda_m^4 R^4}{12R^2} = 0 \tag{2.14.8.12}$$

The solutions are

$$\lambda_m = \pm\left(\frac{12\omega^2}{c_l^2 h^2}\right)^{1/4} = \pm\left[\frac{12\omega^2(1-\nu^2)}{Eh^2}\right]^{1/4} = \pm\left(\frac{\mu\omega^2}{D}\right)^{1/4} \tag{2.14.8.13}$$

$$\lambda_m = \pm i\left(\frac{\mu\omega^2}{D}\right)^{1/4} \tag{2.14.8.14}$$

These solutions represent propagating and evanescent flexural waves on a thin flat plate.

2.14.9 Example 14.9

The sound transmission coefficient τ_d, diffuse incidence, is given by Eq. (14.101) as

$$\tau_d = \sum_{mn} \frac{16\pi\rho_0^2 c^4 (\sigma_r)_{mn}^2}{\mu^2 L_x L_y[(\omega_{mn}^2 - \omega^2)^2 + (\eta_{mn}\omega_{mn}^2)^2]} \tag{2.14.9.1}$$

where $L = L_x$ and $R\varphi_0 = L_y$. For $f > f_c$, $(\sigma_r)_{mn} = 1/\sqrt{1 - f_c/f}$. The frequency average of Eq. (2.14.9.1) is for $f > f_c$

$$\bar{\tau}_d = \frac{1}{\Delta\omega}\int_{\omega-\Delta\omega/2}^{\omega+\Delta\omega/2} \mathrm{d}\omega \frac{16\pi\rho_0^2 c^4}{\mu^2 L_x L_y[(\omega_{mn}^2 - \omega^2)^2 + (\eta_{mn}\omega_{mn}^2)^2](1 - f_c/f)} \tag{2.14.9.2}$$

Terms outside the frequency interval $\omega - \Delta\omega/2 \leqslant \omega \leqslant \omega + \Delta\omega/2$ do not contibute if $\eta \ll 1$.

According to (8.19) $\Delta\omega = \frac{4\pi}{S}\sqrt{\frac{D_0}{\mu}}$. Thus,

$$\bar{\tau}_d = \frac{\rho_0^2 c^4 \sqrt{\mu}}{4\pi^2\mu^2 f_{mn}^3 \eta\sqrt{D_0}(1 - f_c/f_{mn})} \tag{2.14.9.3}$$

By using the relationship $f_c = \frac{c^2}{2\pi}\sqrt{\frac{\mu}{D_0}}$ and by setting $f = f_{mn}$ Eq. (2.14.9.3) is written

$$\bar{\tau}_d = \frac{(\rho_0 c)^2 f_c}{2\pi\mu^2 f^3 \eta(1 - f_c/f)} \tag{2.14.9.4}$$

This is the same result as (13.29) which is valid for a flat plate for $f > f_c$. The sound transmission coefficient is the same for a flat plate as for a curved plate for frequencies well above the ring frequency of the curved plate.

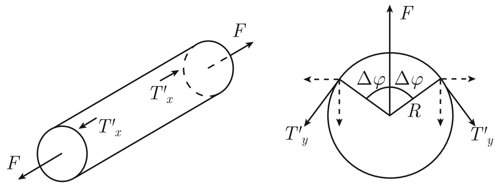


Fig. 2.32 A closed cylinder exposed to an inner overpressure

2.14.10 Example 14.10

The total force on the end structure of the cylinder is $F = \Delta p \cdot \pi R^2$ where R is the radius of the cylinder. The tension per unit width of the shell is T'_x. Thus $F = 2\pi R T'_x$ and $T'_x = \Delta p \cdot R/2$ (Fig. 2.32).
The force F acting on the shell segment is $F_y = 2\Delta p \cdot R \cdot \Delta\varphi$. The forces acting in the opposite direction are $F_y = 2T'_y \sin(\Delta\varphi) \approx 2T'_y \cdot (\Delta\varphi)$. Thus $T'_y = \Delta p \cdot R$.

2.14.11 Example 14.11

The point mobility for a sandwich beam is derived in Problem 14.7. According to Eq. (14.36), the wavenumbers κ_1 and κ_2 approache κ and κ_3 approaches $\left(\frac{G_c S}{2D'_2}\right)^{1/2}$ as $f \to 0$. The parameters Y_1, Y_2 and Y_3 are defined in Eq. (2.14.7.3), Problem 14.7. For $D_2 \to 0$ and $G_c \to \infty$ the parameters Y_1 and Y_2 approaches 1. The parameter Y_3 is for $\kappa_3 = \left(\frac{G_c S}{2D'_2}\right)^{1/2}$ as $f \to 0$

$$Y_3 = \frac{2D'_2 \cdot \frac{G_c S}{2D'_2} - G_c S}{(D'_1 + 2D'_2)\frac{G_c S}{2D'_2} - G_c S} = 0 \tag{2.14.11.1}$$

For $Y_1 = Y_2 = 1$ and $Y_3 = 0$ Eq. (2.14.7.5) Problem 14.7 is reduced to

$$Y = \frac{\omega(-\kappa\kappa_3 + i\kappa\kappa_3)}{2D'_1\kappa^2\kappa_3(-\kappa^2 - \kappa^2)} = \frac{\omega(1-i)}{4D'_1\kappa^3} = \frac{(1-i)\kappa}{4\omega m'} \tag{2.14.11.2}$$

This is the point mobility of an Euler beam given by Eq. (5.39).

2.15 Chapter 15

2.15.1 Example 15.1

The spatial autocorrelation function is defined in Eq. (15.4) as

$$R_{uu}(\xi) = E[u(x)u(x+\xi)] = \lim_{X\to\infty} \frac{1}{X} \int_{-X/2}^{X/2} u(x)u(x+\xi)\mathrm{d}x \tag{2.15.1.1}$$

For a signal $u(x) = A\cos(k_0 x)$ the autocorrelation function is obtained as

$$\begin{aligned} R_{uu}(\xi) &= \lim_{X\to\infty} \frac{A^2}{X} \int_{-X/2}^{X/2} \cos(kx)\cos(k_0 x + k_0\xi) \\ &= \lim_{X\to\infty} \frac{A^2}{2X} \int_{-X/2}^{X/2} [\cos(2k_0 x + \xi k_0) + \cos(k_0\xi)]\,\mathrm{d}x \\ &= \frac{A^2}{2}\cos(k_0\xi) \end{aligned} \tag{2.15.1.2}$$

2.15.2 Example 15.2

The spatial spectral density is defined by Eq. (15.5) as

$$\tilde{S}_{uu}(k) = \int_{-\infty}^{\infty} R_{uu}(\xi)\exp(-ik\xi)\mathrm{d}\xi \tag{2.15.2.1}$$

For a signal $u(x) = A\cos(k_0 x)$ the autocorrelation function is obtained as (Problem 15.1)

$$R_{uu}(\xi) = \frac{A^2}{2}\cos(k_0\xi) \tag{2.15.2.2}$$

Equations (2.15.2.1) and (2.15.2.2) give

$$\tilde{S}_{uu}(k) = \int_{-\infty}^{\infty} R_{uu}(\xi)\cdot e^{-ik\xi}\mathrm{d}\xi = \frac{A^2}{4}\int_{-\infty}^{\infty}\left[e^{i\xi(k_0-k)} + e^{-i\xi(k_0+k)}\right]\mathrm{d}\xi \tag{2.15.2.3}$$

According to Eq. (2.4), the result is

$$\tilde{S}(k) = \frac{\pi A^2}{2}[\delta(k-k_0) + \delta(k+k_0)] \tag{2.15.2.4}$$

2.15.3 Example 15.3

A wave is defined as $u(x) = A \cdot \sin[\omega_0(t - x/c)]$. The autocorrelation function $R_{uu}(\xi, \tau)$ is obtained as

$$\begin{aligned} R_{uu}(\xi,\tau) = \lim_{X\to\infty, T\to\infty} \frac{1}{XT} \int_{-X/2}^{X/2} \mathrm{d}x \\ \int_{-T/2}^{T/2} \mathrm{d}t \sin[\omega_0(t - x/c)] \sin[\omega_0(t + \tau - x/c - \xi/c)] \\ = \frac{A^2}{2} \cos[\omega_0(\tau - \xi/c)] \end{aligned} \tag{2.15.3.1}$$

The corresponding 2D spectral density is defined as

$$\begin{aligned} \tilde{S}_{uu}(k,\omega) &= \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}\tau \cdot R_{uu}(\xi,\tau) \cdot e^{-i(k\xi+\omega\tau)} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} d\tau \\ &\quad \cdot \left[\mathrm{e}^{-i\xi(k+\omega_0/c)+i\tau(\omega_0-\omega)} + e^{-i\xi(k-\omega_0/c)+i\tau(\omega_0+\omega)} \right] \\ &= A^2\pi^2 \left[\delta(\omega_0/c+k) \cdot \delta(\omega-\omega_0) + \delta(\omega_0/c-k) \cdot \delta(\omega+\omega_0) \right] \end{aligned} \tag{2.15.3.2}$$

2.15.4 Example 15.4

The time and space averages $\langle \bar{u}^2 \rangle$ of a signal $u(x,t) = A \cdot \sin[\omega_0(t - x/c)]$ is given by

$$\langle \bar{u}^2 \rangle = R_{uu}(0,0) \tag{2.15.4.1}$$

where

$$R_{uu}(\xi,\tau) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}\omega \tilde{S}_{uu}(k,\omega) e^{i(k\xi+\omega\tau)} \tag{2.15.4.2}$$

From Problem 15.3

$$\tilde{S}_{uu}(k,\omega) = A^2\pi^2 \left[\delta(\omega_0/c + k) \cdot \delta(\omega - \omega_0) + \delta(\omega_0/c - k) \cdot \delta(\omega + \omega_0) \right] \tag{2.15.4.3}$$

Equations (2.15.4.2) and (2.15.4.3) give

$$R_{uu}(\xi,\tau) = \frac{A^2}{2}\cos[\omega_0(\tau - \xi/c)] \tag{2.15.4.4}$$

Thus, from Eq. (2.15.4.1)

$$\langle \bar{u}^2 \rangle = R_{uu}(0,0) = A^2/2 \tag{2.15.4.5}$$

The same result is obtained directly by averaging $u^2(x,t)$ with respect to time and space.

2.15.5 Example 15.5

The FT of the velocity at $\boldsymbol{r}$ is

$$\hat{v}(\boldsymbol{r},\omega) = \int i\omega\hat{p}(\boldsymbol{s},\omega)H(\boldsymbol{r},\boldsymbol{s},\omega)\mathrm{d}^2\boldsymbol{s} \tag{2.15.5.1}$$

where $H(\boldsymbol{r},\boldsymbol{s},\omega)$ is given by (15.46) for a simply supported plate.

The cross-power spectral density $S_{pv}(\boldsymbol{r},\omega)$ of the power input to the plate is

$$\begin{aligned} S_{pv}(\boldsymbol{r},\omega) &= \lim_{T\to\infty}\frac{1}{T}\mathrm{Re}\int \mathrm{d}^2\boldsymbol{r}\,\hat{p}(\boldsymbol{r},\omega)\hat{v}^*(\boldsymbol{r},\omega) \\ &= \lim_{T\to\infty}\frac{1}{T}\mathrm{Re}\int \mathrm{d}^2\boldsymbol{r}\,\hat{p}(\boldsymbol{r},\omega)\int \mathrm{d}^2\boldsymbol{s}(-i\omega)\hat{p}^*(\boldsymbol{s},\omega)H^*(\boldsymbol{r},\boldsymbol{s},\omega) \\ &= \int \mathrm{d}^2\boldsymbol{r}(-i\omega)\int \mathrm{d}^2\boldsymbol{s}\,S_{pp}(\boldsymbol{r},\boldsymbol{s},\omega)H^*(\boldsymbol{r},\boldsymbol{s},\omega) \end{aligned}$$

For "rain on the roof" excitation $S_{pp}(\boldsymbol{r},\boldsymbol{s},\omega) = S_0\cdot\delta(\boldsymbol{r}-\boldsymbol{s})$, see Eq. (15.38). Consequently,

$$\begin{aligned} S_{pv}(\boldsymbol{r},\omega) &= \mathrm{Re}\int \mathrm{d}^2\boldsymbol{r}(-i\omega)\int \mathrm{d}^2\boldsymbol{s}\,H^*(\boldsymbol{r},\boldsymbol{s},\omega)S_0\delta(\boldsymbol{r}-\boldsymbol{s}) \\ &= \mathrm{Re}\int \mathrm{d}^2\boldsymbol{r}(-i\omega)H^*(\boldsymbol{r},\boldsymbol{r},\omega)S_0 \end{aligned} \tag{2.15.5.2}$$

For a simply supported rectangular plate Eqs. (15.45) and (15.46) give

$$H(\boldsymbol{r},\boldsymbol{s},\omega) = \sum_{mn}\frac{4\varphi_{mn}(\boldsymbol{r})\varphi_{mn}(\boldsymbol{s})}{\mu L_x L_y[\omega_{mn}^2(1+i\eta)-\omega^2]} \tag{2.15.5.3}$$

$$H^*(\boldsymbol{r},\boldsymbol{s},\omega) = \sum_{mn}\frac{4[\omega_{mn}^2(1+i\eta)-\omega^2]\varphi_{mn}(\boldsymbol{r})\varphi_{mn}(\boldsymbol{s})}{\mu L_x L_y[(\omega_{mn}^2-\omega^2)^2+(\eta\omega_{mn}^2)^2]} \tag{2.15.5.4}$$

Equations (2.15.5.2) and (2.15.5.4) give

$$S_{pv}(\omega) = \mathrm{Re}\sum_{mn}\int \mathrm{d}^2\boldsymbol{r}\frac{4\eta\omega\omega_{mn}^2\varphi_{mn}^2(\boldsymbol{r})S_0}{\mu L_x L_y[(\omega^2-\omega_{mn}^2)^2+(\eta\omega_{mn}^2)^2]}$$
$$= \sum_{mn}\frac{\eta\omega\omega_{mn}^2 S_0}{\mu[(\omega^2-\omega_{mn}^2)^2+(\eta\omega_{mn}^2)^2]} \tag{2.15.5.5}$$

The time average of the input power is

$$\bar{\Pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\mathrm{Re}S_{pv} = \frac{1}{\pi}\sum_{mn}\int_0^{\infty}\mathrm{d}\omega\frac{\omega\omega_{mn}^2 S_0\eta}{\mu[(\omega^2-\omega_{mn}^2)^2+(\eta\omega_{mn}^2)^2]}$$
$$= \sum_{mn}\frac{S_0}{2\mu} = \sum_{mn}\bar{\Pi}_{mn} \tag{2.15.5.6}$$

Thus,

$$\bar{\Pi}_{mn} = \frac{S_0}{2\mu} \tag{2.15.5.7}$$

The time average of the total energy of mode (m, n) is according to Eq. (15.50) equal to

$$\mathcal{E}_{mn} = \frac{S_0}{2\mu\omega_{mn}\eta} \tag{2.15.5.8}$$

Equations (2.15.5.7) and (2.15.5.8) give

$$\bar{\Pi}_{mn} = \omega_{mn}\eta\bar{\mathcal{E}}_{mn} \tag{2.15.5.9}$$

2.15.6 Example 15.6

The procedure is outlined for a plate in Sect. 15.3. Following the same procedure for a beam the time average of the kinetic energy is

$$\bar{\mathcal{U}} = \frac{m'}{2}\int_0^L \left|\bar{v}^2\right|\mathrm{d}x = \frac{m'}{2}\int_0^L \mathrm{d}x\int_{-\infty}^{\infty}\frac{1}{2\pi}\mathrm{d}\omega S_{vv}(x, x, \omega) \tag{2.15.6.1}$$

The cross-power spectral density between the velocities at x_1 and x_2 is

$$S_{vv}(x_1, x_2, \omega) = \iint d\xi_1\mathrm{d}\xi_2\omega^2 S_{F'F'}(\xi_1, \xi_2, \omega)H^*(x_1, \xi_1, \omega)H(x_2, \xi_2, \omega) \tag{2.15.6.2}$$

The frequency response function is given by

$$H(x_1, \xi_1, \omega) = \sum_m \frac{2\varphi_m(x_1)\varphi_m(\xi_1)}{Lm'[(\omega_m^2 - \omega^2) + i\eta\omega_m^2]} \tag{2.15.6.3}$$

where

$$\varphi_m(x) = \sin(k_m x); \quad k_m = m\pi/L \tag{2.15.6.4}$$

The function $S_{F'F'}(\xi_1, \xi_2, \omega)$ in Eq. (2.15.6.2) is for rain-on-the-roof excitation given by

$$S_{F'F'}(\xi_1, \xi_2, \omega) = S_0\delta(\xi_1 - \xi_2) \tag{2.15.6.5}$$

This expression inserted in (2.15.6.2) yields

$$\begin{aligned} S_{vv}(x_1, x_2, \omega) &= \int_0^L \mathrm{d}\xi_1 \omega^2 S_0 H^*(x_1, \xi_1, \omega) H(x_2, \xi_1, \omega) \\ &= \frac{2\omega^2 S_0}{L(m')^2} \sum_m \frac{\varphi_m(x_1)\varphi_m(x_2)}{(\omega_m^2 - \omega^2)^2 + (\eta\omega_m^2)^2} \end{aligned} \tag{2.15.6.6}$$

Equation (2.15.6.6) inserted in Eq. (2.15.6.6) gives

$$\bar{U} = \frac{S_0}{4\pi m'} \sum_m \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\omega^2}{(\omega_m^2 - \omega^2)^2 + (\eta\omega_m^2)^2} = \sum_m \frac{S_0}{4m'\omega_m\eta} \tag{2.15.6.7}$$

2.15.7 Example 15.7

Consider the integral

$$H = \int_{-\infty}^{\infty} \mathrm{d}\zeta \frac{[1 - \cos(m\pi) \cdot \cos(\zeta L_x)]}{[\zeta^2 - (m\pi/L_x)^2]^2[(\zeta + X)^2 + Y^2]} \tag{2.15.7.1}$$

Writing $\cos(\zeta L_x)$ as $\cos(\zeta L_x) = \left(e^{i\zeta L_x} + e^{-i\zeta L_x}\right)/2$ the expression (2.15.7.1) is rewritten as

$$H = H_1 - H_2 \tag{2.15.7.2}$$

$$H_1 = \frac{1}{2} \oint_{C_1} d\zeta \frac{[1 - \cos(m\pi) \cdot e^{i\zeta L_x}]}{[\zeta^2 - (m\pi/L_x + i\varepsilon)^2]^2[(\zeta + X)^2 + Y^2]} \tag{2.15.7.3}$$

$$H_2 = \frac{1}{2} \oint_{C_2} \mathrm{d}\zeta \frac{[1 - \cos(m\pi) \cdot e^{-i\zeta L_x}]}{[\zeta^2 - (m\pi/L_x + i\varepsilon)^2]^2[(\zeta + X)^2 + Y^2]} \tag{2.15.7.4}$$

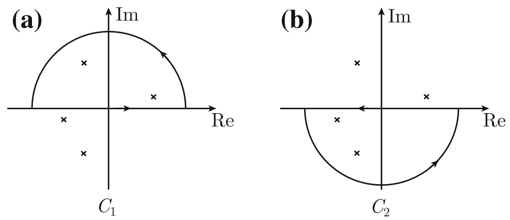


Fig. 2.33 Integration paths in the complex plane

The integration paths are shown in Fig. 2.33. Path C_1 applies to H_1 and path C_2 to H_2. The parameter ε is a small positive quantity introduced to locate the poles off the real axis. The poles in the upper half plane are $\zeta_1 = m\pi/L_x + i\varepsilon$ and $\zeta_2 = -X + iY$. The poles in the lower half plane are $\zeta_3 = -m\pi/L_x - i\varepsilon$ and $\zeta_4 = -X - iY$. Both integrals are zero when integrated along respective semi-circle. Cauchy's integral formula reads

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z)^{n+1}} \tag{2.15.7.5}$$

where $f^{(n)}(z)$ is the nth derivative of $f(z)$. C is any closed path encircling z in the counter clockwise direction. After using Eq. (2.15.7.5) and allowing ε to approach zero the results are

$$\begin{aligned} H_1 =\pi \Bigg\{ &\frac{L_x}{4(m\pi/L_x)^2[(m\pi/L_x + X)^2 + Y^2]} \\ &+ \frac{1 - \cos(m\pi)\cdot\exp(-iL_xX - YL_x)}{2Y[(X - iY)^2 - (m\pi/L_x)^2]}\Bigg\} \end{aligned} \tag{2.15.7.6}$$

$$\begin{aligned} H_2 = -\pi \Bigg\{ &\frac{L_x}{4(m\pi/L_x)^2[(m\pi/L_x - X)^2 - Y^2]} \\ &+ \frac{1 - \cos(m\pi)\cdot\exp(iL_xX - YL_x)}{2Y[(X + iY)^2 - (m\pi/L_x)^2]}\Bigg\} \end{aligned} \tag{2.15.7.7}$$

By inserting the proper values for X and Y the results (15.72) and (15.73) are obtained.

2.15.8 Example 15.8

The average auto spectrum $\bar{S}_{vv}$ of the velocity is given in Eq. (15.78) as

$$\bar{S}_{vv} \approx \frac{1}{\Delta N}\sum_{mn}\frac{C_{mn}}{\mu^2}\sqrt{\frac{\mu}{D}} \propto \frac{1}{\mu^2}\sqrt{\frac{\mu}{D}} \tag{2.15.8.1}$$

The summation is over ΔN modes.

The mass μ is proportional to h and D is proportional to h^3. Thus

$$\bar{S}_{vv} \propto \frac{1}{\mu^2}\sqrt{\frac{\mu}{D}} \propto \frac{1}{h^3} \tag{2.15.8.2}$$

2.15.9 Example 15.9

For a fluid loaded plate $\mu \approx \rho/\kappa_0$. The wavenumber in vacuum is $\kappa_0 = \left(\frac{\mu\omega^2}{D}\right)^{1/4} \propto \frac{1}{\sqrt{h}}$. The bending stiffness $D \propto h^3$. From Eq. (2.15.8.2), Problem 15.8,

$$\bar{S}_{vv} \propto \frac{1}{\mu^2}\sqrt{\frac{\mu}{D}}$$

For $\kappa_0 \propto \frac{1}{\sqrt{h}}$ and $D \propto h^3$ it follows that

$$\bar{S}_{vv} \propto \frac{1}{\mu^2}\sqrt{\frac{\mu}{D}} \propto \frac{\kappa_0^2}{\sqrt{\kappa_0 D}} \propto \frac{1}{h^{9/4}}$$

2.16 Chapter 16

2.16.1 Example 16.1

The potential and kinetic energies of the systems shown in Fig. 16.2 is

$$\mathcal{T} = m_1\dot{y}_1^2/2 + m_2\dot{y}_2^2/2 + m_c(\dot{y}_1 + \dot{y}_2)^2/8 \tag{2.16.1.1}$$

$$\mathcal{U} = k_1 y_1^2/2 + k_2 y_2^2/2 + k_c(y_1 - y_2)^2/2 \tag{2.16.1.2}$$

The external forces, gyroscopic coupling and losses give

$$\mathcal{A} = -F_{1\text{tot}}y_1 - F_{2\text{tot}}y_2 \tag{2.16.1.3}$$

$$F_{1\text{tot}} = F_1 - c_1\dot{y}_1 + G\dot{y}_2 \tag{2.16.1.4}$$

$$F_{2\text{tot}} = F_2 - c_2\dot{y}_2 - G\dot{y}_1 \tag{2.16.1.5}$$

Hamilton's principle, Eq. (9.4) states

$$\delta \int (\mathcal{T} - \mathcal{U} - \mathcal{A})\mathrm{d}t = 0 \tag{2.16.1.6}$$

Thus, Eqs. (2.16.1.1) through (2.16.1.6) give

$$(m_1 + m_c/4)\ddot{y}_1 + (k_1 + k_c)y_1 + (m_c/4)\ddot{y}_2 - k_c y_2 = F_{1\mathrm{tot}} \tag{2.16.1.7}$$

$$(m_2 + m_c/4)\ddot{y}_2 + (k_2 + k_c)y_2 + (m_c/4)\ddot{y}_1 - k_c y_1 = F_{2\mathrm{tot}} \tag{2.16.1.8}$$

Inserting Eqs. (2.16.1.4) and (2.16.1.5) gives

$$(m_1 + m_c/4)\ddot{y}_1 + c_1\dot{y}_1 + (k_1 + k_c)y_1 + (m_c/4)\ddot{y}_2 - G\dot{y}_2 - k_c y_2 = F_1 \tag{2.16.1.9}$$

$$(m_2 + m_c/4)\ddot{y}_2 + c_2\dot{y}_2 + (k_2 + k_c)y_2 + (m_c/4)\ddot{y}_1 - G\dot{y}_1 - k_c y_1 = F_2 \tag{2.16.1.10}$$

2.16.2 Example 16.2

The space average of the pressure squared in the room is according to Eq. (11.141)

$$\begin{aligned}\langle|\hat{p}|^2\rangle &= \omega^2\rho_0^2 \frac{1}{V}\int_V \mathrm{d}V\,|\Phi_0|^2 = \frac{\omega^2\rho_0^2}{V^2}\sum_{l,m,n}\frac{8\,|Q_0|^2\,\varphi_{lmn}^2(\boldsymbol{r}_0)}{\varepsilon_l\varepsilon_m\varepsilon_n\left|k^2 - k_{lmn}^2\right|^2} \\ &= \frac{f^2\rho_0^2}{V^2\pi^2}\sum_{l,m,n}\frac{2c^4\,|Q_0|^2\,\varphi_{lmn}^2(\boldsymbol{r}_0)}{\varepsilon_l\varepsilon_m\varepsilon_n[(f^2 - f_{lmn}^2)^2 + (\delta f_{lmn}^2)^2]}\end{aligned} \tag{2.16.2.1}$$

For the field to be diffuse there must be a number of sources in the room. The space average over the room volume is

$$\langle|\hat{p}|^2\rangle = \frac{f^2\rho_0^2}{V^2\pi^2}\sum_{l,m,n}\frac{2c^4\,|Q_0|^2}{\varepsilon_l\varepsilon_m\varepsilon_n 8[(f^2 - f_{lmn}^2)^2 + (\delta f_{lmn}^2)^2]} \tag{2.16.2.2}$$

The frequency average is

$$\langle\left|\bar{\hat{p}}\right|^2\rangle = \frac{1}{\Delta f}\int_{f-\Delta f/2}^{f+\Delta f/2}\langle|\hat{p}|^2\rangle\mathrm{d}f = \sum_{lmn}\frac{c\rho_0^2 f\,|Q|^2}{2V\delta} \tag{2.16.2.3}$$

According to Eq. (11.159) $\delta = \dfrac{Ac}{4\omega V}$. Thus,

$$\langle \left|\bar{\hat{p}}\right|^2 \rangle = \sum_{lmn} \frac{4\pi\rho_0^2 f^2 \,|Q|^2}{A} \tag{2.16.2.4}$$

The modal energy per volume is

$$\frac{\langle \left|\bar{\hat{p}}_{lmn}\right|^2 \rangle}{(\rho c^2)} = \frac{4\pi\rho_0 f^2 \,|Q|^2}{Ac^2} \tag{2.16.2.5}$$

or

$$G_{pp} = \frac{4\pi\rho_0^2 f^2 G_Q}{A} \tag{2.16.2.6}$$

The modal energy is constant if $A \propto f^2$.

2.16.3 Example 16.3

The last two expressions of Eq. (16.42) are

$$\bar{\Pi}_{12} = \bar{\Pi}_{23} + \bar{\Pi}_{d2}; \quad \bar{\Pi}_{13} + \bar{\Pi}_{23} = \bar{\Pi}_{d3} \tag{2.16.3.1}$$

The energy flow between the systems i and j is

$$\bar{\Pi}_{ij} = \omega\eta_{ij}\bar{\mathcal{E}}_i - \omega\eta_{ji}\bar{\mathcal{E}}_j \tag{2.16.3.2}$$

The power dissipated in system i is

$$\bar{\Pi}_{di} = \omega\eta_{di}\bar{\mathcal{E}}_i \tag{2.16.3.3}$$

Equations (2.16.3.2) and (2.16.3.3) inserted in (2.16.3.1) give

$$\frac{\bar{\mathcal{E}}_1}{\bar{\mathcal{E}}_3} = \frac{(\eta_{d3} + \eta_{31} + \eta_{32})(\eta_{d2} + \eta_{21} + \eta_{23}) - \eta_{32}\eta_{23}}{\eta_{13}(\eta_{d2} + \eta_{21} + \eta_{23}) + \eta_{12}\eta_{23}} \tag{2.16.3.4}$$

The ratio between the pressure squared in the rooms is

$$\frac{\langle|\bar{p}_1|^2\rangle}{\langle|\bar{p}_3|\rangle} = \frac{\bar{\mathcal{E}}_1}{\bar{\mathcal{E}}_3}\frac{V_3}{V_1} = \frac{V_3[(\eta_{d3} + \eta_{31} + \eta_{32})(\eta_{d2} + \eta_{21} + \eta_{23}) - \eta_{32}\eta_{23}]}{V_1\eta_{13}(\eta_{d2} + \eta_{21} + \eta_{23}) + \eta_{12}\eta_{23}} \tag{2.16.3.5}$$

2.16.4 Example 16.4

The ratio between the pressured squared in the rooms 1 and 3 is

$$\frac{\langle|\bar{p}_1|^2\rangle}{\langle|\bar{p}_3|^2\rangle} = \frac{1}{\tau}\frac{A_3}{S} \tag{2.16.4.1}$$

Energy balance requires $\bar{\Pi}_{13} = \bar{\Pi}_{d3}$ or

$$\eta_{13}\bar{\mathcal{E}}_1 = (\eta_{d3} + \eta_{31})\bar{\mathcal{E}}_3 = \eta_{3\text{tot}}\bar{\mathcal{E}}_3 \tag{2.16.4.2}$$

The ratio between the pressured squared in the rooms 1 and 3 is also equal to

$$\frac{\langle|\bar{p}_1|^2\rangle}{\langle|\bar{p}_3|^2\rangle} = \frac{V_3\bar{\mathcal{E}}_1}{V_1\bar{\mathcal{E}}_3} \tag{2.16.4.3}$$

The total losses in room 3 are according to Eq. (16.49)

$$\eta_{3\text{tot}} = \frac{A_3 c}{8\pi f V_3} \tag{2.16.4.4}$$

Equations (2.16.4.1) through (2.16.4.4) give

$$\eta_{13} = \frac{\tau S c}{8\pi f V_1} \tag{2.16.4.5}$$

2.16.5 Example 16.5

In the low-frequency range $f < \frac{c}{2d}$ there are no modes perpendicular to the plate. The modes (m, n) inside the cavity are given by

$$k^2 = \frac{\omega^2}{c^2} = \left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2 \tag{2.16.5.1}$$

Following the technique discussed in Sect. 8.1 the number of modes N for frequencies less than f is $N = \frac{\pi L_x L_y \omega^2}{4c^2}$. The modal density is consequently

$$\mathcal{N}_f = \frac{\mathrm{d}N}{\mathrm{d}f} = \frac{2\pi f S}{c^2} \tag{2.16.5.2}$$

See Eq. (11.137) for high frequencies.

2.16.6 Example 16.6

The vibrational field in beam i is diffuse. The energy flows in both directions of the beam are the same. Consequently, the energy flow $(\bar{\Pi}_i)_{\text{in}}$ towards the junction is according to (3.92) equal to

$$(\bar{\Pi}_i)_{\text{in}} = c_{gi}\bar{\mathcal{E}}_{li}/2 \tag{2.16.6.1}$$

where $\bar{\mathcal{E}}_{li}$ is the total energy per unit length of beam i. The transmitted flow is

$$(\bar{\Pi}_j)_{\text{tr}} = \tau_{ij}(\bar{\Pi}_i)_{\text{in}} = \omega\eta_{ij}\bar{\mathcal{E}}_i = \omega\eta_{ij}\bar{\mathcal{E}}_{li}L_i \tag{2.16.6.2}$$

The total energy $\bar{\mathcal{E}}_i$ of beam i is set to equal $\bar{\mathcal{E}}_i = \bar{\mathcal{E}}_{li}L_i$. Equations (2.16.6.1) and (2.16.6.2) give

$$\eta_{ij} = \frac{c_{gi}\tau_{ij}}{2\omega L_i} = \frac{(D'_{0i})^{1/4}\tau_{ij}}{\pi\omega^{1/2}(m'_i)^{1/4}L_i} \tag{2.16.6.3}$$

2.16.7 Example 16.7

The displacements in the rods, L-waves, are

$$\xi_i = e^{-ik_ix} + R\cdot e^{-ik_ix}; \quad \xi_j = T\cdot e^{-ik_ix} \tag{2.16.7.1}$$

Boundary conditions, displacement and force, at junction at $x = 0$

$$\xi_i(0) = \xi_j(0); \quad E_{0i}A_i\left(\frac{\partial\xi_i}{\partial x}\right)_{x=0} = E_{0j}A_j\left(\frac{\partial\xi_j}{\partial x}\right)_{x=0} \tag{2.16.7.2}$$

Equations (2.16.7.1) and (2.16.7.2) give

$$T = 2\left(1 + \frac{A_j\sqrt{\rho_jE_{0j}}}{A_i\sqrt{\rho_iE_{0i}}}\right)^{-1} \tag{2.16.7.3}$$

The incident and transmitted energy flows are

$$\bar{\Pi}_{\text{in}} = \frac{\omega A_iE_{0i}k_i}{2} \tag{2.16.7.4}$$

$$\bar{\Pi}_{\text{tr}} = \tau_{ij}\bar{\Pi}_{\text{in}} = \tau_{ij}\frac{\omega A_iE_{0i}k_i}{2} = \frac{\omega A_jE_{0j}k_j\,|T|^2}{2} \tag{2.16.7.5}$$

From Eqs. (2.16.7.3) and (2.16.7.5)

$$\tau_{ij} = \frac{A_j k_j E_{0j} |T|^2}{A_i k_i E_{0i}} = \frac{4 A_i A_j \sqrt{\rho_i E_{0i}} \sqrt{\rho_j E_{0j}}}{\left(A_i \sqrt{\rho_i E_{0i}} + A_j \sqrt{\rho_j E_{0j}}\right)^2} \tag{2.16.7.6}$$

The transmitted energy flow is

$$\bar{\Pi}_{\text{tr}} = \bar{\Pi}_{ij} = \omega \eta_{ij} \bar{\mathcal{E}}_{li} = \frac{\omega \eta_{ij} \bar{\mathcal{E}}_i}{L_i} = \frac{\tau_{ij} c_{gi} \bar{\mathcal{E}}_i}{L_i} = \frac{\tau_{ij} \sqrt{E_{0i}} \bar{\mathcal{E}}_i}{L_i \sqrt{\rho_i}} \quad \Rightarrow \tag{2.16.7.7}$$

$$\eta_{ij} = \frac{\tau_{ij}}{2\omega L_i} \sqrt{\frac{E_{0i}}{\rho_i}} \tag{2.16.7.8}$$

2.16.8 Example 16.8

The energy flow for the left-hand case in Fig. 16.13 gives

$$\bar{\Pi}_1^a = \bar{\Pi}_{d1}^a + \bar{\Pi}_{21}^a; \quad \bar{\Pi}_{12}^a = \bar{\Pi}_{d2}^a \tag{2.16.8.1}$$

or

$$\bar{\Pi}_1^a = \omega \eta_{d1} \bar{\mathcal{E}}_1^a + \omega \eta_{12} \bar{\mathcal{E}}_1^a - \omega \eta_{21} \bar{\mathcal{E}}_2^a \tag{2.16.8.2}$$

$$\omega \eta_{12} \bar{\mathcal{E}}_1^a = \omega \eta_{21} \bar{\mathcal{E}}_2^a + \omega \eta_{d2} \bar{\mathcal{E}}_2^a \tag{2.16.8.3}$$

The energy balance for the right-hand system in Fig. 16.3 gives

$$\bar{\Pi}_2^b = \omega \eta_{d2} \bar{\mathcal{E}}_2^b + \omega \eta_{21} \bar{\mathcal{E}}_2^b - \omega \eta_{12} \bar{\mathcal{E}}_1^b \tag{2.16.8.4}$$

$$\omega \eta_{21} \bar{\mathcal{E}}_2^b = \omega \eta_{12} \bar{\mathcal{E}}_1^b + \omega \eta_{d1} \bar{\mathcal{E}}_1^b \tag{2.16.8.5}$$

The results (2.16.8.2) to (2.16.8.5) is in matrix form given by

$$\begin{bmatrix} \bar{\Pi}_1^a & 0 \\ 0 & \bar{\Pi}_2^b \end{bmatrix} = \omega \begin{bmatrix} \eta_{d1} + \eta_{12} & -\eta_{21} \\ -\eta_{12} & \eta_{d2} + \eta_{21} \end{bmatrix} \begin{bmatrix} \bar{\mathcal{E}}_1^a & \bar{\mathcal{E}}_1^b \\ \bar{\mathcal{E}}_2^a & \bar{\mathcal{E}}_2^b \end{bmatrix} \tag{2.16.8.6}$$

which gives

$$\begin{bmatrix} \eta_{d1} + \eta_{12} & -\eta_{21} \\ -\eta_{12} & \eta_{d2} + \eta_{21} \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix} \bar{\Pi}_1^a & 0 \\ 0 & \bar{\Pi}_2^b \end{bmatrix} \begin{bmatrix} \bar{\mathcal{E}}_1^a & \bar{\mathcal{E}}_1^b \\ \bar{\mathcal{E}}_2^a & \bar{\mathcal{E}}_2^b \end{bmatrix}^{-1} \tag{2.16.8.7}$$

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