

Chapter 2

Grassmann Algebra

Abstract The elements of the Grassmann algebra, and the operations addition and multiplication are defined. Distinction is made between even and odd elements. A few remarks on exterior algebra then follow.

2.1 Elements of the Algebra

The elements of the algebra, addition and multiplication are defined.

The use of Grassmann variables in the context of physical problems and an introduction to these variables can be found for example in the books by Zinn-Justin [297] and by Efetov [65]. From the more mathematical side the books by Berezin [25] and by de Witt [55] are recommended.

To begin with, we have a basis of r vectors ζ_i , $i = 1, \dots, r$. This basis is then enlarged by the introduction of products of the vectors ζ_i . This product obeys the associative law and the law of anticommutativity,

$$\zeta_i \zeta_j = -\zeta_j \zeta_i, \quad (2.1)$$

which implies $\zeta_i^2 = 0$. Including the empty product, one obtains 2^r basis elements

$$\begin{aligned} &1 \\ &\zeta_i \quad i = 1 \dots r \\ &\zeta_i \zeta_j = -\zeta_j \zeta_i \quad i < j \\ &\dots \\ &\zeta_1 \zeta_2 \dots \zeta_r. \end{aligned} \quad (2.2)$$

The elements of the algebra are then the linear combinations of these 2^r basis vectors

$$a = a^{(0)} + \sum_i a_i^{(1)} \zeta_i + \sum_{i < j} a_{ij}^{(2)} \zeta_i \zeta_j + \dots, \quad (2.3)$$

where the coefficients $a_{i_1 \dots i_m}^{(m)}$ are complex numbers. We denote the set of elements a given in (2.3) by \mathcal{A} .

Addition is defined as is usual in vector spaces: the coefficients with equal indices m, i_1, \dots, i_m are added. Thus, addition is commutative and associative,

$$a + b = b + a, \quad (a + b) + c = a + (b + c) = a + b + c. \quad (2.4)$$

Multiplication of the monomials

$$a = a^{(k)} \zeta_{i_1} \zeta_{i_2} \dots \zeta_{i_k}, \quad (2.5)$$

$$b = b^{(l)} \zeta_{j_1} \zeta_{j_2} \dots \zeta_{j_l} \quad (2.6)$$

yields

$$ab = a^{(k)} b^{(l)} \zeta_{i_1} \zeta_{i_2} \dots \zeta_{i_k} \zeta_{j_1} \zeta_{j_2} \dots \zeta_{j_l}. \quad (2.7)$$

If at least one factor ζ_i agrees with one factor ζ_j , then ab vanishes. Multiplication of polynomials, like (2.3), follows from the requirement that the law of distributivity holds,

$$(a + b)c = ac + bc, \quad a(b + c) = ab + ac. \quad (2.8)$$

Then it is easy to show that multiplication is associative,

$$(ab)c = a(bc) = abc. \quad (2.9)$$

Example 2.1.1 $a = 5\zeta_2, b = 3\zeta_1\zeta_3$ yield $ab = 15\zeta_2\zeta_1\zeta_3 = -15\zeta_1\zeta_2\zeta_3$.

Example 2.1.2 $a = 3\zeta_1\zeta_3 + 5\zeta_2$ yields $a^2 = (3\zeta_1\zeta_3 + 5\zeta_2)(3\zeta_1\zeta_3 + 5\zeta_2) = 9\zeta_1\zeta_3\zeta_1\zeta_3 + 15\zeta_1\zeta_3\zeta_2 + 15\zeta_2\zeta_1\zeta_3 + 25\zeta_2\zeta_2 = 9 \cdot 0 - 15\zeta_1\zeta_2\zeta_3 - 15\zeta_1\zeta_2\zeta_3 + 25 \cdot 0 = -30\zeta_1\zeta_2\zeta_3$.

2.2 Even and Odd Elements, Graded Algebra

Even and odd elements and their algebraic properties are defined.

Let us introduce the linear parity operator \mathcal{P} ,

$$\mathcal{P}(\zeta) = -\zeta. \quad (2.10)$$

This operator multiplies a monomial of order k in the Grassmann variables by $(-)^k$. Thus a in (2.5) and b in (2.6) obey

$$\mathcal{P}(a) = (-)^k a, \quad \mathcal{P}(b) = (-)^l b. \quad (2.11)$$

For the monomials a, b in (2.5), (2.6) one obtains

$$ab = (-)^{kl}ba. \quad (2.12)$$

A product of an even (odd) number of factors of ζ and their linear combinations are called even (odd) elements of the algebra. Each element, $a \in \mathcal{A}$, can be decomposed uniquely in a sum of an even and an odd element, $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$,

$$a = a_0 + a_1, \quad a_i \in \mathcal{A}_i, \quad (2.13)$$

$$a_0 = a^{(0)} + \sum_{i < j} a_{ij}^{(2)} \zeta_i \zeta_j + \sum_{i < j < k < l} a_{ijkl}^{(4)} \zeta_i \zeta_j \zeta_k \zeta_l + \dots, \quad (2.14)$$

$$a_1 = \sum_i a_i^{(1)} \zeta_i + \sum_{i < j < k} a_{ijk}^{(3)} \zeta_i \zeta_j \zeta_k + \dots \quad (2.15)$$

This decomposition into even and odd elements is the reason that this algebra is called a *graded algebra*. Generally, a graded algebra has the property that all elements can be decomposed into elements of *degree* ν ,

$$a = \sum_{\nu} a_{\nu}, \quad a_{\nu} \in \mathcal{A}_{\nu}. \quad (2.16)$$

Sums of elements in \mathcal{A}_{ν} belong to \mathcal{A}_{ν} . Products of elements in \mathcal{A}_{ν} and $\mathcal{A}_{\nu'}$ belong to $\mathcal{A}_{\nu+\nu'}$.

The following expressions, from (2.14), (2.15), belong to \mathcal{A}_0 and \mathcal{A}_1

$$a_0 + b_0, \quad a_0 b_0 = b_0 a_0, \quad a_1 b_1 = -b_1 a_1 \in \mathcal{A}_0, \quad (2.17)$$

$$a_1 + b_1, \quad a_0 b_1 = b_1 a_0, \quad a_1 b_0 = b_0 a_1 \in \mathcal{A}_1. \quad (2.18)$$

This grading is a \mathbb{Z}_2 -grading, since the degree ν assumes only the values 0 and 1, and calculation is modulo 2.

In the following, we will mainly deal with elements a , which are either even ($a \in \mathcal{A}_0$) or odd ($a \in \mathcal{A}_1$) (Problem 2.3 is an exception). The degree of an element a will often be denoted by $\nu(a)$. Thus (2.12) may be rewritten as

$$ba = (-)^{\nu(a)\nu(b)}ab. \quad (2.19)$$

One observes that $a^2 = 0$ for all $a \in \mathcal{A}_1$.

2.3 Body and Soul, Functions

Body (ordinary part) and soul (nilpotent part) are introduced.

The contribution $a^{(0)}$ from a in (2.3) is called the *body (ordinary part)* of a

$$a^{(0)} = \text{ord } a, \quad (2.20)$$

with everything else being the *soul (nilpotent part)* of a

$$\text{nil } a = a - a^{(0)}, \quad (2.21)$$

since, as one sees immediately

$$(\text{nil } a)^{r+1} = 0. \quad (2.22)$$

Actually, one has already $(\text{nil } a)^p = 0$ for $2p > r + 1$.

If $f : \mathcal{A} \rightarrow \mathcal{A}$ is a function, which can be differentiated sufficiently often, then $f(a)$ can be defined by its Taylor expansion

$$f(a) = f(a^{(0)}) + f'(a^{(0)})\text{nil } a + \frac{1}{2}f''(a^{(0)})(\text{nil } a)^2 + \dots \quad (2.23)$$

Due to (2.22) the expansion only has a finite number of terms.

The generalization to functions of several variables constitutes no problem if the variables are even, since these variables and their nilpotent parts commute. Functions of even and odd variables can always be represented as polynomials in the odd variables, where the coefficients are functions of the even variables.

2.4 Exterior Algebra I

A short excursion (first part) to exterior algebra is given.

From

$$a(\zeta) = a_1\zeta_1 + a_2\zeta_2 + a_3\zeta_3, \quad b(\zeta) = b_1\zeta_1 + b_2\zeta_2 + b_3\zeta_3, \quad (2.24)$$

and similarly for c , one obtains the exterior products

$$\begin{aligned} (ab)(\zeta) &= (a_1b_2 - a_2b_1)\zeta_1\zeta_2 + (a_2b_3 - a_3b_2)\zeta_2\zeta_3 + (a_3b_1 - a_1b_3)\zeta_3\zeta_1, \\ (abc)(\zeta) &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \zeta_1\zeta_2\zeta_3. \end{aligned} \quad (2.25)$$

Usually the elements of the exterior algebra refer to some vectors or antisymmetric tensors in an n -dimensional space. The examples in (2.24) are 1-vectors with the products (ab) and (abc) being 2-vector and 3-vector, respectively.

Exterior algebra uses a grading different from that introduced in Sect. 2.2. Linear combinations of the monomials in (2.5) with fixed k are k -vectors. This set of k -vectors is denoted by \mathcal{E}_k . The grading decomposes the elements into the sets \mathcal{E}_k for $k = 0 \dots n$. Thus the vectors a and b in (2.5), (2.6) are k - and l -vectors, respectively. Generally, the product ab of $a \in \mathcal{E}_k$, $b \in \mathcal{E}_l$ is a $k + l$ -product $ab \in \mathcal{E}_{k+l}$.

Often the basis vectors \mathbf{e}_i are used instead of ξ_i . To indicate that the product is anticommutative one may use a wedge, \wedge , as the sign for multiplication and the product is called the *wedge product*, for example

$$a(\mathbf{e}) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3, \quad (2.26)$$

$$(ab)(\mathbf{e}) = (a_1b_2 - a_2b_1)\mathbf{e}_1 \wedge \mathbf{e}_2 + (a_2b_3 - a_3b_2)\mathbf{e}_2 \wedge \mathbf{e}_3 + (a_3b_1 - a_1b_3)\mathbf{e}_3 \wedge \mathbf{e}_1.$$

Obviously the coefficients of the product ab are the coefficients of the cross product $a \times b$. The coefficient of $\xi_1\xi_2\xi_3$ in the product abc is the triple product of the three vectors a, b, c , if $\mathbf{a} = \sum_i a_i\mathbf{e}_i$ with the orthonormal vectors \mathbf{e}_i (similarly for b and c).

Quite generally, such products are called wedge products and the algebra is called an exterior algebra. Such ideas in general dimensions were the basis of Grassmann's extension calculus.

Problems

2.1 Calculate

$$\left(\sum_{i=1}^3 \sum_{j=i+1}^4 a_{ij} \xi_i \xi_j \right)^2, \quad a_{ij} \in \mathcal{A}_0, \quad \xi_i \in \mathcal{A}_1.$$

2.2 Solve $x^2 = a^2 + 2\xi_1\xi_2$ for x with $a \in \mathbb{C}$, $\xi_1, \xi_2 \in \mathcal{A}_1$. Is there a solution for $a = 0$?

2.3 Solve $x^2 = \xi_1\xi_2\xi_3$ for x with $\xi_i \in \mathcal{A}_1$. The result shall depend only on the Grassmannians ξ_i , $i = 1..3$.

2.4 Substitute $x = z + \alpha\xi_1 + \beta\xi_2 + c\xi_1\xi_2$, $x, z, c \in \mathcal{A}_0$, $\alpha, \beta, \xi_i \in \mathcal{A}_1$ in $g(x, \xi_1, \xi_2) = g_0(z) + g_1(z)\gamma_1\xi_1 + g_2(z)\gamma_2\xi_2 + g_{12}(z)\xi_1\xi_2$, $\gamma_i \in \mathcal{A}_1$, $g_i(z) \in \mathcal{A}_0$ and determine $f(z, \xi_1, \xi_2) = g(x, \xi_1, \xi_2)$ in the form $f(z, \xi_1, \xi_2) = f_0(z) + f_1(z)\xi_1 + f_2(z)\xi_2 + f_{12}(z)\xi_1\xi_2$.

2.5 Calculate A^{-1} from $A = a_0 + \xi_1\eta_1 + \xi_2\eta_2 + a_2\xi_1\xi_2\eta_1\eta_2$, $a_0 \in \mathbb{C}$, $a_0 \neq 0$, $a_2 \in \mathcal{A}_0$, $\xi_i, \eta_i \in \mathcal{A}_1$.

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