Introduction [1]

The Aim of this Lecture Course

In recent years[5], a far reaching interest has arisen among university teachers of mathematics and natural sciences directed toward an appropriate *training of future teachers at secondary schools*, satisfying all desirable needs. This is really quite a new phenomenon. For a long time prior to its appearance, university men were practicing exclusively research of optimal quality, without giving a thought to the needs of the schools, without even caring to establish a connection with school mathematics. What is the result of this practice? The young university student finds himself, at the outset, confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. Naturally he forgets all these things quickly and thoroughly. When, after finishing his course of study, he becomes a teacher, he suddenly finds himself expected to teach the traditional elementary mathematics according to school practice; and, since he will be scarcely able, unaided, to discern any connection between this task and his university mathematics, he will soon fell in with the time honoured way of teaching, and his university studies remain only a more or less pleasant memory which has no influence upon his teaching.

There is now a movement to abolish this *double discontinuity*, never having been helpful either to the school or to the university. On the one hand, there is an effort to impregnate the subject matter, which the schools teach with new ideas derived from modern developments of science and in accord with modern culture. We shall often have occasion to go into this. On the other hand, the attempt is made to take into account, in university teaching, the needs of the school teacher. And it is precisely in such summarising lecture courses as I am about to deliver to you that I see one of the most important tools. I shall by no means address myself to beginners, but I shall take for granted that you are all acquainted with the main features of the [2] most important disciplines of mathematics. I shall often have to talk of problems of

[5] [Attention is again drawn to the fact that the wording of the text is, almost throughout, that of the lithographed volume of 1908 and that comments which refer to later years have been put into the footnotes and appendices.]

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algebra, of number theory, of function theory, etc., without being able to go into details. You must, therefore, be moderately familiar with these fields, in order to follow me. My task will always be to show you the *mutual connection between problems in the various disciplines*, these connections use not to be sufficiently considered in the specialised lecture courses, and I want more especially to emphasize the relation of these problems to those of school mathematics. In this way I hope to make it easier for you to acquire that ability which I look upon as the real goal of your academic study: the ability to draw (in ample measure) from the great body of knowledge taught to you here as vivid stimuli for your teaching.

Let me now put before you some documents of recent date which give evidence of widespread interest in the training of teachers and which contain valuable material for us. Above all I think here of the talks given at the last *Meeting of Naturalists* held September 16, 1907, in Dresden, to which body we submitted the "Proposals for the Scientific Training of Prospective Teachers of Mathematics and Science" of the Committee on Instruction of the Society of German Naturalists and Physicians. You will find these Proposals as the last section in the Complete Report of this Committee[6] which, since 1904, has been considering the entire complex of questions concerning teaching of mathematics and natural sciences and has now ended its activity; I urge you to take notice, not only of these Proposals, but also of the other parts of this very interesting report. Shortly after the Dresden meeting there occurred a similar debate at the Meeting of German Philologists and Schoolmen in Basel, September 25, in which, to be sure, the mathematical-scientific reform movement was discussed only as one link in the chain of parallel movements occurring in the spheres of philologists. After a report by me concerning our aims in mathematical-natural science reform there were talks by Paul Wendland (Breslau) on questions in *classical and ancient studies*, Almut Brandl (Berlin) on *modern languages* and, finally, *Adolf Harnack* (Berlin) on *History and religion*. These four talks appeared together in one brochure[7] to which I particularly refer you. I appreci-
[3] ate the thus initiated joint proceeding of our sciences with the philologists as highly fruitful, since it will bring about friendly feeling and mutual understanding between two groups who face each other as aliens, or even adversely. Let us endeavour always to foster such good relations even if we do among ourselves occasionally drop a critical word about the philologists, just as they may about us. Bear in mind that you will later be called upon in the schools to work together with the philologists for the common good and that this requires mutual understanding and appreciation – always beyond the particularism for the proper discipline.

[6] *Die Tätigkeit der Unterrichtskommission der Gesellschaft deutscher Naturforscher und Ärzte*, edited by August Gutzmer. Leipzig and Berlin, 1908. [The German title of the last section is: *Vorschläge für die wissenschaftliche Ausbildung der Lehramtskandidaten der Mathematik und Naturwissenschaften*.]

[7] *Universität und Schule*. Vorträge ... gehalten von F. Klein, Paul Wendland, Almut Brandl, Adolf Harnack. Leipzig 1907.

Relevant Literature

Along with this evidence of efforts which reach beyond the borders of our field, I should like to mention *a few books* which aim in the same direction, in particular *in the mathematical field* and which will therefore become very important for these lectures. Three years ago I gave, for the first time, *a course of lectures with a similar purpose*. My assistant at that time, Rudolf Schimmack, worked the material up and the first part has recently appeared in print[8]. In it are considered the different kinds of schools, including higher education, the general methods to teach mathematics in them, their mutual interacting, and other similar matters. In what follows I shall from time to time refer to things which appear there without repeating them. The more extensive I will discuss here – somehow as continuation of those observations on the organisation of mathematics teaching – the mathematical subject matter, which in whatever way can be relevant for teaching in schools. If I frequently refer to the actual method of teaching in the schools, my remarks will be based not merely upon indefinite pictures of how the thing might be done or even upon dim recollections of my own school days; for I am constantly in touch with Schimmack, who is now teaching in the Göttingen Gymnasium and who keeps me informed as to the present state of teaching, which has, in fact, advanced substantially beyond what it was in earlier years. During this winter semester I shall discuss "the three great A's", that is arithmetic, algebra, and analysis, withholding geometry for a continuation of the lecture course during the coming summer term. Let me remark that, in the language of the secondary schools, these three subjects are classed together as "arithmetic", and that we shall more often note deviations in the mathematical terminology of the schools as compared with that in higher **[4]** education. This small illustration shows you, that only vivid interlocking can bring about agreement.

As a second reference I shall mention the three volume *Enzyklopädie der Elementarmathematik* by Heinrich Weber and Josef Wellstein, the work which, among recent publications, most nearly accords with my own tendencies. For this semester, the first volume, *Enzyklopädie der elementaren Algebra und Analysis*, prepared by H. Weber[9], will be the most important. I shall indicate at once certain differences between this work and the plan of my lecture course. In Weber-Wellstein, the entire structure of elementary mathematics is built up systematically and logically in the mature language accessible to the advanced student. No account is taken of how these things actually may come up in school teaching. The presentation in the schools, however, should be *psychological* – to use a 'catch word' – and not *systematic*. The teacher so to speak, must be a *diplomat*. He must take account of the psychic processes in the boy in order to grip his interest; and he will succeed

[8] Klein, F., *Vorträge über den mathematischen Unterricht an höheren Schulen*. Prepared by Rudolf Schimmack. Part 1. *Von der Organisation des mathematischen Unterrichts*. Leipzig 1907. This book is referred to later as "Klein-Schimmack".

[9] Second edition. Leipzig 1906. [Fourth edition, 1922, revised by P. Epstein. – Referred to as "Weber-Wellstein I"].

only if he presents things in a form *intuitively comprehensible*. A more abstract presentation will be possible only in the upper grades. For example: The child cannot possibly understand if numbers are explained axiomatically as abstract things devoid of meaning, with which one can operate according to formal rules. On the contrary, he associates numbers with *concrete representations*. They are nothing else than quantities of nuts, apples, and other good things, and in the beginning they can be and should be put before him only in such tangible form. While this goes without saying, one should – mutatis mutandis – take it to heart, that in all teaching, even in higher education, mathematics should be associated with everything that is seriously interesting to a person at the particular stage of his development and that can in any way be brought into relation with mathematics. It is just this which is aimed by the recent efforts to emphasise applied mathematics at the university. This need has never been overlooked in the schools so much as it has at the university. It is just this '*psychological moment*' which I shall try to emphasize especially in my lectures.

Another difference between Weber-Wellstein and myself has to do with *delimiting the content of school mathematics*. Weber and Wellstein are disposed to be "*conservative*", while I am "*progressive*". These things are thoroughly discussed
[5] in Klein-Schimmack. We, who use to be called the "*reformers*", would put the *function concept* at the very centre of teaching, because, of all the concepts of the mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used. We would introduce it into teaching as early as possible *with constant use of the graphical method*, the representation of functional relations in the x-y-system, which is used today as a matter of course in every practical application of mathematics. In order to make this innovation possible, we would abolish much of the traditional subject matters of teaching, topics which may in themselves be interesting, but which are less essential from the standpoint of their significance in connection with the entire modern culture. *Intense formation of space intuition*, above all, will always be a prime task. In its upper reaches, however, teaching should enter far enough into the *elements of infinitesimal calculus* for the natural scientist or insurance specialist to get at school the tools which will be indispensable to him. As opposed to these comparatively recent ideas, Weber-Wellstein adhere essentially to the traditional limitations of the subject matter. In this lecture course I shall of course be a protagonist of the new conception.

My third reference will be to a very stimulating book: *Didaktik und Methodik des Rechnens und der Mathematik*[10] by Max Simon, who like Weber and Wellstein is working at Strassburg. Simon is often in agreement with our views, but there are also many opposite standpoints; and inasmuch as he is a very subjective, temperamental, personality he often clothes these contrasting views in vivid words. To give one example, the proposals of the Committee on Instruction of the Natural Scientists require one hour of geometric propaedeutic in the second year of the Gymnasium, whereas at the present time this usually begins only in the third year.

[10] Second edition, München 1908. Separate reprint from Baumeister's *Handbuch der Erziehungs- und Unterrichtslehre für höhere Schulen*, first edition, 1895.

It has long been a matter of discussion which plan is the better; and the custom in the schools has often changed. But Simon declares the position taken by the Commission, which, mind you, is at worst open to argument, to be "worse than a crime", and that without in the least substantiating his judgment. One could find many passages of this sort. As a precursor of this book I might mention Simon's *Methodik der elementaren Arithmetik in Verbindung mit algebraischer Analysis*[11]. After this brief introduction let us go over to the subject proper, which I shall consider under the three headings, as above indicated. **[6]**

[11] Leipzig 1906. [Regarding more recent literature, confer appendices 1 and 2, pp. 290–303].

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