Thus to every generalized function $F \in \mathcal{D}'(\Gamma)$ we assign the Fourier series

$$F \sim \sum_{-\infty}^{\infty} c_k(F)\mathrm{e}^{iks}.$$

Show that $\delta \sim \sum_{-\infty}^{\infty} \frac{1}{2\pi}\mathrm{e}^{iks}$.

c) Prove the following fact, which is remarkable for its simplicity and the freedom of action that it provides: the Fourier series of every generalized function $F \in \mathcal{D}'(\Gamma)$ converges to $F$ (in the sense of convergence in the space $\mathcal{D}'(\Gamma)$).

d) Show that the Fourier series of a function $F \in \mathcal{D}'(\Gamma)$ (like the function $F$ itself, and like every convergent series of generalized functions) can be differentiated termwise any number of times.

e) Starting from the equality $\delta = \sum_{-\infty}^{\infty} \frac{1}{2\pi}\mathrm{e}^{iks}$, find the Fourier series of $\delta'$.

f) Let us now return from the circle $\Gamma$ to the line $\mathbb{R}$ and study the functions $\mathrm{e}^{iks}$ as regular generalized functions in $\mathcal{D}'(\mathbb{R})$ (that is, as continuous linear functionals on the space $\mathcal{D}(\mathbb{R})$ of functions in the class $C_0^{(\infty)}(\mathbb{R})$ of infinitely differentiable functions of compact support in $\mathbb{R}$).

Every locally integrable function $f$ can be regarded as an element of $\mathcal{D}'(\mathbb{R})$ (a *regular generalized function in* $\mathcal{D}'(\mathbb{R})$) whose effect on the function $\varphi \in C_0^{(\infty)}(\mathbb{R}, \mathbb{C})$ is given by the rule $f(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x$. Convergence in $\mathcal{D}'(\mathbb{R})$ is defined in the standard way:

$$\Bigl(\lim_{n\to\infty} F_n = F\Bigr) := \forall\varphi \in \mathcal{D}(\mathbb{R})\ \Bigl(\lim_{n\to\infty} F_n(\varphi) = F(\varphi)\Bigr).$$

Show that the equality

$$\frac{1}{2\pi}\sum_{-\infty}^{\infty} \mathrm{e}^{ikx} = \sum_{-\infty}^{\infty} \delta(x - 2\pi k)$$

holds in the sense of convergence in $\mathcal{D}'(\mathbb{R})$. In both sides of this equality a limiting passage is assumed as $n \to \infty$ over symmetric partial sums $\sum_{-n}^{n}$, and $\delta(x - x_0)$, as always, denotes the $\delta$-function of $\mathcal{D}'(\mathbb{R})$ shifted to the point $x_0$, that is, $\delta(x - x_0)(\varphi) = \varphi(x_0)$.

## 18.3 The Fourier Transform

### *18.3.1 Representation of a Function by Means of a Fourier Integral*

#### a. The Spectrum and Harmonic Analysis of a Function

Let $f(t)$ be a $T$-periodic function, for example a periodic signal with frequency $\frac{1}{T}$ as a function of time. We shall assume that the function $f$ is absolutely integrable over a period. Expanding $f$ in a Fourier series (when $f$ is sufficiently regular, as we know, the Fourier series converges to $f$) and transforming that series,

$$f(t) = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} a_k(f)\cos k\omega_0 t + b_k(f)\sin k\omega_0 t =$$

$$= \sum_{-\infty}^{\infty} c_k(f)\mathrm{e}^{ik\omega_0 t} = c_0 + 2\sum_{k=1}^{\infty} |c_k|\cos(k\omega_0 t + \arg c_k), \tag{18.78}$$

we obtain a representation of $f$ as a sum of a constant term $\frac{a_0}{2} = c_0$ – the *mean value of $f$ over a period* – and *sinusoidal components* with frequencies $\nu_0 = \frac{1}{T}$ (the *fundamental frequency*), $2\nu_0$ (the *second harmonic frequency*), and so on. In general the $k$th *harmonic component* $2|c_k|\cos(k\frac{2\pi}{T}t + \arg c_k)$ of the signal has *frequency* $k\nu_0 = \frac{k}{T}$, *cyclic frequency* $k\omega_0 = 2\pi k\nu_0 = \frac{2\pi}{T}k$, *amplitude* $2|c_k| = \sqrt{a_k^2 + b_k^2}$, and *phase* $\arg c_k = -\arctan\frac{b_k}{a_k}$.

The expansion of a periodic function (signal) into a sum of simple harmonic oscillations is called the *harmonic* analysis of $f$. The numbers $\{c_k(f); k \in \mathbb{Z}\}$ or $\{a_0(f), a_k(f), b_k(f); k \in \mathbb{N}\}$ are called the *spectrum of the function* (signal) $f$. A periodic function thus has a *discrete spectrum*.

Let us now set out (on a heuristic level) what happens to the expansion (18.78) when the period $T$ of the signal increases without bound.

Simplifying the notation by writing $l = \frac{T}{2}$ and $\alpha_k = k\frac{\pi}{l}$, we rewrite the expansion

$$f(t) = \sum_{-\infty}^{\infty} c_k \mathrm{e}^{ik\frac{\pi}{l}t}$$

as follows:

$$f(t) = \sum_{-\infty}^{\infty}\left(c_k\frac{l}{\pi}\right)\mathrm{e}^{ik\frac{\pi}{l}t}\frac{\pi}{l}, \tag{18.79}$$

where

$$c_k = \frac{1}{2l}\int_{-l}^{l} f(t)\mathrm{e}^{-i\alpha_k t}\,\mathrm{d}t$$

and hence

$$c_k\frac{l}{\pi} = \frac{1}{2\pi}\int_{-l}^{l} f(t)\mathrm{e}^{-i\alpha_k t}\,\mathrm{d}t.$$

Assuming that in the limit as $l \to +\infty$ we arrive at an arbitrary function $f$ that is absolutely integrable over $\mathbb{R}$, we introduce the auxiliary function

$$c(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\alpha t}\,\mathrm{d}t, \tag{18.80}$$

whose values at points $\alpha = \alpha_k$ differ only slightly from the quantities $c_k \frac{l}{\pi}$ in formula (18.79). In that case

$$f(t) \approx \sum_{-\infty}^{\infty} c(\alpha_k) \mathrm{e}^{i\alpha_k t} \frac{\pi}{l}, \tag{18.81}$$

where $\alpha_k = k\frac{\pi}{l}$ and $\alpha_{k+1} - \alpha_k = \frac{\pi}{l}$. This last integral resembles a Riemann sum, and as the partition is refined, which occurs as $l \to \infty$, we obtain

$$f(t) = \int_{-\infty}^{\infty} c(\alpha) \mathrm{e}^{i\alpha t} \, \mathrm{d}\alpha. \tag{18.82}$$

Thus, following Fourier, we have arrived at the expansion of the function $f$ into a continuous linear combination of harmonics of variable frequency and phase.

The integral (18.82) will be called the Fourier integral below. It is the continuous equivalent of a Fourier series. The function $c(\alpha)$ in it is the analog of the Fourier coefficient, and will be called the Fourier transform of the function $f$ (defined on the entire line $\mathbb{R}$). Formula (18.80) for the Fourier transform is completely equivalent to the formula for the Fourier coefficients. It is natural to regard the function $c(\alpha)$ as the *spectrum of the function* (signal) $f$. In contrast to the case of a periodic signal $f$ considered above and the discrete spectrum (of Fourier coefficients) corresponding to it, the spectrum $c(\alpha)$ of an arbitrary signal may be nonzero on whole intervals and even on the entire line (*continuous spectrum*).

*Example 1* Let us find the function having the following spectrum of compact support:

$$c(\alpha) = \begin{cases} h, & \text{if } |\alpha| \le a, \\ 0, & \text{if } |\alpha| > a. \end{cases} \tag{18.83}$$

*Proof* By formula (18.82) we find, for $t \neq 0$

$$f(t) = \int_{-a}^{a} h\mathrm{e}^{i\alpha t} \, \mathrm{d}\alpha = h \frac{\mathrm{e}^{iat} - \mathrm{e}^{-iat}}{it} = 2h \frac{\sin at}{t}, \tag{18.84}$$

and when $t = 0$, we obtain $f(0) = 2ha$, which equals the limit of $2h\frac{\sin at}{t}$ as $t \to 0$. □

The representation of a function in the form (18.82) is called its *Fourier integral representation*. We shall discuss below the conditions under which such a representation is possible. Right now, we consider another example.

*Example 2* Let $P$ be a device having the following properties: it is a linear signal transform, that is, $P(\sum_j a_j f_j) = \sum_j a_j P(f_j)$, and it preserves the periodicity of a signal, that is, $P(e^{i\omega t}) = p(\omega)\mathrm{e}^{i\omega t}$, where the coefficient $p(\omega)$ depends on the frequency $\omega$ of the periodic signal $\mathrm{e}^{i\omega t}$.

We use the compact complex notation, although of course everything could be rewritten in terms of $\cos \omega t$ and $\sin \omega t$.

The function $p(\omega) =: R(\omega)e^{i\varphi(\omega)}$ is called the *spectral characteristic of the device $P$*. Its absolute value $R(\omega)$ is usually called the *frequency characteristic* and its argument $\varphi(\omega)$ the *phase characteristic* of the device $P$. A signal $e^{i\omega t}$, after passing through the device, emerges transformed into the signal $R(\omega)e^{i(\omega t+\varphi(\omega))}$, its amplitude changed as a result of the factor $R(\omega)$ and its phase shifted due to the presence of the term $\varphi(\omega)$.

Let us assume that we know the spectral characteristic $p(\omega)$ of the device $P$ and the signal $f(t)$ that enters the device; we ask how to find the signal $x(t) = P(f)(t)$ that emerges from the device.

Representing the signal $f(t)$ as the Fourier integral (18.82) and using the linearity of the device and the integral, we find

$$x(t) = P(f)(t) = \int_{-\infty}^{\infty} c(\omega)p(\omega)e^{i\omega t}\,d\omega.$$

In particular, if

$$p(\omega) = \begin{cases} 1 & \text{for } |\omega| \le \Omega, \\ 0 & \text{for } |\omega| > \Omega, \end{cases} \tag{18.85}$$

then

$$x(t) = \int_{-\Omega}^{\Omega} c(\omega)e^{i\omega t}\,d\omega$$

and, as one can see from the spectral characteristics of the device,

$$P\left(e^{i\omega t}\right) = \begin{cases} e^{i\omega t} & \text{for } |\omega| \le \Omega, \\ 0 & \text{for } |\omega| > \Omega. \end{cases}$$

A device $P$ with the spectral characteristic (18.85) transmits (filters) frequencies not greater than $\Omega$ without distortion and truncates all of the signal involved with higher frequencies (larger than $\Omega$). For that reason, such a device is called an *ideal low-frequency filter* (with *upper frequency limit* $\Omega$) in radio technology.

Let us now turn to the mathematical side of the matter and to a more careful study of the concepts that arise.

### b. Definition of the Fourier Transform and the Fourier Integral

In accordance with formulas (18.80) and (18.82) we make the following definition.

**Definition 1** The function

$$\mathcal{F}[f](\xi) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx \tag{18.86}$$

is the *Fourier transform* of the function $f : \mathbb{R} \to \mathbb{C}$.

The integral here is understood in the sense of the principal value

$$\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x := \lim_{A\to+\infty}\int_{-A}^{A} f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x,$$

and we assume that it exists.

If $f:\mathbb{R}\to\mathbb{C}$ is absolutely integrable on $\mathbb{R}$, then, since $|f(x)\mathrm{e}^{-ix\xi}| = |f(x)|$ for $x,\xi\in\mathbb{R}$, the Fourier transform (18.86) is defined, and the integral (18.86) converges absolutely and uniformly with respect to $\xi$ on the entire line $\mathbb{R}$.

**Definition 2** If $c(\xi) = \mathcal{F}[f](\xi)$ is the Fourier transform of $f:\mathbb{R}\to\mathbb{C}$, then the integral assigned to $f$,

$$f(x) \sim \int_{-\infty}^{\infty} c(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi, \tag{18.87}$$

understood as a principal value, is called the *Fourier integral* of $f$.

The Fourier coefficients and the Fourier series of a periodic function are thus the discrete analog of the Fourier transform and the Fourier integral respectively.

**Definition 3** The following integrals, understood as principal values,

$$\mathcal{F}_c[f](\xi) := \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\xi x\,\mathrm{d}x, \tag{18.88}$$

$$\mathcal{F}_s[f](\xi) := \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\xi x\,\mathrm{d}x, \tag{18.89}$$

are called respectively the *Fourier cosine transform* and the *Fourier sine transform* of the function $f$.

Setting $c(\xi) = \mathcal{F}[f](\xi)$, $a(\xi) = \mathcal{F}_c[f](\xi)$, and $b(\xi) = \mathcal{F}_s[f](\xi)$, we obtain the relation that is already partly familiar to us from Fourier series

$$c(\xi) = \frac{1}{2}\big(a(\xi) - ib(\xi)\big). \tag{18.90}$$

As can be seen from relations (18.88) and (18.89),

$$a(-\xi) = a(\xi), \qquad b(-\xi) = -b(\xi). \tag{18.91}$$

Formulas (18.90) and (18.91) show that Fourier transforms are completely determined on the entire real line $\mathbb{R}$ if they are known for nonnegative values of the argument.

From the physical point of view this is a completely natural fact – the spectrum of a signal needs to be known for frequencies $\omega \geq 0$; the negative frequencies $\alpha$ in (18.80) and (18.82) – result from the form in which they are written. Indeed,

$$\int_{-A}^{A} c(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi = \biggl(\int_{-A}^{0} + \int_{0}^{A}\biggr) c(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi = \int_{0}^{A} \bigl(c(\xi)\mathrm{e}^{ix\xi} + c(-\xi)\mathrm{e}^{ix\xi}\bigr)\,\mathrm{d}\xi =$$
$$= \int_{0}^{A} \bigl(a(\xi)\cos x\xi + b(\xi)\sin x\xi\bigr)\,\mathrm{d}\xi,$$

and hence the Fourier integral (18.87) can be represented as

$$\int_{0}^{\infty} \bigl(a(\xi)\cos x\xi + b(\xi)\sin x\xi\bigr)\,\mathrm{d}\xi, \tag{18.87'}$$

which is in complete agreement with the classical form of a Fourier series. If the function $f$ is real-valued, it follows from formulas (18.90) and (18.91) that

$$c(-\xi) = \overline{c(\xi)}, \tag{18.92}$$

since in this case $a(\xi)$ and $b(\xi)$ are real-valued functions on $\mathbb{R}$, as one can see from their definitions (18.88) and (18.89). On the other hand, under the assumption $\overline{f}(x) = f(x)$, Eq. (18.92) can be obtained immediately from the definition (18.86) of the Fourier transform, if we take into account that the conjugation sign can be moved under the integral sign. This last observation allows us to conclude that

$$\mathcal{F}[\overline{f}](-\xi) = \overline{\mathcal{F}[f](\xi)} \tag{18.93}$$

for every function $f : \mathbb{R} \to \mathbb{C}$.

It is also useful to note that if $f$ is a real-valued even function, that is, $\overline{f(x)} = f(x) = f(-x)$, then

$$\begin{aligned} \overline{\mathcal{F}_c[f](\xi)} &= \mathcal{F}_c[f](\xi), \qquad \mathcal{F}_s[f](\xi) \equiv 0,\\ \overline{\mathcal{F}[f](\xi)} &= \mathcal{F}[f](\xi) = \mathcal{F}[f](-\xi); \end{aligned} \tag{18.94}$$

and if $f$ is a real-valued odd function, that is, $\overline{f(x)} = f(x) = -f(-x)$, then

$$\begin{aligned} \mathcal{F}_c[f](\xi) &\equiv 0, \qquad \overline{\mathcal{F}_s[f](\xi)} = \mathcal{F}_s[f](\xi),\\ \overline{\mathcal{F}[f](\xi)} &= -\mathcal{F}[f](\xi) = \mathcal{F}[f](-\xi); \end{aligned} \tag{18.95}$$

and if $f$ is a purely imaginary function, that is, $\overline{f(x)} = -f(x)$, then

$$\mathcal{F}[\overline{f}](-\xi) = -\overline{\mathcal{F}[f](\xi)}. \tag{18.96}$$

We remark that if $f$ is a real-valued function, its Fourier integral (18.87′) can also be written as

$$\int_0^\infty \sqrt{a^2(\xi)+b^2(\xi)}\cos\bigl(x\xi+\varphi(\xi)\bigr)\,\mathrm{d}\xi = 2\int_0^\infty \bigl|c(\xi)\bigr|\cos\bigl(x\xi+\varphi(\xi)\bigr)\,\mathrm{d}\xi,$$

where $\varphi(\xi) = -\arctan\frac{b(\xi)}{a(\xi)} = \arg c(\xi)$.

*Example 3* Let us find the Fourier transform of $f(t) = \frac{\sin at}{t}$ (assuming $f(0) = a \in \mathbb{R}$).

$$\begin{aligned}
\mathcal{F}[f](\alpha) &= \lim_{A\to+\infty}\frac{1}{2\pi}\int_{-A}^{A}\frac{\sin at}{t}\mathrm{e}^{-i\alpha t}\,\mathrm{d}t = \\
&= \lim_{A\to+\infty}\frac{1}{2\pi}\int_{-A}^{A}\frac{\sin at\cos\alpha t}{t}\,\mathrm{d}t = \frac{2}{2\pi}\int_0^{+\infty}\frac{\sin at\cos\alpha t}{t}\,\mathrm{d}t = \\
&= \frac{1}{2\pi}\int_0^{+\infty}\left(\frac{\sin(a+\alpha)t}{t}+\frac{\sin(a-\alpha)t}{t}\right)\mathrm{d}t = \\
&= \frac{1}{2\pi}\bigl(\operatorname{sgn}(a+\alpha)+\operatorname{sgn}(a-\alpha)\bigr)\int_0^\infty\frac{\sin u}{u}\,\mathrm{d}u = \begin{cases}\frac{1}{2}\operatorname{sgn} a, & \text{if } |\alpha|\le|a|,\\ 0, & \text{if } |\alpha|>|a|,\end{cases}
\end{aligned}$$

since we know the value of the Dirichlet integral

$$\int_0^\infty \frac{\sin u}{u}\,\mathrm{d}u = \frac{\pi}{2}. \tag{18.97}$$

Hence if we assume $a \ge 0$ and take the function $f(t) = 2h\frac{\sin at}{t}$ of Eq. (18.84), we find, as we should have expected, that the Fourier transform is the spectrum of this function exhibited in relations (18.83).

The function $f$ in Example 3 is not absolutely integrable on $\mathbb{R}$, and its Fourier transform has discontinuities. That the Fourier transform of an absolutely integrable function has no discontinuities is attested by the following lemma.

**Lemma 1** *If the function $f : \mathbb{R} \to \mathbb{C}$ is locally integrable and absolutely integrable on $\mathbb{R}$, then*

a) *its Fourier transform $\mathcal{F}[f](\xi)$ is defined for every value $\xi \in \mathbb{R}$;*
b) $\mathcal{F}[f] \in C(\mathbb{R},\mathbb{C})$;
c) $\sup_\xi |\mathcal{F}[g](\xi)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}|f(x)|\,\mathrm{d}x$;
d) $\mathcal{F}[f](\xi) \to 0$ *as* $\xi \to \infty$.

*Proof* We have already noted that $|f(x)\mathrm{e}^{ix\xi}| \le |f(x)|$, from which it follows that the integral (18.86) converges absolutely and uniformly with respect to $\xi \in \mathbb{R}$. This fact simultaneously proves parts a) and c).

Part d) follows from the Riemann–Lebesgue lemma (see Sect. 18.2).

For a fixed finite $A \geq 0$, the estimate

$$\left|\int_{-A}^{A} f(x)\left(\mathrm{e}^{-ix(\xi+h)} - \mathrm{e}^{-ix\xi}\right) \mathrm{d}x\right| \leq \sup_{|x|\leq A} \left|\mathrm{e}^{-ixh} - 1\right| \int_{-A}^{A} \left|f(x)\right| \mathrm{d}x$$

establishes that the integral

$$\frac{1}{2\pi} \int_{-A}^{A} f(x)\mathrm{e}^{-ix\xi} \,\mathrm{d}x,$$

is continuous with respect to $\xi$; and the uniform convergence of this integral as $A \to +\infty$ enables us to conclude that $\mathcal{F}[f] \in C(\mathbb{R}, \mathbb{C})$. □

*Example 4* Let us find the Fourier transform of the function $f(t) = \mathrm{e}^{-t^2/2}$:

$$\mathcal{F}[f](\alpha) = \int_{-\infty}^{+\infty} \mathrm{e}^{-t^2/2}\mathrm{e}^{-i\alpha t} \,\mathrm{d}t = \int_{-\infty}^{+\infty} \mathrm{e}^{-t^2/2} \cos \alpha t \,\mathrm{d}t.$$

Differentiating this last integral with respect to the parameter $\alpha$ and then integrating by parts, we find that

$$\frac{\mathrm{d}\mathcal{F}[f]}{\mathrm{d}\alpha}(\alpha) + \alpha\mathcal{F}[f](\alpha) = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \ln \mathcal{F}[f](\alpha) = -\alpha.$$

It follows that $\mathcal{F}[f](\alpha) = c\mathrm{e}^{-\alpha^2/2}$, where $c$ is a constant which, using the Euler–Poisson integral (see Example 17 of Sect. 17.2) we find from the relation

$$c = \mathcal{F}[f](0) = \int_{-\infty}^{+\infty} \mathrm{e}^{-t^2/2} \,\mathrm{d}t = \sqrt{2\pi}.$$

Thus we have found that $\mathcal{F}[f](\alpha) = \sqrt{2\pi}\mathrm{e}^{-\alpha^2/2}$, and simultaneously shown that $\mathcal{F}_c[f](\alpha) = \sqrt{2\pi}\mathrm{e}^{-\alpha^2/2}$ and $\mathcal{F}_s[f](\alpha) \equiv 0$.

### c. Normalization of the Fourier Transform

We obtained the Fourier transform (18.80) and the Fourier integral (18.82) as the natural continuous analogs of the Fourier coefficients $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\mathrm{e}^{-ikx} \,\mathrm{d}x$ and the Fourier series $\sum_{-\infty}^{\infty} c_k \mathrm{e}^{ikx}$ of a periodic function $f$ in the trigonometric system $\{\mathrm{e}^{ikx}; k \in \mathbb{Z}\}$. This system is not orthonormal, and only the ease of writing a trigonometric Fourier series in it has caused it to be used traditionally instead of the more natural orthonormal system $\{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{ikx}; k \in \mathbb{Z}\}$. In this normalized system the Fourier series has the form $\sum_{-\infty}^{\infty} \hat{c}_k \frac{1}{\sqrt{2\pi}} \mathrm{e}^{ikx}$, and the Fourier coefficients are defined by the formulas $\hat{c}_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\mathrm{e}^{-ikx}\,\mathrm{d}x$.

The continuous analogs of such natural Fourier coefficients and such a Fourier series would be the Fourier transform

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ix\xi}\,\mathrm{d}x \tag{18.98}$$

and the Fourier integral

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi, \tag{18.99}$$

which differ from those considered above only in the normalizing coefficient.

In the symmetric formulas (18.98) and (18.99) the Fourier "coefficient" and the Fourier "series" practically coalesce, and so in the future we shall essentially be interested only in the properties of the integral transform (18.98), calling it the *normalized Fourier transform* or, where no confusion can arise, simply the *Fourier transform* of the function $f$.

In general the name *integral operator* or *integral transform* is customarily given to an operator $A$ that acts on a function $f$ according to a rule

$$A(f)(y) = \int_X K(x, y) f(x)\,\mathrm{d}x,$$

where $K(x, y)$ is a given function called the *kernel of the integral operator*, and $X \subset \mathbb{R}^n$ is the set over which the integration extends and on which the integrands are assumed to be defined. Since $y$ is a free parameter in some set $Y$, it follows that $A(f)$ is a function on $Y$.

In mathematics there are many important integral transforms, and among them the Fourier transform occupies one of the most key positions. The reasons for this situation go very deep and involve the remarkable properties of the transformation (18.98), which we shall to some extent describe and illustrate in action in the remaining part of this section.

Thus, we shall study the normalized Fourier transform (18.98).

Along with the notation $\hat{f}$ for the normalized Fourier transform, we introduce the notation

$$\tilde{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{i\xi x}\,\mathrm{d}x, \tag{18.100}$$

that is, $\tilde{f}(\xi) = \hat{f}(-\xi)$.

Formulas (18.98) and (18.99) say that

$$\tilde{\hat{f}} = \hat{\tilde{f}} = f, \tag{18.101}$$

that is, the integral transforms (18.98) and (18.99) are mutually inverse to each other. Hence if (18.98) is the Fourier transform, then it is natural to call the integral operator (18.100) the *inverse Fourier transform*.

We shall discuss in detail below certain remarkable properties of the Fourier transform and justify them. For example

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n \hat{f}(\xi),$$
$$\widehat{f * g} = \sqrt{2\pi}\,\hat{f}\cdot\hat{g},$$
$$\|\hat{f}\| = \|f\|.$$

That is, the Fourier transform maps the operator of differentiation into the operator of multiplication by the independent variable; the Fourier transform of the convolution of functions amounts to multiplying the transforms; the Fourier transform preserves the norm (Parseval's equality), and is therefore an isometry of the corresponding function space.

But we shall begin with the inversion formula (18.101).

For another convenient normalization of the Fourier transform see Problem 10 below.

### d. Sufficient Conditions for a Function to be Representable as a Fourier Integral

We shall now prove a theorem that is completely analogous in both form and content to the theorem on convergence of a trigonometric Fourier series at a point. To preserve the familiar appearance of our earlier formulas and transformations to the maximum extent, we shall use the nonnormalized Fourier transform $c(\xi)$ in the present part of this subsection, together with its rather cumbersome but sometimes convenient notation $\mathcal{F}[f](\xi)$. Afterwards, when studying the integral Fourier transform as such, we shall as a rule work with the normalized Fourier transform $\hat{f}$ of the function $f$.

**Theorem 1** (Convergence of the Fourier integral at a point) *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function that is piecewise continuous on each finite closed interval of the real axis $\mathbb{R}$.*

*If the function $f$ satisfies the Dini conditions at a point $x \in \mathbb{R}$, then its Fourier integral* (18.82), (18.87), (18.87′), (18.99) *converges at that point to the value $\frac{1}{2}(f(x_-) + f(x_+))$, equal to half the sum of the left and right-hand limits of the function at that point.*

*Proof* By Lemma 1 the Fourier transform $c(\xi) = \mathcal{F}[f](\xi)$ of the function $f$ is continuous on $\mathbb{R}$ and hence integrable on every interval $[-A, A]$. Just as we transformed the partial sum of the Fourier series, we now carry out the following transformations of the partial Fourier integral:

$$S_A(x) = \int_{-A}^{A} c(\xi) \mathrm{e}^{ix\xi}\,\mathrm{d}\xi = \int_{-A}^{A}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-it\xi}\,\mathrm{d}t\right)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\Big(\int_{-A}^{A} e^{i(x-t)\xi}\,d\xi\Big)\,dt =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\frac{e^{i(x-t)A}-e^{-i(x-t)A}}{i(x-t)}\,dt =$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin(x-t)A}{x-t}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+u)\frac{\sin Au}{u}\,du =$$
$$= \frac{1}{\pi}\int_{0}^{\infty}\big(f(x-u)+f(x+u)\big)\frac{\sin Au}{u}\,du.$$

The change in the order of integration at the second equality from the beginning of the computation is legal. In fact, in view of the piecewise continuity of $f$, for every finite $B > 0$ we have the equality

$$\int_{-A}^{A}\Big(\frac{1}{2\pi}\int_{-B}^{B} f(t)e^{-it\xi}\,dt\Big)e^{ix\xi}\,d\xi = \frac{1}{2\pi}\int_{-B}^{B} f(t)\Big(\int_{-A}^{A} e^{i(x-t)\xi}\,d\xi\Big)\,dt,$$

from which as $B \to +\infty$, taking account of the uniform convergence of the integral $\int_{-B}^{B} f(x)e^{-it\xi}\,dt$ with respect to $\xi$, we obtain the equality we need.

We now use the value of the Dirichlet integral (18.97) and complete our transformation:

$$S_A(x) - \frac{f(x_-)+f(x_+)}{2} =$$
$$= \frac{1}{\pi}\int_{0}^{+\infty}\frac{(f(x-u)-f(x_-))+(f(x+u)-f(x_+))}{u}\sin Au\,du.$$

The resulting integral tends to zero as $A \to \infty$. We shall explain this and thereby finish the proof of the theorem.

We represent this integral as the sum of the integrals over the interval $]0, 1]$ and over the interval $[1, +\infty[$. The first of these two integrals tends to zero as $A \to +\infty$ in view of the Dini conditions and the Riemann–Lebesgue lemma. The second integral is the sum of four integrals corresponding to the four terms $f(x-u)$, $f(x+u)$, $f(x_-)$ and $f(x_+)$. The Riemann–Lebesgue lemma applies to the first two of these four integrals, and the last two can be brought into the following form, up to a constant factor:

$$\int_{1}^{+\infty}\frac{\sin Au}{u}\,du = \int_{A}^{+\infty}\frac{\sin v}{v}\,dv.$$

But as $A \to +\infty$ this last integral tends to zero, since the Dirichlet integral (18.97) converges. □

*Remark 1* In the proof of Theorem 1 we have actually studied the convergence of the integral as a principal value. But if we compare the notations (18.87) and (18.87′), it becomes obvious that it is precisely this interpretation of the integral that corresponds to convergence of the integral (18.87′).

From this theorem we obtain in particular

**Corollary 1** *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous absolutely integrable function.*

*If the function $f$ is differentiable at each point $x \in \mathbb{R}$ or has finite one-sided derivatives or satisfies a Hölder condition, then it is represented by its Fourier integral.*

Hence for functions of these classes both equalities (18.80) and (18.82) or (18.98) and (18.99) hold, and we have thus proved the inversion formula for the Fourier transform for such functions.

Let us consider several examples.

*Example 5* Assume that the signal $v(t) = P(f)(t)$ emerging from the device $P$ considered in Example 2 is known, and we wish to find the input signal $f(t)$ entering the device $P$.

In Example 2 we have shown that $f$ and $v$ are connected by the relation

$$v(t) = \int_{-\infty}^{\infty} c(\omega) p(\omega) \mathrm{e}^{i\omega t} \,\mathrm{d}\omega,$$

where $c(\omega) = \mathcal{F}[f](\omega)$ is the spectrum of the signal $F$ (the nonnormalized Fourier transform of the function $f$) and $p$ is the spectral characteristic of the device $P$. Assuming all these functions are sufficiently regular, from the theorem just proved we conclude that then

$$c(\omega) p(\omega) = \mathcal{F}[v](\omega).$$

From this we find $c(\omega) = \mathcal{F}[f](\omega)$. Knowing $c(\omega)$, we find the signal $f$ using the Fourier integral (18.87).

*Example 6* Let $a > 0$ and

$$f(x) = \begin{cases} \mathrm{e}^{-ax} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

Then

$$\mathcal{F}[f](\xi) = \frac{1}{2\pi} \int_0^{+\infty} \mathrm{e}^{-ax} \mathrm{e}^{-i\xi x} \,\mathrm{d}x = \frac{1}{2\pi} \frac{1}{a + i\xi}.$$

In discussing the definition of the Fourier transform, we have already noted a number of its obvious properties in Part b of the present subsection. We note further that if $f_-(x) := f(-x)$, then $\mathcal{F}[f_-](\xi) = \mathcal{F}[f](-\xi)$. This is an elementary change of variable in the integral.

We now take the function $\mathrm{e}^{-a|x|} = f(x) + f(-x) =: \varphi(x)$.

Then

$$\mathcal{F}[\varphi](\xi) = \mathcal{F}[f](\xi) + \mathcal{F}[f](-\xi) = \frac{1}{\pi} \frac{a}{a^2 + \xi^2}.$$

If we now take the function $\psi(x) = f(x) - f(-x)$, which is an odd extension of the function $e^{-ax}$, $x > 0$, to the entire real line, then

$$\mathcal{F}[\psi](\xi) = \mathcal{F}[f](\xi) - \mathcal{F}[f](-\xi) = -\frac{i}{\pi}\frac{\xi}{a^2+\xi^2}.$$

Using Theorem 1, or more precisely the corollary to it, we find that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{a+i\xi}\,d\xi = \begin{cases} e^{-ax}, & \text{if } x > 0, \\ \frac{1}{2}, & \text{if } x = 0, \\ 0, & \text{if } x < 0; \end{cases}$$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{a e^{ix\xi}}{a^2+\xi^2}\,d\xi = e^{-a|x|};$$

$$\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{\xi e^{ix\xi}}{a^2+\xi^2}\,d\xi = \begin{cases} e^{-ax}, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -e^{ax}, & \text{if } x < 0. \end{cases}$$

All the integrals here are understood in the sense of the principal value, although the second one, in view of its absolute convergence, can also be understood in the sense of an ordinary improper integral.

Separating the real and imaginary parts in these last two integrals, we find the Laplace integrals we have encountered earlier

$$\int_0^{+\infty}\frac{\cos x\xi}{a^2+\xi^2}\,d\xi = \frac{\pi}{2a}e^{-a|x|},$$

$$\int_0^{+\infty}\frac{\sin x\xi}{a^2+\xi^2}\,d\xi = \frac{\pi}{2}e^{-a|x|}\operatorname{sgn} x.$$

*Example 7* On the basis of Example 4 it is easy to find (by an elementary change of variable) that if

$$f(x) = e^{-a^2x^2}, \quad \text{then} \quad \hat{f}(\xi) = \frac{1}{\sqrt{2}a}e^{-\frac{\xi^2}{4a^2}}.$$

It is very instructive to trace the simultaneous evolution of the graphs of the functions $f$ and $\hat{f}$ as the parameter $a$ varies from $1/\sqrt{2}$ to 0. The more "concentrated" one of the functions is, the more "smeared" the other is. This circumstance is closely connected with the Heisenberg uncertainty principle in quantum mechanics. (In this connection see Problems 6 and 7.)

*Remark 2* In completing the discussion of the question of the possibility of representing a function by a Fourier integral, we note that, as Examples 1 and 3 show, the conditions on $f$ stated in Theorem 1 and its corollary are sufficient but not necessary for such a representation to be possible.

## *18.3.2 The Connection of the Differential and Asymptotic Properties of a Function and Its Fourier Transform*

### a. Smoothness of a Function and the Rate of Decrease of Its Fourier Transform

It follows from the Riemann–Lebesgue lemma that the Fourier transform of any absolutely integrable function on $\mathbb{R}$ tends to zero at infinity. This has already been noted in Lemma 1 proved above. We now show that, like the Fourier coefficients, the smoother the function, the faster its Fourier transform tends to zero. The dual fact is that the faster a function tends to zero, the smoother its Fourier transform.

We begin with the following auxiliary proposition.

**Lemma 2** *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function having a locally piecewise continuous derivative $f'$ on $\mathbb{R}$. Given this,*

a) *if the function $f'$ is integrable on $\mathbb{R}$, then $f(x)$ has a limit both as $x \to -\infty$ and as $\to +\infty$;*

b) *if the functions f and $f'$ are integrable on $\mathbb{R}$, then $f(x) \to 0$ as $\to \infty$.*

*Proof* Under these restrictions on the functions $f$ and $f'$ the Newton–Leibniz formula holds

$$f(x) = f(0) + \int_0^x f'(t)\,\mathrm{d}t.$$

In conditions a) the right-hand side of this equality has a limit both as $x \to +\infty$ and as $x \to -\infty$.

If a function $f$ having a limit at infinity is integrable on $\mathbb{R}$, then both of these limits must obviously be zero. $\square$

We now prove

**Proposition 1** (Connection between the smoothness of a function and the rate of decrease of its Fourier transform) *If $f \in C^{(k)}(\mathbb{R}, \mathbb{C})$ $(k = 0, 1, \dots)$ and all the functions $f, f', \dots, f^{(k)}$ are absolutely integrable on $\mathbb{R}$, then*

a) *for every $n \in \{0, 1, \dots, k\}$*

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n \hat{f}(\xi), \tag{18.102}$$

b) $\hat{f}(\xi) = o\left(\frac{1}{\xi^k}\right)$ *as $\xi \to 0$.*

*Proof* If $k = 0$, then a) holds trivially and b) follows from the Riemann–Lebesgue lemma.

Let $k > 0$. By Lemma 2 the functions $f, f', \dots, f^{(k-1)}$ tend to zero as $x \to \infty$. Taking this into account, we integrate by parts,

$$\begin{aligned}\widehat{f^{(k)}}(\xi) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{(k)}(x) \mathrm{e}^{-i\xi x}\, \mathrm{d}x = \\ &= \frac{1}{\sqrt{2\pi}} \left( f^{(k-1)}(x) \mathrm{e}^{-i\xi x} \Big|_{x=-\infty}^{+\infty} + (i\xi) \int_{-\infty}^{\infty} f^{(k-1)}(x) \mathrm{e}^{-i\xi x}\, \mathrm{d}x \right) = \\ &= \dots = \frac{(i\xi)^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-i\xi x}\, \mathrm{d}x = (i\xi)^k \hat{f}(\xi).\end{aligned}$$

Thus Eq. (18.102) is established. This is a very important relation, and we shall return to it.

We have shown that $\hat{f}(\xi) = (i\xi)^{-k} \widehat{f^{(k)}}(\xi)$, but by the Riemann–Lebesgue lemma $\widehat{f^{(k)}}(\xi) \to 0$ as $\xi \to 0$ and hence b) is also proved. □

## b. The Rate of Decrease of a Function and the Smoothness of Its Fourier Transform

In view of the nearly complete identity of the direct and inverse Fourier transforms the following proposition, dual to Proposition 1, holds.

**Proposition 2** (The connection between the rate of decrease of a function and the smoothness of its Fourier transform) *If a locally integrable function $f : \mathbb{R} \to \mathbb{C}$ is such that the function $x^k f(x)$ is absolutely integrable on $\mathbb{R}$, then*

a) *the Fourier transform of $f$ belongs to $C^{(k)}(\mathbb{R}, \mathbb{C})$.*
b) *the following equality holds*:

$$\hat{f}^{(k)}(\xi) = (-i)^k \widehat{x^k f(x)}(\xi). \tag{18.103}$$

*Proof* For $k = 0$ relation (18.103) holds trivially, and the continuity of $\hat{f}(\xi)$ has already been proved in Lemma 1. If $k > 0$, then for $n < k$ we have the estimate $|x^n f(x)| \le |x^k f(x)|$ at infinity, from which it follows that $x^n f(x)$ is absolutely integrable. But $|x^n f(x) \mathrm{e}^{-i\xi x}| = |x^n f(x)|$, which enables us to invoke the uniform convergence of these integrals with respect to the parameter $\xi$ and successively differentiate them under the integral sign:

$$\begin{aligned}\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-i\xi x}\, \mathrm{d}x, \\ \hat{f}'(\xi) &= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) \mathrm{e}^{-i\xi x}\, \mathrm{d}x, \\ &\vdots\end{aligned}$$

$$\hat{f}^{(k)}(\xi) = \frac{(-i)^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k f(x) \mathrm{e}^{-i\xi x} \, \mathrm{d}x.$$

By Lemma 1 this last integral is continuous on the entire real line. Hence indeed $\hat{f} \in C^{(k)}(\mathbb{R}, \mathbb{C})$. □

### c. The Space of Rapidly Decreasing Functions

**Definition 4** We denote the set of functions $f \in C^{(\infty)}(\mathbb{R}, \mathbb{C})$ satisfying the condition

$$\sup_{x \in \mathbb{R}} \left| x^\beta f^{(\alpha)}(x) \right| < \infty$$

for all nonnegative integers $\alpha$ and $\beta$ by $S(\mathbb{R}, \mathbb{C})$ or more briefly by $S$. Such functions are called *rapidly decreasing functions* (as $x \to \infty$).

The set of rapidly decreasing functions obviously forms a vector space under the standard operations of addition of functions and multiplication of a function by a complex number.

*Example 8* The function $\mathrm{e}^{-x^2}$ or, for example, all functions of compact support in $C_0^{(\infty)}(\mathbb{R}, \mathbb{C})$ belong to $S$.

**Lemma 3** *The restriction of the Fourier transform to $S$ is a vector-space automorphism of $S$.*

*Proof* We first show that $(f \in S) \Rightarrow (\hat{f} \in S)$.

To do this we first remark that by Proposition 2a we have $\hat{f} \in C^{(\infty)}(\mathbb{R}, \mathbb{C})$.

We then remark that the operation of multiplication by $x^\alpha$ $(\alpha \geq 0)$ and the operation $D$ of differentiation do not lead outside the class of rapidly decreasing functions. Hence, for any nonnegative integers $\alpha$ and $\beta$ the relation $f \in S$ implies that the function $D^\beta(x^\alpha f(x))$ belongs to the space $S$. Its Fourier transform tends to zero at infinity by the Riemann–Lebesgue lemma. But by formulas (18.102) and (18.103)

$$D^\beta \widehat{\left(x^\alpha f(x)\right)}(\xi) = i^{\alpha+\beta} \xi^\beta \hat{f}^{(\alpha)}(\xi),$$

and we have shown that $\xi^\beta \hat{f}^{(\alpha)}(\xi) \to 0$ as $\xi \to \infty$, that is, $\hat{f} \in S$.

We now show that $\hat{S} = S$, that is, that the Fourier transform maps $S$ onto the whole space $S$.

We recall that the direct and inverse Fourier transforms are connected by the simple relation $\hat{f}(\xi) = \tilde{f}(-\xi)$. Reversing the sign of the argument of the function obviously is an operation that maps the set $S$ into itself. Hence the inverse Fourier transform also maps $S$ into itself.

Finally, if $f$ is an arbitrary function in $S$, then by what has been proved $\varphi = \tilde{f} \in S$ and by the inversion formula (18.101) we find that $f = \hat{\varphi}$.

The linearity of the Fourier transform is obvious, so that Lemma 3 is now completely proved. □

### 18.3.3 The Main Structural Properties of the Fourier Transform

#### a. Definitions, Notation, Examples

We have made a rather detailed study above of the Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ defined on the real line. In particular, we have clarified the connection that exists between the regularity properties of a function and the corresponding properties of its Fourier transform. Now that this question has been theoretically answered, we shall study the Fourier transform only of sufficiently regular functions so as to exhibit the fundamental technical properties of the Fourier transform in concentrated form and without technical complications. In compensation we shall consider not only one-dimensional but also the multi-dimensional Fourier transform and derive its basic properties practically independently of what was discussed above.

Those wishing to confine themselves to the one-dimensional case may assume that $n = 1$ below.

**Definition 5** Suppose $f : \mathbb{R}^n \to \mathbb{C}$ is a locally integrable function on $\mathbb{R}^n$. The function

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-i(\xi,x)} \, \mathrm{d}x \tag{18.104}$$

is called the *Fourier transform of the function* $f$.

Here we mean that $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$, $(\xi, x) = \xi_1 x_1 + \dots + \xi_n x_n$, and the integral is regarded as convergent in the following sense of principal value:

$$\int_{\mathbb{R}^n} \varphi(x_1, \dots, x_n) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n := \lim_{A \to +\infty} \int_{-A}^{A} \cdots \int_{-A}^{A} \varphi(x_1, \dots, x_n) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n .$$

In this case the multidimensional Fourier transform (18.104) can be regarded as $n$ one-dimensional Fourier transforms carried out with respect to each of the variables $x_1, \dots, x_n$.

Then, when the function $f$ is absolutely integrable, the question of the sense in which the integral (18.104) is to be understood does not arise at all.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be multi-indices consisting of nonnegative integers $\alpha_j, \beta_j$, $j = 1, \dots, n$, and suppose, as always, that $D^\alpha$ denotes the differentiation operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ of order $|\alpha| := \alpha_1 + \dots + \alpha_n$ and $x^\beta := x_1^{\beta_1} \cdot \ldots \cdot x_n^{\beta_n}$.

**Definition 6** We denote the set of functions $f \in C^{(\infty)}(\mathbb{R}^n, \mathbb{C})$ satisfying the condition

$$\sup_{x \in \mathbb{R}^n} \left| x^\beta D^\alpha f(x) \right| < \infty$$

for all nonnegative multi-indices $\alpha$ and $\beta$ by the symbol $S(\mathbb{R}^n, \mathbb{C})$, or by $S$ where no confusion can arise. Such functions are said to be *rapidly decreasing* (as $x \to \infty$).

The set $S$ with the algebraic operations of addition of functions and multiplication of a function by a complex number is obviously a vector space.

*Example 9* The function $e^{-|x|^2}$, where $|x|^2 = x_1^2 + \cdots + x_n^2$, and all the functions in $C_0^{(\infty)}(\mathbb{R}^n, \mathbb{C})$ of compact support belong to $S$.

If $f \in S$, then integral in relation (18.104) obviously converges absolutely and uniformly with respect to $\xi$ on the entire space $\mathbb{R}^n$. Moreover, if $f \in S$, then by standard rules this integral can be differentiated as many times as desired with respect to any of the variables $\xi_1, \dots, \xi_n$. Thus if $f \in S$, then $\hat{f} \in C^{(\infty)}(\mathbb{R}, \mathbb{C})$.

*Example 10* Let us find the Fourier transform of the function $\exp(-|x|^2/2)$. When integrating rapidly decreasing functions one can obviously use Fubini's theorem and if necessary change the order of improper integrations without difficulty.

In the present case, using Fubini's theorem and Example 4, we find

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|x|^2/2} \cdot e^{-i(\xi,x)} \, dx =$$
$$= \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x_j^2/2} e^{-i\xi_j x_j} \, dx_j = \prod_{j=1}^{n} e^{-\xi_j^2/2} = e^{-|\xi|^2/2}.$$

We now state and prove the basic structural properties of the Fourier transform, assuming, so as to avoid technical complications, that the Fourier transform is being applied to functions of class $S$. This is approximately the same as learning to operate (compute) with rational numbers rather than the entire space $\mathbb{R}$ all at once. The process of completion is of the same type. On this account, see Problem 5.

### b. Linearity

The linearity of the Fourier transform is obvious; it follows from the linearity of the integral.

### c. The Relation Between Differentiation and the Fourier Transform

The following formulas hold

$$\widehat{D^\alpha f}(\xi) = i^{|\alpha|} \xi^\alpha \hat{f}(\xi), \tag{18.105}$$

$$\big(\widehat{x^\alpha f(x)}\big)(\xi) = i^{|\alpha|} D^\alpha \hat{f}(\xi). \tag{18.106}$$

*Proof* The first of these can be obtained, like formula (18.102), via integration by parts (of course, with a preliminary use of Fubini's theorem in the case of a space $\mathbb{R}^n$ of dimension $n > 1$).

Formula (18.106) generalizes relation (18.103) and is obtained by direct differentiation of (18.104) with respect to the parameters $\xi_1, \dots, \xi_n$. □

*Remark 3* In view of the obvious estimate

$$\big|\hat{f}(\xi)\big| \le \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \big|f(x)\big| \, \mathrm{d}x < +\infty,$$

it follows from (18.105) that $\hat{f}(\xi) \to 0$ as $\xi \to \infty$ for every function $f \in S$, since $D^\alpha f \in S$.

Next, the simultaneous use of formulas (18.105) and (18.106) enables us to write that

$$D^\beta \big(\widehat{x^\alpha f(x)}\big)(\xi) = (i)^{|\alpha|+|\beta|} \xi^\beta D^\alpha \hat{f}(\xi),$$

from which it follows that if $f \in S$, then for any nonnegative multi-indices $\alpha$ and $\beta$ we have $\xi^\beta D^\alpha \hat{f}(\xi) \to 0$ when $\xi \to \infty$ in $\mathbb{R}^n$. Thus we have shown that

$$(f \in S) \Rightarrow (\hat{f} \in S).$$

### d. The Inversion Formula

**Definition 7** The operator defined (together with its notation) by the equality

$$\tilde{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{i(\xi,x)} \, \mathrm{d}x, \tag{18.107}$$

is called the *inverse Fourier transform*.

The following *Fourier inversion formula* holds:

$$\tilde{\hat{f}} = \hat{\tilde{f}} = f, \tag{18.108}$$

or in the form of the Fourier integral:

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) \mathrm{e}^{i(x,\xi)} \, \mathrm{d}\xi. \tag{18.109}$$

Using Fubini's theorem one can immediately obtain formula (18.108) from the corresponding formula (18.101) for the one-dimensional Fourier transform, but, as promised, we shall give a brief independent proof of the formula.

*Proof* We first show that

$$\int_{\mathbb{R}^n} g(\xi)\hat{f}(\xi)\mathrm{e}^{i(x,\xi)}\,\mathrm{d}\xi = \int_{\mathbb{R}^n} \hat{g}(\xi)f(x+y)\,\mathrm{d}y \tag{18.110}$$

for any functions $f, g \in S(\mathbb{R}, \mathbb{C})$. Both integrals are defined, since $f, g \in S$ and so by Remark 3 we also have $\hat{f}, \hat{g} \in S$.

Let us transform the integral on the left-hand side of the equality to be proved:

$$\begin{aligned}
&\int_{\mathbb{R}^n} g(\xi)\hat{f}(\xi)\mathrm{e}^{i(x,\xi)}\,\mathrm{d}\xi = \\
&= \int_{\mathbb{R}^n} g(\xi)\left(\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(y)\mathrm{e}^{-i(\xi,y)}\,\mathrm{d}y\right)\mathrm{e}^{i(x,\xi)}\,\mathrm{d}\xi = \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} g(\xi)\mathrm{e}^{-i(\xi,y-x)}\,\mathrm{d}\xi\right)f(y)\,\mathrm{d}y = \\
&= \int_{\mathbb{R}^n} \hat{g}(y-x)f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \hat{g}(y)f(x+y)\,\mathrm{d}y.
\end{aligned}$$

There is no doubt as to the legitimacy of the reversal in the order of integration, since $f$ and $g$ are rapidly decreasing functions. Thus (18.110) is now verified.

We now remark that for every $\varepsilon > 0$

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} g(\varepsilon\xi)\mathrm{e}^{i(y,\xi)}\,\mathrm{d}\xi = \frac{1}{(2\pi)^{n/2}\varepsilon^n}\int_{\mathbb{R}^n} g(u)\mathrm{e}^{-i(y,u/\varepsilon)}\,\mathrm{d}u = \varepsilon^{-n}\hat{g}(y/\varepsilon),$$

so that, by Eq. (18.110)

$$\int_{\mathbb{R}^n} g(\varepsilon\xi)\hat{f}(\xi)\mathrm{e}^{i(x,\xi)}\,\mathrm{d}\xi = \int_{\mathbb{R}^n} \varepsilon^{-n}\hat{g}(y/\varepsilon)f(x+y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \hat{g}(u)f(x+\varepsilon u)\,\mathrm{d}u.$$

Taking account of the absolute and uniform convergence with respect to $\varepsilon$ of the extreme integrals in the last chain of equalities, we find, as $\varepsilon \to 0$,

$$g(0)\int_{\mathbb{R}^n} \hat{f}(\xi)\mathrm{e}^{i(x,\xi)}\,\mathrm{d}\xi = f(x)\int_{\mathbb{R}^n} \hat{g}(u)\,\mathrm{d}u.$$

Here we set $g(x) = \mathrm{e}^{-|x|^2/2}$. In Example 10 we saw that $\hat{g}(u) = \mathrm{e}^{-|u|^2/2}$. Recalling the Euler–Poisson integral $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$ and using Fubini's theorem, we conclude that $\int_{\mathbb{R}^n} \mathrm{e}^{-|u|^2/2}\,\mathrm{d}u = (2\pi)^{n/2}$, and as a result, we obtain Eq. (18.109). □

*Remark 4* In contrast to the single equality (18.109), which means that $\tilde{\hat{f}} = f$, relations (18.108) also contain the equality $\hat{\tilde{f}}$. But this relation follows immediately from the one proved, since $\tilde{f}(\xi) = \hat{f}(-\xi)$ and $\widetilde{f(-x)} = \widehat{f(x)}$.

*Remark 5* We have already seen (see Remark 3) that if $f \in S$, then $\hat{f} \in S$, and hence $\tilde{f} \in S$ also, that is, $\hat{S} \subset S$ and $\widetilde{S} \subset S$. We now conclude from the relations $\hat{\tilde{f}} = \tilde{\hat{f}} = f$ that $\widetilde{S} = \hat{S} = S$.

### e. Parseval's Equality

This is the name given to the relation

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \tag{18.111}$$

which in expanded form means that

$$\int_{\mathbb{R}^n} f(x)\overline{g}(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}}(\xi)\,\mathrm{d}\xi. \tag{18.111'}$$

It follows in particular from (18.111) that

$$\|f\|^2 = \langle f, f \rangle = \langle \hat{f}, \hat{f} \rangle = \|\hat{f}\|^2. \tag{18.112}$$

From the geometric point of view, Eq. (18.111) means that the Fourier transform preserves the inner product between functions (vectors of the space $S$), and hence is an isometry of $S$.

The name "Parseval's equality" is also sometimes given to the relation

$$\int_{\mathbb{R}^n} \hat{f}(\xi)g(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} f(x)\hat{g}(x)\,\mathrm{d}x, \tag{18.113}$$

which is obtained from (18.110) by setting $x = 0$. The main Parseval equality (18.111) is obtained from (18.113) by replacing $g$ with $\overline{\hat{g}}$ and using the fact that $\hat{(\overline{\hat{g}})} = \overline{g}$, since $\overline{\hat{g}} = \tilde{\overline{g}}$ and $\tilde{\hat{g}} = \hat{\tilde{g}} = g$.

### f. The Fourier Transform and Convolution

The following important relations hold

$$(\widehat{f * g}) = (2\pi)^{n/2}\hat{f} \cdot \hat{g}, \tag{18.114}$$

$$(\widehat{f \cdot g}) = (2\pi)^{-n/2}\hat{f} * \hat{g} \tag{18.115}$$

(sometimes called *Borel's formulas*), which connect the operations of convolution and multiplication of functions through the Fourier transform.

Let us prove these formulas:

*Proof*

$$(\widehat{f * g})(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (f * g)(x)\mathrm{e}^{-i(\xi,x)}\,\mathrm{d}x =$$

$$
\begin{aligned}
&= \frac{1}{(2\pi^{n/2}} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(x-y) g(y)\, \mathrm{d}y \right) \mathrm{e}^{-i(\xi,x)}\, \mathrm{d}x = \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(y) \mathrm{e}^{-i(\xi,y)} \left( \int_{\mathbb{R}^n} f(x-y) \mathrm{e}^{-i(\xi,x-y)}\, \mathrm{d}x \right) \mathrm{d}y = \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(y) \mathrm{e}^{-i(\xi,y)} \left( \int_{\mathbb{R}^n} f(u) \mathrm{e}^{-i(\xi,u)}\, \mathrm{d}u \right) \mathrm{d}y = \\
&= \int_{\mathbb{R}^n} g(y) \mathrm{e}^{-i(\xi,y)} \hat{f}(\xi)\, \mathrm{d}y = (2\pi)^{n/2} \hat{f}(\xi) \hat{g}(\xi).
\end{aligned}
$$

The legitimacy of the change in order of integration is not in doubt, given that $f, g \in S$.

Formula (18.115) can be obtained by a similar computation if we use the inversion formula (18.109). However, Eq. (18.115) can be derived from relation (18.114) already proved if we recall that $\hat{\tilde{f}} = \tilde{\hat{f}} = f$, $\overline{\tilde{f}} = \overline{\hat{f}}$, $\overline{\tilde{f}} = \hat{\overline{f}}$, and that $\overline{u \cdot v} = \overline{u} \cdot \overline{v}$, $\overline{u * v} = \overline{u} * \overline{v}$. $\square$

*Remark 6* If we set $\tilde{f}$ and $\tilde{g}$ in place of $f$ and $g$ in formulas (18.114) and (18.115) and apply the inverse Fourier transform to both sides of the resulting equalities, we arrive at the relations

$$\widetilde{f \cdot g} = (2\pi)^{-n/2} (\tilde{f} * \tilde{g}), \tag{18.114$'$}$$

$$\widetilde{f * g} = (2\pi)^{n/2} (\tilde{f} \cdot \tilde{g}). \tag{18.115$'$}$$

## 18.3.4 Examples of Applications

Let us now illustrate the Fourier transform (and some of the machinery of Fourier series) in action.

### a. The Wave Equation

The successful use of the Fourier transform in the equations of mathematical physics is bound up (in its mathematical aspect) primarily with the fact that the Fourier transform replaces the operation of differentiation with the algebraic operation of multiplication.

For example, suppose we are seeking a function $u : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$a_0 u^{(n)}(x) + a_1 u^{(n-1)}(x) + \cdots + a_n u(x) = f(x),$$

where $a_0, \ldots, a_n$ are constant coefficients and $f$ is a known function. Applying the Fourier transform to both sides of this equation (assuming that the functions $u$ and

$f$ are sufficiently regular), by relation (18.105) we obtain the algebraic equation

$$\big(a_0(i\xi)^n + a_1(i\xi)^{n-1} + \cdots + a_n\big)\hat{u}(\xi) = \hat{f}(\xi)$$

for $\hat{u}$. After finding $\hat{u}(\xi) = \frac{\hat{f}(\xi)}{P(i\xi)}$ from the equation, we obtain $u(x)$ by applying the inverse Fourier transform.

We now apply this idea to the search for a function $u = u(x,t)$ satisfying the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (a > 0)$$

and the initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

in $\mathbb{R} \times \mathbb{R}$.

Here and in the next example we shall not take the time to justify the intermediate computations because, as a rule, it is easier simply to find the required function and verify directly that it solves the problem posed than to justify and overcome all the technical difficulties that arise along the way. As it happens, generalized functions, which have already been mentioned, play an essential role in the theoretical struggle with these difficulties.

Thus, regarding $t$ as a parameter, we carry out a Fourier transform on $x$ on both sides of the equation. Then, assuming on the one hand that differentiation with respect to the parameter under the integral sign is permitted and using formula (18.105) on the other hand, we obtain

$$\hat{u}''_{tt}(\xi,t) = -a^2\xi^2\hat{u}(\xi,t),$$

from which we find

$$\hat{u}(\xi,t) = A(\xi)\cos a\xi t + B(\xi)\sin a\xi t.$$

By the initial conditions, we have

$$\hat{u}(\xi,0) = \hat{f}(\xi) = A(\xi),$$
$$\hat{u'_t}(\xi,0) = \widehat{\big(u'_t\big)}(\xi,0) = \hat{g}(\xi) = a\xi B(\xi).$$

Thus,

$$\begin{aligned}\hat{u}(\xi,t) &= \hat{f}(\xi)\cos a\xi t + \frac{\hat{g}(\xi)}{a\xi}\sin a\xi t = \\ &= \frac{1}{2}\hat{f}(\xi)\big(\mathrm{e}^{ia\xi t} + \mathrm{e}^{-ia\xi t}\big) + \frac{1}{2}\frac{\hat{g}(\xi)}{ia\xi}\big(\mathrm{e}^{ia\xi t} - \mathrm{e}^{-ia\xi t}\big).\end{aligned}$$

Multiplying this equality by $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{ix\xi}$ and integrating with respect to $\xi$ – in short, taking the inverse Fourier transform – and using formula (18.105) we obtain immediately

$$u(x,t)=\frac{1}{2}\big(f(x-at)+f(x+at)\big)+\frac{1}{2}\int_0^t\big(g(x-a\tau)+g(x+a\tau)\big)\,\mathrm{d}\tau.$$

**b. The Heat Equation**

Another element of the machinery of Fourier transforms (specifically, formulas (18.114′) and (18.115′)) which remained in the background in the preceding example, manifests itself quite clearly when we seek a function $u=u(x,t)$, $x\in\mathbb{R}^n$, $t\geq 0$, that satisfies the heat equation

$$\frac{\partial u}{\partial t}=a^2\Delta u \quad (a>0)$$

and the initial condition $u(x,0)=f(x)$ on all of $\mathbb{R}^n$.

Here, as always $\Delta=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}$.

Carrying out a Fourier transform with respect to the variable $x\in\mathbb{R}^n$, (assuming that this is possible to do) we find by (18.105) the ordinary equation

$$\frac{\partial\hat{u}}{\partial t}(\xi,t)=a^2(i)^2\big(\xi_1^2+\cdots+\xi_n^2\big)\hat{u}(\xi,t),$$

from which it follows that

$$\hat{u}(\xi,t)=c(\xi)\mathrm{e}^{-a^2|\xi|^2t},$$

where $|\xi|^2=\xi_1^2+\cdots+\xi_n^2$. Taking into account the relation $\hat{u}(\xi,0)=\hat{f}(\xi)$, we find

$$\hat{u}(\xi,t)=\hat{f}(\xi)\cdot\mathrm{e}^{-a^2|\xi|^2t}.$$

Now applying the inverse Fourier transform, taking account of (18.114′), we obtain

$$u(x,t)=(2\pi)^{-n/2}\int_{\mathbb{R}_n}f(y)E_0(y-x,t)\,\mathrm{d}y,$$

where $E_0(x,t)$ is the function whose Fourier transform with respect to $x$ is $\mathrm{e}^{-a^2|\xi|^2t}$. The inverse Fourier transform with respect to $\xi$ of the function $\mathrm{e}^{-a^2|\xi|^2t}$ is essentially already known to us from Example 10. Making an obvious change of variable, we find

$$E_0(x,t)=\frac{1}{(2\pi)^{n/2}}\left(\frac{\sqrt{\pi}}{a\sqrt{t}}\right)^n\mathrm{e}^{-\frac{|x|^2}{4a^2\mathrm{t}}}.$$

Setting $E(x,t) = (2\pi)^{-n/2}E_0(x,t)$, we find the fundamental solution

$$E(x,t) = (2a\sqrt{\pi t})^{-n}\mathrm{e}^{-\frac{|x|^2}{4a^2t}} \quad (t > 0),$$

of the heat equation, which was already familiar to us (see Example 15 of Sect. 17.4), and the formula

$$u(x,t) = (f * E)(x,t)$$

for the solution satisfying the initial condition $u(x,0) = f(x)$.

### c. The Poisson Summation Formula

This is the name given to the following relation

$$\sqrt{2\pi}\sum_{n=-\infty}^{\infty}\varphi(2\pi n) = \sum_{n=-\infty}^{\infty}\hat{\varphi}(n) \tag{18.116}$$

between a function $\varphi : \mathbb{R} \to \mathbb{C}$ (assume $\varphi \in S$) and its Fourier transform $\hat{\varphi}$. Formula (18.116) is obtained by setting $x = 0$ in the equality

$$\sqrt{2\pi}\sum_{n=-\infty}^{\infty}\varphi(x + 2\pi n) = \sum_{n=-\infty}^{\infty}\hat{\varphi}(n)\mathrm{e}^{inx}, \tag{18.117}$$

which we shall prove assuming that $\varphi$ is a rapidly decreasing function.

*Proof* Since $\varphi$ and $\hat{\varphi}$ both belong to $S$, the series on both sides of (18.117) converge absolutely (and so they can be summed in any order), and uniformly with respect to $x$ on the entire line $\mathbb{R}$. Moreover, since the derivatives of a rapidly decreasing function are themselves in class $S$, we can conclude that the function $f(x) = \sum_{n=-\infty}^{\infty}\varphi(x + 2\pi n)$ belongs to $C^{(\infty)}(\mathbb{R}, \mathbb{C})$. The function $f$ is obviously of period $2\pi$. Let $\{\hat{c}_k(f)\}$ be its Fourier coefficients in the orthonormal system $\{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{ikx}; k \in \mathbb{Z}\}$, then

$$\hat{c}_k(f) := \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(x)\mathrm{e}^{-ikx}\,\mathrm{d}x = \sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\varphi(x + 2\pi n)\mathrm{e}^{-ikx}\,\mathrm{d}x =$$
$$= \sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{2\pi n}^{2\pi(n+1)}\varphi(x)\mathrm{e}^{-ikx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x)\mathrm{e}^{-ikx}\,\mathrm{d}x =: \hat{\varphi}(k).$$

But $f$ is a smooth $2\pi$-periodic function and so its Fourier series converges to it at every point $x \in \mathbb{R}$. Hence, at every point $x \in \mathbb{R}$ we have the relation

$$\sum_{n=-\infty}^{\infty}\varphi(x + 2\pi n) = f(x) = \sum_{n=-\infty}^{\infty}\hat{c}_n(f)\frac{\mathrm{e}^{inx}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\hat{\varphi}(n)\mathrm{e}^{inx}. \qquad \square$$

*Remark 7* As can be seen from the proof, relations (18.116) and (18.117) by no means hold only for functions of class $S$. But if $\varphi$ does happen to belong to $S$, then Eq. (18.117) can be differentiated termwise with respect to $x$ any number of times, yielding as a corollary new relations between $\varphi, \varphi', \dots,$ and $\hat{\varphi}$.

### d. Kotel'nikov's Theorem (Whittaker–Shannon Sampling Theorem)[25]

This example, based like the preceding one on a beautiful combination of the Fourier series and the Fourier integral, has a direct relation to the theory of information transmission in a communication channel. To keep it from appearing artificial, we recall that because of the limited capabilities of our sense organs, we are able to perceive signals only in a certain range of frequencies. For example, the ear "hears" in the range from 20 Hz to 20 kHz. Thus, no matter what the signals are, we, like a filter (see Sect. 18.3.1) cut out only a bounded part of their spectra and perceive them as band-limited signals (having a bounded spectrum).

For that reason, we shall assume from the outset that the transmitted or received signal $f(t)$ (where $t$ is time, $-\infty < t < \infty$) is band-limited, the spectrum being nonzero only for frequencies whose magnitudes do not exceed a certain critical value $a > 0$. Thus $\hat{f}(\omega) \equiv 0$ for $|\omega| > a$, and so for a band-limited function the representation

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega$$

reduces to the integral over just the interval $[-a, a]$:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \hat{f}(\omega) e^{i\omega t}\, d\omega. \tag{18.118}$$

On the closed interval $[-a, a]$ we expand the function $\hat{f}(\omega)$ in a Fourier series

$$\hat{f}(\omega) = \sum_{-\infty}^{\infty} c_k(\hat{f}) e^{i\frac{\pi\omega}{a}k} \tag{18.119}$$

in the system $\{e^{i\frac{\pi\omega}{a}k}; k \in \mathbb{Z}\}$ which is orthogonal and complete in that interval. Taking account of formula (18.118), we find the following simple expression for the coefficients $c_k(\hat{f})$ of this series:

$$c_k(\hat{f}) := \frac{1}{2a} \int_{-a}^{a} \hat{f}(\omega) e^{-i\frac{\pi\omega}{a}k}\, d\omega - a = \frac{\sqrt{2\pi}}{2a} f\left(-\frac{\pi}{a}k\right). \tag{18.120}$$

[25] V.A. Kotel'nikov (b. 1908) – Soviet scholar, a well-known specialist in the theory of radio communication.

J.M. Whittaker (1905–1984) – British mathematician who worked mainly in complex analysis.

C.E. Shannon (1916–2001) – American mathematician and engineer, one of the founders of information theory and inventor of the term "bit" as an abbreviation of "binary digit".

Substituting the series (18.119) into the integral (18.118), taking account of relations (18.120), we find

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \left( \frac{\sqrt{2\pi}}{2a} \sum_{k=-\infty}^{\infty} f\left(\frac{\pi}{a}k\right) e^{i\omega t - i\frac{\pi k}{a}\omega} \right) d\omega =$$

$$= \frac{1}{2a} \sum_{k=-\infty}^{\infty} f\left(\frac{\pi}{a}k\right) \int_{-a}^{a} e^{i\omega(t-\frac{\pi}{a}k)}\, d\omega.$$

Calculating these elementary integrals, we arrive at *Kotel'nikov's formula*

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{\pi}{a}k\right) \frac{\sin a(t-\frac{\pi}{a}k)}{a(t-\frac{\pi}{a}k)}. \tag{18.121}$$

Formula (18.121) shows that, in order to reconstruct a message described by a band-limited function $f(t)$ whose spectrum is concentrated in the frequency range $|\omega| \le a$, it suffices to transmit over the channel only the values $f(k\Delta)$ (called *marker values*) of the function at equal time intervals $\Delta = \pi/a$.

This proposition, together with formula (18.121) is due to V.A. Kotel'nikov and is called *Kotel'nikov's theorem* or the *sampling theorem*.

*Remark 8* The interpolation formula (18.121) itself was known in mathematics before Kotel'nikov's 1933 paper, but this paper was the first to point out the fundamental significance of the expansion (18.121) for the theory of transmission of continuous messages over a communication channel. The idea of the derivation of formula (18.121) given above is also due to Kotel'nikov. In the general case this question was later studied by the outstanding American engineer and mathematician Claude Shannon, whose work in 1948 provided the fundamentals the information theory.

*Remark 9* In reality the transmission and receiving time of a communication is also limited, so that instead of the entire series (18.121) we take one of its partial sums $\sum_{-N}^{N}$. Special research has been devoted to estimating the errors that thereby arise.

*Remark 10* If we assume that the amount of information transmitted over the communication channel is proportional to the amount of reference values, then accordingly to formula (18.121) the communication channel capacity is proportional to its bandwidth frequency.

### *18.3.5 Problems and Exercises*

**1.** a) Write out the proof of relations (18.93)–(18.96) in detail.

b) Regarding the Fourier transform as a mapping $f \mapsto \hat{f}$, show that it has the following frequently used properties:

$$f(at) \mapsto \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right)$$

(the change of scale rule);

$$f(t-t_0) \mapsto \hat{f}(\omega)\mathrm{e}^{-i\omega t_0}$$

(time shift of the input signal – the Fourier pre-image – or the *translation theorem*)

$$\big[f(t+t_0) \pm f(t-t_0)\big] \mapsto \begin{cases} \hat{f}(\omega)2\cos\omega t_0, \\ \hat{f}(\omega)2\sin\omega t_0; \end{cases}$$

$$f(t)\mathrm{e}^{\pm i\omega_0 t} \mapsto \hat{f}(\omega \pm \omega_0)$$

(frequency shift of the Fourier transform);

$$f(t)\cos\omega_0 t \mapsto \frac{1}{2}\big[\hat{f}(\omega-\omega_0)+\hat{f}(\omega+\omega_0)\big],$$

$$f(t)\sin\omega_0 t \mapsto \frac{1}{2}\big[\hat{f}(\omega-\omega_0)-\hat{f}(\omega+\omega_0)\big]$$

(amplitude modulation of a harmonic signal);

$$f(t)\sin^2\frac{\omega_0 t}{2} \mapsto \frac{1}{4}\big[2\hat{f}(\omega)-\hat{f}(\omega-\omega_0)-\hat{f}(\omega+\omega_0)\big].$$

c) Find the Fourier transforms (or, as we say, the *Fourier images*) of the following functions:

$$\Pi_A(t) = \begin{cases} \frac{1}{2A} & \text{for } |t| \le A, \\ 0 & \text{for } |t| > A \end{cases}$$

(the *rectangular pulse*);

$$\Pi_A(t)\cos\omega_0 t$$

(a harmonic signal modulated by a rectangular pulse);

$$\Pi_A(t+2A)+\Pi_A(t-2A)$$

(two rectangular pulses of the same polarity);

$$\Pi_A(t-A)-\Pi_A(t+A)$$

(two rectangular pulses of opposite polarity);

$$\Lambda_A(t) = \begin{cases} \frac{1}{A}(1-\frac{|t|}{A}) & \text{for } |t| \le A, \\ 0 & \text{for } |t| > A \end{cases}$$

(a triangular pulse);

$$\cos at^2 \quad \text{and} \quad \sin at^2 \quad (a > 0);$$
$$|t|^{-\frac{1}{2}} \quad \text{and} \quad |t|^{-\frac{1}{2}}\mathrm{e}^{-a|t|} \quad (a > 0).$$

d) Find the Fourier pre-images of the following functions:

$$\operatorname{sinc}\frac{\omega A}{\pi}, \quad 2i\,\frac{\sin^2\omega A}{\omega A}, \quad 2\operatorname{sinc}^2\frac{\omega A}{\pi},$$

where $\operatorname{sinc}\frac{x}{\pi} := \frac{\sin x}{x}$ is the *sample function* (*cardinal sine*).

e) Using the preceding results, find the values of the following integrals, which we have already encountered:

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,\mathrm{d}x, \quad \int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}\,\mathrm{d}x, \quad \int_{-\infty}^{\infty}\cos x^2\,\mathrm{d}x, \quad \int_{-\infty}^{\infty}\sin x^2\,\mathrm{d}x.$$

f) Verify that the Fourier integral of a function $f(t)$ can be written in any of the following forms:

$$f(t) \sim \int_{-\infty}^{\infty}\hat{f}(\omega)\mathrm{e}^{it\omega}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\int_{-\infty}^{\infty}f(x)\mathrm{e}^{-i\omega(x-t)}\,\mathrm{d}x =$$
$$= \frac{1}{\pi}\int_{0}^{\infty}\mathrm{d}\omega\int_{-\infty}^{\infty}f(x)\cos 2\omega(x-t)\,\mathrm{d}x.$$

**2.** Let $f = f(x, y)$ be a solution of the two-dimensional Laplace equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ in the half-plane $y \geq 0$ satisfying the conditions $f(x, 0) = g(x)$ and $f(x, y) \to 0$ as $y \to +\infty$ for every $x \in \mathbb{R}$.

a) Verify that the Fourier transform $\hat{f}(\xi, y)$ of $f$ on the variable $x$ has the form $\hat{g}(\xi)\mathrm{e}^{-y|\xi|}$.

b) Find the Fourier pre-image of the function $\mathrm{e}^{-y|\xi|}$ on the variable $\xi$.

c) Now obtain the representation of the function $f$ as a Poisson integral

$$f(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-\xi)^2 + y^2}g(\xi)\,\mathrm{d}\xi,$$

which we have met in Example 5 of Sect. 17.4.

**3.** We recall that the $n$th *moment* of the function $f : \mathbb{R} \to \mathbb{C}$ is the quantity $M_n(f) = \int_{-\infty}^{\infty} x^n f(x)\,\mathrm{d}x$. In particular, if $f$ is the density of a probability distribution, that is, $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$, then $x_0 = M_1(f)$ is the mathematical expectation of a random variable $x$ with the distribution $f$ and the variance $\sigma^2 := \int_{-\infty}^{\infty}(x - x_0)^2 f(x)\,\mathrm{d}x$ of this random variable can be represented as $\sigma^2 = M_2(f) - M_1^2(f)$.

Consider the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x$$

of the function $f$. By expanding $\mathrm{e}^{-i\xi x}$ in a series, show that

a) $\hat{f}(\xi) = \sum_{n=0}^{\infty} \frac{(-i)^n M_n(f)}{n!}\xi^n$ if, for example, $f \in S$.

b) $M_n(f) = (i)^n \hat{f}^{(n)}(0)$, $n = 0, 1, \dots$.

c) Now let $f$ be real-valued, and let $\hat{f}(\xi) = A(\xi)\mathrm{e}^{i\varphi(\xi)}$, where $A(\xi)$ is the absolute value of $\hat{f}(\xi)$ and $\varphi(\xi)$ is its argument; then $A(\xi) = A(-\xi)$ and $\varphi(-\xi) = -\varphi(\xi)$. To normalize the problem, assume that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$. Verify that in that case

$$\hat{f}(\xi) = 1 + i\varphi'(0)\xi + \frac{A''(0) - (\varphi'(0))^2}{2}\xi^2 + o\big(\xi^2\big) \quad (\xi \to 0)$$

and

$$x_0 := M_1(f) = -\varphi'(0), \quad \text{and} \quad \sigma^2 = M_2(f) - M_1^2(f) = -A''(0).$$

**4.** a) Verify that the function $\mathrm{e}^{-a|x|}$ $(a > 0)$, like all its derivatives, which are defined for $x \neq 0$, decreases at infinity faster than any negative power of $|x|$ and yet this function does not belong to the class $S$.

b) Verify that the Fourier transform of this function is infinitely differentiable on $\mathbb{R}$, but does not belong to $S$ (and all because $\mathrm{e}^{-a|x|}$ is not differentiable at $x = 0$).

**5.** a) Show that the functions of class $S$ are dense in the space $\mathcal{R}_2(\mathbb{R}^n, \mathbb{C})$ of functions $f : \mathbb{R}^n \to \mathbb{C}$ whose squares are absolutely integrable, endowed with the inner product $\langle f, g\rangle = \int_{\mathbb{R}^n}(f \cdot \overline{g})(x)\,\mathrm{d}x$ and the norm it generates $\|f\| = \big(\int_{\mathbb{R}^n} |f|^2(x)\,\mathrm{d}x\big)^{1/2}$ and the metric $d(f, g) = \|f - g\|$.

b) Now let us regard $S$ as a metric space $(S, d)$ with this metric $d$ (convergence in the mean-square sense on $\mathbb{R}^n$). Let $L_2(\mathbb{R}^n, \mathbb{C})$ or, more briefly, $L_2$, denote the completion of the metric space $(S, d)$ (see Sect. 9.5). Each element $f \in L_2$ is determined by a sequence $\{\varphi_k\}$ of functions $\varphi_k \in S$ that is a Cauchy sequence in the sense of the metric $d$.

Show that in that case the sequence $\{\hat{\varphi}\}$ of Fourier images of the functions $\varphi_k$ is also a Cauchy sequence in $S$ and hence defines a certain element $\hat{f} \in L_2$, which it is natural to call the Fourier transform of $f \in L_2$.

c) Show that a vector-space structure and an inner product can be introduced in a natural way on $L_2$, and in these structures the Fourier transform $L_2 \xrightarrow{\widehat{\ }} L_2$ turns out to be a linear isometry of $L_2$ onto itself.

d) Using the example of the function $f(x) = \frac{1}{\sqrt{1+x^2}}$, one can see that if $f \in \mathcal{R}_2(\mathbb{R}, \mathbb{C})$ we do not necessarily have $f \in \mathcal{R}(\mathbb{R}, \mathbb{C})$. Nevertheless, if $f \in \mathcal{R}_2(\mathbb{R}, \mathbb{C})$, then, since $f$ is locally integrable, one can consider the function

$$\hat{f}_A(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x.$$

Verify that $\hat{f}_A \in C(\mathbb{R}, \mathbb{C})$ and $\hat{f}_A \in \mathcal{R}_2(\mathbb{R}, \mathbb{C})$.

e) Prove that $\hat{f}_A$ converges in $L_2$ to some element $\hat{f} \in L_2$ and $\|\hat{f}_A\| \to \|\hat{f}\| = \|f\|$ as $A \to +\infty$ (this is Plancherel's theorem[26]).

**6.** *The uncertainty principle*. Let $\varphi(x)$ and $\psi(p)$ be functions of class $S$ (or elements of the space $L_2$ of Problem 5), with $\psi = \hat{\varphi}$ and $\int_{-\infty}^{\infty} |\varphi|^2(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} |\psi|^2(p)\,\mathrm{d}p = 1$. In this case the functions $|\varphi|^2$ and $|\psi|^2$ can be regarded as probability densities for random variables $x$ and $p$ respectively.

a) Show that by a shift in the argument of $\varphi$ (a special choice of the point from which the argument is measured) one can obtain a new function $\varphi$ such that $M_1(|\varphi|) = \int_{-\infty}^{\infty} x|\varphi|^2(x)\,\mathrm{d}x = 0$ without changing the value of $\|\hat{\varphi}\|$, and then, without changing the relation $M_1(|\varphi|) = 0$ one can, by a similar shift in the argument of $\psi$ arrange that $M_1(|\psi|) = \int_{-\infty}^{\infty} p|\psi|^2(p)\,\mathrm{d}p = 0$.

b) For real values of the parameter $\alpha$ consider the quantity

$$\int_{-\infty}^{\infty} \left|\alpha x\varphi(x) + \varphi'(x)\right|^2 \mathrm{d}x \geq 0$$

and, using Parseval's equality and the formula $\hat{\varphi}'(p) = ip\hat{\varphi}(p)$, show that $\alpha^2 M_2(|\varphi|) - \alpha + M_2(|\psi|) \geq 0$. (For the definitions of $M_1$ and $M_2$ see Problem 3.)

c) Obtain from this the relation

$$M_2\big(|\varphi|\big) \cdot M_2\big(|\psi|\big) \geq 1/4.$$

This relation shows that the more "concentrated" the function $\varphi$ itself is, the more "smeared" its Fourier transform, and vice versa (see Examples 1 and 7 and Problem 7b).

In quantum mechanics this relation, called the *uncertainty principle*, assumes a specific physical meaning. For example, it is impossible to measure precisely both the coordinate of a quantum particle and its momentum. This fundamental fact (called *Heisenberg's*[27] *uncertainty principle*), is mathematically the same as the relation between $M_2(|\varphi|)$ and $M_2(|\psi|)$ found above.

The next three problems give an elementary picture of the Fourier transform of generalized functions.

**7.** a) Using Example 1, find the spectrum of the signal expressed by the functions

$$\Delta_\alpha(t) = \begin{cases} \frac{1}{2\alpha} & \text{for } |t| \leq \alpha, \\ 0 & \text{for } |t| > \alpha. \end{cases}$$

b) Examine the variation of the function $\Delta_\alpha(t)$ and its spectrum as $\alpha \to +0$ and tell what, in your opinion, should be regarded as the spectrum of a unit pulse, expressed by the $\delta$-function.

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[26]M. Plancherel (1885–1967) – Swiss mathematician.

[27]W. Heisenberg (1901–1976) – German physicist, one of the founders of quantum mechanics.

c) Using Example 2, now find the signal $\varphi(t)$ emerging from an ideal low-frequency filter (with upper frequency limit $a$) in response to a unit pulse $\delta(t)$.

d) Using the result just obtained, now explain the physical meaning of the terms in the Kotel'nikov series (18.121) and propose a theoretical scheme for transmitting a band-limited signal $f(t)$, based on Kotel'nikov's formula (18.121).

**8.** *The space of L. Schwartz.* Verify that

a) If $\varphi \in S$ and $P$ is a polynomial, then $(P \cdot \varphi) \in S$.

b) If $\varphi \in S$, then $D^\alpha \varphi \in S$ and $D^\beta(PD^\alpha \varphi) \in S$, where $\alpha$ and $\beta$ are nonnegative multi-indices and $P$ is a polynomial.

c) We introduce the following notion of convergence in $S$. A sequence $\{\varphi_k\}$ of functions $\varphi_k \in S$ converges to zero if for all nonnegative multi-indices $\alpha$ and $\beta$ the sequence of functions $\{x^\beta D^\alpha \varphi(x)\}$ converges uniformly to zero on $\mathbb{R}^n$. The relation $\varphi_k \to \varphi \in S$ will mean that $(\varphi - \varphi_k) \to 0$ in $S$.

The vector space $S$ of rapidly decreasing functions with this convergence is called the *Schwartz space*.

Show that if $\varphi_k \to \varphi$ in $S$, then $\hat{\varphi}_k \to \hat{\varphi}$ in $S$ as $k \to \infty$. Thus the Fourier transform is a continuous linear operator on the Schwartz space.

**9.** *The space $S'$ of tempered distributions.* The continuous linear functionals defined on the space $S$ of rapidly decreasing functions are called *tempered distributions*. The vector space of such functionals (the conjugate of $S$) is denoted $S'$. The value of the functional $F \in S'$ on a function $\varphi \in S$ will be denoted $F(\varphi)$.

a) Let $P : \mathbb{R}^n \to \mathbb{C}$ be a polynomial in $n$ variables and $f : \mathbb{R}^n \to \mathbb{C}$ a locally integrable function admitting the estimate $|f(x)| \le |P(x)|$ at infinity (that is, it may increase as $x \to \infty$, but only moderately: not faster than power growth). Show that $f$ can then be regarded as a (regular) element of $S'$ if we set

$$f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,\mathrm{d}x \quad (\varphi \in S).$$

b) Multiplication of a tempered distribution $F \in S'$ by an ordinary function $f : \mathbb{R}^n \to \mathbb{C}$ is defined, as always, by the relation $(fF)(\varphi) := F(f\varphi)$. Verify that for tempered distributions multiplication is well defined, not only by functions $f \in S$, but also by polynomials $P : \mathbb{R}^n \to \mathbb{C}$.

c) Differentiation of tempered distributions $F \in S'$ is defined in the traditional way: $(D^\alpha F)(\varphi) := (-1)^{|\alpha|} F(D^\alpha \varphi)$.

Show that this is correctly defined, that is, if $F \in S'$, then $D^\alpha F \in S'$ for every nonnegative integer multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$.

d) If $f$ and $\varphi$ are sufficiently regular functions (for example, functions in $S$), then, as relation (18.113) shows, the following equality holds:

$$\hat{f}(\varphi) = \int_{\mathbb{R}^n} \hat{f}(x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\hat{\varphi}(x)\,\mathrm{d}x = f(\hat{\varphi}).$$

This equality (Parseval's equality) is made the basis of the definition of the Fourier transform $\hat{F}$ of a tempered distribution $F \in S'$. By definition we set $\hat{F}(\varphi) := F(\hat{\varphi})$.

Due to the invariance of $S$ under the Fourier transform, this definition is correct for every element $F \in S'$.

Show that it is not correct for generalized functions in $\mathcal{D}'(\mathbb{R}^n)$ mapping the space $\mathcal{D}(\mathbb{R}^n)$ of smooth functions of compact support. This fact explains the role of the Schwartz space $S$ in the theory of the Fourier transform and its application to generalized functions.

e) In Problem 7 we acquired a preliminary idea of the Fourier transform of the $\delta$-function. The Fourier transform of the $\delta$-function could have been sought directly from the definition of the Fourier transform of a regular function. In that case we would have found that

$$\hat{\delta}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \delta(x) \mathrm{e}^{-i(\xi,x)}\, \mathrm{d}x = \frac{1}{(2\pi)^{n/2}}.$$

Now show that when we seek the Fourier transform of the tempered distribution $\delta \in S'(\mathbb{R}^n)$ correctly, that is, starting from the equality $\hat{\delta}(\varphi) = \delta(\hat{\varphi})$, the result (still the same) is that $\delta(\hat{\varphi}) = \hat{\varphi}(0) = \frac{1}{(2\pi)^{n/2}}$. (One can renormalize the Fourier transform so that this constant equals 1; see Problem 10.)

f) Convergence in $S'$, as always in generalized functions, is understood in the following sense: $(F_n \to F)$ in $S'$ as $n \to \infty := (\forall \varphi \in S\ (F_n(\varphi) \to F(\varphi)$ as $n \to \infty))$.

Verify the Fourier inversion formula (the Fourier integral formula) for the $\delta$-function:

$$\delta(x) = \lim_{A \to +\infty} \frac{1}{(2\pi)^{n/2}} \int_{-A}^{A} \cdots \int_{-A}^{A} \hat{\delta}(\xi) \mathrm{e}^{i(x,\xi)}\, \mathrm{d}\xi.$$

g) Let $\delta(x - x_0)$, as usual, denote the shift of the $\delta$-function to the point $x_0$, that is, $\delta(x - x_0)(\varphi) = \varphi(x_0)$. Verify that the series

$$\sum_{n=-\infty}^{\infty} \delta(x - n) \quad \left( = \lim_{N \to \infty} \sum_{-N}^{N} \delta(x - n) \right)$$

converges in $S'(\mathbb{R}^n)$. (Here $\delta \in S'(\mathbb{R}^n)$ and $n \in \mathbb{Z}$.)

h) Using the possibility of differentiating a convergent series of generalized functions termwise and taking account of the equality from Problem 13f of Sect. 18.2, show that if $F = \sum_{n=-\infty}^{\infty} \delta(x - n)$, then

$$\hat{F} = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n).$$

i) Using the relation $\hat{F}(\varphi) = F(\hat{\varphi})$, obtain the Poisson summation formula from the preceding result.

j) Prove the following relation (the $\theta$-*formula*)

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{-tn^2} = \sqrt{\frac{\pi}{t}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi^2}{t}n^2} \quad (t > 0),$$

which plays an important role in the theory of elliptic functions and the theory of heat conduction.

**10.** If the Fourier transform $\check{\mathcal{F}}[f]$ of a function $f : \mathbb{R} \to \mathbb{C}$ is defined by the formulas

$$\check{f}(\nu) := \check{\mathcal{F}}[f](\nu) := \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-2\pi i\nu t}\,\mathrm{d}t,$$

many of the formulas relating to the Fourier transform become particularly simple and elegant.

a) Verify that $\hat{f}(u) = \frac{1}{\sqrt{2\pi}}\check{f}(\frac{u}{2\pi})$.

b) Show that $\check{\mathcal{F}}[\check{\mathcal{F}}[f]](t) = f(-t)$, that is,

$$f(t) = \int_{-\infty}^{\infty} \check{f}(\nu)\mathrm{e}^{2\pi i\nu t}\,\mathrm{d}\nu.$$

This is the most natural form of the expansion of $f$ in harmonics of different frequencies $\nu$, and $\check{f}(\nu)$ in this expansion is the *frequency spectrum of* $f$.

c) Verify that $\check{\delta} = 1$ and $\check{1} = \delta$.

d) Verify that the Poisson summation formula (18.116) now assumes the particularly elegant form

$$\sum_{n=-\infty}^{\infty} \varphi(n) = \sum_{n=-\infty}^{\infty} \check{\varphi}(n).$$

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