Chapter 2
Matrix Games with Payoffs of Triangular Fuzzy Numbers

2.1 Introduction

The matrix game theory gives a mathematical background for dealing with competitive or antagonistic situations arise in many parts of real life. Matrix games have been extensively studied and successfully applied to many fields such as economics, business, management, and e-commerce as well as advertising. As stated in Chap. 1, however, the assumption that all payoffs are precise common knowledge to both the players is not realistic in many antagonistic decision occasions. In fact, more often than not, in real antagonistic situations, the players are not able to exactly estimate payoffs in the game due to lack of adequate information and/or imprecision of the available information on the environment [1, 2]. This lack of precision and certainty may be appropriately modeled by using the fuzzy set [3–6]. As a special case of fuzzy sets, intervals which are also called fuzzy intervals or interval-valued fuzzy sets are used to deal with fuzziness in matrix games. Consequently, we have extensively studied interval-valued matrix games. From now on, we focus on studying fuzzy matrix games with payoffs represented by fuzzy numbers such as triangular fuzzy numbers and trapezoidal fuzzy numbers.

Fuzzy matrix games were firstly solved by developing the fuzzy linear programming method based on ranking functions of fuzzy numbers and auxiliary linear programming models [7–9]. However, Campos' methods [7–9] provided only crisp solutions with interpretation of fuzzy semantics. Their results were generalized to multi-objective matrix games with fuzzy payoffs and fuzzy goals [10, 11]. Bector and Chandra [12], Bector et al. [13, 14], and Vijay et al. [15] proposed linear programming methods for solving fuzzy matrix games based on certain duality for linear programming with fuzzy parameters. These works cannot provide membership functions of the gain-floor and loss-ceiling for the players even though they are very much desirable. The above methods were essentially the same as that of

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Campos [7] but certain modifications were made to help in having a better understanding of the same. Obviously, all the aforementioned methods are defuzzification ones based on suitable ranking functions, which are not easily chosen. In these methods, the obtained solutions closely depend on ranking functions and more or less involve in subjective factors such as attitudes and preference. On the other hand, these methods provided only defuzzification ones of the gain-floor and loss-ceiling for the players, whose membership functions cannot be explicitly obtained even though they are very much desirable. Moreover, it is not always sure that the obtained defuzzification gain-floor and loss-ceiling for the players are identical. This case is not rational and effective. From viewpoints of logic and the concept of matrix games with fuzzy payoffs, the gain-floor and loss-ceiling for the players should be fuzzy and identical since the expected payoffs are a linear combination of fuzzy payoffs and the matrix games are zero-sum.

Li [16] (with reference to [17]) proposed the two-level linear programming method for solving matrix games with payoffs of triangular fuzzy numbers, which was called as Li's model by Bector and Chandra [12] and Larbani [18]. In Li's model [16], the obtained gain-floor and loss-ceiling for the players are fuzzy and their membership functions can be explicitly obtained. However, Li's model cannot always guarantee that the gain-floor and loss-ceiling for the players are identical and hereby any fuzzy matrix game with payoffs of triangular fuzzy numbers has a fuzzy value, which is not rational since the matrix game is zero-sum. As far as we know, there is no method which can always guarantee that the gain-floor and loss-ceiling for the players are identical and hereby the matrix game with fuzzy payoffs has a fuzzy value, whose membership functions can be explicitly obtained. In this chapter, we will focus on studying matrix games with payoffs of triangular fuzzy numbers. Selecting triangular fuzzy numbers to express fuzzy payoffs stems from the fact that in many management applications they provide a very convenient object for the representation of imprecision and uncertain information in payoffs. On the one hand, triangular fuzzy numbers allow the modeling of a wide class of fuzzy numbers. Intervals and real numbers are special cases of triangular fuzzy numbers. On the other hand, triangular fuzzy numbers are easily extended to trapezoidal fuzzy numbers. Using triangular fuzzy numbers, we also have the freedom of being or not being symmetric. Another positive feature of the triangular fuzzy numbers is the ease of acquiring the necessary parameters. An additional consideration in using the triangular fuzzy number is the ease with which it can be manipulated in the context of the application.

In this chapter, we will propose some important concepts of solutions of matrix games with payoffs of triangular fuzzy numbers and develop auxiliary linear programming models and methods for solving matrix games with payoffs of triangular fuzzy numbers. Stated as earlier, it is easy to see that some linear programming models and methods proposed in this chapter are easily extended to establish those for matrix games with payoffs of trapezoidal fuzzy numbers.

2.2 Triangular Fuzzy Numbers and Alfa-Cut Sets

A fuzzy number $\tilde{a}$ with the membership function $\mu_{\tilde{a}}(x)$ is a special fuzzy subset of the real number set R, which satisfies the following two conditions [3]:

1. there exists at least a real number $x_0 \in \mathrm{R}$ so that $\mu_{\tilde{a}}(x_0) = 1$;
2. the membership function $\mu_{\tilde{a}}(x)$ is left and right continuous, depicted as in Fig. 2.1.

In the following, we mainly review a special and an important forms of fuzzy numbers: triangular fuzzy numbers.

Triangular fuzzy numbers are a special case of fuzzy numbers. A triangular fuzzy number $\tilde{a} = (a^l, a^m, a^r)$ is a special fuzzy number [3], whose membership function is given as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a^l}{a^m-a^l} & \text{if } a^l \leq x < a^m \\ 1 & \text{if } x = a^m \\ \frac{a^r-x}{a^r-a^m} & \text{if } a^m < x \leq a^r \\ 0 & \text{else,} \end{cases} \tag{2.1}$$

where a^m is the mean of $\tilde{a}$, a^l and a^r are the lower and upper limits (bounds) of $\tilde{a}$, respectively, depicted as in Fig. 2.2. The set of triangular fuzzy numbers is denoted by $T(\mathrm{R})$.

Obviously, if $a^l = a^m = a^r$, then the triangular fuzzy number $\tilde{a} = (a^l, a^m, a^r)$ is reduced to a real number. Conversely, a real number is easily rewritten as a

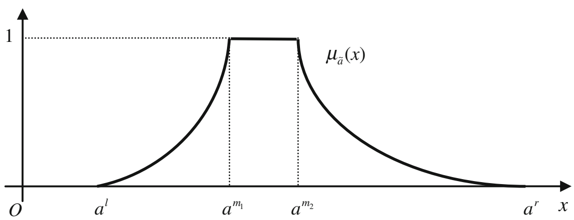


Fig. 2.1 A fuzzy number

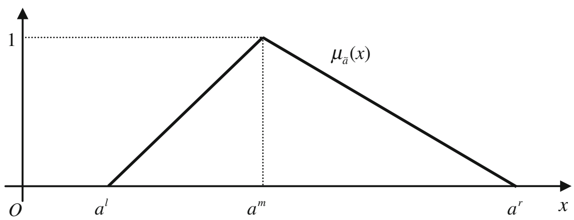


Fig. 2.2 A triangular fuzzy number

triangular fuzzy number. Thus, the triangular fuzzy number can be flexible to represent various semantics of uncertainty such as ill-quantity [5].

If $a^l \geq 0$ and $a^r > 0$, then $\tilde{a} = (a^l, a^m, a^r)$ is called a non-negative triangular fuzzy number, denoted by $\tilde{a} \geq 0$. If $a^l > 0$, then $\tilde{a}$ is called a positive triangular fuzzy number, denoted by $\tilde{a} > 0$. Conversely, if $a^r \leq 0$ and $a^l < 0$, then $\tilde{a}$ is called a non-positive triangular fuzzy number, denoted by $\tilde{a} \leq 0$. If $a^r < 0$, then $\tilde{a}$ is called a negative triangular fuzzy number, denoted by $\tilde{a} < 0$.

Let $\tilde{a} = (a^l, a^m, a^r)$ and $\tilde{b} = (b^l, b^m, b^r)$ be two triangular fuzzy numbers. Then, their arithmetical operations can be expressed as follows:

$$\tilde{a} + \tilde{b} = (a^l + b^l, a^m + b^m, a^r + b^r) \tag{2.2}$$

and

$$\lambda\tilde{a} = \begin{cases} (\lambda a^l, \lambda a^m, \lambda a^r) & \text{if } \lambda \geq 0 \\ (\lambda a^r, \lambda a^m, \lambda a^l) & \text{if } \lambda < 0, \end{cases} \tag{2.3}$$

where $\lambda \in \mathrm{R}$ is a real number.

A α-cut set of the triangular fuzzy number $\tilde{a} = (a^l, a^m, a^r)$ is defined as $\tilde{a}(\alpha) = \{x | \mu_{\tilde{a}}(x) \geq \alpha\}$, where $\alpha \in [0, 1]$. Thus, for any $\alpha \in [0, 1]$, we can obtain a α-cut set of the triangular fuzzy number $\tilde{a}$, which is an interval, denoted by $\tilde{a}(\alpha) = [a^L(\alpha), a^R(\alpha)]$. It is easily derived from Eq. (2.1) that

$$a^L(\alpha) = \alpha a^m + (1 - \alpha)a^l$$

and

$$a^R(\alpha) = \alpha a^m + (1 - \alpha)a^r.$$

In particular, we have

$$\tilde{a}(1) = [a^L(1), a^R(1)] = [a^m, a^m] = a^m$$

and

$$\tilde{a}(0) = [a^L(0), a^R(0)] = [a^l, a^r].$$

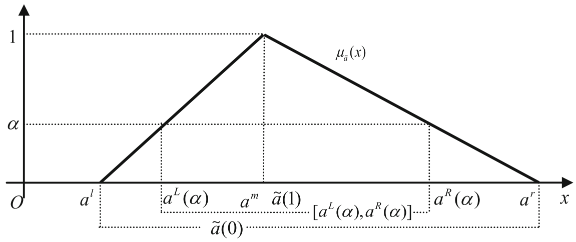
According to the operations over intervals [19], we can easily have:

$$[a^L(\alpha), a^R(\alpha)] = \alpha[a^m, a^m] + (1 - \alpha)[a^l, a^r] = \alpha\tilde{a}(1) + (1 - \alpha)\tilde{a}(0), \tag{2.4}$$

which means that any α-cut set of an arbitrary triangular fuzzy number can be directly obtained from its 1-cut set and 0-cut set, depicted as in Fig. 2.3.

According to the representation theorem for the fuzzy set [5], using Eq. (2.4), any triangular fuzzy number $\tilde{a} = (a^l, a^m, a^r)$ can be expressed as follows:

Fig. 2.3 α-cut sets of a triangular fuzzy number



$$\tilde{a} = \bigcup_{\alpha \in [0,1]} \{\alpha \otimes \tilde{a}(\alpha)\} = \bigcup_{\alpha \in [0,1]} \{\alpha \otimes [\alpha \tilde{a}(1) + (1-\alpha)\tilde{a}(0)]\}, \tag{2.5}$$

where $\alpha \otimes \tilde{a}(\alpha)$ is defined as a fuzzy set, whose membership function is given as follows:

$$\mu_{\alpha \otimes \tilde{a}(\alpha)}(x) = \begin{cases} \alpha & \text{if } x \in \tilde{a}(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

Equation (2.5) means that any triangular fuzzy number can be directly constructed through using its 1-cut set and 0-cut set.

From the aforementioned discussion, we summarize the conclusion as in Theorem 2.1, which will be used to construct the fuzzy values of matrix games with payoffs of triangular fuzzy numbers.

Theorem 2.1 *Any triangular fuzzy number and its α-cuts have the relations (1) and (2) as follows:*

1. *Any α-cut of a triangular fuzzy number can be directly obtained from both its 1-cut and 0-cut;*
2. *Any triangular fuzzy number can be directly constructed by using both its 1-cut and 0-cut.*

Proof According to the concept of α-cuts of triangular fuzzy numbers and the representation theorem for the fuzzy set, it is easy to prove that (1) and (2) of Theorem 2.1 are valid (omitted).

2.3 Fuzzy Multi-Objective Programming Models of Matrix Games with Payoffs of Triangular Fuzzy Numbers

2.3.1 Order Relations of Triangular Fuzzy Numbers

In contrast with the intervals' ranking or order relation as stated in Sects. 1.3 and 1.4, it is very difficult to rank (or compare) fuzzy numbers. Ramik and Rimanek [20] gave the definition of the order relation "$\tilde{\leq}$" for general fuzzy numbers. In this section, the order relations "$\tilde{\leq}$" and "$\tilde{\geq}$" are used only for triangular fuzzy numbers, not for general fuzzy numbers as stated in Sect. 2.2. To be more precisely, we give the meaning of the order relations "$\tilde{\leq}$" and "$\tilde{\geq}$" for the triangular fuzzy numbers in Definition 2.1 as follows.

Definition 2.1 Let $\tilde{a} = (a^l, a^m, a^r)$ and $\tilde{b} = (b^l, b^m, b^r)$ be two triangular fuzzy numbers. Then, $\tilde{a} \tilde{\leq} \tilde{b}$ if and only if $a^l \leq b^l$, $a \leq b$, and $a^r \leq b^r$. Similarly, $\tilde{a} \tilde{\geq} \tilde{b}$ if and only if $a^l \geq b^l$, $a \geq b$, and $a^r \geq b^r$.

The validity of Definition 2.1 may be discussed in a similar way to that of fuzzy numbers [20].

"$\tilde{\leq}$" and "$\tilde{\geq}$" are fuzzy versions of the order relations "$\leq$" and "$\geq$" in the three-dimension Euclidean space R^3, and have the linguistic interpretation "essentially less than or equal to" and "essentially greater than or equal to", respectively.

Analogously, $\tilde{a} \tilde{<} \tilde{b}$ if and only if $\tilde{a} \tilde{\leq} \tilde{b}$ and $\tilde{a} \neq \tilde{b}$. $\tilde{a} \tilde{>} \tilde{b}$ if and only if $\tilde{a} \tilde{\geq} \tilde{b}$ and $\tilde{a} \neq \tilde{b}$.

From Definition 2.1, a triangular fuzzy number $\tilde{a} \in T(\mathrm{R})$ may be regarded as a three-dimension vector and the order relations "$\tilde{\leq}$" and "$\tilde{\geq}$" are similar to those in the three-dimension Euclidean space R^3. Thus, the definition of maximizing and minimizing triangular fuzzy numbers can be given as follows.

Definition 2.2 Let $\tilde{a} = (a^l, a^m, a^r)$ be any triangular fuzzy number. A maximization problem of triangular fuzzy numbers is expressed as follows:

$$\max\{\tilde{a} | \tilde{a} \in \Omega_3 \cap T(\mathrm{R})\},$$

which is equivalent to the multi-objective mathematical programming model as follows:

$$\begin{aligned}
&\max\{a^l\} \\
&\max\{a^m\} \\
&\max\{a^r\} \\
&\text{s.t.} \begin{cases} \tilde{a} \in \Omega_3 \\ a^l \leq a^m \leq a^r \\ a^l, a^m, \text{ and } a^r \text{ unrestricted in sign,} \end{cases}
\end{aligned}$$

where $T(\mathrm{R})$ is the set of triangular fuzzy numbers as stated in Sect. 2.2, Ω_3 is the set of constraints in which the variable $\tilde{a}$ should be satisfied according to requirements in the real situation.

Definition 2.3 Let $\tilde{a} = (a^l, a^m, a^r)$ be any triangular fuzzy number. A minimization problem of triangular fuzzy numbers is described as follows:

$$\min\{\tilde{a}|\tilde{a} \in \Omega_4 \cap T(\mathrm{R})\},$$

which is equivalent to the multi-objective mathematical programming model as follows:

$$\begin{aligned}
&\min\{a^l\} \\
&\min\{a^m\} \\
&\min\{a^r\} \\
&\text{s.t.} \begin{cases} \tilde{a} \in \Omega_4 \\ a^l \le a^m \le a^r \\ a^l, a^m, \text{ and } a^r \text{ unrestricted in sign,} \end{cases}
\end{aligned}$$

where Ω_4 is the set of constraints in which the variable $\tilde{a}$ should be satisfied according to requirements in the real situation.

Definitions 2.2 and 2.3 can be used to transform corresponding fuzzy optimization problems of matrix games with payoffs of triangular fuzzy numbers into multi-objective linear programming models, which may be solved by using the existing multi-objective programming methods [21, 22].

2.3.2 *Concepts of Solutions of Matrix Games with Payoffs of Triangular Fuzzy Numbers*

Let us consider matrix games with payoffs of triangular fuzzy numbers, where the sets of pure strategies and the sets of mixed strategies for the players I and II respectively are S_1, S_2, Y, and Z defined as in Sect. 1.2. Assume that the payoff matrix of the player I is given as follows:

$$\tilde{\boldsymbol{A}} = (\tilde{a}_{ij})_{m\times n} = \begin{array}{c} \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{array} \begin{array}{c} \begin{array}{cccc} \beta_1 & \beta_2 & \cdots & \beta_n \end{array} \\ \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mn} \end{pmatrix} \end{array},$$

where $\tilde{a}_{ij} = (a^l_{ij}, a^m_{ij}, a^r_{ij})$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are triangular fuzzy numbers defined as in Sect. 2.2. Then, a matrix game with payoffs of triangular fuzzy numbers is expressed with $\tilde{\boldsymbol{A}}$ for short.

According to Eqs. (2.2) and (2.3), the fuzzy expected payoff (or value) of the player I can be computed as follows:

$$\tilde{E}(\tilde{\boldsymbol{A}}) = \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{A}}\boldsymbol{z} = \sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{a}_{ij}y_i z_j = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{l}y_i z_j, \sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{m}y_i z_j, \sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{r}y_i z_j\right),$$

which is a triangular fuzzy number.

As the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers is zero-sum, according to Eq. (2.3), the fuzzy expected payoff of the player II is equal to

$$\begin{aligned}\tilde{E}(-\tilde{\boldsymbol{A}}) &= \boldsymbol{y}^{\mathrm{T}}(-\tilde{\boldsymbol{A}})\boldsymbol{z} = \sum_{i=1}^{m}\sum_{j=1}^{n}(-\tilde{a}_{ij})y_i z_j \\ &= \left(-\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{r}y_i z_j, -\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{m}y_i z_j, -\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{l}y_i z_j\right),\end{aligned}$$

which is also a triangular fuzzy number. Thus, in general, the player I's gain-floor and the player II's loss-ceiling should be triangular fuzzy numbers, denoted by $\tilde{\upsilon} = (\upsilon^l, \upsilon^m, \upsilon^r)$ and $\tilde{\omega} = (\omega^l, \omega^m, \omega^r)$, respectively.

Since the fuzzy expected payoffs of the players and the player I's gain-floor and the player II's loss-ceiling are triangular fuzzy numbers, thus according to Definitions 2.2 and 2.3, the concept of solutions of matrix games with payoffs of triangular fuzzy numbers may be given by using that of the Pareto optimal solution as follows. Bector et al. [13, 14] firstly introduced the notion of reasonable solutions of fuzzy matrix games, which is a generalization of that of fuzzy matrix games [23].

Definition 2.4 Let $\tilde{\upsilon} = (\upsilon^l, \upsilon^m, \upsilon^r)$ and $\tilde{\omega} = (\omega^l, \omega^m, \omega^r)$ be triangular fuzzy numbers. Assume that there exist mixed strategies $\boldsymbol{y}^* \in Y$ and $\boldsymbol{z}^* \in Z$. Then, $(\boldsymbol{y}^*, \boldsymbol{z}^*, \tilde{\upsilon}, \tilde{\omega})$ is called a reasonable solution of the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers if it satisfies both the following conditions:

1. $\boldsymbol{y}^{*\mathrm{T}}\tilde{\boldsymbol{A}}\boldsymbol{z} \tilde{\geq} \tilde{\upsilon}$
 and
2. $\boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{A}}\boldsymbol{z}^* \tilde{\leq} \tilde{\omega}$
 for any $\boldsymbol{z} \in Z$ and $\boldsymbol{y} \in Y$.

If $(\boldsymbol{y}^*, \boldsymbol{z}^*, \tilde{\upsilon}, \tilde{\omega})$ is a reasonable solution of the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers, then $\tilde{\upsilon}$ and $\tilde{\omega}$ are called reasonable values for the players I and II, $\boldsymbol{y}^*$ and $\boldsymbol{z}^*$ are called reasonable (mixed) strategies for the players I and II, respectively.

The sets of all reasonable values $\tilde{\upsilon}$ and $\tilde{\omega}$ for the players I and II are denoted by U and W, respectively.

As stated earlier, Definition 2.4 only gives the notion of reasonable solutions of matrix games with payoffs of triangular fuzzy numbers rather than the notion of optimal solutions. Thus, we give the concept of solutions of matrix games with payoffs of triangular fuzzy numbers as in the following Definition 2.5.

Definition 2.5 Assume that there exist $\tilde{\upsilon}^* \in U$ and $\tilde{\omega} \in W$. If there do not exist any $\tilde{\upsilon} \in U$ ($\tilde{\upsilon} \neq \tilde{\upsilon}^*$) and $\tilde{\omega} \in W$ ($\tilde{\omega} \neq \tilde{\omega}^*$) so that

1. $\tilde{\upsilon}^* \tilde{\leq} \tilde{\upsilon}$
and
2. $\tilde{\omega}^* \tilde{\geq} \tilde{\omega}$,

then, $(\boldsymbol{y}^*, \boldsymbol{z}^*, \tilde{\upsilon}^*, \tilde{\omega}^*)$ is called a solution of the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers, $\boldsymbol{y}^*$ and $\boldsymbol{z}^*$ are called a maximin (mixed) strategy and a minimax (mixed) strategy for the players I and II, $\tilde{\upsilon}^*$ and $\tilde{\omega}^*$ are called the player I's gain-floor and the player II's loss-ceiling (or fuzzy values for the players I and II), respectively.

Let

$$\tilde{V}^* = \tilde{\upsilon}^* \wedge \tilde{\omega}^*$$

with the membership function

$$\mu_{\tilde{V}^*}(x) = \min_x\{\mu_{\tilde{\upsilon}^*}(x), \mu_{\tilde{\omega}^*}(x)\}.$$

Then, $\tilde{V}^*$ is called a fuzzy equilibrium value of the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers, depicted as in Fig. 2.4.

It is easy to see from Fig. 2.4 that a fuzzy value $\tilde{V}^*$ of the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers must not be always a (normal) triangular fuzzy number.

2.3.3 *Fuzzy Linear Programming Method of Matrix Games with Payoffs of Triangular Fuzzy Numbers*

According to Definitions 2.4 and 2.5, the maximin (mixed) strategy $\boldsymbol{y}^* \in Y$ and gain-floor $\tilde{\upsilon}^*$ for the player I and the minimax (mixed) strategy $\boldsymbol{z}^* \in Z$ and loss-ceiling $\tilde{\omega}^*$ for the player II can be generated by solving the fuzzy mathematical programming models:

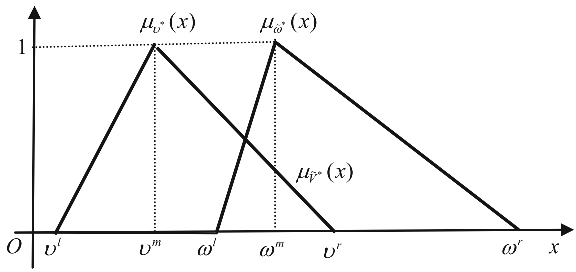


Fig. 2.4 A fuzzy equilibrium value $\tilde{V}^*$

$$\max\{\tilde{\upsilon}\}$$
$$\text{s.t.}\begin{cases} \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{A}}\boldsymbol{z} \tilde{\geq} \tilde{\upsilon} \quad \text{for all } \boldsymbol{z} \in Z \\ \boldsymbol{y} \in Y \\ \tilde{\upsilon} \in T(\mathrm{R}) \\ \tilde{\upsilon} \text{ unrestricted in sign} \end{cases} \tag{2.6}$$

and

$$\min\{\tilde{\omega}\}$$
$$\text{s.t.}\begin{cases} \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{A}}\boldsymbol{z} \tilde{\leq} \tilde{\omega} \quad \text{for all } \boldsymbol{y} \in Y \\ \boldsymbol{z} \in Z \\ \tilde{\omega} \in T(\mathrm{R}) \\ \tilde{\omega} \text{ unrestricted in sign,} \end{cases} \tag{2.7}$$

respectively.

It makes sense to consider only the extreme points of the sets Y and Z in the constraints of Eqs. (2.6) and (2.7) since "$\tilde{\leq}$" and "$\tilde{\geq}$" preserve the ranking order when triangular fuzzy numbers are multiplied by positive scalars according to Eq. (2.3) and Definition 2.1. Then, Eqs. (2.6) and (2.7) can be converted into the fuzzy mathematical programming models as follows:

$$\max\{\tilde{\upsilon}\}$$
$$\text{s.t.}\begin{cases} \sum_{i=1}^{m} \tilde{a}_{ij} y_i \tilde{\geq} \tilde{\upsilon} \quad (j = 1, 2, \ldots, n) \\ \sum_{i=1}^{m} y_i = 1 \\ y_i \geq 0 \quad (i = 1, 2, \ldots, m) \\ \tilde{\upsilon} \in T(\mathrm{R}) \\ \tilde{\upsilon} \text{ unrestricted in sign} \end{cases} \tag{2.8}$$

and

$$\min\{\tilde{\omega}\}$$
$$\text{s.t.}\begin{cases} \sum_{j=1}^{n} \tilde{a}_{ij} z_j \tilde{\leq} \tilde{\omega} \quad (i = 1, 2, \ldots, m) \\ \sum_{j=1}^{n} z_j = 1 \\ z_j \geq 0 \quad (j = 1, 2, \ldots, n) \\ \tilde{\omega} \in T(\mathrm{R}) \\ \tilde{\omega} \text{ unrestricted in sign,} \end{cases} \tag{2.9}$$

respectively, where $\tilde{\upsilon}$ and $\tilde{\omega}$ are fuzzy variables, y_i $(i = 1, 2, \ldots, m)$ and z_j $(j = 1, 2, \ldots, n)$ are decision variables.

According to the operations of triangular fuzzy numbers, in general, we can draw an important conclusion, which is summarized as in Theorem 2.2.

Theorem 2.2 *Assume that* $(\boldsymbol{y}^*, \tilde{\upsilon}^*)$ *and* $(\boldsymbol{z}^*, \tilde{\omega}^*)$ *are optimal solutions of Eqs.* (2.8) *and* (2.9)*, respectively. Then,* $\tilde{\upsilon}^*$ *and* $\tilde{\omega}^*$ *are triangular fuzzy numbers and* $\tilde{\upsilon}^* \tilde{\leq} \tilde{\omega}^*$.

Proof Due to the assumption that $(\boldsymbol{y}^*, \tilde{\upsilon}^*)$ and $(\boldsymbol{z}^*, \tilde{\omega}^*)$ respectively are optimal solutions of Eqs. (2.8) and (2.9), then according to Eqs. (2.2) and (2.3), it follows that $\tilde{\upsilon}^*$ and $\tilde{\omega}^*$ are triangular fuzzy numbers. Furthermore, it follows from Eqs. (2.8) and (2.9) that

$$\begin{aligned} \tilde{\upsilon}^* &= \sum_{j=1}^{n} \tilde{\upsilon}^* z_j^* \tilde{\leq} \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} \tilde{a}_{ij} y_i^*\Big) z_j^* \\ &= \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \tilde{a}_{ij} z_j^*\Big) y_i^* \tilde{\leq} \sum_{i=1}^{m} \tilde{\omega}^* y_i^* = \tilde{\omega}^*, \end{aligned}$$

i.e., $\tilde{\upsilon}^* \tilde{\leq} \tilde{\omega}^*$. Thus, we have finished the proof of Theorem 2.2.

Theorem 2.2 means that the player I's gain-floor "essentially cannot exceed" the player II's loss-ceiling in the sense of Definition 2.1.

Equations (2.8) and (2.9) are general fuzzy mathematical programming models which may involve in different solutions [24, 25]. But in this section, the fuzzy optimization is made in the sense of Definition 2.2 or Definition 2.3. In the following, we will focus on studying the solving method and procedure of Eqs. (2.8) and (2.9).

According to Definitions 2.1–2.3, Eqs. (2.8) and (2.9) can be converted into the multi-objective mathematical programming models as follows:

$$\begin{aligned} &\max\{\upsilon^l\} \\ &\max\{\upsilon^m\} \\ &\max\{\upsilon^r\} \\ &\text{s.t.} \begin{cases} \sum_{i=1}^{m} a_{ij}^l y_i \geq \upsilon^l \quad (j = 1, 2, \ldots, n) \\ \sum_{i=1}^{m} a_{ij}^m y_i \geq \upsilon^m \quad (j = 1, 2, \ldots, n) \\ \sum_{i=1}^{m} a_{ij}^r y_i \geq \upsilon^r \quad (j = 1, 2, \ldots, n) \\ \upsilon^l \leq \upsilon^m \leq \upsilon^r \\ \sum_{i=1}^{m} y_i = 1 \\ y_i \geq 0 \quad (i = 1, 2, \ldots, m) \\ \upsilon^l, \upsilon^m, \ \text{ and } \ \upsilon^r \text{ unrestricted in sign} \end{cases} \end{aligned} \tag{2.10}$$

and

$$\min\{\omega^l\}$$
$$\min\{\omega^m\}$$
$$\min\{\omega^r\}$$
$$\text{s.t.}\begin{cases}\sum_{j=1}^{n} a_{ij}^l z_j \le \omega^l \quad (i=1,2,\ldots,m)\\ \sum_{j=1}^{n} a_{ij}^m z_j \le \omega^m \quad (i=1,2,\ldots,m)\\ \sum_{j=1}^{n} a_{ij}^r z_j \le \omega^r \quad (i=1,2,\ldots,m)\\ \omega^l \le \omega^m \le \omega^r\\ \sum_{j=1}^{n} z_j = 1\\ z_j \ge 0 \quad (j=1,2,\ldots,n)\\ \omega^l, \omega^m, \text{ and } \omega^r \text{ unrestricted in sign,}\end{cases} \tag{2.11}$$

respectively.

For the above multi-objective mathematical programming models, there are few standard ways of defining a solution. Normally, the concept of Pareto optimal solutions/efficient solutions is commonly-used [4, 21, 22]. There exist several solution methods for them such as utility theory, goal programming, fuzzy programming, and interactive approaches. However, in the following, we develop a fuzzy linear programming method based on Zimmermann's fuzzy programming method [24] with our normalization process.

Firstly, we can compute the positive ideal solution and negative ideal solution of Eq. (2.10) through solving three linear programming models with different objective functions, respectively. Specifically, using the simplex method of linear programming, we solve the linear programming model as follows:

$$\max\{\upsilon^l\}$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^{m} a_{ij}^l y_i \ge \upsilon^l \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^{m} a_{ij}^m y_i \ge \upsilon^m \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^{m} a_{ij}^r y_i \ge \upsilon^r \quad (j=1,2,\ldots,n)\\ \upsilon^l \le \upsilon^m \le \upsilon^r\\ \sum_{i=1}^{m} y_i = 1\\ y_i \ge 0 \quad (i=1,2,\ldots,m)\\ \upsilon^l, \upsilon^m, \text{ and } \upsilon^r \text{ unrestricted in sign,}\end{cases}$$

denoted its optimal solution by $(\boldsymbol{y}^{1+}, \upsilon^{l1+}, \upsilon^{m1+}, \upsilon^{r1+})$.

Analogously, using the simplex method of linear programming, we solve the linear programming model as follows:

$$\max\{\upsilon^m\}$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^m a_{ij}^l y_i \ge \upsilon^l \quad (j = 1, 2, \ldots, n)\\ \sum_{i=1}^m a_{ij}^m y_i \ge \upsilon^m \quad (j = 1, 2, \ldots, n)\\ \sum_{i=1}^m a_{ij}^r y_i \ge \upsilon^r \quad (j = 1, 2, \ldots, n)\\ \upsilon^l \le \upsilon^m \le \upsilon^r\\ \sum_{i=1}^m y_i = 1\\ y_i \ge 0 \quad (i = 1, 2, \ldots, m)\\ \upsilon^l, \upsilon^m, \text{ and } \upsilon^r \text{ unrestricted in sign,}\end{cases}$$

denoted its optimal solution by $(\boldsymbol{y}^{2+}, \upsilon^{l2+}, \upsilon^{m2+}, \upsilon^{r2+})$. We solve the linear programming model as follows:

$$\max\{\upsilon^r\}$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^m a_{ij}^l y_i \ge \upsilon^l \quad (j = 1, 2, \ldots, n)\\ \sum_{i=1}^m a_{ij}^m y_i \ge \upsilon^m \quad (j = 1, 2, \ldots, n)\\ \sum_{i=1}^m a_{ij}^r y_i \ge \upsilon^r \quad (j = 1, 2, \ldots, n)\\ \upsilon^l \le \upsilon^m \le \upsilon^r\\ \sum_{i=1}^m y_i = 1\\ y_i \ge 0 \quad (i = 1, 2, \ldots, m)\\ \upsilon^l, \upsilon^m, \text{ and } \upsilon^r \text{ unrestricted in sign,}\end{cases}$$

denoted its optimal solution by $(\boldsymbol{y}^{3+}, \upsilon^{l3+}, \upsilon^{m3+}, \upsilon^{r3+})$.

Thus, the positive ideal solution of Eq. (2.10) can be obtained as $(\upsilon^{l+}, \upsilon^{m+}, \upsilon^{r+}) = (\upsilon^{l1+}, \upsilon^{m2+}, \upsilon^{r3+})$. The negative ideal solution of Eq. (2.10) can be defined as follows:

$$(\upsilon^{l-}, \upsilon^{m-}, \upsilon^{r-}) = (\min\{\upsilon^{lt+} | t = 1, 2, 3\},$$
$$\min\{\upsilon^{mt+} | t = 1, 2, 3\}, \min\{\upsilon^{r3+} | t = 1, 2, 3\}).$$

Hereby, the relative membership functions of the three objective functions in Eq. (2.10) can be defined as follows:

$$\eta_{\upsilon^l}(\upsilon^l) = \begin{cases} 1 & \text{if } \upsilon^l \geq \upsilon^{l+} \\ \dfrac{\upsilon^l - \upsilon^{l-}}{\upsilon^{l+} - \upsilon^{l-}} & \text{if } \upsilon^{l-} \leq \upsilon^l < \upsilon^{l+} \\ 0 & \text{if } \upsilon^l < \upsilon^{l-}, \end{cases}$$

$$\eta_{\upsilon^m}(\upsilon^m) = \begin{cases} 1 & \text{if } \upsilon^m \geq \upsilon^{m+} \\ \dfrac{\upsilon^m - \upsilon^{m-}}{\upsilon^{m+} - \upsilon^{m-}} & \text{if } \upsilon^{m-} \leq \upsilon^m < \upsilon^{m+} \\ 0 & \text{if } \upsilon^m < \upsilon^{m-} \end{cases}$$

and

$$\eta_{\upsilon^r}(\upsilon^r) = \begin{cases} 1 & \text{if } \upsilon^r \geq \upsilon^{r+} \\ \dfrac{\upsilon^r - \upsilon^{r-}}{\upsilon^{r+} - \upsilon^{r-}} & \text{if } \upsilon^{r-} \leq \upsilon^r < \upsilon^{r+} \\ 0 & \text{if } \upsilon^r < \upsilon^{r-}, \end{cases}$$

respectively.

Using Zimmermann's fuzzy programming method [24], Eq. (2.10) can be converted into the linear programming model as follows:

$$\max\{\eta\}$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{m} a_{ij}^l y_i \geq \upsilon^l \quad (j = 1, 2, \ldots, n) \\ \sum_{i=1}^{m} a_{ij}^m y_i \geq \upsilon^m \quad (j = 1, 2, \ldots, n) \\ \sum_{i=1}^{m} a_{ij}^r y_i \geq \upsilon^r \quad (j = 1, 2, \ldots, n) \\ \upsilon^l - \upsilon^{l-} \geq (\upsilon^{l+} - \upsilon^{l-})\eta \\ \upsilon^m - \upsilon^{m-} \geq (\upsilon^{m+} - \upsilon^{m-})\eta \\ \upsilon^r - \upsilon^{r-} \geq (\upsilon^{r+} - \upsilon^{r-})\eta \\ \upsilon^l \leq \upsilon^m \leq \upsilon^r \\ \sum_{i=1}^{m} y_i = 1 \\ 0 \leq \eta \leq 1 \\ y_i \geq 0 \quad (i = 1, 2, \ldots, m) \\ \upsilon^l, \upsilon^m, \text{ and } \upsilon^r \text{ unrestricted in sign,} \end{cases} \tag{2.12}$$

where $\eta = \min\{\eta_{\upsilon^l}(\upsilon^l), \eta_{\upsilon^m}(\upsilon^m), \eta_{\upsilon^r}(\upsilon^r)\}$.

Solving Eq. (2.12) by using the simplex method of linear programming, we can obtain the optimal or maximin (mixed) strategy $\boldsymbol{y}^*$ and gain-floor $\tilde{\upsilon}^*$ for the player I.

In the same way to the above consideration of Eq. (2.10), according to Eq. (2.11), using the simplex method of linear programming, we can solve the linear programming model as follows:

$$
\min\{\omega^l\}
$$
$$
\text{s.t.}\begin{cases}
\sum_{j=1}^{n} a_{ij}^l z_j \le \omega^l \quad (i=1,2,\ldots,m) \\
\sum_{j=1}^{n} a_{ij}^m z_j \le \omega^m \quad (i=1,2,\ldots,m) \\
\sum_{j=1}^{n} a_{ij}^r z_j \le \omega^r \quad (i=1,2,\ldots,m) \\
\omega^l \le \omega^m \le \omega^r \\
\sum_{j=1}^{n} z_j = 1 \\
z_j \ge 0 \quad (j=1,2,\ldots,n) \\
\omega^l, \omega^m, \text{ and } \omega^r \text{ unrestricted in sign,}
\end{cases}
$$

denoted its optimal solution by $(\boldsymbol{z}^{1+}, \omega^{l1+}, \omega^{m1+}, \omega^{r1+})$. Analogously, we can solve the linear programming model as follows:

$$
\min\{\omega^m\}
$$
$$
\text{s.t.}\begin{cases}
\sum_{j=1}^{n} a_{ij}^l z_j \le \omega^l \quad (i=1,2,\ldots,m) \\
\sum_{j=1}^{n} a_{ij}^m z_j \le \omega^m \quad (i=1,2,\ldots,m) \\
\sum_{j=1}^{n} a_{ij}^r z_j \le \omega^r \quad (i=1,2,\ldots,m) \\
\omega^l \le \omega^m \le \omega^r \\
\sum_{j=1}^{n} z_j = 1 \\
z_j \ge 0 \quad (j=1,2,\ldots,n) \\
\omega^l, \omega^m, \text{ and } \omega^r \text{ unrestricted in sign,}
\end{cases}
$$

denoted its optimal solution by $(\boldsymbol{z}^{2+}, \omega^{l2+}, \omega^{m2+}, \omega^{r2+})$. We can solve the linear programming model as follows:

$$\min\{\omega^r\}$$
$$\text{s.t.}\begin{cases} \sum_{j=1}^{n} a_{ij}^{l} z_j \le \omega^l \quad (i=1,2,\ldots,m) \\ \sum_{j=1}^{n} a_{ij}^{m} z_j \le \omega^m \quad (i=1,2,\ldots,m) \\ \sum_{j=1}^{n} a_{ij}^{r} z_j \le \omega^r \quad (i=1,2,\ldots,m) \\ \omega^l \le \omega^m \le \omega^r \\ \sum_{j=1}^{n} z_j = 1 \\ z_j \ge 0 \quad (j=1,2,\ldots,n) \\ \omega^l, \omega^m, \text{ and } \omega^r \text{ unrestricted in sign,} \end{cases}$$

denoted its optimal solution by $(\boldsymbol{z}^{3+}, \omega^{l3+}, \omega^{m3+}, \omega^{r3+})$.

Then, the positive ideal solution of Eq. (2.11) can be obtained as $(\omega^{l+}, \omega^{m+}, \omega^{r+}) = (\omega^{l1+}, \omega^{m2+}, \omega^{r3+})$. The negative ideal solution of Eq. (2.11) can be defined as follows:

$$\begin{aligned}(\omega^{l-}, \omega^{m-}, \omega^{r-}) = (&\max\{\omega^{lt+} | t=1,2,3\}, \\ &\max\{\omega^{mt+} | t=1,2,3\}, \max\{\omega^{r3+} | t=1,2,3\}).\end{aligned}$$

Hereby, the relative membership functions of the three objective functions in Eq. (2.11) can be defined as follows:

$$\rho_{\omega^l}(\omega^l) = \begin{cases} 1 & \text{if } \omega^l \le \omega^{l+} \\ \dfrac{\omega^l - \omega^{l+}}{\omega^{l-} - \omega^{l+}} & \text{if } \omega^{l+} < \omega^l \le \omega^{l-} \\ 0 & \text{if } \omega^l > \omega^{l-}, \end{cases}$$

$$\rho_{\omega^m}(\omega^m) = \begin{cases} 1 & \text{if } \omega^m \le \omega^{m+} \\ \dfrac{\omega^m - \omega^{m+}}{\omega^{m-} - \omega^{m+}} & \text{if } \omega^{m+} < \omega^m \le \omega^{m-} \\ 0 & \text{if } \omega^m > \omega^{m-} \end{cases}$$

and

$$\rho_{\omega^r}(\omega^r) = \begin{cases} 1 & \text{if } \quad \omega^r \le \omega^{r+} \\ \dfrac{\omega^r - \omega^{r+}}{\omega^{r-} - \omega^{r+}} & \text{if } \quad \omega^{r+} < \omega^r \le \omega^{r-} \\ 0 & \text{if } \quad \omega^r > \omega^{r-}, \end{cases}$$

respectively.

Using Zimmermann's fuzzy programming method [24], Eq. (2.11) can be converted into the linear programming model as follows:

$$
\begin{aligned}
&\max\{\rho\}\\
&\text{s.t.}\begin{cases}
\sum_{j=1}^{n} a_{ij}^{l} z_j \le \omega^{l} \quad (i=1,2,\ldots,m)\\
\sum_{j=1}^{n} a_{ij}^{m} z_j \le \omega^{m} \quad (i=1,2,\ldots,m)\\
\sum_{j=1}^{n} a_{ij}^{r} z_j \le \omega^{r} \quad (i=1,2,\ldots,m)\\
\omega^{l}-\omega^{l+} \ge (\omega^{l-}-\omega^{l+})\rho\\
\omega^{m}-\omega^{m+} \ge (\omega^{m-}-\omega^{m+})\rho\\
\omega^{r}-\omega^{r+} \ge (\omega^{r-}-\omega^{r+})\rho\\
\omega^{l} \le \omega^{m} \le \omega^{r}\\
\sum_{j=1}^{n} z_j = 1\\
0 \le \rho \le 1\\
z_j \ge 0 \quad (j=1,2,\ldots,n)\\
\omega^{l}, \omega^{m}, \text{ and } \omega^{r} \text{ unrestricted in sign,}
\end{cases}
\end{aligned}
\tag{2.13}
$$

where $\rho = \min\{\rho_{\omega^l}(\omega^l), \rho_{\omega^m}(\omega^m), \rho_{\omega^r}(\omega^r)\}$.

Solving Eq. (2.13) by using the simplex method of linear programming, we can obtain the optimal or minimax (mixed) strategy $\boldsymbol{z}^*$ and loss-ceiling $\tilde{\omega}^*$ for the player II.

Example 2.1 Let us consider a simple numerical example of matrix games with payoffs of triangular fuzzy numbers. Assume that the payoff matrix for the player I is given as follows:

$$
\tilde{\boldsymbol{A}}_1 = \begin{array}{c} \\ \delta_1 \\ \delta_2 \end{array}
\begin{array}{c}
\begin{array}{cc} \qquad\beta_1 \qquad\qquad & \qquad\beta_2 \end{array}\\
\begin{pmatrix} (18,20,23) & (-21,-18,-16) \\ (-33,-32,-27) & (38,40,43) \end{pmatrix}
\end{array}.
$$

According to Eqs. (2.12) and (2.13), we can construct two linear programming models for the players I and II, respectively. Using the simplex method of linear programming, we can easily obtain their optimal solutions whose components are given as follows:

$$
\begin{aligned}
\boldsymbol{y}_1^* &= (0.648, 0.352)^{\mathrm{T}},\\
\tilde{\upsilon}_1^* &= (-0.254, 1.715, 4.746),\\
\eta_1^* &= 0.501,\\
\boldsymbol{z}_1^* &= (0.534, 0.466)^{\mathrm{T}},\\
\tilde{\omega}_1^* &= (0.241, 2.303, 5.601)
\end{aligned}
$$

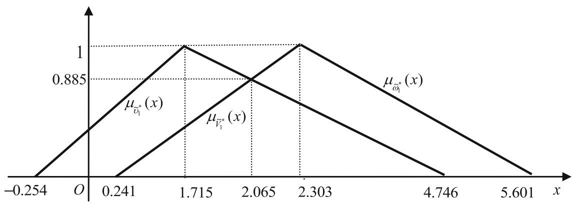


Fig. 2.5 The fuzzy equilibrium value $\tilde{V}_1^*$

and

$$\rho_1^* = 0.500,$$

respectively. Furthermore, we have

$$\mu_{\tilde{V}_1^*}(x) = \begin{cases} \dfrac{x - 0.241}{2.062} & \text{if } 0.241 \le x < 2.065 \\ 0.885 & \text{if } x = 2.065 \\ \dfrac{4.746 - x}{3.031} & \text{if } 2.065 < x \le 4.746 \\ 0 & \text{else.} \end{cases}$$

Therefore, there exists a fuzzy equilibrium value 2.065 with the possibility of 0.885. In other words, the fuzzy value of the matrix game $\tilde{\boldsymbol{A}}_1$ with payoffs of triangular fuzzy numbers is "around 2.065". Or the player I's minimum reward is 0.241 while his/her maximum reward is 4.746. The player I can win any intermediate value x between 0.241 and 4.746 with the possibility $\mu_{\tilde{V}_1^*}(x)$, depicted as in Fig. 2.5.

2.4 Two-Level Linear Programming Models of Matrix Games with Payoffs of Triangular Fuzzy Numbers

Stated as in Sect. 2.3, Eqs. (2.10) and (2.11) are multi-objective linear programming models, which may be solved by several methods [21, 22]. However, in this section, we develop a two-level linear programming method for solving Eqs. (2.10) and (2.11).

In Eq. (2.10), the three objective functions (i.e., υ^l, υ^m, and υ^r) should have different priority. In fact, the objective functions may be written as the triangular fuzzy number $\tilde{\upsilon} = (\upsilon^l, \upsilon^m, \upsilon^r)$, where υ^m is the mean (or center) of the triangular fuzzy number $\tilde{\upsilon}$, and υ^l and υ^r are lower and upper limits (or bounds) of the triangular fuzzy number $\tilde{\upsilon}$, respectively. The priority of the objective function υ^m

should be higher than that of both the objective functions υ^l and υ^r, and the priority of υ^l and υ^r may be identical because the priority of the mean of the triangular fuzzy number is much higher than that of its lower and upper limits according to the fuzzy sets [3, 4, 24]. Hence, Eq. (2.10) may be regarded as a two-level linear programming problem. Its first priority is given to the objective function υ^m. Its second priority is given to the objective functions υ^l and υ^r. Thus, solving Eq. (2.10) becomes solving the following linear programming models [i.e., Eqs. (2.14) and (2.15)] successively. To be more specific, we give its procedure as follows.

According to Eq. (2.10), the linear programming model in the first level is constructed as follows:

$$\max\{\upsilon^m\}$$
$$\text{s.t.}\begin{cases} \sum_{i=1}^{m} a_{ij}^l y_i \geq \upsilon^l \quad (j=1,2,\ldots,n) \\ \sum_{i=1}^{m} a_{ij}^m y_i \geq \upsilon^m \quad (j=1,2,\ldots,n) \\ \sum_{i=1}^{m} a_{ij}^r y_i \geq \upsilon^r \quad (j=1,2,\ldots,n) \\ \upsilon^l \leq \upsilon^m \leq \upsilon^r \\ \sum_{i=1}^{m} y_i = 1 \\ y_i \geq 0 \quad (i=1,2,\ldots,m) \\ \upsilon^l, \upsilon^m, \text{ and } \upsilon^r \text{ unrestricted in sign,} \end{cases} \tag{2.14}$$

where y_i $(i=1,2,\ldots,m)$, υ^l, υ^m, and υ^r are decision variables. Using the simplex method of linear programming, we can obtain its optimal solution by $(\boldsymbol{y}^*, \upsilon^{l0}, \upsilon^{m*}, \upsilon^{r0})$, where $\boldsymbol{y}^* = (y_1^*, y_2^*, \ldots, y_m^*)^{\mathrm{T}}$.

Combining with Eq. (2.10), the linear programming model in the second level is constructed as follows:

$$\max\{\upsilon^l\}$$
$$\max\{\upsilon^r\}$$
$$\text{s.t.}\begin{cases} \sum_{i=1}^{m} a_{ij}^l y_i^* \geq \upsilon^l \quad (j=1,2,\ldots,n) \\ \sum_{i=1}^{m} a_{ij}^r y_i^* \geq \upsilon^r \quad (j=1,2,\ldots,n) \\ \upsilon^l \geq \upsilon^{l0} \\ \upsilon^r \geq \upsilon^{r0} \\ \upsilon^l \text{ and } \upsilon^r \text{ unrestricted in sign,} \end{cases} \tag{2.15}$$

where υ^l and υ^r are decision variables.

In Eq. (2.15), adding the constraints $\upsilon^l \geq \upsilon^{l0}$ and $\upsilon^r \geq \upsilon^{r0}$ aim to improve the objective functions υ^l and υ^r, respectively. It is the real reason why the second-level linear programming model [i.e., Eq. (2.15)] is introduced after the first-level linear programming model [i.e., Eq. (2.14)].

It is easy to see from Eq. (2.15) that the constraints of the variable υ^l are independent of those of the variable υ^r. Therefore, Eq. (2.15) can be decompounded into the two linear programming models as follows:

$$\max\{\upsilon^l\}$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{m} a_{ij}^l y_i^* \geq \upsilon^l \quad (j = 1, 2, \ldots, n) \\ \upsilon^l \geq \upsilon^{l0} \\ \upsilon^l \text{ unrestricted in sign} \end{cases} \tag{2.16}$$

and

$$\max\{\upsilon^r\}$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{m} a_{ij}^l y_i^* \geq \upsilon^r \quad (j = 1, 2, \ldots, n) \\ \upsilon^r \geq \upsilon^{r0} \\ \upsilon^r \text{ unrestricted in sign.} \end{cases} \tag{2.17}$$

Solving Eqs. (2.16) and (2.17) by using the simplex method of linear programming, we can obtain their optimal solutions υ^{l*} and υ^{r*}, respectively.

It is not difficult to prove that $(\boldsymbol{y}^*, \tilde{\upsilon}^*)$ is a Pareto optimal solution of Eq. (2.10), where $\tilde{\upsilon}^* = (\upsilon^{l*}, \upsilon^{m*}, \upsilon^{r*})$ is a triangular fuzzy number. Thus, the optimal (or maximin) mixed strategy $\boldsymbol{y}^*$ and the gain-floor $\tilde{\upsilon}^*$ for the player I can be obtained.

In the same way to the above consideration of Eq. (2.10), the three objective functions ω^l, ω^m, and ω^r of Eq. (2.11) should have different priority. Namely, the priority of the objective function ω^m should be higher than that of both the objective functions ω^l, and ω^r, and the priority of ω^l and ω^r should be assumed to be identical in that ω^m, ω^l, and ω^r are the mean and the lower and upper limits of the triangular fuzzy number $\tilde{\omega} = (\omega^l, \omega^m, \omega^r)$, respectively. Thus, Eq. (2.11) may be regarded as a two-level linear programming problem. Its first priority is given to the objective function ω^m. Its second priority is given to the objective functions ω^l and ω^r. As a result, solving Eq. (2.11) turns into solving the following two linear programming models [i.e., Eqs. (2.18) and (2.19)] successively.

According to Eq. (2.11), the linear programming model in the first level is constructed as follows:

$$\min\{\omega^m\}$$
$$\text{s.t.}\begin{cases}\sum_{j=1}^{n} a_{ij}^{l} z_j \le \omega^l \quad (i = 1,2,\ldots,m)\\ \sum_{j=1}^{n} a_{ij}^{m} z_j \le \omega^m \quad (i = 1,2,\ldots,m)\\ \sum_{j=1}^{n} a_{ij}^{r} z_j \le \omega^r \quad (i = 1,2,\ldots,m)\\ \omega^l \le \omega^m \le \omega^r\\ \sum_{j=1}^{n} z_j = 1\\ z_j \ge 0 \quad (j = 1,2,\ldots,n)\\ \omega^l, \omega^m, \text{ and } \omega^r \text{ unrestricted in sign,}\end{cases} \tag{2.18}$$

where z_j $(j = 1,2,\ldots,n)$, ω^l, ω^m, and ω^r are decision variables. Solving Eq. (2.18) by using the simplex method of linear programming, we can easily obtain its optimal solution $(z^*, \omega^{l0}, \omega^{m*}, \omega^{r0})$, where $z^* = (z_1^*, z_2^*, \ldots, z_n^*)^{\mathrm{T}}$.

Combining with Eq. (2.11), the linear programming model in the second level is constructed as follows:

$$\min\{\omega^l\}$$
$$\min\{\omega^r\}$$
$$\text{s.t.}\begin{cases}\sum_{j=1}^{n} a_{ij}^{l} z_j^* \le \omega^l \quad (i = 1,2,\ldots,m)\\ \sum_{j=1}^{n} a_{ij}^{r} z_j^* \le \omega^r \quad (i = 1,2,\ldots,m)\\ \omega^l \le \omega^{l0}\\ \omega^r \le \omega^{r0}\\ \omega^l \text{ and } \omega^r \text{ unrestricted in sign,}\end{cases} \tag{2.19}$$

where ω^l and ω^r are decision variables.

Analogously, adding the constraints $\omega^l \le \omega^{l0}$ and $\omega^r \le \omega^{r0}$ in Eq. (2.19) aim to improve ω^l and ω^r, respectively.

It is easy to see from Eq. (2.19) that the constraints of the variable ω^l are independent of those of the variable ω^r. Therefore, Eq. (2.19) can be decompounded into the linear programming models as follows:

$$\min\{\omega^l\}$$
$$\text{s.t.}\begin{cases}\sum_{j=1}^{n} a_{ij}^l z_j^* \le \omega^l \quad (i = 1, 2, \ldots, m) \\ \omega^l \le \omega^{l0} \\ \omega^l \ \text{ unrestricted in sign}\end{cases} \tag{2.20}$$

and

$$\min\{\omega^r\}$$
$$\text{s.t.}\begin{cases}\sum_{j=1}^{n} a_{ij}^r z_j^* \le \omega^r \quad (i = 1, 2, \cdots, m) \\ \omega^r \le \omega^{r0} \\ \omega^r \text{ unrestricted in sign.}\end{cases} \tag{2.21}$$

Solving Eqs. (2.20) and (2.21) through using the simplex method of linear programming, we can easily obtain their solutions ω^{l*} and ω^{r*}, respectively.

It is not difficult to prove that $(\boldsymbol{z}^*, \tilde{\omega}^*)$ is a Pareto optimal solution of Eq. (2.11), where $\tilde{\omega}^* = (\omega^{l*}, \omega^{m*}, \omega^{r*})$ is a triangular fuzzy number. Thus, the optimal (or minimax) mixed strategy $\boldsymbol{z}^*$ and the loss-ceiling $\tilde{\omega}^*$ for the player II can be obtained.

Hence, $(\boldsymbol{y}^*, \boldsymbol{z}^*, \tilde{\upsilon}^*, \tilde{\omega}^*)^{\mathrm{T}}$ and $\tilde{V}^* = \tilde{\upsilon}^* \wedge \tilde{\omega}^*$ are a solution and a fuzzy equilibrium value of the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers, respectively.

Example 2.2 Let us consider a simple numerical example which is taken from Campos [7]. Suppose that the payoff matrix for the player I is given as follows:

$$\tilde{\boldsymbol{A}}_2 = \begin{array}{c} \\ \delta_1 \\ \delta_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \quad\beta_1 \quad\quad & \quad\quad\beta_2 \end{array} \\ \begin{pmatrix} (175, 180, 190) & (150, 156, 158) \\ (80, 90, 100) & (175, 180, 190) \end{pmatrix} \end{array},$$

where all elements of the above payoff matrix $\tilde{\boldsymbol{A}}_2$ are triangular fuzzy numbers.

According to Eq. (2.14), the linear programming model in the first level can be constructed as follows:

$$\max\{\upsilon^m\}$$
$$\text{s.t.}\begin{cases}175y_1 + 80y_2 \ge \upsilon^l \\ 150y_1 + 175y_2 \ge \upsilon^l \\ 180y_1 + 90y_2 \ge \upsilon^m \\ 156y_1 + 180y_2 \ge \upsilon^m \\ 190y_1 + 100y_2 \ge \upsilon^r \\ 158y_1 + 190y_2 \ge \upsilon^r \\ \upsilon^l \le \upsilon^m \le \upsilon^r \\ y_1 + y_2 = 1 \\ y_1 \ge 0, y_2 \ge 0 \\ \upsilon^l, \upsilon^m, \text{ and } \upsilon^r \text{ unrestricted in sign.}\end{cases}$$

Solving the above linear programming model by using the simplex method of linear programming, we can obtain its optimal solution $(\boldsymbol{y}^*, \upsilon^{l0}, \upsilon^{m*}, \upsilon^{r0})$, where $\boldsymbol{y}^* = (0.7895,\ 0.2105)^{\mathrm{T}}$, $\upsilon^{l0} = 61.398$, $\upsilon^{m*} = 161.05$, and $\upsilon^{r0} = 163.063$.

According to Eqs. (2.16) and (2.17), the two linear programming models in the second level can be constructed as follows:

$$\max\{\upsilon^l\}$$
$$\text{s.t.}\begin{cases} \upsilon^l \le 154.9996 \\ \upsilon^l \le 155.2633 \\ \upsilon^l \ge 61.398 \\ \upsilon^l \text{ unrestricted in sign} \end{cases}$$

and

$$\max\{\upsilon^r\}$$
$$\text{s.t.}\begin{cases} \upsilon^r \le 171.0523 \\ \upsilon^r \le 164.737 \\ \upsilon^r \ge 163.063 \\ \upsilon^r \text{ unrestricted in sign,} \end{cases}$$

respectively. It is easy to see that $\upsilon^{l*} = 154.9996$ and $\upsilon^{r*} = 164.737$ are the solutions of the above two linear programming models, respectively.

Therefore, the optimal (or maximin) mixed strategy and the gain-floor for the player I are $\boldsymbol{y}^* = (0.7895,\ 0.2105)^{\mathrm{T}}$ and $\tilde{\upsilon}^* = (154.9996,\ 161.05, 164.737)$, respectively.

Analogously, according to Eq. (2.18), the linear programming model in the first level can be constructed as follows:

$$\min\{\omega^m\}$$
$$\text{s.t.}\begin{cases} 175z_1 + 150z_2 \le \omega^l \\ 80z_1 + 175z_2 \le \omega^l \\ 180z_1 + 156z_2 \le \omega^m \\ 90z_1 + 180z_2 \le \omega^m \\ 190z_1 + 158z_2 \le \omega^r \\ 100z_1 + 190z_2 \le \omega^r \\ \omega^l \le \omega^m \le \omega^r \\ z_1 + z_2 = 1 \\ z_1 \ge 0, z_2 \ge 0 \\ \omega^l, \omega^m, \text{ and } \omega^r \text{ unrestricted in sign.} \end{cases}$$

Solving the above linear programming model by using the simplex method of linear programming, we can obtain its optimal solution $(z^*, \omega^{l0}, \omega^{m*}, \omega^{r0})$, where $z^* = (0.2105, 0.7895)^{\mathrm{T}}$, $\omega^{l0} = 158.8984$, $\omega^{m*} = 161.05$, and $\omega^{r0} = 339.61$.

According to Eqs. (2.20) and (2.21), the two linear programming models in the second level can be constructed as follows:

$$\min\{\omega^l\}$$
$$\text{s.t.}\begin{cases} \omega^l \geq 155.2633 \\ \omega^l \geq 154.9997 \\ \omega^l \leq 158.8984 \\ \omega^l \text{ unrestricted in sign} \end{cases}$$

and

$$\min\{\omega^r\}$$
$$\text{s.t.}\begin{cases} \omega^r \geq 164.737 \\ \omega^r \geq 171.0523 \\ \omega^r \leq 339.61 \\ \omega^r \text{ unrestricted in sign,} \end{cases}$$

respectively. It is easy to see that $\omega^{l*} = 155.2633$ and $\omega^{r*} = 171.0523$ are the solutions of the above linear programming models, respectively.

Thus, the optimal (or minimax) mixed strategy and the loss-ceiling for the player II are obtained as $z^* = (0.2105, 0.7895)^{\mathrm{T}}$ and $\tilde{\omega}^* = (155.2633, 161.05, 171.0523)$, respectively. Furthermore, we can obtain the fuzzy equilibrium value of the matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers as follows:

$$\tilde{V}^* = \tilde{v}^* \wedge \tilde{\omega}^* = (155.2633, 161.05, 164.737),$$

which means that the fuzzy value of the matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers is “around 161.05”. In other words, the player I’s minimum reward is 155.2633 while his/her maximum reward is 164.737. He/she could win any intermediate value x between 155.2633 and 164.737 with the possibility $\mu_{\tilde{V}^*}(x)$ as follows:

$$\mu_{\tilde{V}^*}(x) = \begin{cases} \dfrac{x - 155.2633}{5.7867} & \text{if } 155.2633 \leq x < 161.05 \\ 1 & \text{if } x = 161.05 \\ \dfrac{164.737 - x}{3.687} & \text{if } 161.05 < x \leq 164.737 \\ 0 & \text{else,} \end{cases}$$

depicted as in Fig. 2.6.

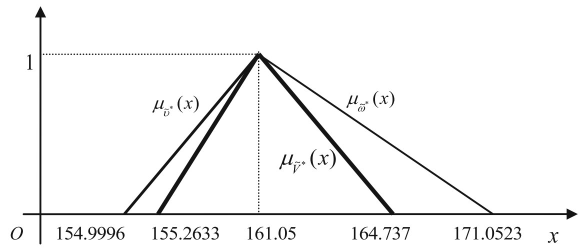


Fig. 2.6 The fuzzy equilibrium value $\tilde{V}^*$

It is easy to see from Fig. 2.6 that the fuzzy equilibrium value $\tilde{V}^*$ is a triangular fuzzy number.

Campos [7] solved the above matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers by deriving two auxiliary fuzzy linear programming models according to four different kinds of ranking methods for fuzzy numbers, and obtained its four fuzzy values and optimal mixed strategies, respectively. The optimal mixed strategies for both the players provided by Campos [7] are almost the same as that generated by using the two-level linear programming method proposed in this section. However, the ranking method for fuzzy numbers needs to be determined a priori, when the method proposed by Campos [7] is employed to solve the matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers. Obviously, it is difficult for the players to determine what kind of ranking methods should be chosen. Moreover, the fuzzy values generated by using the method proposed by Campos [7] closely depend on some additional parameters which are not easy to be chosen for the players.

2.5 The Lexicographic Method of Matrix Games with Payoffs of Triangular Fuzzy Numbers

Let us continue to develop an effective method for solving Eqs. (2.10) and (2.11) stated as in Sect. 2.3.

As stated in Sect. 2.4, the three objective functions υ^l, υ^m, and υ^r in Eq. (2.10) have different priority. Consequently, solving Eq. (2.10) becomes solving the following linear programming problem which consists of the two linear programming models [i.e., Eqs. (2.14) and (2.22)].

Firstly, we solve Eq. (2.14) by using the simplex method of linear programming and obtain its optimal solution, denoted by $(\boldsymbol{y}^0, \upsilon^{l0}, \upsilon^{m*}, \upsilon^{r0})$, where $\boldsymbol{y}^0 = (y_1^0, y_2^0, \ldots, y_m^0)^{\mathrm{T}}$.

Then, combining with Eq. (2.10), the bi-objective linear programming model is constructed as follows:

$$\begin{aligned}&\max\{\upsilon^l\}\\&\max\{\upsilon^r\}\\&\text{s.t.}\begin{cases}\sum_{i=1}^m a_{ij}^l y_i \geq \upsilon^l \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^m a_{ij}^m y_i \geq \upsilon^{m*} \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^m a_{ij}^r y_i \geq \upsilon^r \quad (j=1,2,\ldots,n)\\ \upsilon^l \leq \upsilon^{m*} \leq \upsilon^r\\ \upsilon^l \geq \upsilon^{l0}\\ \upsilon^r \geq \upsilon^{r0}\\ \sum_{i=1}^m y_i = 1\\ y_i \geq 0 \quad (i=1,2,\ldots,m)\\ \upsilon^l \text{ and } \upsilon^r \text{ unrestricted in sign,}\end{cases}\end{aligned} \tag{2.22}$$

where y_i $(i=1,2,\ldots,m)$, υ^l, and υ^r are decision variables.

The objective functions υ^l and υ^r in Eq. (2.22) may be regarded as equal importance, i.e., they have identical weights. Therefore, Eq. (2.22) can be aggregated into the linear programming model as follows:

$$\begin{aligned}&\max\left\{\frac{\upsilon^l+\upsilon^r}{2}\right\}\\&\text{s.t.}\begin{cases}\sum_{i=1}^m a_{ij}^l y_i \geq \upsilon^l \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^m a_{ij}^m y_i \geq \upsilon^{m*} \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^m a_{ij}^r y_i \geq \upsilon^r \quad (j=1,2,\ldots,n)\\ \upsilon^l \leq \upsilon^{m*} \leq \upsilon^r\\ \upsilon^l \geq \upsilon^{l0}\\ \upsilon^r \geq \upsilon^{r0}\\ \sum_{i=1}^m y_i = 1\\ y_i \geq 0 \quad (i=1,2,\ldots,m)\\ \upsilon^l \text{ and } \upsilon^r \text{ unrestricted in sign.}\end{cases}\end{aligned} \tag{2.23}$$

Using the simplex method of linear programming, we can obtain the optimal solution of Eq. (2.23), denoted by $(\boldsymbol{y}^*, \upsilon^{l*}, \upsilon^{r*})$, where $\boldsymbol{y}^* = (y_1^*, y_2^*, \ldots, y_m^*)^{\mathrm{T}}$.

It is not difficult to prove that $(\boldsymbol{y}^*, \tilde{\upsilon}^*)$ is a Pareto optimal solution of Eq. (2.10), where $\tilde{\upsilon}^* = (\upsilon^{l*}, \upsilon^{m*}, \upsilon^{r*})$ is a triangular fuzzy number. Thus, the maximin (or optimal) mixed strategy $\boldsymbol{y}^*$ and the gain-floor $\tilde{\upsilon}^*$ for the player I can be obtained.

In the similar way, solving Eq. (2.11) turns into solving the following linear programming problem which consists of Eqs. (2.18) and (2.24).

Solving Eq. (2.18) by using the simplex method of linear programming, we can easily obtain its optimal solution $(\boldsymbol{z}^0, \omega^{l0}, \omega^{m*}, \omega^{r0})$, where $\boldsymbol{z}^0 = (z_1^0, z_2^0, \ldots, z_n^0)^{\mathrm{T}}$.

Combining with Eq. (2.16), the bi-objective linear programming model is constructed as follows:

$$
\begin{aligned}
&\min\{\omega^l\}\\
&\min\{\omega^r\}\\
&\text{s.t.}\begin{cases}
\sum_{j=1}^{n} a_{ij}^l z_j \le \omega^l \quad (i = 1, 2, \ldots, m)\\
\sum_{j=1}^{n} a_{ij}^m z_j \le \omega^{m*} \quad (i = 1, 2, \ldots, m)\\
\sum_{j=1}^{n} a_{ij}^r z_j \le \omega^r \quad (i = 1, 2, \ldots, m)\\
\omega^l \le \omega^{m*} \le \omega^r\\
\omega^l \le \omega^{l0}\\
\omega^r \le \omega^{r0}\\
\sum_{j=1}^{n} z_j = 1\\
z_j \ge 0 \quad (j = 1, 2, \ldots, n)\\
\omega^l \text{ and } \omega^r \text{ unrestricted in sign,}
\end{cases}
\end{aligned}
\tag{2.24}
$$

where z_j $(j = 1, 2, \ldots, n)$, ω^l, and ω^r are decision variables.

Analogously, the objective functions ω^l and ω^r in Eq. (2.24) may be regarded as equal importance, i.e., they have identical weights. Then, Eq. (2.24) can be aggregated into the linear programming model as follows:

$$\min\left\{\frac{\omega^l+\omega^r}{2}\right\}$$
$$\text{s.t.}\begin{cases}\sum_{j=1}^{n} a_{ij}^l z_j \le \omega^l \quad (i=1,2,\ldots,m)\\ \sum_{j=1}^{n} a_{ij}^m z_j \le \omega^{m*} \quad (i=1,2,\ldots,m)\\ \sum_{j=1}^{n} a_{ij}^r z_j \le \omega^r \quad (i=1,2,\ldots,m)\\ \omega^l \le \omega^{m*} \le \omega^r\\ \omega^l \le \omega^{l0}\\ \omega^r \le \omega^{r0}\\ \sum_{j=1}^{n} z_j = 1\\ z_j \ge 0 \quad (j=1,2,\ldots,n)\\ \omega^l \text{ and } \omega^r \text{ unrestricted in sign.}\end{cases} \tag{2.25}$$

Solving Eq. (2.25) by using the simplex method of linear programming, we can easily obtain its optimal solution $(\boldsymbol{z}^*, \omega^{l*}, \omega^{r*})$, where $\boldsymbol{z}^* = (z_1^*, z_2^*, \ldots, z_n^*)^{\mathrm{T}}$.

It is easily proved that $(\boldsymbol{z}^*, \tilde{\omega}^*)$ is a Pareto optimal solution of Eq. (2.11), where $\tilde{\omega}^* = (\omega^{l*}, \omega^{m*}, \omega^{r*})$ is a triangular fuzzy number. Thus, the minimax (or optimal) mixed strategy $\boldsymbol{z}^*$ and the loss-ceiling $\tilde{\omega}^*$ for the player II can be obtained.

From the above discussion, we can summarize the process of the lexicographic method of matrix games with payoffs of triangular fuzzy numbers as follows.

Step 1: Construct the linear programming model according to Eq. (2.14), and solve it by using the simplex method of linear programming;
Step 2: Construct the linear programming model according to Eq. (2.23), and solve it by using the simplex method of linear programming;
Step 3: Construct the linear programming model according to Eq. (2.18), and solve it by using the simplex method of linear programming;
Step 4: Construct the linear programming model according to Eq. (2.25), and solve it by using the simplex method of linear programming;
Step 5: Obtain the solution of the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers, stop.

Example 2.3 Let us employ the above lexicographic method to solve the matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers given in Example 2.2. Namely, the payoff matrix for the player I is given as follows:

$$\tilde{\boldsymbol{A}}_2 = \begin{array}{c} \\ \delta_1 \\ \delta_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{pmatrix} (175, 180, 190) & (150, 156, 158) \\ (80, 90, 100) & (175, 180, 190) \end{pmatrix} \end{array}.$$

According to Eq. (2.14), the linear programming model can be constructed as follows:

$$\max\{\upsilon^m\}$$
$$\text{s.t.}\begin{cases} 175y_1 + 80y_2 \le \upsilon^l \\ 150y_1 + 175y_2 \le \upsilon^l \\ 180y_1 + 90y_2 \le \upsilon^m \\ 156y_1 + 180y_2 \le \upsilon^m \\ 190y_1 + 100y_2 \le \upsilon^r \\ 158y_1 + 190y_2 \le \upsilon^r \\ \upsilon^l \le \upsilon^m \le \upsilon^r \\ y_1 + y_2 = 1 \\ y_1 \ge 0, y_2 \ge 0 \\ \upsilon^l, \upsilon^m, \text{ and } \upsilon^r \text{ unrestricted in sign.} \end{cases}$$

Solving the above linear programming model by using the simplex method of linear programming, we can obtain its optimal solution $(\boldsymbol{y}^0, \upsilon^{l0}, \upsilon^{m*}, \upsilon^{r0})$ whose components are given as follows:

$$\boldsymbol{y}^0 = (0.789, 0.211)^{\mathrm{T}},\ \upsilon^{l0} = 61.408,\ \upsilon^{m*} = 161.06,\ \upsilon^{r0} = 163.073.$$

According to Eq. (2.23), the linear programming model can be constructed as follows:

$$\max\left\{\frac{\upsilon^l + \upsilon^r}{2}\right\}$$
$$\text{s.t.}\begin{cases} 175y_1 + 80y_2 \ge \upsilon^l \\ 150y_1 + 175y_2 \ge \upsilon^l \\ 180y_1 + 90y_2 \ge 161.06 \\ 156y_1 + 180y_2 \ge 161.06 \\ 190y_1 + 100y_2 \ge \upsilon^r \\ 158y_1 + 190y_2 \ge \upsilon^r \\ \upsilon^l \le 161.06 \le \upsilon^r \\ \upsilon^l \ge 61.408 \\ \upsilon^r \ge 163.073 \\ y_1 + y_2 = 1 \\ y_1 \ge 0, y_2 \ge 0 \\ \upsilon^l \text{ and } \upsilon^r \text{ unrestricted in sign.} \end{cases}$$

Solving the above linear programming model by using the simplex method of linear programming, we can obtain its optimal solution $(\boldsymbol{y}^*, \upsilon^{l*}, \upsilon^{r*})$ whose components are given as follows:

$$\boldsymbol{y}^* = (0.789, 0.211)^{\mathrm{T}},\ \upsilon^{l*} = 154.955,\ \upsilon^{r*} = 164.752.$$

Therefore, the maximin (or optimal) mixed strategy and the gain-floor for the player I are obtained as $\boldsymbol{y}^* = (0.789, 0.211)^{\mathrm{T}}$ and $\tilde{\upsilon}^* = (154.955, 161.06, 164.752)$, respectively.

Analogously, according to Eq. (2.18), the linear programming model can be obtained as follows:

$$\min\{\omega^m\}$$
$$\text{s.t.}\begin{cases} 175z_1 + 150z_2 \le \omega^l \\ 80z_1 + 175z_2 \le \omega^l \\ 180z_1 + 156z_2 \le \omega^m \\ 90z_1 + 180z_2 \le \omega^m \\ 190z_1 + 158z_2 \le \omega^r \\ 100z_1 + 190z_2 \le \omega^r \\ \omega^l \le \omega^m \le \omega^r \\ z_1 + z_2 = 1 \\ z_1 \ge 0, z_2 \ge 0 \\ \omega^l, \omega^m, \text{ and } \omega^r \text{ unrestricted in sign.} \end{cases}$$

Solving the above linear programming model by using the simplex method of linear programming, we can obtain its optimal solution $(\boldsymbol{z}^0, \omega^{l0}, \omega^{m*}, \omega^{r0})$ whose components are given as follows:

$$\boldsymbol{z}^0 = (0.211, 0.789)^{\mathrm{T}},\ \omega^{l0} = 158.908,\ \omega^{m*} = 161.06,\ \omega^{r0} = 339.62.$$

According to Eq. (2.25), the linear programming model can be obtained as follows:

$$\min\left\{\frac{\omega^l+\omega^r}{2}\right\}$$
$$\text{s.t.}\begin{cases} 175z_1+150z_2\leq\omega^l \\ 80z_1+175z_2\leq\omega^l \\ 180z_1+156z_2\leq 161.06 \\ 90z_1+180z_2\leq 161.06 \\ 190z_1+158z_2\leq\omega^r \\ 100z_1+190z_2\leq\omega^r \\ \omega^l\leq 161.06\leq\omega^r \\ \omega^l\leq 158.908 \\ \omega^r\leq 339.62 \\ z_1+z_2=1 \\ z_1\geq 0, z_2\geq 0 \\ \omega^l \text{ and } \omega^r \text{ unrestricted in sign.} \end{cases}$$

Solving the above linear programming model by using the simplex method of linear programming, we can obtain its optimal solution $(z^*, \omega^{l*}, \omega^{r*})$ whose components are given as follows:

$$z^*=(0.211, 0.789)^{\mathrm{T}},\ \omega^{l*}=155.275,\ \omega^{r*}=171.01.$$

Thus, the minimax (or optimal) mixed strategy and the loss-ceiling for the player II are obtained as $z^*=(0.211, 0.789)^{\mathrm{T}}$ and $\tilde{\omega}^*=(155.275, 161.06, 171.01)$, respectively.

Furthermore, the fuzzy equilibrium value of the matrix game $\tilde{A}_2$ with payoffs of triangular fuzzy numbers can be obtained as follows:

$$\tilde{V}^*=\tilde{\upsilon}^*\wedge\tilde{\omega}^*=(155.275, 161.06, 164.752),$$

which means that the fuzzy value of the matrix game $\tilde{A}_2$ with payoffs of triangular fuzzy numbers is "around 161.06". In other words, the player I's minimum reward is 155.275 while his/her maximum reward is 164.752. He/she could win any intermediate value x between 155.275 and 164.752 with the possibility $\mu_{\tilde{V}^*}(x)$ as follows:

$$\mu_{\tilde{V}^*}(x)=\begin{cases} \dfrac{x-155.275}{5.785} & \text{if } 155.275\leq x<161.06 \\ 1 & \text{if } x=161.06 \\ \dfrac{164.752-x}{3.692} & \text{if } 161.06<x\leq 164.752 \\ 0 & \text{else,} \end{cases}$$

depicted as in Fig. 2.7.

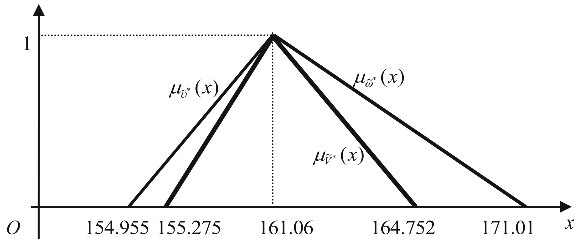


Fig. 2.7 The fuzzy equilibrium value $\tilde{V}^*$

2.6 Alfa-Cut-Based Primal-Dual Linear Programming Models of Matrix Games with Payoffs of Triangular Fuzzy Numbers

We firstly discuss a simple example of matrix games with payoffs of triangular fuzzy numbers.

Example 2.4 Let us consider a specific matrix game $\tilde{\boldsymbol{A}}^0$ with payoffs of triangular fuzzy numbers in which the player I's payoff matrix is given as follows:

$$\tilde{\boldsymbol{A}}^0 = \begin{array}{c} \\ \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{array} \begin{array}{c} \begin{array}{cccc} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{array} \\ \begin{pmatrix} (8, 8.5, 10) & (5, 7, 8) & (14, 16, 18) & (5, 7, 8) \\ (14, 16, 18) & (3.5, 4, 5) & (-5, -3, -1) & (2, 3, 3.5) \\ (11, 12, 14) & (5, 7, 8) & (8, 9, 11) & (5, 7, 8) \\ (-5, -3, -2) & (-1, 0, 2) & (20, 21, 25) & (3.5, 4, 5) \end{pmatrix} \end{array}.$$

By intuition observation or using the ranking relation of triangular fuzzy numbers and in the same way to crisp matrix games, it is easy to see from the minimax/maximin criteria [4, 26] that there are four pure strategy saddle points (δ_1, β_2), (δ_1, β_4), (δ_3, β_2), (δ_3, β_4) [or (1, 2), (1, 4), (3, 2), (3, 4)] and the matrix game $\tilde{\boldsymbol{A}}^0$ with payoffs of triangular fuzzy numbers has a fuzzy value $\tilde{V}^0 = (5, 7, 8)$, which is also a triangular fuzzy number. The fuzzy value means that the player I wins (5, 7, 8) whereas the player II loses (5, 7, 8) [or II wins $-\tilde{V}^0 = (-8, -7, -5)$] when I and II use the optimal pure strategies δ_1 (or δ_3) and β_2 (or β_4), respectively.

Unfortunately, in general, it is not always sure that there are pure strategy saddle points in matrix games with payoffs of triangular fuzzy numbers. Therefore, in the same way to crisp matrix games, we need to consider the players' mixed strategies $\boldsymbol{y}$ and $\boldsymbol{z}$ as stated in Sect. 1.2 or Sect. 2.3. Thus, stated as in Sect. 2.3.2, the player I's gain-floor $\tilde{\upsilon} = (\upsilon^l, \upsilon^m, \upsilon^r)$ and the player II's loss-ceiling $\tilde{\omega} = (\omega^l, \omega^m, \omega^r)$ are triangular fuzzy numbers. Moreover, it is always sure that $\tilde{\upsilon} \leq \tilde{\omega}$ according to Theorem 2.2.

In a similar way to Definition of the value of crisp matrix games [26], if $\tilde{\upsilon} = \tilde{\omega}$, then their common value is called the fuzzy value of the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers, i.e., $\tilde{V} = \tilde{\upsilon} = \tilde{\omega}$. In other words, the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers has a fuzzy value $\tilde{V}$. Obviously, $\tilde{V}$ is a triangular fuzzy number also, denoted by $\tilde{V} = (V^l, V^m, V^r)$.

2.6.1 Interval-Valued Matrix Games Based on Alfa-Cut Sets of Triangular Fuzzy Numbers

Stated as earlier, for any $\alpha \in [0, 1]$, α-cut sets of the triangular fuzzy numbers $\tilde{a}_{ij} = (a_{ij}^l, a_{ij}^m, a_{ij}^r)$ are intervals, which are easily obtained by using Eq. (2.4) as follows:

$$\tilde{a}_{ij}(\alpha) = [a_{ij}^L(\alpha), a_{ij}^R(\alpha)] = [\alpha a_{ij}^m + (1-\alpha)a_{ij}^l, \alpha a_{ij}^m + (1-\alpha)a_{ij}^r]. \tag{2.26}$$

Let us consider an interval-valued matrix game $\tilde{\boldsymbol{A}}(\alpha)$ with the payoff matrix $\tilde{\boldsymbol{A}}(\alpha) = (\tilde{a}_{ij}(\alpha))_{m \times n}$, whose elements $\tilde{a}_{ij}(\alpha)$ are the intervals given by Eq. (2.26). $\tilde{a}_{ij}(\alpha)$ represents the interval-valued payoff of the player I when the players I and II use the pure strategies $\delta_i \in S_1$ and $\beta_j \in S_2$, respectively. Naturally, the player II's payoff is the interval $-\tilde{a}_{ij}(\alpha) = [-a_{ij}^R(\alpha), -a_{ij}^L(\alpha)]$ according to the arithmetic operations over intervals in Sect. 1.3.1.

Taking any value $a_{ij}(\alpha)$ in the interval-valued payoffs $\tilde{a}_{ij}(\alpha) = [a_{ij}^L(\alpha), a_{ij}^R(\alpha)]$, we consider a (crisp) matrix game $\boldsymbol{A}(\alpha)$ with the payoff matrix $\boldsymbol{A}(\alpha) = (a_{ij}(\alpha))_{m \times n}$. It is easy to from Eqs. (1.3) and (1.4) that the player I's gain-floor $v(\alpha)$ in the matrix game $\boldsymbol{A}(\alpha)$ is closely related to all $a_{ij}(\alpha)$. That is to say, $v(\alpha)$ is a function of $a_{ij}(\alpha)$ in the interval-valued payoffs $\tilde{a}_{ij}(\alpha)$, denoted by $v(\alpha) = \upsilon((a_{ij}(\alpha)))$. Similarly, the optimal mixed strategy $\boldsymbol{y}^*(\alpha)$ for the player I is a function of all $a_{ij}(\alpha)$ also, denoted by $\boldsymbol{y}^*(\alpha) = \boldsymbol{y}^*((a_{ij}(\alpha)))$.

In the same way to the above analysis, it is easy to see from Eqs. (1.6) and (1.7) that the loss-ceiling $\mu(\alpha)$ and corresponding optimal mixed strategy $\boldsymbol{z}^*(\alpha)$ for the player II in the matrix game $\boldsymbol{A}(\alpha)$ are functions of all $a_{ij}(\alpha)$ in the interval-valued payoffs $\tilde{a}_{ij}(\alpha)$, denoted by $\mu(\alpha) = \omega((a_{ij}(\alpha)))$ and $\boldsymbol{z}^*(\alpha) = \boldsymbol{z}^*((a_{ij}(\alpha)))$.

According to Eqs. (1.3) and (1.4), we can easily prove that the player I's gain-floor $\upsilon((a_{ij}(\alpha)))$ in the matrix game $\boldsymbol{A}(\alpha)$ is a non-decreasing function of all $a_{ij}(\alpha)$ in the interval-valued payoffs $\tilde{a}_{ij}(\alpha)$. In fact, for any $a_{ij}(\alpha)$ and $a'_{ij}(\alpha)$ in the interval-valued payoffs $\tilde{a}_{ij}(\alpha)$, if $a_{ij}(\alpha) \le a'_{ij}(\alpha)$, then

$$\sum_{i=1}^{m} y_i a_{ij}(\alpha) \le \sum_{i=1}^{m} y_i a'_{ij}(\alpha)$$

due to $y_i \ge 0$ $(i = 1, 2, \ldots, m)$ and $\sum_{i=1}^{m} y_i = 1$, where $\boldsymbol{y}$ is any mixed strategy of the player I. Hence, we have

$$\min_{1\le j\le n}\left\{\sum_{i=1}^{m} y_i a_{ij}(\alpha)\right\} \le \min_{1\le j\le n}\left\{\sum_{i=1}^{m} y_i a'_{ij}(\alpha)\right\},$$

which directly infers that

$$\max_{\boldsymbol{y}\in Y}\min_{1\le j\le n}\left\{\sum_{i=1}^{m} y_i a_{ij}(\alpha)\right\} \le \max_{\boldsymbol{y}\in Y}\min_{1\le j\le n}\left\{\sum_{i=1}^{m} y_i a'_{ij}(\alpha)\right\},$$

i.e.,

$$\upsilon((a_{ij}(\alpha))) \le \upsilon((a'_{ij}(\alpha))),$$

where $\boldsymbol{A}'(\alpha) = (a'_{ij}(\alpha))_{m\times n}$ is the payoff matrix of the player I in the matrix game $\boldsymbol{A}'(\alpha)$.

According to the minimax theorem of matrix games [4, 26], the matrix game $\boldsymbol{A}(\alpha)$ has a value, denoted by $V(\alpha) = V((a_{ij}(\alpha)))$. Obviously, $V(\alpha) = \nu(\alpha) = \mu(\alpha)$. From the above discussion, $V((a_{ij}(\alpha)))$ is a non-decreasing function of all $a_{ij}(\alpha)$ in the interval-valued payoffs $\tilde{a}_{ij}(\alpha)$.

Stated as earlier, the value of the interval-valued matrix game $\tilde{\boldsymbol{A}}(\alpha)$ is an interval. The upper bound $\nu^R(\alpha)$ of the player I's gain-floor in the interval-valued matrix game $\tilde{\boldsymbol{A}}(\alpha)$ and corresponding optimal mixed strategy $\boldsymbol{y}^{R*}(\alpha)$ are $\nu^R(\alpha) = \upsilon^R((a_{ij}^R(\alpha)))$ and $\boldsymbol{y}^{R*} = \boldsymbol{y}^{R*}((a_{ij}^R(\alpha)))$, respectively. According to Eq. (1.5), $(\nu^R(\alpha), \boldsymbol{y}^{R*}(\alpha))$ is an optimal solution to the linear programming model as follows:

$$\max\{\upsilon^R(\alpha)\}$$
$$\text{s.t.}\begin{cases} \sum_{i=1}^{m} a_{ij}^R(\alpha) y_i^R(\alpha) \ge \upsilon^R(\alpha) \quad (j = 1, 2, \ldots, n) \\ \sum_{i=1}^{m} y_i^R(\alpha) = 1 \\ y_i^R(\alpha) \ge 0 \quad (i = 1, 2, \ldots, m) \\ \upsilon^R(\alpha) \text{ unrestricted in sign,} \end{cases} \tag{2.27}$$

where $y_i^R(\alpha)$ $(i = 1, 2, \ldots, m)$ and $\upsilon^R(\alpha)$ are decision variables.

Without loss of generality [26], assume that $\upsilon^R(\alpha) > 0$. Let

$$x_i^R(\alpha) = \frac{y_i^R(\alpha)}{\upsilon^R(\alpha)} \quad (i = 1, 2, \ldots, m). \tag{2.28}$$

Then, $x_i^R(\alpha) \ge 0$ $(i = 1, 2, \ldots, m)$ and

$$\sum_{i=1}^{m} x_i^R(\alpha) = \frac{1}{\upsilon^R(\alpha)}. \tag{2.29}$$

Combining with Eq. (2.26), Eq. (2.27) can be transformed into the linear programming model as follows:

$$\begin{aligned} &\min\left\{\sum_{i=1}^{m} x_i^R(\alpha)\right\} \\ &\text{s.t.}\begin{cases} \sum\limits_{i=1}^{m} [\alpha a_{ij}^m + (1-\alpha)a_{ij}^r] x_i^R(\alpha) \geq 1 \quad (j = 1, 2, \ldots, n) \\ x_i^R(\alpha) \geq 0 \qquad (i = 1, 2, \ldots, m), \end{cases} \end{aligned} \tag{2.30}$$

where $x_i^R(\alpha)$ $(i = 1, 2, \ldots, m)$ are decision variables.

Solving Eq. (2.30) by using the simplex method of linear programming, we can obtain its optimal solution, denoted by $\boldsymbol{x}^{R*}(\alpha) = (x_1^{R*}(\alpha), x_2^{R*}(\alpha), \ldots, x_m^{R*}(\alpha))^{\mathrm{T}}$. According to Eqs. (2.28) and (2.29), the upper bound $v^R(\alpha)$ and the optimal mixed strategy $\boldsymbol{y}^{R*}(\alpha) = (y_1^{R*}(\alpha), y_2^{R*}(\alpha), \ldots, y_m^{R*}(\alpha))^{\mathrm{T}}$ can be obtained, respectively, where

$$v^R(\alpha) = \frac{1}{\sum_{i=1}^{m} x_i^{R*}(\alpha)} \tag{2.31}$$

and

$$y_i^{R*}(\alpha) = v^R(\alpha) x_i^{R*}(\alpha) \quad (i = 1, 2, \ldots, m). \tag{2.32}$$

Analogously, the lower bound $v^L(\alpha)$ of the player I's gain-floor in the interval-valued matrix game $\tilde{\boldsymbol{A}}(\alpha)$ and corresponding optimal mixed strategy $\boldsymbol{y}^{L*}(\alpha)$ are $v^L(\alpha) = \upsilon^L((a_{ij}^L(\alpha)))$ and $\boldsymbol{y}^{L*} = \boldsymbol{y}^{L*}((a_{ij}^L(\alpha)))$, respectively. Then, according to Eq. (1.5), $(v^L(\alpha), \boldsymbol{y}^{L*}(\alpha))$ is an optimal solution to the linear programming model as follows:

$$\begin{aligned} &\max\{\upsilon^L(\alpha)\} \\ &\text{s.t.}\begin{cases} \sum\limits_{i=1}^{m} a_{ij}^L(\alpha) y_i^L(\alpha) \geq \upsilon^L(\alpha) \quad (j = 1, 2, \ldots, n) \\ \sum\limits_{i=1}^{m} y_i^L(\alpha) = 1 \\ y_i^L(\alpha) \geq 0 \quad (i = 1, 2, \ldots, m) \\ \upsilon^L(\alpha) \text{ unrestricted in sign,} \end{cases} \end{aligned} \tag{2.33}$$

where $y_i^L(\alpha)$ $(i = 1, 2, \ldots, m)$ and $\upsilon^L(\alpha)$ are decision variables.

Without loss of generality [26], assume that $\upsilon^L(\alpha) > 0$. Let

$$x_i^L(\alpha) = \frac{y_i^L(\alpha)}{\upsilon^L(\alpha)} \quad (i = 1, 2, \ldots, m). \tag{2.34}$$

Then, $x_i^L(\alpha) \geq 0$ $(i = 1, 2, \ldots, m)$ and

$$\sum_{i=1}^{m} x_i^L(\alpha) = \frac{1}{\upsilon^L(\alpha)}. \tag{2.35}$$

Combining with Eq. (2.26), Eq. (2.33) can be transformed into the linear programming model as follows:

$$\begin{aligned} &\min\left\{\sum_{i=1}^{m} x_i^L(\alpha)\right\} \\ &\text{s.t.} \begin{cases} \sum\limits_{i=1}^{m} [\alpha a_{ij}^m + (1-\alpha) a_{ij}^l] x_i^L(\alpha) \geq 1 & (j = 1, 2, \ldots, n) \\ x_i^L(\alpha) \geq 0 & (i = 1, 2, \ldots, m), \end{cases} \end{aligned} \tag{2.36}$$

where $x_i^L(\alpha)$ $(i = 1, 2, \ldots, m)$ are decision variables.

Solving Eq. (2.36) by using the simplex method of linear programming, we can obtain its optimal solution, denoted by $\boldsymbol{x}^{L*}(\alpha) = (x_1^{L*}(\alpha), x_2^{L*}(\alpha), \ldots, x_m^{L*}(\alpha))^{\mathrm{T}}$. According to Eqs. (2.34) and (2.35), the lower bound $v^L(\alpha)$ and the optimal mixed strategy $\boldsymbol{y}^{L*}(\alpha) = (y_1^{L*}(\alpha), y_2^{L*}(\alpha), \ldots, y_m^{L*}(\alpha))^{\mathrm{T}}$ can be obtained, respectively, where

$$v^L(\alpha) = \frac{1}{\sum_{i=1}^{m} x_i^{L*}(\alpha)} \tag{2.37}$$

and

$$y_i^{L*}(\alpha) = v^L(\alpha) x_i^{L*}(\alpha) \quad (i = 1, 2, \ldots, m). \tag{2.38}$$

Thus, the lower bound $v^L(\alpha)$ and upper bound $v^R(\alpha)$ and corresponding optimal mixed strategies can be obtained. Hence, the player I's gain-floor in the interval-valued matrix game $\tilde{\boldsymbol{A}}(\alpha)$ is obtained as an interval $\tilde{v}(\alpha) = [v^L(\alpha), v^R(\alpha)]$, which is a α-cut set of $\tilde{\upsilon}$, i.e., $\tilde{v}(\alpha) = \tilde{\upsilon}(\alpha)$.

In the same analysis, the upper bound $\mu^R(\alpha)$ of the player II's loss-ceiling in the interval-valued matrix game $\tilde{\boldsymbol{A}}(\alpha)$ and corresponding optimal mixed strategy $\boldsymbol{z}^{R*}(\alpha)$ are $\mu^R(\alpha) = \omega^R((a_{ij}^R(\alpha)))$ and $\boldsymbol{z}^{R*}(\alpha) = \boldsymbol{z}^{R*}((a_{ij}^R(\alpha)))$, respectively. According to Eq. (1.8), $(\mu^R(\alpha), \boldsymbol{z}^{R*}(\alpha))$ is an optimal solution to the linear programming model as follows:

$$\min\{\omega^R(\alpha)\}$$
$$\text{s.t.}\begin{cases} \sum_{j=1}^{n} a_{ij}^R(\alpha) z_j^R(\alpha) \le \omega^R(\alpha) \quad (i = 1, 2, \ldots, m) \\ \sum_{j=1}^{n} z_j^R(\alpha) = 1 \\ z_j^R(\alpha) \ge 0 \quad (j = 1, 2, \ldots, n) \\ \omega^R(\alpha) \text{ unrestricted in sign,} \end{cases} \tag{2.39}$$

where $\omega^R(\alpha)$ and $z_j^R(\alpha)$ $(j = 1, 2, \ldots, n)$ are decision variables.

Without loss of generality [26], assume that $\omega^R(\alpha) > 0$. Let

$$t_j^R(\alpha) = \frac{z_j^R(\alpha)}{\omega^R(\alpha)} \quad (j = 1, 2, \ldots, n), \tag{2.40}$$

thus, we have

$$\sum_{j=1}^{n} t_j^R(\alpha) = \frac{1}{\omega^R(\alpha)}. \tag{2.41}$$

Combining with Eq. (2.26), Eq. (2.39) can be converted into the linear programming model as follows:

$$\max\left\{\sum_{j=1}^{n} t_j^R(\alpha)\right\}$$
$$\text{s.t.}\begin{cases} \sum_{j=1}^{n} [\alpha a_{ij}^m + (1-\alpha) a_{ij}^r] t_j^R(\alpha) \le 1 \quad (i = 1, 2, \ldots, m) \\ t_j^R(\alpha) \ge 0 \qquad (j = 1, 2, \ldots, n), \end{cases} \tag{2.42}$$

where $t_j^R(\alpha)$ $(j = 1, 2, \ldots, n)$ are decision variables.

Solving Eq. (2.42) by using the simplex method of linear programming, we can obtain its optimal solution, denoted by $\boldsymbol{t}^{R*}(\alpha) = (t_1^{R*}(\alpha), t_2^{R*}(\alpha), \ldots, t_n^{R*}(\alpha))^{\mathrm{T}}$. According to Eqs. (2.40) and (2.41), the upper bound $\mu^R(\alpha)$ and the optimal mixed strategy $\boldsymbol{z}^{R*}(\alpha) = (z_1^{R*}(\alpha), z_2^{R*}(\alpha), \ldots, z_n^{R*}(\alpha))^{\mathrm{T}}$ can be obtained, respectively, where

$$\mu^R(\alpha) = \frac{1}{\sum_{j=1}^{n} t_j^{R*}(\alpha)} \tag{2.43}$$

and

$$z_j^{R*}(\alpha) = \mu^R(\alpha) t_j^{R*}(\alpha) \quad (j = 1, 2, \ldots, n). \tag{2.44}$$

Analogously, the lower bound $\mu^L(\alpha)$ of the player II's loss-ceiling in the interval-valued matrix game $\tilde{\boldsymbol{A}}(\alpha)$ and corresponding optimal mixed strategy $\boldsymbol{z}^{L*}(\alpha)$ are $\mu^L(\alpha) = \omega^L((a_{ij}^L(\alpha)))$ and $\boldsymbol{z}^{L*}(\alpha) = \boldsymbol{z}^{L*}((a_{ij}^L(\alpha)))$, respectively. Then, according to Eq. (1.8), $(\mu^L(\alpha), \boldsymbol{z}^{L*}(\alpha))$ is an optimal solution to the linear programming model as follows:

$$\min\{\omega^L(\alpha)\}$$
$$\text{s.t.}\begin{cases} \sum_{j=1}^{n} a_{ij}^L(\alpha) z_j^L(\alpha) \leq \omega^L(\alpha) \quad (i = 1, 2, \ldots, m) \\ \sum_{j=1}^{n} z_j^L(\alpha) = 1 \\ z_j^L(\alpha) \geq 0 \quad (j = 1, 2, \ldots, n) \\ \omega^L(\alpha) \text{ unrestricted in sign,} \end{cases} \tag{2.45}$$

where $\omega^L(\alpha)$ and $z_j^L(\alpha)$ $(j = 1, 2, \ldots, n)$ are decision variables.

Without loss of generality [26], assume that $\omega^L(\alpha) > 0$. Let

$$t_j^L(\alpha) = \frac{z_j^L(\alpha)}{\omega^L(\alpha)} \quad (j = 1, 2, \ldots, n), \tag{2.46}$$

then

$$\sum_{j=1}^{n} t_j^L(\alpha) = \frac{1}{\omega^L(\alpha)}. \tag{2.47}$$

Combining with Eq. (2.26), Eq. (2.45) can be converted into the linear programming model as follows:

$$\max\left\{\sum_{j=1}^{n} t_j^L(\alpha)\right\}$$
$$\text{s.t.}\begin{cases} \sum_{j=1}^{n} [\alpha a_{ij}^m + (1-\alpha) a_{ij}^l] t_j^L(\alpha) \leq 1 \quad (i = 1, 2, \ldots, m) \\ t_j^L(\alpha) \geq 0 \qquad (j = 1, 2, \ldots, n), \end{cases} \tag{2.48}$$

where $t_j^L(\alpha)$ $(j = 1, 2, \ldots, n)$ are decision variables.

Solving Eq. (2.48) by using the simplex method of linear programming, we can obtain its optimal solution, denoted by $\boldsymbol{t}^{L*}(\alpha) = (t_1^{L*}(\alpha), t_2^{L*}(\alpha), \ldots, t_n^{L*}(\alpha))^{\mathrm{T}}$. According to Eqs. (2.46) and (2.47), the lower bound $\mu^L(\alpha)$ and the optimal mixed strategy $\boldsymbol{z}^{L*}(\alpha) = (z_1^{L*}(\alpha), z_2^{L*}(\alpha), \ldots, z_n^{L*}(\alpha))^{\mathrm{T}}$ can be obtained, respectively, where

$$\mu^L(\alpha) = \frac{1}{\sum_{j=1}^{n} t_j^{L*}(\alpha)} \tag{2.49}$$

and

$$z_j^{L*}(\alpha) = \mu^L(\alpha) t_j^{L*}(\alpha) \quad (j = 1, 2, \ldots, n). \tag{2.50}$$

Thus, the lower bound $\mu^L(\alpha)$ and upper bound $\mu^R(\alpha)$ and corresponding optimal mixed strategies for the player II can be obtained. Hereby, the player II's loss-ceiling in the interval-valued matrix game $\tilde{\boldsymbol{A}}(\alpha)$ is obtained as an interval $\tilde{\mu}(\alpha) = [\mu^L(\alpha), \mu^R(\alpha)]$, which is a α-cut set of $\tilde{\omega}$, i.e., $\tilde{\mu}(\alpha) = \tilde{\omega}(\alpha)$.

It is easy to see that Eqs. (2.30) and (2.42) are a pair of primal-dual linear programming models. Therefore, the minimum of $\sum_{i=1}^{m} x_i^R(\alpha)$ (i.e., the maximum of $\upsilon^R(\alpha)$) is equal to the maximum of $\sum_{j=1}^{n} t_j^R(\alpha)$ (i.e., the minimum of $\omega^R(\alpha)$) by the duality theorem of linear programming, i.e.,

$$v^R(\alpha) = \mu^R(\alpha).$$

In the same way, Eqs. (2.36) and (2.48) are a pair of primal-dual linear programming models. Hence, we have

$$v^L(\alpha) = \mu^L(\alpha).$$

Therefore, the player I's gain-floor $\tilde{v}(\alpha) = [v^L(\alpha), v^R(\alpha)]$ is equal to the player II's loss-ceiling $\tilde{\mu}(\alpha) = [\mu^L(\alpha), \mu^R(\alpha)]$, i.e., $\tilde{v}(\alpha) = \tilde{\mu}(\alpha)$. Namely, the players' gain-floor and loss-ceiling have a common interval-type value. According to Definition of the value of matrix games [26], the interval-valued matrix game $\tilde{\boldsymbol{A}}(\alpha)$ has an interval-type value, denoted by the interval $\tilde{V}(\alpha) = [V^L(\alpha), V^R(\alpha)]$, where $\tilde{V}(\alpha) = \tilde{v}(\alpha) = \tilde{\mu}(\alpha)$. Essentially, $\tilde{V}(\alpha)$ is a α-cut set of $\tilde{V}$ of the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers. Noticing the fact that $\tilde{V}(\alpha) = \tilde{\upsilon}(\alpha) = \tilde{\omega}(\alpha)$ for any $\alpha \in [0, 1]$. According to the concept of α-cuts and the representation theorem for fuzzy sets [5], we directly have $\tilde{V} = \tilde{\upsilon} = \tilde{\omega}$, which infers that the player I's gain-floor $\tilde{\upsilon}$ is equal to the player II's loss-ceiling $\tilde{\omega}$ (or the players' gain-floor and loss-ceiling have a common value) and hereby the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers has the fuzzy value $\tilde{V}$, which is also a triangular fuzzy number as stated in Sect. 2.2.

Example 2.5 Let us again consider the matrix game $\tilde{\boldsymbol{A}}^0$ with payoffs of triangular fuzzy numbers, which is given in Example 2.4.

For any $\alpha \in [0, 1]$, we can obtain the interval-valued matrix game $\tilde{\boldsymbol{A}}^0(\alpha)$ whose interval-valued payoff matrix is given as follows:

$$\tilde{\boldsymbol{A}}^0(\alpha) = \begin{array}{c} \\ \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{array} \begin{array}{c} \begin{array}{cccc} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{array} \\ \begin{pmatrix} [8+0.5\alpha, 10-1.5\alpha] & \underline{[5+2\alpha, 8-\alpha]} & [14+2\alpha, 18-2\alpha] & \underline{[5+2\alpha, 8-\alpha]} \\ [14+2\alpha, 18-2\alpha] & [3.5+0.5\alpha, 5-\alpha] & [-5+2\alpha, -1-2\alpha] & [2+\alpha, 3.5-0.5\alpha] \\ [11+\alpha, 14-2\alpha] & \underline{[5+2\alpha, 8-\alpha]} & [8+\alpha, 11-2\alpha] & \underline{[5+2\alpha, 8-\alpha]} \\ [-5+2\alpha, -1-2\alpha] & [-1+\alpha, 2-\alpha] & [20+\alpha, 25-4\alpha] & [3.5+0.5\alpha, 5-\alpha] \end{pmatrix} \end{array}.$$

According to the minimax/maximin criteria and the ranking methods of intervals, it is easy to see that the players' gain-floor and loss-ceiling have a common interval-type value, i.e., $\tilde{\nu}^0(\alpha) = \tilde{\rho}^0(\alpha) = [5+2\alpha, 8-\alpha]$. Therefore, there are still the four pure strategy saddle points (δ_1, β_2), (δ_1, β_4), (δ_3, β_2), (δ_3, β_4) [or (1, 2), (1, 4), (3, 2), (3, 4)] and the interval-valued matrix game $\tilde{\boldsymbol{A}}^0(\alpha)$ has an interval-type value $\tilde{V}^0(\alpha) = [5+2\alpha, 8-\alpha]$. Noticing that $\alpha \in [0, 1]$ is arbitrary. Hence, the player I's gain-floor in the aforementioned matrix game $\tilde{\boldsymbol{A}}^0$ with payoffs of triangular fuzzy numbers is equal to the player II's loss-ceiling, i.e., $\tilde{\upsilon}^0 = \tilde{\omega}^0 = (5, 7, 8)$. Thus, the matrix game $\tilde{\boldsymbol{A}}^0$ with payoffs of triangular fuzzy numbers has a fuzzy value $\tilde{V}^0$ at the pure strategy saddle points (δ_1, β_2), (δ_1, β_4), (δ_3, β_2), and (δ_3, β_4), where $\tilde{V}^0 = \tilde{\upsilon}^0 = \tilde{\omega}^0 = (5, 7, 8)$. Obviously, these results are the same as those obtained in Example 2.4.

Likewise, for the aforementioned interval-valued matrix game $\tilde{\boldsymbol{A}}^0(\alpha)$, according to Eqs. (2.30), (2.36), (2.42), and (2.48), we can easily obtain the player I's gain-floor $\tilde{\nu}^0(\alpha) = [5+2\alpha, 8-\alpha]$ and optimal mixed strategy $\boldsymbol{y}^* = (0.5, 0, 0.5, 0)^{\mathrm{T}}$ as well as the player II's loss-ceiling $\tilde{\mu}^0(\alpha) = [5+2\alpha, 8-\alpha]$ and optimal mixed strategy $\boldsymbol{z}^* = (0, 0.5, 0, 0.5)^{\mathrm{T}}$. Then, the interval-valued matrix game $\tilde{\boldsymbol{A}}^0(\alpha)$ has an interval-type value $\tilde{V}^0(\alpha)$, where $\tilde{V}^0(\alpha) = \tilde{\nu}^0(\alpha) = \tilde{\mu}^0(\alpha)$. Hereby, the matrix game $\tilde{\boldsymbol{A}}^0$ with payoffs of triangular fuzzy numbers has the fuzzy value $\tilde{V}^0 = (5, 7, 8)$ and corresponding optimal mixed strategies for the players I and II are $\boldsymbol{y}^* = (0.5, 0, 0.5, 0)^{\mathrm{T}}$ and $\boldsymbol{z}^* = (0, 0.5, 0, 0.5)^{\mathrm{T}}$, respectively, where $\tilde{V}^0 = \tilde{\upsilon}^0 = \tilde{\omega}^0 = (5, 7, 8)$.

Example 2.6 Let us use the proposed method in this section to solve the specific matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers given in Example 2.2. The payoff matrix of the player I is $\tilde{\boldsymbol{A}}_2$ given as in Example 2.2 and the pure and mixed strategies of the players I and II are crisp.

According to Eqs. (2.30) and (2.42), the linear programming models are constructed as follows:

$$\min\{x_1^R(\alpha) + x_2^R(\alpha)\}$$
$$\text{s.t.} \begin{cases} [180\alpha + 190(1-\alpha)]x_1^R(\alpha) + [90\alpha + 100(1-\alpha)]x_2^R(\alpha) \geq 1 \\ [156\alpha + 158(1-\alpha)]x_1^R(\alpha) + [180\alpha + 190(1-\alpha)]x_2^R(\alpha) \geq 1 \\ [x_1^R(\alpha) \geq 0, x_2^R(\alpha) \geq 0 \end{cases}$$

Table 2.1 Upper and lower bounds of interval-type values of the interval-valued matrix games and the players' optimal strategies

α	0	0.1	0.2	0.3
$V^R(\alpha)$	166.3934	165.8317	165.2757	164.7257
$(y_1^{R*}(\alpha), y_2^{R*}(\alpha))^{\mathrm{T}}$	(0.7377, 0.2623)	(0.7426, 0.2574)	(0.7475, 0.2525)	(0.7525, 0.2475)
$(z_1^{R*}(\alpha), z_2^{R*}(\alpha))^{\mathrm{T}}$	(0.2623, 0.7377)	(0.2574, 0.7426)	(0.2525, 0.7475)	(0.2475, 0.7525)
$V^L(\alpha)$	155.2083	155.7927	156.3771	156.9615
$(y_1^{L*}(\alpha), y_2^{L*}(\alpha))^{\mathrm{T}}$	(0.7917, 0.2083)	(0.7915, 0.2085)	(0.7912, 0.2088)	(0.7910, 0.2090)
$(z_1^{L*}(\alpha), z_2^{L*}(\alpha))^{\mathrm{T}}$	(0.2083, 0.7917)	(0.2085, 0.7915)	(0.2088, 0.7912)	(0.2090, 0.7910)
$\tilde{V}(\alpha) = [V^L(\alpha), V^R(\alpha)]$	[155.2083, 166.3934]	[155.7927, 165.8317]	[156.3771, 165.2757]	[156.9615, 164.7257]
α	0.4	0.5	0.6	0.7
$V^R(\alpha)$	164.1818	163.6441	163.1126	162.5876
$(y_1^{R*}(\alpha), y_2^{R*}(\alpha))^{\mathrm{T}}$	(0.7576, 0.2424)	(0.7627, 0.2373)	(0.7679, 0.2321)	(0.7732, 0.2268)
$(z_1^{R*}(\alpha), z_2^{R*}(\alpha))^{\mathrm{T}}$	(0.2424, 0.7576)	(0.2373, 0.7627)	(0.2321, 0.7679)	(0.2268, 0.7732)
$V^L(\alpha)$	157.5459	158.1303	158.7148	159.2992
$(y_1^{L*}(\alpha), y_2^{L*}(\alpha))^{\mathrm{T}}$	(0.7908, 0.2092)	(0.7906, 0.2094)	(0.7904, 0.2096)	(0.7902, 0.2098)
$(z_1^{L*}(\alpha), z_2^{L*}(\alpha))^{\mathrm{T}}$	(0.2092, 0.7908)	(0.2094, 0.7906)	(0.2096, 0.7904)	(0.2098, 0.7902)
$\tilde{V}(\alpha) = [V^L(\alpha), V^R(\alpha)]$	[157.5459, 164.1818]	[158.1303, 163.6441]	[158.7148, 163.1126]	[159.2992, 162.5876]
α	0.8	0.9	1.0	
$V^R(\alpha)$	162.0692	161.5575	161.0526	
$(y_1^{R*}(\alpha), y_2^{R*}(\alpha))^{\mathrm{T}}$	(0.7785, 0.2215)	(0.7840, 0.2160)	(0.7895, 0.2105)	
$(z_1^{R*}(\alpha), z_2^{R*}(\alpha))^{\mathrm{T}}$	(0.2215, 0.7785)	(0.2160, 0.7840)	(0.2105, 0.7895)	
$V^L(\alpha)$	159.8837	160.4682	161.0526	
$(y_1^{L*}(\alpha), y_2^{L*}(\alpha))^{\mathrm{T}}$	(0.7899, 0.2101)	(0.7897, 0.2103)	(0.7895, 0.2105)	
$(z_1^{L*}(\alpha), z_2^{L*}(\alpha))^{\mathrm{T}}$	(0.2101, 0.7899)	(0.2103, 0.7897)	(0.2105, 0.7895)	
$\tilde{V}(\alpha) = [V^L(\alpha), V^R(\alpha)]$	[159.8837, 162.0692]	[160.4682, 161.5575]	161.0526	

and

$$\max\{t_1^R(\alpha)+t_2^R(\alpha)\}$$
$$\text{s.t.}\begin{cases}[180\alpha+190(1-\alpha)]t_1^R(\alpha)+[156\alpha+158(1-\alpha)]t_2^R(\alpha)\leq 1\\ [90\alpha+100(1-\alpha)]t_1^R(\alpha)+[180\alpha+190(1-\alpha)]t_2^R(\alpha)\leq 1\\ t_1^R(\alpha)\geq 0, t_2^R(\alpha)\geq 0,\end{cases}$$

where $x_1^R(\alpha)$, $x_2^R(\alpha)$, $t_1^R(\alpha)$, and $t_2^R(\alpha)$ are decision variables.

For some given special values of $\alpha\in[0,1]$, solving the above two linear programming models by using the simplex method of linear programming, we can obtain their optimal solutions $\boldsymbol{x}^{R*}(\alpha)=(x_1^{R*}(\alpha),x_2^{R*}(\alpha))^{\mathrm{T}}$ and $\boldsymbol{t}^{R*}(\alpha)=(t_1^{R*}(\alpha),t_2^{R*}(\alpha))^{\mathrm{T}}$, respectively. Combining with Eqs. (2.31), (2.32), (2.43), and (2.44), we obtain the upper bounds of the interval-type values of the interval-valued matrix games and corresponding optimal strategies for the players I and II, depicted as in Table 2.1.

Analogously, according to (2.36) and (2.48), the linear programming models are constructed as follows:

$$\min\{x_1^L(\alpha)+x_2^L(\alpha)\}$$
$$\text{s.t.}\begin{cases}[180\alpha+175(1-\alpha)]x_1^L(\alpha)+[90\alpha+80(1-\alpha)]x_2^L(\alpha)\geq 1\\ [156\alpha+150(1-\alpha)]x_1^L(\alpha)+[180\alpha+175(1-\alpha)]x_2^L(\alpha)\geq 1\\ x_1^L(\alpha)\geq 0, x_2^L(\alpha)\geq 0\end{cases}$$

and

$$\max\{t_1^L(\alpha)+t_2^L(\alpha)\}$$
$$\text{s.t.}\begin{cases}[180\alpha+175(1-\alpha)]t_1^L(\alpha)+[156\alpha+150(1-\alpha)]t_2^L(\alpha)\leq 1\\ [90\alpha+80(1-\alpha)]t_1^L(\alpha)+[180\alpha+175(1-\alpha)]t_2^L(\alpha)\leq 1\\ t_1^L(\alpha)\geq 0, t_2^L(\alpha)\geq 0,\end{cases}$$

where $x_1^L(\alpha)$, $x_2^L(\alpha)$, $t_1^L(\alpha)$, and $t_2^L(\alpha)$ are decision variables.

For the given special values of $\alpha\in[0,1]$, solving the above linear programming models by using the simplex method of linear programming, we can obtain their optimal solutions $\boldsymbol{x}^{L*}(\alpha)=(x_1^{L*}(\alpha),x_2^{L*}(\alpha))^{\mathrm{T}}$ and $\boldsymbol{t}^{L*}(\alpha)=(t_1^{L*}(\alpha),t_2^{L*}(\alpha))^{T}$, respectively. Combining with Eqs. (2.37), (2.38), (2.49), and (2.50), we obtain the lower bounds of the interval-type values of the interval-valued matrix games and corresponding optimal strategies for the players I and II, depicted as in Table 2.1.

For $\alpha=1$, it is easy to see from Table 2.1 that the value of the interval-valued matrix game is $\tilde{V}(1)=161.0526$ when the player I employs the optimal strategy $(0.7895,0.2105)^{\mathrm{T}}$ and the player II employs the optimal strategy $(0.2105,0.7895)^{\mathrm{T}}$, respectively. It is noticed that the upper and lower bounds of the interval-type value of the interval-valued matrix game are identical, i.e., $V^L(1) = V^R(1) = 161.0526$. Namely, the interval-type value $\tilde{V}(1)$ degenerates to the real number 161.0526. Moreover, the player I's optimal strategies $\boldsymbol{y}^{R*}(1) = (y_1^{R*}(1), y_2^{R*}(1))^{\mathrm{T}}$ and $\boldsymbol{y}^{L*}(1) = (y_1^{L*}(1), y_2^{L*}(1))^{\mathrm{T}}$ are identical, i.e., $\boldsymbol{y}^{R*}(1) = \boldsymbol{y}^{L*}(1) = (0.7895, 0.2105)^{\mathrm{T}}$. The player II's optimal strategies $\boldsymbol{z}^{R*}(1) = (z_1^{R*}(1), z_2^{R*}(1))^{\mathrm{T}}$ and $\boldsymbol{z}^{L*}(1) = (z_1^{L*}(1), z_2^{L*}(1))^{\mathrm{T}}$ are identical, i.e., $\boldsymbol{z}^{R*}(1) = \boldsymbol{z}^{L*}(1) = (0.2105, 0.7895)^{\mathrm{T}}$.

In the same way, for $\alpha = 0$, it is easy to see from Table 2.1 that the value of the interval-valued matrix game is the interval $\tilde{V}(0) = [155.2083, 166.3934]$. The player I wins (i.e., the player II loses) the upper bound $V^R(0) = 166.3934$ of the value $\tilde{V}(0)$ when the player I employs the optimal strategy $\boldsymbol{y}^{R*}(0) = (0.7377, 0.2623)^{\mathrm{T}}$ and the player II employs the optimal strategy $\boldsymbol{z}^{R*}(0) = (0.2623, 0.7377)^{\mathrm{T}}$, respectively. The player I wins (i.e., the player II loses) the lower bound $V^L(0) = 155.2083$ of the value $\tilde{V}(0)$ when the player I employs the optimal strategy $\boldsymbol{y}^{L*}(0) = (0.7917, 0.2083)^{\mathrm{T}}$ and the player II employs the optimal strategy $\boldsymbol{z}^{L*}(0) = (0.2083, 0.7917)^{\mathrm{T}}$, respectively.

For $\alpha = 0.6$, it is easy to see from Table 2.1 that the value of the interval-valued matrix game is the interval $\tilde{V}(0.6) = [158.7158, 163.1126]$. The player I wins (i.e., the player II loses) the upper bound $V^R(0.6) = 163.1126$ of the value $\tilde{V}(0.6)$ when the player II employs the optimal strategy $\boldsymbol{y}^{R*}(0.6) = (0.7679, 0.2321)^{\mathrm{T}}$ and the player II employs the optimal strategy $\boldsymbol{z}^{R*}(0.6) = (0.2321, 0.7679)^{\mathrm{T}}$, respectively. Likewise, the player I wins (i.e., the player II loses) the lower bound $V^L(0.6) = 158.7148$ of the value $\tilde{V}(0.6)$ when the player I employs the optimal strategy $\boldsymbol{y}^{L*}(0.6) = (0.7904, 0.2096)^{\mathrm{T}}$ and the player II employs the optimal strategy $\boldsymbol{z}^{L*}(0.6) = (0.2096, 0.7904)^{\mathrm{T}}$, respectively. The obtained results in Table 2.1 for the other values $\alpha \in [0, 1]$ are similarly explained.

2.6.2 Linear Programming Method of Matrix Games with Payoffs of Triangular Fuzzy Numbers

Usually, computing fuzzy values of matrix games with payoffs of triangular fuzzy numbers is not easier than that in Example 2.5. In the sequent, we focus on developing an effective and a simple method which can explicitly and quickly compute fuzzy values of matrix games $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers.

For $\alpha = 1$, according to Eqs. (2.30) and (2.48), the linear programming models are constructed as follows:

$$\min\left\{\sum_{i=1}^{m} x_i^R(1)\right\}$$
$$\text{s.t.}\begin{cases}\sum\limits_{i=1}^{m} a_{ij}^m x_i^R(1) \geq 1 \quad (j = 1, 2, \ldots, n)\\ x_i^R(1) \geq 0 \quad (i = 1, 2, \ldots, m)\end{cases} \tag{2.51}$$

and

$$\max\left\{\sum_{j=1}^{n} t_j^L(1)\right\}$$
$$\text{s.t.}\begin{cases}\sum\limits_{j=1}^{n} a_{ij}^m t_j^L(1) \leq 1 \quad (i = 1, 2, \ldots, m)\\ t_j^L(1) \geq 0 \quad (j = 1, 2, \ldots, n),\end{cases} \tag{2.52}$$

where $x_i^R(1)$ and $t_j^L(1)$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ are decision variables.

Obviously, Eqs. (2.51) and (2.52) are a pair of primal-dual linear programming models. Then, the minimum of $\sum_{i=1}^{m} x_i^R(1)$ (i.e., the maximum of $\upsilon^R(1)$) is equal to the maximum of $\sum_{j=1}^{n} t_j^L(1)$ (i.e., the minimum of $\omega^L(1)$) by the duality theorem of linear programming [26], i.e., $v^R(1) = \mu^L(1)$.

Analogously, for $\alpha = 1$, according to Eqs. (2.36) and (2.42), the linear programming models are constructed as follows:

$$\min\left\{\sum_{i=1}^{m} x_i^L(1)\right\}$$
$$\text{s.t.}\begin{cases}\sum\limits_{i=1}^{m} a_{ij}^m x_i^L(1) \geq 1 \quad (j = 1, 2, \ldots, n)\\ x_i^L(1) \geq 0 \quad (i = 1, 2, \ldots, m)\end{cases} \tag{2.53}$$

and

$$\max\left\{\sum_{j=1}^{n} t_j^R(1)\right\}$$
$$\text{s.t.}\begin{cases}\sum\limits_{j=1}^{n} a_{ij}^m t_j^R(1) \leq 1 \quad (i = 1, 2, \ldots, m)\\ t_j^R(1) \geq 0 \quad (j = 1, 2, \ldots, n),\end{cases} \tag{2.54}$$

where $x_i^L(1)$ and $t_j^R(1)$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ are decision variables.

Obviously, Eqs. (2.53) and (2.54) are a pair of primal-dual linear programming models. According to the duality theorem of linear programming, we have

$v^L(1) = \mu^R(1)$. Combining with the above discussion, it directly follows that $v^L(1) = v^R(1) = \mu^L(1) = \mu^R(1)$. Thus, $[v^L(1), v^R(1)] = [\mu^L(1), \mu^R(1)]$ degenerates to a real number. Hence, $V^L(1) = V^R(1) = v^L(1)$, i.e., $\tilde{V}(1)$ is a real number. It is derived from the notation of the triangular fuzzy number $\tilde{V} = (V^l, V^m, V^r)$ that $V^m = V^L(1) = V^R(1)$. Namely, the mean of the fuzzy value $\tilde{V}$ can be directly obtained by solving one of Eqs. (2.51)–(2.54).

In the same way, for $\alpha = 0$, according to Eqs. (2.30) and (2.42), the linear programming models are constructed as follows:

$$\begin{aligned} &\min\left\{\sum_{i=1}^{m} x_i^R(0)\right\} \\ &\text{s.t.}\begin{cases} \sum\limits_{i=1}^{m} a_{ij}^r x_i^R(0) \geq 1 \quad (j = 1, 2, \ldots, n) \\ x_i^R(0) \geq 0 \quad (i = 1, 2, \ldots, m) \end{cases} \end{aligned} \tag{2.55}$$

and

$$\begin{aligned} &\max\left\{\sum_{j=1}^{n} t_j^R(0)\right\} \\ &s.t.\begin{cases} \sum\limits_{j=1}^{n} a_{ij}^r t_j^R(0) \leq 1 \quad (i = 1, 2, \ldots, m) \\ t_j^R(0) \geq 0 \quad (j = 1, 2, \ldots, n), \end{cases} \end{aligned} \tag{2.56}$$

which infer that $v^R(0) = \mu^R(0)$.

Analogously, according to Eqs. (2.36) and (2.48), the linear programming models are constructed as follows:

$$\begin{aligned} &\min\left\{\sum_{i=1}^{m} x_i^L(0)\right\} \\ &\text{s.t.}\begin{cases} \sum\limits_{i=1}^{m} a_{ij}^l x_i^L(0) \geq 1 \quad (j = 1, 2, \ldots, n) \\ x_i^L(0) \geq 0 \quad (i = 1, 2, \ldots, m) \end{cases} \end{aligned} \tag{2.57}$$

and

$$\begin{aligned} &\max\left\{\sum_{j=1}^{n} t_j^L(0)\right\} \\ &\text{s.t.}\begin{cases} \sum\limits_{j=1}^{n} a_{ij}^l t_j^L(0) \leq 1 \quad (i = 1, 2, \ldots, m) \\ t_j^L(0) \geq 0 \quad (j = 1, 2, \ldots, n), \end{cases} \end{aligned} \tag{2.58}$$

which infer that $v^L(0) = \mu^L(0)$.

It is easily derived from the above discussion that $\tilde{V}(0) = \tilde{v}(0) = \bar{\mu}(0)$. According to the notation of the triangular fuzzy number $\tilde{V} = (V^l, V^m, V^r)$, it follows that $V^l = V^L(0) = v^L(0)$ and $V^r = V^R(0) = v^R(0)$, which mean that the lower and upper bounds of the fuzzy value $\tilde{V}$ can be directly obtained by solving either Eqs. (2.55) and (2.57) or Eqs. (2.56) and (2.58). Obviously, $\tilde{V}(0) = [V^L(0), V^R(0)] = [V^l, V^r]$.

Thus, according to Eq. (2.4), any α-cut set of the fuzzy value $\tilde{V}$ of the matrix game $\tilde{\boldsymbol{A}}$ with payoffs of triangular fuzzy numbers can be obtained as

$$[V^L(\alpha), V^R(\alpha)] = [\alpha V^m + (1-\alpha)V^l, \alpha V^m + (1-\alpha)V^r].$$

Hereby, according to Eq. (2.5) or the representation theorem for the fuzzy set [5], the fuzzy value $\tilde{V}$ can be expressed as

$$\tilde{V} = \bigcup_{\alpha\in[0,1]} \{\alpha \otimes \tilde{V}(\alpha)\} = \bigcup_{\alpha\in[0,1]} \{\alpha \otimes [\alpha V^m + (1-\alpha)V^l, \alpha V^m + (1-\alpha)V^r]\},$$

which means that $\tilde{V}$ can be explicitly obtained by using both its 1-cut set and 0-cut set of fuzzy payoffs.

2.6.3 Computational Analysis of a Real Example

Let us continue to consider the specific matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers given in Example 2.2. The players' pure and mixed strategies are crisp and the player I' payoff matrix is $\tilde{\boldsymbol{A}}_2$ as stated in Example 2.2.

1. Computational results obtained by the proposed Alfa-cut-based primal-dual linear programming method

   Using Eq. (2.51), the linear programming model is constructed as follows:

$$\min\{x_1^R(1) + x_2^R(1)\}$$
$$\text{s.t.}\begin{cases} 180x_1^R(1) + 90x_2^R(1) \geq 1 \\ 156x_1^R(1) + 180x_2^R(1) \geq 1 \\ x_1^R(1) \geq 0,\ x_2^R(1) \geq 0, \end{cases}$$

where $x_1^R(1)$ and $x_2^R(1)$ are decision variables. Solving the above linear programming model by using the simplex method of linear programming, we obtain its optimal solution $\boldsymbol{x}^{R*}(1) = (x_1^{R*}(1), x_2^{R*}(1))^{\mathrm{T}}$, where

$$x_1^{R*}(1) = \frac{1}{204} \approx 0.0049, \quad x_2^{R*}(1) = \frac{1}{765} \approx 0.0013.$$

According to Eqs. (2.31) and (2.32), we obtain V^m and corresponding optimal mixed strategy $\boldsymbol{y}^{R*}(1) = (y_1^{R*}(1), y_2^{R*}(1))^{\mathrm{T}}$ for the player I, where

$$\begin{aligned} V^m &= v^R(1) = \frac{1}{\frac{1}{204} + \frac{1}{765}} = \frac{52020}{323} \approx 161.0526, \\ y_1^{R*}(1) &= \frac{52020}{323} \times \frac{1}{204} = \frac{255}{323} \approx 0.7895 \end{aligned}$$

and

$$y_2^{R*}(1) = \frac{52020}{323} \times \frac{1}{765} = \frac{68}{323} \approx 0.2105.$$

Analogously, according to Eq. (2.55), the linear programming model is constructed as follows:

$$\min\{x_1^R(0) + x_2^R(0)\}$$
$$\text{s.t.} \begin{cases} 190x_1^R(0) + 100x_2^R(0) \geq 1 \\ 158x_1^R(0) + 190x_2^R(0) \geq 1 \\ x_1^R(0) \geq 0, \ x_2^R(0) \geq 0, \end{cases}$$

where $x_1^R(0)$ and $x_2^R(0)$ are decision variables. Solving the above linear programming model, we obtain its optimal solution $\boldsymbol{x}^{R*}(0) = (x_1^{R*}(0), x_2^{R*}(0))^{\mathrm{T}}$, where

$$x_1^{R*}(0) = \frac{9}{2030} \approx 0.0044, \quad x_2^{R*}(0) = \frac{8}{5075} \approx 0.0016.$$

According to Eqs. (2.31) and (2.32), we obtain V^r and corresponding optimal mixed strategy $\boldsymbol{y}^{R*}(0) = (y_1^{R*}(0), y_2^{R*}(0))^{\mathrm{T}}$ for the player I, where

$$\begin{aligned} V^r &= v^R(0) = \frac{1}{\dfrac{9}{2030} + \dfrac{8}{5075}} = \frac{2060450}{12383} \approx 166.3934, \\ y_1^{R*}(0) &= \frac{2060450}{12383} \times \frac{9}{2030} = \frac{9135}{12383} \approx 0.7377 \end{aligned}$$

and

$$y_2^{R*}(0) = \frac{2060450}{12383} \times \frac{8}{5075} = \frac{3248}{12383} \approx 0.2623.$$

According to Eq. (2.57), the linear programming model is constructed as follows:

$$\min\{x_1^L(0)+x_2^L(0)\}$$
$$\text{s.t.}\begin{cases}175x_1^L(0)+80x_2^L(0)\geq 1\\ 150x_1^L(0)+175x_2^L(0)\geq 1\\ x_1^L(0)\geq 0, x_2^L(0)\geq 0,\end{cases}$$

where $x_1^L(0)$ and $x_2^L(0)$ are decision variables. Solving the above linear programming model, we obtain its optimal solution $\boldsymbol{x}^{L*}(0)=(x_1^{L*}(0), x_2^{L*}(0))^{\mathrm{T}}$, where

$$x_1^{L*}(0)=\frac{19}{3725}\approx 0.0051,\quad x_2^{L*}(0)=\frac{1}{745}\approx 0.0013.$$

According to Eqs. (2.37) and (2.38), we obtain V^l and the optimal mixed strategy $\boldsymbol{y}^{L*}(0)=(y_1^{L*}(0), y_2^{L*}(0))^{\mathrm{T}}$ for the player I, where

$$V^l=v^L(0)=\frac{1}{\frac{19}{3725}+\frac{1}{745}}=\frac{3725}{24}\approx 155.2083,$$
$$y_1^{L*}(0)=\frac{3725}{24}\times\frac{19}{3725}=\frac{19}{24}\approx 0.7917$$

and

$$y_2^{L*}(0)=\frac{3725}{24}\times\frac{1}{745}=\frac{5}{24}\approx 0.2083.$$

Therefore, the fuzzy value of the matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers can be directly obtained as $\tilde{V}'^{*}=(V^l, V^m, V^r)=(155.2083, 161.0526, 166.3934)$, whose membership function is given as follows:

$$\mu_{\tilde{V}'^{*}}(x)=\begin{cases}\dfrac{x-155.2083}{5.8443} & \text{if } 155.2083\leq x<161.0526\\ 1 & \text{if } x=161.0526\\ \dfrac{166.3934-x}{5.3408} & \text{if } 161.0526<x\leq 166.3934\\ 0 & \text{else,}\end{cases}$$

depicted as in Fig. 2.8.

2. Computational results obtained by other methods and analysis

The above numerical example was solved by the two-level linear programming method proposed in Sect. 2.4 and the lexicographic method proposed in Sect. 2.5.

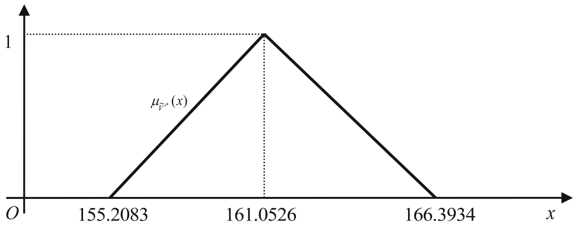


Fig. 2.8 The fuzzy value $\tilde{V}'^*$

In this subsection, this matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers is solved by other methods [7, 14]. The computational results are analyzed and compared to show the validity, applicability, and superiority of the proposed method in this Section.

(2a) Computational results obtained by Campos' method

Taking the players' gain-floor and loss-ceiling as crisp values, following a similar way to crisp matrix games [4, 26], i.e., according to Eqs. (2.8) and (2.9), using a suitable defuzzification (i.e., linear ranking) function of fuzzy numbers, Campos [7] constructed the auxiliary linear programming models as follows:

$$\begin{array}{l} \min\left\{\sum\limits_{i=1}^{m} u_i^C\right\} \\ \text{s.t.}\begin{cases} \sum\limits_{i=1}^{m}(a_{ij}^l + a_{ij}^m + a_{ij}^r)u_i^C \geq 3-(1-\lambda)(p_j^l + p_j^m + p_j^r) \quad (j=1,2,\ldots,n) \\ u_i^C \geq 0 \quad (i=1,2,\ldots,m) \end{cases} \end{array} \tag{2.59}$$

and

$$\begin{array}{l} \max\left\{\sum\limits_{j=1}^{n} v_j^C\right\} \\ \text{s.t.}\begin{cases} \sum\limits_{j=1}^{n}(a_{ij}^l + a_{ij}^m + a_{ij}^r)v_j^C \leq 3+(1-\tau)(q_i^l + q_i^m + q_i^r) \quad (i=1,2,\ldots,m) \\ v_j^C \geq 0 \quad (j=1,2,\ldots,n), \end{cases} \end{array} \tag{2.60}$$

where $\lambda \in [0,1]$ and $\tau \in [0,1]$, $\tilde{p}_j = (p_j^l, p_j^m, p_j^r)$ and $\tilde{q}_i = (q_i^l, q_i^m, q_i^r)$ are triangular fuzzy numbers, and

$$u_i^C = \frac{y_i^C}{v^C} \quad (i=1,2,\ldots,m) \tag{2.61}$$

and

$$v_j^C = \frac{z_j^C}{\omega^C} \quad (j = 1, 2, \ldots, n) \tag{2.62}$$

are decision variables.

For the aforementioned matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers, according to Eqs. (2.59) and (2.60), the linear programming models are constructed as follows:

$$\begin{aligned} &\min\{u_1^C + u_2^C\} \\ &\text{s.t.} \begin{cases} 545u_1^C + 270u_2^C \geq 3 - 0.29(1-\lambda) \\ 464u_1^C + 545u_2^C \geq 3 - 0.29(1-\lambda) \\ u_1^C \geq 0, u_2^C \geq 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\max\{v_1^C + v_2^C\} \\ &\text{s.t.} \begin{cases} 545v_1^C + 464v_2^C \leq 3 + 0.46(1-\tau) \\ 270v_1^C + 545v_2^C \leq 3 + 0.46(1-\tau) \\ v_1^C \geq 0, v_2^C \geq 0, \end{cases} \end{aligned}$$

where $\tilde{p}_1 = \tilde{p}_2 = (0.08, 0.10, 0.11)$ and $\tilde{q}_1 = \tilde{q}_2 = (0.14, 0.15, 0.17)$ are taken from Campos [7].

Solving the above linear programming models by using the simplex method of linear programming, and combining with Eqs. (2.61) and (2.62), we obtain the player I's gain-floor and the player II' loss-ceiling and their optimal mixed strategies as follows:

$$\begin{aligned} v^{*C}(\lambda) &= \frac{171745}{356[3 - 0.29(1-\lambda)]} \approx \frac{160.8099}{1 - 0.0967(1-\lambda)}, \\ (y_1^{*C}, y_2^{*C})^{\mathrm{T}} &= (\frac{275}{356}, \frac{81}{356})^{\mathrm{T}} \approx (0.7725, 0.2275)^{\mathrm{T}}, \\ \omega^{*C}(\tau) &= \frac{171745}{356[3 + 0.46(1-\tau)]} \approx \frac{160.8099}{1 + 0.1533(1-\tau)} \end{aligned}$$

and

$$(z_1^{*C}, z_2^{*C})^{\mathrm{T}} = (\frac{81}{356}, \frac{275}{356})^{\mathrm{T}} \approx (0.2275, 0.7725)^{\mathrm{T}},$$

respectively. Obviously, $v^{*C}(\lambda) \geq \omega^{*C}(\tau)$. Moreover, $\omega^{*C}(\tau)$ is an increasing function of $\tau \in [0,1]$ whereas $v^{*C}(\lambda)$ is a decreasing function of $\lambda \in [0,1]$. It easily follows that $v^{*C}(1) = \omega^{*C}(1) = 160.8099$ when $\lambda = \tau = 1$. Thus, Campos [7] argued that the matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers has the fuzzy value "close to 160.8099".

(2b) Computational results obtained by Bector et al.'s method

Taking the players' gain-floor and loss-ceiling as fuzzy numbers, using a suitable defuzzification function F, according to Eqs. (2.8) and (2.9) and the concept of double fuzzy constraints [7], Bector et al. [14] (with reference to [12, 13]) suggested the mathematical programming models for the players I and II as follows:

$$\max\{F(\tilde{v}^B)\}$$
$$\text{s.t.}\begin{cases} \sum_{i=1}^{m} F(\tilde{a}_{ij}) y_i^B \geq F(\tilde{v}^B) - (1-\lambda)F(\tilde{p}_j) \quad (j = 1,2,\ldots,n) \\ \sum_{i=1}^{m} y_i^B = 1 \\ y_i^B \geq 0 \quad (i = 1,2,\ldots,m) \end{cases} \tag{2.63}$$

and

$$\min\{F(\tilde{\omega}^B)\}$$
$$\text{s.t.}\begin{cases} \sum_{j=1}^{n} F(\tilde{a}_{ij}) z_j^B \leq F(\tilde{\omega}^B) + (1-\tau)F(\tilde{q}_i) \quad (i = 1,2,\ldots,m) \\ \sum_{j=}^{n} z_j^B = 1 \\ z_j^B \geq 0 \quad (j = 1,2,\ldots,n), \end{cases} \tag{2.64}$$

respectively, where $\tilde{p}_j$ and $\tilde{q}_i$ $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$ are fuzzy numbers, $\lambda \in [0,1]$, $\tau \in [0,1]$.

In the case that $\tilde{\boldsymbol{A}}$ is the matrix game with payoffs of triangular fuzzy numbers, i.e., all $\tilde{v}^B = (v^{Bl}, v^{Bm}, v^{Br})$, $\tilde{\omega}^B = (\omega^{Bl}, \omega^{Bm}, \omega^{Br})$, $\tilde{a}_{ij} = (a_{ij}^l, a_{ij}^m, a_{ij}^r)$, $\tilde{p}_j = (p_j^l, p_j^m, p_j^r)$, and $\tilde{q}_i = (q_i^l, q_i^m, q_i^r)$ $(i = 1,2,\ldots,m; j = 1,2,\ldots,n)$ are triangular fuzzy numbers, using Yager's index [27], Bector et al. [14] transformed Eqs. (2.63) and (2.64) into the following linear programming models:

$$\max\{v^B\}$$
$$\text{s.t.}\begin{cases} \sum\limits_{i=1}^{m}(a_{ij}^l + a_{ij}^m + a_{ij}^r)y_i^B \geq 3v^B - (1-\lambda)(p_j^l + p_j^m + p_j^r) \quad (j = 1, 2, \ldots, n) \\ \sum\limits_{i=1}^{m} y_i^B = 1 \\ y_i^B \geq 0 \quad (i = 1, 2, \ldots, m) \end{cases} \tag{2.65}$$

and

$$\min\{\omega^B\}$$
$$\text{s.t.}\begin{cases} \sum\limits_{j=1}^{n}(a_{ij}^l + a_{ij}^m + a_{ij}^r)z_j^B \leq 3\omega^B + (1-\tau)(q_i^l + q_i^m + q_i^r) \quad (i = 1, 2, \ldots, m) \\ \sum\limits_{j=}^{n} z_j^B = 1 \\ z_j^B \geq 0 \quad (j = 1, 2, \ldots, n), \end{cases} \tag{2.66}$$

respectively, where

$$v^B = F(\tilde{v}^B) = \frac{v^{Bl} + v^{Bm} + v^{Br}}{3} \tag{2.67}$$

and

$$\omega^B = F(\tilde{\omega}^B) = \frac{\omega^{Bl} + \omega^{Bm} + \omega^{Br}}{3}. \tag{2.68}$$

For the aforementioned matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers, according to Eqs. (2.65) and (2.66) with $\tilde{p}_1 = \tilde{p}_2 = (0.08, 0.10, 0.11)$ and $\tilde{q}_1 = \tilde{q}_2 = (0.14, 0.15, 0.17)$, the linear programming models are constructed as follows:

$$\max\{v^B\}$$
$$\text{s.t.}\begin{cases} 545y_1^B + 270y_2^B \geq 3v^B - 0.29(1-\lambda) \\ 464y_1^B + 545y_2^B \geq 3v^B - 0.29(1-\lambda) \\ y_1^B + y_2^B = 1 \\ y_1^B \geq 0, y_2^B \geq 0 \end{cases}$$

and

$$\min\{\omega^B\}$$
$$\text{s.t.}\begin{cases} 545z_1^B + 464z_2^B \le 3\omega^B + 0.46(1-\tau) \\ 270z_1^B + 545z_2^B \le 3\omega^B + 0.46(1-\tau) \\ z_1^B + z_2^B = 1 \\ z_1^B \ge 0, z_2^B \ge 0, \end{cases}$$

respectively. Simply computing/solving the above linear programming models, we can obtain the player I's gain-floor, the player II' loss-ceiling, and their optimal mixed strategies as follows:

$$\begin{aligned} v^{*B}(\lambda) &= \frac{171745}{1068} + \frac{0.29(1-\lambda)}{3} \approx 160.8099 + 0.0967(1-\lambda), \\ (y_1^{*B}, y_2^{*B})^{\mathrm{T}} &= (\frac{275}{356}, \frac{81}{356})^{\mathrm{T}} = (0.7725, 0.2275)^{\mathrm{T}}, \\ \omega^{*B}(\tau) &= \frac{171745}{1068} - \frac{0.46(1-\tau)}{3} \approx 160.8099 - 0.1533(1-\tau) \end{aligned}$$

and

$$(z_1^{*B}, z_2^{*B})^{\mathrm{T}} = (\frac{81}{356}, \frac{275}{356})^{\mathrm{T}} = (0.2275, 0.7725)^{\mathrm{T}},$$

respectively.

Obviously, $v^{*B}(\lambda)$ and $\omega^{*B}(\tau)$ remarkably differ from $v^{*C}(\lambda)$ and $\omega^{*C}(\tau)$ when $\lambda \neq 1$ and $\tau \neq 1$.

3. Computational result comparison and conclusions

Comparing the aforementioned modeling, methods, and computational results, we can easily draw the following conclusions.

(3a) Modeling. The players' gain-floor and loss-ceiling were regarded as triangular fuzzy numbers in the proposed methods in this section and Sects. 2.4, 2.5 and Bector et al.'s [14]. However, they were regarded as real numbers in Campos's method [7]. This case is not rational since the players' expected payoffs are a linear combination of fuzzy payoffs which are expressed with triangular fuzzy numbers.

(3b) Process and methods. The proposed method in this section is developed on the monotonicity of values of matrix games. It always ensures that any matrix game with payoffs of triangular fuzzy numbers has a fuzzy value, which is a triangular fuzzy number also. Moreover, the fuzzy value can be directly and explicitly obtained by solving the derived three linear programming models with data taken from 1-cut set and 0-cut set of fuzzy payoffs. Li's model as stated in Sect. 2.4 was developed on the ordering relation of triangular fuzzy numbers [20] and multi-objective programming. The derived six linear programming models were used to compute the players' gain-floor and loss-ceiling. Obviously, Li's model in Sect. 2.4 depended on the ordering relation. Following a similar way to crisp matrix games, based on the concept of double fuzzy constraints and ranking functions, Campos's method [7] regarded the players' gain-floor and loss-ceiling as real numbers and hereby suggested two auxiliary linear programming models. Bector et al.'s method [14] was developed on certain duality of linear programming with fuzzy parameters. As Bector et al. [12] themselves pointed out, Bector et al.'s method [14] was essentially the same as that of Campos [7]. Campos's method and Bector et al.'s method are defuzzification approaches, which not only closely depend on ranking functions, parameters, and adequacies but also cannot explicitly obtain membership functions of the players' gain-floor and loss-ceiling.

(3c) Computational results. The proposed method in this section can explicitly obtain the fuzzy value $\tilde{V}'^* = (155.2083, 161.0526, 166.3934)$ of the matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers. Li's model in Sect. 2.4 can explicitly obtain the player I's gain-floor $\tilde{\upsilon}^* = (154.9996, 161.05, 164.737)$ and the player II's loss-ceiling $\tilde{\omega}^* = (155.2633, 161.05, 171.0523)$ which are not identical. This case is not rational since the matrix game is zero-sum. Moreover, it is intuitively seen from Fig. 2.6 that $\tilde{V}^* = (155.2633, 161.05, 164.737)$ is better than $\tilde{\upsilon}^*$ and $\tilde{\omega}^*$. In fact, using Yager's index F [27], i.e., Eq. (2.67) or Eq. (2.68), we have

$$F(\tilde{\upsilon}^*) = \frac{154.9996 + 161.05 + 164.737}{3} = 160.2622,$$
$$F(\tilde{V}^*) = \frac{155.2633 + 161.05 + 164.737}{3} = 160.3501$$

and

$$F(\tilde{\omega}^*) = \frac{155.2633 + 161.05 + 171.0523}{3} = 162.4552,$$

which infers that $F(\tilde{\upsilon}^*) < F(\tilde{V}^*) < F(\tilde{\omega}^*)$. Therefore, $\tilde{\upsilon}^* < \tilde{V}^* < \tilde{\omega}^*$.

Campos's method [7] provided crisp values for the players' gain-floor and loss-ceiling in the matrix game $\tilde{\boldsymbol{A}}_2$ with payoffs of triangular fuzzy numbers. Bector et al.'s method [14] provided defuzzification values of the players' gain-floor and loss-ceiling. Namely, these two methods cannot explicitly obtain membership functions of the players' gain-floor and loss-ceiling even though these are very much desirable. Moreover, these methods cannot always guarantee that the defuzzification values are identical and the matrix game $\tilde{A}_2$ with payoffs of triangular fuzzy numbers has a defuzzification value. On the other hand, the defuzzification values closely depend on not only choice of ranking functions but also the parameters and adequacies, which are difficult to be appropriately determined a priori.

(3d) Computational complexity. The proposed method in this section needs to solve three linear programming models. Li's model proposed in Sect. 2.4 needs to solve six linear programming models with additional decision variables and constraints, which usually may be superabundant and even contradictable. However, Campos's method [7] and Bector et al.'s method [14] need to solve a series of linear programming models for different parameters and adequacies. Therefore, the computational amount and complexity of the proposed method in this section are less than those of Li's model, Campos's method, and Bector et al.'s method.

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