Chapter 2
General Relativity

General Relativity is our current theory for the description of the gravitational force. It is the first ingredient in the Standard Model of cosmology. General Relativity has successfully passed a large number of tests, mainly in Earth's gravitational field, Solar System, and by studying the orbital motion of binary pulsars (Will 2006). Today, the observed accelerating expansion rate of the Universe is questioning the validity of the theory at very large scales, but at present the phenomenon may be explained with a small positive cosmological constant.

The basic idea of General Relativity is that the gravitational force can be interpreted as a deformation of the geometry of the spacetime, which is not flat any more. The kinematics, namely how particles move in the spacetime, is determined by the geodesic equations and it is a relatively easy problem. The dynamics, i.e. how the energy makes the spacetime curved, is regulated by the Einstein equations. The latter are second order non-linear partial differential equations for the metric coefficients and it is highly non-trivial to find a solution. Analytical solutions are thus possible only in special cases, in which the spacetime possesses some nice symmetries.

This section provides a short review on General Relativity, focusing on the concepts necessary for an introductory course on cosmology. More details can be found in standard textbooks like Hartle 2003, Landau and Lifshitz 1975, Stephani 2004.

2.1 Scalars, Vectors and Tensors

Let us start considering the usual Euclidean space in 3 dimensions. The coordinate system can be indicated by $\mathbf{x}$ or $\{x^i\}$, with $i = 1, 2,$ and 3. In the case of Cartesian coordinates, we have $\{x^i\} = \{x, y, z\}$. If we have a curve γ from the point A to the point B, its length is given by

$$I = \int_\gamma ds, \tag{2.1}$$

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where ds is the line element. The curve γ can be parametrized in terms of a chosen coordinate system as $\gamma(\lambda) = \{x(\lambda), y(\lambda), z(\lambda)\}$, where λ is an affine parameter running along the curve. Equation (2.1) becomes

$$I = \int_{\lambda_1}^{\lambda_2} \left[\left(\frac{dx}{d\lambda} \right)^2 + \left(\frac{dy}{d\lambda} \right)^2 + \left(\frac{dz}{d\lambda} \right)^2 \right]^{1/2} d\lambda, \tag{2.2}$$

where $\gamma(\lambda_1)$ and $\gamma(\lambda_2)$ correspond, respectively, to the point A and B. Equation (2.2) can be written in a more compact way by introducing the *metric tensor* g_{ij}

$$I = \int_{\lambda_1}^{\lambda_2} \left[g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right]^{1/2} d\lambda, \tag{2.3}$$

where we have used the Einstein convention of summation over repeated indices; that is,

$$g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}. \tag{2.4}$$

In this case, g_{ij} is 1 for $i = j$ and 0 for $i \neq j$. In the case of spherical coordinates $\{r, \theta, \phi\}$, the line element is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{2.5}$$

and therefore $g_{11} = 1$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2 \theta$, and all the off-diagonal terms vanish.

If we go from the coordinate system $\{x^i\}$ to the coordinate system $\{x'^i\}$, the infinitesimal displacements change as

$$dx^i \to dx'^i = \frac{\partial x'^i}{\partial x^a} dx^a. \tag{2.6}$$

Since the length of the curve and the line element must be independent of the choice of the coordinate system, the metric tensor changes as

$$g_{ij} \to g'_{ij} = \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^j} g_{ab}. \tag{2.7}$$

It is easy to verify that this is indeed the case for the metric tensor in Cartesian and spherical coordinates.

In general, we call *vector* an object V with components V^is changing according to the rule

$$V^i \to V'^i = \frac{\partial x'^i}{\partial x^a} V^a, \tag{2.8}$$

when we go from the coordinate system $\{x^i\}$ to the coordinate system $\{x'^i\}$.

The *dual vector* of V is the object with components given by

$$V_i = g_{ij} V^j, \tag{2.9}$$

and, under a coordinate transformation, its components change with the opposite rule; that is,

$$V_i \to V'_i = \frac{\partial x^a}{\partial x'^i} V_a. \tag{2.10}$$

Upper indices are used for components that obey the rule in Eq. (2.8), lower indices when the transformation rule is given by Eq. (2.10).

A *scalar* is an object invariant under a coordinate transformation. For instance, a scalar is

$$S = V_i V^i. \tag{2.11}$$

From the transformation rules of V_i and V^i, we see that $S \to S' = S$.

The derivative operator, $\partial_i \equiv \partial / \partial x^i$, is a dual vector because

$$\partial_i \to \partial'_i = \frac{\partial x^a}{\partial x'^i} \partial_a. \tag{2.12}$$

In general, upper indices can be lowered with the use of g_{ij}, as shown in Eq. (2.9), and lower indices can be raised with the use of g^{ij}, which is the inverse matrix of g_{ij}, so

$$V^i = g^{ij} V_j. \tag{2.13}$$

Indeed, $V^i = g^{ij} V_j = g^{ij} g_{jk} V^k = \delta^i_k V^k = V^i$, where δ^i_k is the Kronecker delta and $g^{ij} g_{jk} = \delta^i_k$ by definition of inverse matrix. We also note that the metric tensor with an upper and a lower index is the Kronecker delta, $g^i_k = \delta^i_k$.

Tensors are a multi-index generalization of vectors and dual vectors. The metric tensor g_{ij} is an example of tensor with special properties. In general, the components of a tensor can have some upper and some lower indices; examples are T_{ij}, T^{ij}, $T^i{}_{jk}$, $T_{ij}{}^k{}_l$, etc. In the case of a change of coordinates, upper indices transform according to the rule in Eq. (2.8), while lower indices follow the rule of Eq. (2.10). For example, in the case of a tensor with components $T_{ij}{}^k{}_l$, we have

$$T_{ij}{}^k{}_l \to T'^{\,k}_{ij\ \ l} = \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^j} \frac{\partial x'^k}{\partial x^c} \frac{\partial x^d}{\partial x'^l} T_{ab}{}^c{}_d. \tag{2.14}$$

Upper indices can be lowered with g_{ij}, lower indices can be raised with g^{ij}. For instance,

$$T_{ijkl} = g_{ka} T_{ij}{}^{a}{}_{l}, \quad T_{ij}{}^{kl} = g^{la} T_{ij}{}^{k}{}_{a}, \quad \text{etc.} \tag{2.15}$$

2.2 Geodesic Equations

The geodesic equations determine the kinematics, namely how test-particles move in space. With the introduction of the metric tensor, we can use the same formalism for Newtonian and relativistic mechanics.

2.2.1 Newtonian Mechanics

In Newtonian mechanics, the Principle of Least Action plays a very important role. It can be used to obtain in an elegant way the equations of motion for a system when its action is known. In the case of a free point-like particle, the Lagrangian is simply given by the kinetic energy of the particle

$$L = \frac{1}{2} m \mathbf{v}^2 = \frac{1}{2} m g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}, \tag{2.16}$$

where m is the mass of the particle, $\mathbf{v} = (v^1, v^2, v^3)$ is the particle velocity, g_{ij} is the metric tensor, $\{x^i\}$ are the particle coordinates, and t is the time. The action is

$$S = \int L dt. \tag{2.17}$$

From the Principle of Least Action, we find the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0, \tag{2.18}$$

where the dot indicates the derivative with respect to t.

If we plug the Lagrangian in Eq. (2.16) into the Euler-Lagrange equations (2.18), we obtain the *geodesic equations*

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \tag{2.19}$$

where Γ^i_{jk}s are the *Christoffel symbols*

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right). \tag{2.20}$$

We note that the Christoffel symbols are not the components of a tensor. Indeed, if we consider the coordinate transformation $\{x^i\} \to \{x'^i\}$, the Christoffel symbols change according to the rule

$$\Gamma^i_{jk} \to \Gamma'^i_{jk} = \frac{\partial x'^i}{\partial x^a}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\Gamma^a_{bc} + \frac{\partial x'^i}{\partial x^a}\frac{\partial^2 x^a}{\partial x'^j \partial x'^k}. \tag{2.21}$$

Γ^i_{jk} transforms as a tensor only in the special case of linear transformations. In Cartesian coordinates, all the Christoffel symbols vanish, and therefore the geodesic equations simply reduce to $\ddot{x} = \ddot{y} = \ddot{z} = 0$ (First Newton's Law). In spherical coordinates, we have

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = 0, \tag{2.22}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \cos\theta\sin\theta\dot{\phi}^2 = 0, \tag{2.23}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0. \tag{2.24}$$

2.2.2 Relativistic Mechanics

In Special and General Relativity, time and space are not two independent entities any more and the Newtonian 3-dimensional space becomes a 4-dimensional spacetime. The coordinates are usually indicated by $\{x^\mu\}$, with $\mu = 0$, 1, 2, and 3, where the 0 component refers to the temporal one and the 1, 2, and 3 components refer to the space ones. Greek letters μ, ν, ρ,...are commonly used for the spacetime indices ranging from 0 to 3, while Latin letters i, j, k,...are for the space components only, ranging from 1 to 3.

In Special Relativity and in Cartesian coordinates $\{t, x, y, z\}$, or $\{t, \mathbf{x}\}$ with $\mathbf{x} = \{x, y, z\}$, the metric tensor is indicated by $\eta_{\mu\nu}$ and the line element of the spacetime ds is[1]

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2 = dt^2 - d\mathbf{x}^2. \tag{2.25}$$

The metric coefficients are thus $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$, and all the off-diagonal components vanish. The Principle of Least Action can be naturally extended to relativistic mechanics. The action for a free point-like particle can now be written as

$$S = -m\int_\gamma ds, \tag{2.26}$$

[1] In this book, we use the metric signature convention $(+ - - -)$, which is common in particle physics. In the General Relativity community, it is more common the convention $(- + + +)$.

where m is the particle mass, γ is the particle trajectory, and ds is the line element of the spacetime. From the action (2.26) and the line element (2.25), we find that the Lagrangian is

$$S = \int L dt \quad \Rightarrow \quad L = -m\sqrt{1 - \mathbf{v}^2}, \tag{2.27}$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the particle velocity. In the non-relativistic limit $\mathbf{v}^2 \ll 1$, we recover the Newtonian Lagrangian (modulo a constant)

$$L \approx -m + \frac{1}{2}m\mathbf{v}^2, \tag{2.28}$$

and therefore we obtain the correct Newtonian equations of motion.

In general, the metric tensor $g_{\mu\nu}$ has not the simple form of $\eta_{\mu\nu}$. The line element is an invariant; that is, it is independent of the choice of the coordinates. With the terminology of the previous section, ds^2 is a scalar. We can thus define the following coordinate independent types of trajectories:

$$\begin{aligned} ds^2 &> 0 \quad \text{time-like trajectories,} \\ ds^2 &= 0 \quad \text{light-like trajectories,} \\ ds^2 &< 0 \quad \text{space-like trajectories.} \end{aligned} \tag{2.29}$$

In particular, massless particles like photons will follow light-like trajectories with $ds^2 = 0$; that is, massless particles move with the speed of light. The equations of motion for a massless particle can still be obtained from the action in (2.26), but now m cannot be the mass but just a constant with the dimensions of mass.

In the case of massive particles, it is convenient to use as affine parameter λ their "proper time" τ, i.e. the time measured in the rest-frame of the particle. Since ds^2 is an invariant, $d\tau^2 = ds^2$, because the coordinate system is anchored on the particle and therefore there is no motion along the spatial directions. With this choice of the affine parameter, $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$, where the dot indicates the derivative with respect to τ.

In Newtonian mechanics, the motion of a test-particle in a gravitational field can be described by adding the correct gravitational potential to the Lagrangian of the free particle. One of the key-points in General Relativity is that the gravitational field can be absorbed into the metric tensor $g_{\mu\nu}$: in other words, we have still a free particle, but now it lives in a curved spacetime and follows the geodesics of that spacetime. If the metric of the spacetime $g_{\mu\nu}$ is known, we can obtain the geodesic equations from the action in Eq. (2.26) with $\eta_{\mu\nu}$ replaced by $g_{\mu\nu}$. Equivalently, one can write the Euler-Lagrange equations for the Lagrangian

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{2.30}$$

The geodesic equations have the same form as the ones in the Newtonain theory, with Latin letters replaced by Greek letters

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0 \quad \text{with} \quad \Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\frac{\partial g_{\sigma\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\rho} - \frac{\partial g_{\nu\rho}}{\partial x^\sigma}\right). \tag{2.31}$$

The fact that the motion is only determined by the background geometry and not by specific features of the body meets the well-known *Weak Equivalence Principle*, which asserts that the trajectory of a test-particle is independent of its internal structure and composition (Will 2006).

It is instructive to see how we can recover the Newtonian limit. We use Cartesian coordinates and we require that: (i) the gravitational field is weak, (ii) the gravitational field is stationary, and (iii) the motion of the particle is non-relativistic. These three conditions are given, respectively, by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{with} \quad |h_{\mu\nu}| \ll 1, \tag{2.32}$$

$$\frac{\partial g_{\mu\nu}}{\partial t} = 0, \tag{2.33}$$

$$\frac{dt}{d\lambda} \gg \frac{dx^i}{d\lambda}. \tag{2.34}$$

Within these approximations, the geodesic equations reduce to

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\lambda}\right)^2 \approx 0 \qquad \text{with} \quad \Gamma^\mu_{00} \approx \frac{1}{2}\eta^{\mu\nu}\frac{\partial h_{00}}{\partial x^\nu}. \tag{2.35}$$

After a simple integration, we find

$$\frac{d^2x^i}{dt^2} \approx -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i}. \tag{2.36}$$

If we compare Eq. (2.36) with the Newtonian formula $m\ddot{\mathbf{x}} = -m\nabla\Phi$, where Φ is the Newtonian gravitational potential, and we require that the spacetime is flat at infinity, we find

$$g_{00} = 1 + 2\Phi. \tag{2.37}$$

2.3 Energy and Momentum in Flat Spacetime

From the Lagrangian in Eq. (2.27), we obtain the particle 3-momentum

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{m\mathbf{v}}{\sqrt{1 - v^2}}, \tag{2.38}$$

and the particle Hamiltonian

$$\mathscr{H} = \mathbf{p}\mathbf{v} - L = \frac{m}{\sqrt{1-\mathbf{v}^2}}. \tag{2.39}$$

$\mathscr{H}$ corresponds to the energy of the particle, i.e. $E = \mathscr{H}$. For $\mathbf{v}^2 = 0$, we get the particle *rest-energy* $E = m$. We note that, if this particle is made of a number of elementary particles, its rest-energy is not just the sum of the masses of all the elementary particles, but it includes also the kinetic energy and the interaction energy of all the constituents. In the non-relativistic limit $\mathbf{v}^2 \ll 1$, the particle energy is

$$E \approx m + \frac{1}{2}m\mathbf{v}^2. \tag{2.40}$$

The correct Newtonian kinetic energy is thus recovered subtracting the rest-energy m from the total energy E. Massive particles cannot reach the speed of light $\mathbf{v}^2 = 1$ because it would require an infinite energy.

The 4-momentum of a massive particle can be introduced as

$$p^\mu = m\dot{x}^\mu = (E, \mathbf{p})\,. \tag{2.41}$$

The scalar $p_\mu p^\mu = m^2$ corresponds to the well-known relativistic formula relating the energy, the mass, and the 3-momentum of a particle

$$E^2 = m^2 + \mathbf{p}^2. \tag{2.42}$$

2.4 Energy-Momentum Tensor in Flat Spacetime

Let us consider a system with action

$$S = \int L dt \quad \text{with} \;\; L = \int \mathscr{L} d^3V, \tag{2.43}$$

where $\mathscr{L} = \mathscr{L}(\phi, \partial_\mu\phi)$ is the *Lagrangian density* and depends on the field $\phi(t, \mathbf{x})$ and on its first derivatives, while $d^3V = dx\,dy\,dz$ is the volume element in Cartesian coordinates. Since $\mathscr{L}$ does not explicitly depend on the coordinates x^μ, the system is closed, and its energy and momentum are conserved. If we apply the Principle of Least Action, namely we consider small variations of ϕ and $\partial_\mu\phi$ and demand that $\delta S = 0$, we get the equations of motion

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu\phi\right)} - \frac{\partial \mathscr{L}}{\partial \phi} = 0. \tag{2.44}$$

Using the equations of motion and the fact that $\partial_\mu \partial_\nu \phi = \partial_\nu \partial_\mu \phi$, we find

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial x^\mu} &= \frac{\partial \mathscr{L}}{\partial \phi} \left(\partial_\mu \phi\right) + \frac{\partial \mathscr{L}}{\partial \left(\partial_\nu \phi\right)} \partial_\mu \left(\partial_\nu \phi\right) \\ &= \left[\frac{\partial}{\partial x^\nu} \frac{\partial \mathscr{L}}{\partial \left(\partial_\nu \phi\right)}\right] \left(\partial_\mu \phi\right) + \frac{\partial \mathscr{L}}{\partial \left(\partial_\nu \phi\right)} \partial_\nu \left(\partial_\mu \phi\right) \\ &= \frac{\partial}{\partial x^\nu} \left[\frac{\partial \mathscr{L}}{\partial \left(\partial_\nu \phi\right)} \left(\partial_\mu \phi\right)\right]. \end{aligned} \tag{2.45}$$

We define the *energy-momentum tensor* of the system as

$$T^\nu_\mu = \frac{\partial \mathscr{L}}{\partial \left(\partial_\nu \phi\right)} \left(\partial_\mu \phi\right) - \eta^\nu_\mu \mathscr{L}, \tag{2.46}$$

and Eq. (2.45) reduces to

$$\partial_\nu T^\nu_\mu = 0. \tag{2.47}$$

A few comments are in order here. First, Eq. (2.46) looks like the Legendre transformation of the Lagrangian density, so T^{00} should be the energy density of the system and T^{0i}s should be the momentum densities of the system. Second, Eq. (2.47) is an equation of conservation. Indeed, if we integrate Eq. (2.47) over the volume V and we apply Gauss' theorem, we find

$$\frac{d}{dt} \int_V T^{00} d^3V = - \int_\Sigma T^{0i} d^2\sigma_i, \tag{2.48}$$

$$\frac{d}{dt} \int_V T^{0i} d^3V = - \int_\Sigma T^{ij} d^2\sigma_j, \tag{2.49}$$

where $d^2\sigma_j$ represents the surface element of the surface Σ and it is outwardly perpendicular to Σ. Third, such a definition of the energy-momentum tensor has some ambiguity: if $T^{\mu\nu}$ is our energy-momentum tensor, the tensor

$$T^{\mu\nu} + \partial_\rho A^{\mu\nu\rho} \quad \text{with} \quad A^{\mu\nu\rho} = -A^{\mu\rho\nu} \tag{2.50}$$

satisfies Eq. (2.47) as well. It turns out that such an ambiguity can be removed by imposing that $T^{\mu\nu}$ is a symmetric tensor, namely $T^{\mu\nu} = T^{\nu\mu}$. If the initial energy-momentum tensor is not symmetric, it is always possible to make it symmetric with a suitable choice of $A^{\mu\nu\rho}$. This requirement can be inferred by imposing the conservation of the angular momentum of the system in Special Relativity, which can be constructed from $T^{\mu\nu}$ as

$$M^{\mu\nu} = \int_V \left(x^\mu T^{\nu 0} - x^\nu T^{\mu 0}\right) d^3V. \tag{2.51}$$

It is easy to see that

$$\partial_\mu M^{\mu\nu} = 0 \quad \Rightarrow \quad T^{\mu\nu} = T^{\nu\mu}. \tag{2.52}$$

2.5 Curved Spacetime

In curved spacetime, but even in flat spacetime in curved coordinates, the derivative of a scalar is a vector, but the derivative of a vector is not a tensor. The generalization of the ordinary derivative ∂_μ in curved spacetime, or in curved coordinates in flat spacetime, is the *covariant derivative* ∇_μ. For a generic vector V^μ, the action of the covariant derivative is

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho. \tag{2.53}$$

It can be checked that for a dual vector V_μ the action of the covariant derivative is

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho. \tag{2.54}$$

One can see that $\nabla_\mu V^\nu$ and $\nabla_\mu V_\nu$ are tensors and that ∇_μ is the natural generalization of ∂_μ. In the case of a generic tensor with upper and lower indices, the rule is

$$\begin{aligned} \nabla_\rho T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = \partial_\rho T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} & \\ + \Gamma^{\mu_1}_{\rho\sigma} T^{\sigma \dots \mu_m}{}_{\nu_1 \dots \nu_n} + \dots + \Gamma^{\mu_m}_{\rho\sigma} T^{\mu_1 \dots \sigma}{}_{\nu_1 \dots \nu_n} & \\ - \Gamma^{\sigma}_{\rho\nu_1} T^{\mu_1 \dots \mu_m}{}_{\sigma \dots \nu_n} - \dots - \Gamma^{\sigma}_{\rho\nu_n} T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \sigma}. & \end{aligned} \tag{2.55}$$

With the covariant derivative, we can introduce the *Riemann tensor* as the commutator of the derivatives

$$\nabla_\mu \nabla_\nu V_\rho - \nabla_\nu \nabla_\mu V_\rho = R^\sigma{}_{\rho\mu\nu} V_\sigma, \tag{2.56}$$

for any vector V^μ. It turns out that the Riemann tensor can be written as

$$R^\mu{}_{\nu\rho\sigma} = \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\rho} - \frac{\partial \Gamma^\mu_{\nu\rho}}{\partial x^\sigma} + \Gamma^\mu_{\lambda\rho} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma} \Gamma^\lambda_{\nu\rho}. \tag{2.57}$$

Since it is a tensor, under a coordinate transformation $\{x^\mu\} \to \{x'^\mu\}$ it changes as

$$R^\mu{}_{\nu\rho\sigma} \to R'^\mu{}_{\nu\rho\sigma} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\rho} \frac{\partial x^\delta}{\partial x'^\sigma} R^\alpha{}_{\beta\gamma\delta}. \tag{2.58}$$

With the Riemann tensor, we can introduce the *Ricci tensor* $R_{\mu\nu}$ and the *scalar curvature* R

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}\,, \quad R = g^{\mu\nu} R_{\mu\nu}. \tag{2.59}$$

R is a scalar, namely it is invariant under coordinate transformations. $R^{\sigma}{}_{\rho\mu\nu}$ and $R_{\mu\nu}$ are tensors. If all their components vanish in a coordinate system, they vanish in any coordinate system, as can be seen from Eq. (2.58). In particular in flat spacetime $R^{\sigma}{}_{\rho\mu\nu} = 0$.

An important issue concerns how the laws of physics formulated in Special Relativity change when we pass to General Relativity. In flat spacetime, under a coordinate transformation from Cartesian to other coordinates, it is easy to see that one has to replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$ and ordinary derivatives with covariant derivatives:

$$\eta_{\mu\nu} \to g_{\mu\nu}\,, \quad \partial_\mu \to \nabla_\mu. \tag{2.60}$$

The integral over $d^4x = dtd^3V$ is replaced by $\sqrt{-g}d^4x$, where g is the determinant of the metric tensor

$$d^4x \to \sqrt{-g}d^4x. \tag{2.61}$$

These rules directly follow from the coordinate transformation and they are easy to check, for instance for the transformation from Cartesian to spherical or cylindrical coordinates. In the case of curved spacetime, the issue is more tricky. In principle, there may appear some interaction terms with the Riemann tensor, the Ricci tensor, or the scalar curvature. These terms are called non-minimal couplings (an example is given in Sect. 2.6). It turns out that, for the time being, experiments and observations are consistent with the simple rules of Eqs. (2.60) and (2.61). Such a prescription is not demanded by any fundamental principle and sometimes it may not be unique, but it just seems to work. Lastly, we note that the conservation of the energy-momentum tensor in Eq. (2.47) becomes

$$\nabla_\mu T^{\mu\nu} = 0 \tag{2.62}$$

in curved spacetime. However, since we have now the covariant rather than the ordinary derivative, there is no conservation of $T^{\mu\nu}$. The reason is that matter can exchange energy and momentum with the gravitational field.

2.6 Field Theory in Flat and Curved Spacetimes

In flat spacetime and Cartesian coordinates, the action of a field can be conveniently written in the form

$$S = \int \mathscr{L} d^4x, \tag{2.63}$$

where $\mathscr{L}$ is the Lagrangian density, as introduced in Sect. 2.4. In the case of the electromagnetic field, the Lagrangian density is

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{2.64}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength, $A^\mu = (\phi, \mathbf{A})$ is the 4-potential, ϕ is the scalar potential, and $\mathbf{A}$ is the vector potential. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are related to the 4-potential A^μ by

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla\phi\,, \quad \mathbf{B} = \nabla \wedge \mathbf{A}. \tag{2.65}$$

From the definition of $F_{\mu\nu}$, it follows that

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{2.66}$$

If we write Eq. (2.66) in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, we find the second and the third Maxwell equations in the usual form

$$\nabla \cdot \mathbf{B} = 0\,, \quad \nabla \wedge \mathbf{E} = \partial_t \mathbf{B}. \tag{2.67}$$

Applying the Principle of Least Action, we consider small variations of A^μ and of its first derivatives in the action and we get the field equations of the electromagnetic field in covariant form

$$\partial_\mu F^{\mu\nu} = 0. \tag{2.68}$$

These equations in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ reduce to the first and the fourth Maxwell equations in vacuum, namely

$$\nabla \cdot \mathbf{E} = 0\,, \quad \nabla \wedge \mathbf{B} = \partial_t \mathbf{E}. \tag{2.69}$$

It is straightforward to write the action and the field equations for the electromagnetic field in curved spacetime following the recipe of Sect. 2.5. We replace ordinary derivatives with covariant derivatives. However, the field strength of the electromagnetic field is unaltered

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.70}$$

because the Maxwell tensor $F_{\mu\nu}$ is antisymmetric with respect to interchange of μ and ν and symmetric Christoffel symbols disappear from the difference. $F^{\mu\nu}$ is now obtained by raising the indices with $g^{\mu\nu}$, not with $\eta^{\mu\nu}$. The Lagrangian density is still given by Eq. (2.64), while the action is

$$S = \int \mathscr{L} \sqrt{-g} d^4 x. \tag{2.71}$$

Equations (2.66) and (2.68) become, respectively,

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0,$$
$$\nabla_\mu F^{\mu\nu} = 0. \tag{2.72}$$

Let us now consider a scalar field, which is the simplest field and it is widely used in cosmology. In flat spacetime and Cartesian coordinates, the action is given by Eq. (2.63) and the Lagrangian density is

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) - \frac{1}{2}m^2\phi^2, \tag{2.73}$$

where ϕ is the scalar field and m is the mass of the particles associated to this field. From the variation of the action, we get the field equation (Klein-Gordon equation)

$$\left(\partial_\mu\partial^\mu - m^2\right)\phi = 0, \tag{2.74}$$

where $\partial_\mu\partial^\mu = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$ is the D'Alembert operator. In curved spacetime, the action is given by Eq. (2.71) and the Lagrangian density becomes

$$\mathscr{L} = \frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) - \frac{1}{2}m^2\phi^2. \tag{2.75}$$

The field equation in curved spacetime is

$$\left(\nabla_\mu\partial^\mu - m^2\right)\phi = 0. \tag{2.76}$$

As discussed in Sect. 2.5, it is not guaranteed that the Lagrangian density in the presence of a gravitational field is given by Eq. (2.75) and there are no interaction terms. In cosmology, it is common to introduce some non-minimal couplings. In the simplest case, the Lagrangian density can be taken as

$$\mathscr{L} = \frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) - \frac{1}{2}m^2\phi^2 + \xi R\phi^2, \tag{2.77}$$

where ξ is a dimensionless coupling constant. We note that in Eq. (2.77) we have $\partial_\mu\phi$ and not $\nabla_\mu\phi$. This is because ϕ is a scalar and therefore $\partial_\mu = \nabla_\mu$.

2.7 Einstein Equations

In the previous sections, we have seen that the motion of test-particles is determined by the geodesic equations and the non-gravitational laws of physics in flat spacetime can be easily translated for a curved spacetime with the prescription given in

Eqs. (2.60) and (2.61). In all these cases, we just need to know the background metric $g_{\mu\nu}$. The latter depends on the coordinate system, but it takes into account the gravitational field as well, and therefore it is determined by the matter distribution. The *Einstein equations* are the master equations of General Relativity and they relate the spacetime geometry to the matter content. They can be obtained by imposing a number of "reasonable" requirements, and *a posteriori* one can check that its predictions are consistent with observations. One can thus require that

1. They are tensor equations, to be independent of the coordinate system.
2. They are partial differential equations at most of second order in the variable of the gravitational field, namely $g_{\mu\nu}$, in analogy with the other field equations in physics.
3. They must have the correct Newtonian limit.
4. $T^{\mu\nu}$ is the source of the gravitational field.
5. If $T^{\mu\nu} = 0$, the spacetime is flat.

From the requirements 1 and 4, the Einstein equations must have the form

$$G^{\mu\nu} = \kappa T^{\mu\nu}, \tag{2.78}$$

where $G_{\mu\nu}$ is the *Einstein tensor* and κ is the Einstein constant. Since $\nabla_\mu T^{\mu\nu} = 0$, we need that

$$\nabla_\mu G^{\mu\nu} = 0. \tag{2.79}$$

From the conditions 2 and 5, it follows that the simplest choice is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.80}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the scalar curvature. The tensor in Eq. (2.80) satisfies the condition in (2.79), called Bianchi identity. To find the Newtonian limit, we assume the approximations (2.32) and (2.33), as well as that in our coordinate system all the components of the matter energy-momentum tensor are negligible, except the 00 one, which describes the energy density and reduces to the matter density in the Newtonian limit, so

$$T_{00} = \rho \qquad T_{\mu\nu} = 0 \;\text{ for }\; \mu, \nu \neq 0. \tag{2.81}$$

After some passages, we find

$$R_{00} = \frac{1}{2} \Delta h_{00} = \kappa \rho, \tag{2.82}$$

where Δ is the Laplace operator. The Poisson equation of Newtonian gravity is recovered by replacing h_{00} with 2Φ, where Φ is the Newtonian gravitational potential, as found in Sect. 2.2.2. The Einstein constant is thus

$$\kappa = 8\pi G_{\rm N} = \frac{8\pi}{M_{\rm Pl}^2}, \tag{2.83}$$

where $G_{\rm N}$ is the Newton constant and $M_{\rm Pl} = G_{\rm N}^{-1/2}$ is the Planck mass.[2] In the next chapters, we will use the Planck mass instead of $G_{\rm N}$, as it is more common in particle cosmology.

If we relax the assumption 5, the Einstein equations can have the form

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = 8\pi G_{\rm N} T^{\mu\nu}, \tag{2.84}$$

where Λ is called *cosmological constant*. For a non-vanishing Λ, we do not recover the flat spacetime in the absence of matter. However, a sufficiently small Λ cannot be ruled out by experiments.

Lastly, like any other known field equation of physics, even the Einstein equations can be derived from the Principle of Least Action. The total action of the system has the form $S_{\rm tot} = S_{\rm EH} + S_{\rm matter}$, where $S_{\rm EH}$ is the Einstein-Hilbert action describing the gravitational field

$$S_{\rm EH} = \frac{1}{16\pi G_{\rm N}} \int R \sqrt{-g}\, d^4x, \tag{2.85}$$

while $S_{\rm matter}$ is the action of the matter sector. If we consider small variations of the metric coefficients and of their first derivatives, we get the Einstein equations. Such a procedure allows to define the matter energy-momentum tensor as

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\rm matter}}{\delta g_{\mu\nu}}, \tag{2.86}$$

which is automatically a symmetric tensor (see the discussion at the end of Sect. 2.4) and reduces to the one of Special Relativity in the absence of gravitational fields. If we consider small variations of the fundamental variables of the matter sector and of their derivatives, we get the field equations of matter (e.g. the Maxwell equations in the case of an electromagnetic field).

The covariant conservation (2.62) of the energy-momentum tensor (2.86) follows from the invariance of the action with respect to arbitrary coordinate transformations, according to the Noether theorem. This property is compatible with the Einstein equations (2.84), with $\Lambda = 0$ due to the Bianchi identity (2.79). This identity is automatically fulfilled in General Relativity, as follows from the definition of the curvature tensors and the Christoffel symbols. If $\Lambda \neq 0$, Λ must be constant, and this is why it has the name "cosmological constant".

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[2]We remind the reader that we are using units in which $c = \hbar = 1$. If we reintroduce c and $\hbar$, we have $\kappa = \frac{8\pi G_{\rm N}}{c^4} = \frac{8\pi \hbar}{M_{\rm Pl}^2 c^3}$.

There is an interesting analogy between the automatic conservation of the left hand side of the Maxwell and the Einstein equations. The Maxwell equations in the presence of electric current have the form

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{2.87}$$

Because of the antisymmetry of $F^{\mu\nu}$ with respect to the interchange of μ and ν, the derivative of the left hand side vanishes, $\partial_\nu \partial_\mu F^{\mu\nu} = 0$. So it implies the current conservation.

Problems

2.1 The exterior gravitational field of a spherically symmetric object is described by the Schwarzschild solution. The line element is

$$ds^2 = \left(1 - \frac{2G_{\rm N}M}{r}\right) dt^2 - \left(1 - \frac{2G_{\rm N}M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 . \tag{2.88}$$

Here M is the gravitational mass of the object.

(a) Write the geodesic equations. [Hint: write the Euler-Lagrange equations for the Lagrangian in (2.30) with $g_{\mu\nu}$ of the Schwarzschild solution and then arrange these equations in the form (2.31).]
(b) Find the non-vanishing Christoffel symbols. [Hint: they can be gotten from the geodesic equations rather than from their definition in terms of the metric tensor.]
(c) What is the value of the Riemann tensor, Ricci tensor, and scalar curvature for $r \to +\infty$?

2.2 The Friedmann-Robertson-Walker metric describes the spacetime geometry of a homogeneous and isotropic universe. The line element is given by

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \tag{2.89}$$

where $a(t)$ is called scale factor and it is a function of t only, while k is a constant.

(a) Answer the questions (a) and (b) of the previous problem for the metric in (2.89).
(b) Is the energy of a test-particle conserved? And its angular momentum?

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