Preface to the 2016 Edition

The present volume contains the translation of Felix Klein's "Präzisions- und Approximationsmathematik", which appeared in 1928 as volume III of the series *Elementarmathematik von einem höheren Standpunkte aus*[1] (Elementary mathematics from a higher standpoint). In turn, the 1928 edition was a re-edition of the lecture course delivered by Felix Klein in 1901 with the title "Anwendung der Differential- und Integralrechnung auf die Geometrie: eine Revision der Prinzipien", published in 1902 in lithographic form[2] (a reprint of 1908, edited by Conrad Heinrich Müller, left the text essentially unchanged).

Klein participated, together with Fritz Seyfarth, in the whole project of re-editing the three volumes on *Elementary mathematics from a higher standpoint*, but he died in 1925, after the first two volumes had appeared. The third volume was, therefore, edited only by Seyfarth; changes and insertions had been nevertheless discussed with Klein, as Seyfarth writes in his preface. The notes added are probably only by Seyfarth. We left, as in the 1928 edition, the notes by Seyfarth in square brackets, and labelled "translator's note" the notes added in the present edition. Of course, the notes by Klein himself are included without any additional marks.

In the text the reader will find, again in square brackets, *the page numbering of the original edition*. Cross references in notes and in the text *refer to this numbering*, as well as *the name index and the subject index* (that is, the original text has not been changed to this respect). Similarly for the references to Volume 1 and 2 of 1924/25, whose original layout is marked in the current new edition. Moreover, we left, as in the German edition, the comma to separate the integer part from the fractional part of a decimal number.

[1] Felix Klein, 1928. *Elementarmathematik vom höheren Standpunkte aus, 3: Präzisions- und Approximationsmathematik*; ausgearbeitet von C. H. Müller; für den Druck fertig gemacht und mit Zusätzen versehen von Fr. Seyfarth. 3. Aufl., Berlin: J. Springer, 1928 Series: Die Grundlehren der mathematischen Wissenschaften, 16. Reprint in 1968.

[2] Felix Klein, 1902. *Anwendung der Differential- und Integralrechnung auf die Geometrie: eine Revision der Prinzipien: Vorlesung gehalten während des Sommersemesters 1901*; ausgearbeitet von C. Müller, Leipzig: Teubner.

In the present translation we have added, when possible, the first names of the persons mentioned. In the German edition, as it was customary at that time, the first names were indicated only with the initials. The bibliographic references in the notes have also been completed, when needed.

"Präzisions- und Approximationsmathematik" was never translated into English, unlike the first two volumes; although the translators of the first two volumes were aware of the existence of volume III, they gave no reason for not having translated it. While the Spanish, the Russian and the Japanese translations also did not include volume III[3], there is a complete translation of all the three volumes into Chinese – first published in 1989 by Hubei Educational Press, Beijing, and published again by Chiu Chan Publishing (Republic of China, Taipei) in 1996.

On the occasion of the 13th International Congress on Mathematics Education (ICME-13, Hamburg 2016), the will of the *International Commission on Mathematical Instruction* (ICMI), together with Springer, is of closing this gap in the English version. We also proudly note that Felix Klein was the first president of ICMI, which was founded in Rome in 1908.

A translation after nearly a century, joined with a new edition of the two first volumes, might seem strange. Undoubtedly, this translation has an historical value, since Klein is one of the greatest mathematicians of history[4]. It also has a mathematical value – because of the interesting approaches and the links with the applications. But, above all, it has a didactical value, concerning the training of mathematics teachers. The volumes on *Elementary Mathematics from a higher standpoint* are still apt to this aim, for the simple reason that today's school mathematics is not very different from the school mathematics proposed at the time of Klein. Even if today we can find some snapshots of the mathematics of the 20th century (which is indeed too specialised) and also contents of probability and statistics, the remaining curricular contents have been developed well before the 20th century. The *higher mathematics* for these contents is mainly found in the period of the systematization of the mathematical theories, which corresponds to the period of Klein's writings. And, indeed, most of the contents of Klein's volumes are still offered in the many courses (which often bear the same name of *Elementary mathematics from a higher standpoint*) that all over the world are devoted to pre-service training of mathematics teachers.

For this third volume in particular, the motivations for the kind of study it proposes are given by Klein himself during his lifelong career. The third volume focuses on those properties that applied mathematicians take for granted when studying certain phenomena from a mathematical point of view. These properties must be seen as supplementary conditions (and constraints) to be required for the ideal objects of pure mathematics. However, in the meantime, these very proper-

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[3] There is a recent translation of Volumes I and II into Portuguese and it might be extended to volume III.

[4] Books by Felix Klein are still in use today: Klein's *Nicht-euklidische Geometrie* has been re-edited in 2006, the English version of the *Lectures on the Ikosahedron* re-edited in 2007, and his *Development of mathematics in the nineteenth century* was translated in 1979.

ties prove to be the more intuitive ones. Therefore the comparison moves towards another field: it is a comparison between properties that can be considered only in the theoretical field of abstract mathematics and properties that can be grasped by intuition. Here the problem proves to become pertinent for mathematics teaching.

Klein had always been convinced of the importance of maintaining the link of mathematics with its applications and had manifested this interest ever since his initial work at the university of Erlangen, in 1872. The necessary link between pure and applied mathematics has been aptly characterised by Klein, in the 6th of his American Conferences (The Evanston Colloquium, 1894, p. 46):

> *I should lay particular stress on the heuristic value of the applied sciences as an aid to discovering new truths in mathematics. Thus I have shown (in my little book on Riemann's theories) that the Abelian integrals can best be understood and illustrated by considering electric currents on closed surfaces. In an analogous way, theorems concerning differential equations can be derived from the consideration of sound-vibrations; and so on.*

In this way Klein stresses the importance of the connection of mathematics with other disciplines that can foster intuition; and, in particular, the importance of maintaining this connection for (university) teaching:

> *It is not only admissible, but absolutely necessary, to be less abstract at the start, to have constant regard to the applications, and to refer to the refinements only gradually as the student becomes able to understand them. This is of course nothing but a universal pedagogical principle to be observed in all mathematical instruction* (ibid., 50).

Given Klein's intense plea for the applications, one should remark, furthermore, that he had not only alerted, already in the first volume, in the context of the emergence of set theory, against pushing a formalist program for the foundations of mathematics too far, but he also had taken up the issue again here in volume III, advising against searching for the new only for the sake of doing it:

> *Provided that a deep epistemological need exists, which will be satisfied by the study of a new problem, then it is justified to study it; but if one does it only to do something new, then the extension is not desirable* (p. [157]).

In the preface of the 1928 edition of *Präzisions- und Approximationsmathematik* we do not find explanations of the fact that this lecture course, conceived with the purpose of filling the gap that was created between pure mathematicians and applied mathematicians, had become a teacher training course within the series of *Elementary mathematics from a higher standpoint*. Indeed, the re-editing of this lecture course was completed after Klein's death.

In any case, a reference to teaching in schools can be found in the preface to Volume III of 1902, but we find nothing explicit concerning Klein's general vision on applications. Klein only claims that teaching – for beginners as well as for those students who want to use mathematics only as an auxiliary means – has to make a naïve use of the intuitive approaches.

In place of Klein, we can quote the Italian geometer Guido Castelnuovo, who was very close to Klein's points of view. In 1913/14 at the University of Rome Castelnuovo used Klein's 1902 lecture course, recognising its significance for the students who were to become high school teachers. At the beginning of his lectures Castelnuovo asserted that the teaching and learning of mathematics would be more successful if it included, besides the logical procedures that lead to the theorems, also the way in which concepts are formed starting from observations, and the way in which these same theorems can be verified in practice. Castelnuovo therefore sees the necessity to distinguish, as does Felix Klein, between precision mathematics and approximation mathematics: the first one includes all the propositions that can be logically deduced from the axioms of geometry or of analysis – obtained by abstraction from experience; the second one includes the results that can be obtained from experience with a certain degree of approximation.[5]

The school that Castelnuovo had in mind was in particular the *modern lyceé*, which – in the wake of the reform movements of many European countries – had to include new topics, in particular differential and integral calculus, with wider application in the life sciences. Mathematics was there presented as an appropriate language for describing natural phenomena in which the concept of function assumes particular importance.

Still today, this kind of high schools that devote a greater number of hours to mathematics and sciences is widespread, and it is considered important to maintain the link between the different scientific fields and to show how mathematics arises from observation.

Maybe some of the examples offered by Klein will seem "old", like for instance the harmonic analyser for the calculation of the Fourier coefficients. In any case the functioning of this instrument can represent an historical means to clarify certain concepts. The motivations for proposing this kind of study are still, and possibly are even more so today, those given by Klein.

On the other hand, even though he starts from an intuitive and sometimes practical approach, Klein develops more abstract concepts working in rich "mathematical environments", which form the core of a pertinent program for mathematics teacher education at secondary schools.

We thank Leo Rogers for his careful re-reading of the book, and the many colleagues to whom we have asked advice, in particular Geoffrey Howson and Claudio Bernardi.

Marta Menghini and Gert Schubring

[5] See Menghini, Marta (2016), Precision and approximation mathematics for teacher education: the lecture course of Guido Castelnuovo and the influence of Felix Klein, in *Dig where you stand 4* (Bjarnadottir, Furinghetti, Menghini, Prytz and Schubring, eds), Proceedings of the 4th International Conference on the History of Mathematics Education (ICHME4), Turin, 23–26 September, 2015, Rome: Edizioni Nuova Cultura.

Preface to the First Edition

In publishing in lithographic form the lecture course that I gave last summer, I would like to take a new step in the main direction of my efforts over the last years: namely to give back to mathematical science – seen as a connected whole – its value with respect to all its branches, as it was taken for granted before there were specializations. But, in particular, once again to achieve a better mutual understanding among the representatives of abstract and applied mathematics, compared to that which presently exists. I have often emphasised that, in this respect, it would be of crucial importance to have a clear grasp of the differences and also of the mutual relations between the two parts of mathematics that I shall call *precision mathematics* and *approximation mathematics*. Similar thoughts have been expressed in recent years by Messrs. Heinrich Burkhardt and Karl Heun; see for instance the inaugural speech of the former (Zürich, 1897[6]), which recently became accessible to a wider circle of colleagues thanks to its reprint in the first issue of vol. 11 of the *Jahresbericht der deutschen Mathematiker-Vereinigung*, and the report of Mr. Heun on the kinetic problems in technology in Vol. 9 of the same journal (1900–1901). As far as I know, the term "Approximation mathematics" appears for the first time in this report; in any case I took it from there. A detailed explanation of the contrasts to be considered, like those presented for the field of geometry in the following lecture course, should thus be better understood and appreciated by a wider circle of people.

At the same time, with this lecture course, I complete those discussions that I often presented on the methods of mathematics teaching, particularly in higher education (see for instance *Jahresbericht*, vol. 8, 1898–1899). My opinion is still that the teaching for beginners and for those students who want to use mathematics only as an auxiliary means for other studies has to make a naïve use of the intuitive approaches; the conviction that this is necessary for pedagogic reasons, considering the disposition of the majority of students, has become noticeably stronger in the last few years, here and abroad.

[6] Mathematisches und naturwissenschaftliches Denken.

But I am no less convinced (something I never omitted to state) that – in accordance with today's development of science – such a form of teaching will not suffice for the training of the professional mathematician in high school; rather, that here must prevail, in addition to the facts of intuition, the central importance of the modern concept of number and of the far-reaching related developments. What I miss now, in the textbooks and the lecture courses that I have encountered, is the transition from one conception to the other. In this sense the following presentation needs to be inserted as a complement; its major aim will be reached if one day it will prove to be superfluous, because in this case the considerations, which it offers, will have become an obvious part of teaching in all the branches of higher mathematics.

Göttingen, February 28, 1902.

Felix Klein

Preface to the Second Edition

The following is essentially an unchanged reprint of the first edition of the lecture notes in 1902. Only some inaccuracies have been amended, and occasional references to recent publications that are intimately connected to the topics developed in the lecture have been added. At the end, the “Report of the Göttingen Faculty of Philosophy concerning the task for the Beneke prize 1901” is added, to which different remarks in the text refer.

Göttingen, January 5, 1907.

Conrad Heinrich Müller

Preface to the Third Edition

The lecture notes of F. *Klein*, which are now appearing as volume III of "Elementary mathematics from a higher standpoint"[7] with the subtitle "Precision and Approximation Mathematics", were written some years before the lecture courses to be found in the two first volumes. Like these latter, this course is intended for a wide audience because of its goals and the manner of presentation. The lecture notes, in their lithographic form, have for a long time been cited in mathematical literature with the title "applications of differential and integral calculus to geometry (a revision of the principles)". The change of title is due to the personal request of *Felix Klein*, with whom I had – in the last two months before his death – a series of conversations about the work required for their publication. Klein believed that the new title would better meet the goals of the notes than the former.

In editing the lecture notes, the same principles, which have inspired the publication of the first two volumes, have been followed. The original presentation was essentially maintained, but single changes and insertions were added when practical and stylistic considerations were considered necessary. The additions, some of which were due to the need to take into account more recent publications, were inserted in the text when this was required by the organization of the lecture course, otherwise they were added in the form of footnotes. The figures were often improved and expanded, particularly in the final part of the lecture, which concerns gestalt relations of curves in space and surfaces of the third degree. This section, which shows *Klein* to be the master of the art of impressive representation of geometric forms, as he is widely recognised, was originally conceived only for those readers for whom the discussed models were available. Here, it was necessary to structure the text in such a way, that the reader needs no longer to refer to these models, by adding figures and explanations. For the additions, I made much use of a lecture by *Klein* in 1907 about the theory of surfaces and spatial curves. The

[7] Translator's note: The German title is *Elementarmathematik vom höheren Standpunkte aus. Dritter Band: Präzisions- und Approximationsmathematik.*

"report of the Göttingen University Faculty of Philosophy concerning the task for the Beneke prize 1901", which was added to the second edition of 1907, is now omitted, as it has already been reprinted in volume II of *Klein*'s Gesammelte mathematische Abhandlungen.

In the edition of the text I was assisted by Mr. *Alwin Walther*, Göttingen, who made many valuable suggestions. Also Mr. *Hermann Vermeil* contributed to the reading of the proofs and he prepared the new indexes. My colleague *H. Homann* assisted me by photographing a series of models in the mathematical collections in Göttingen. With Prof. *Richard Courant,* in particular, I had some conversations that were important for the progress of my work. I owe all the above persons my hearty thanks. I am also very grateful to the publishing company Julius Springer for the willingness with which they met my requests.

Göttingen, January 31, 1928.

Fritz Seyfarth

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