

II

Introduction to Continuum Physics

In continuum physics, material bodies are modeled as continuous media whose motion and equilibrium is governed by balance laws and constitutive relations.

The list of balance laws identifies the theory, for example mechanics, thermomechanics, electrodynamics, etc. The referential (Lagrangian) and the spatial (Eulerian) formulation of the typical balance law will be presented. The balance laws of mass, momentum, energy, and the Clausius-Duhem inequality, which demarcate continuum thermomechanics, will be recorded.

The type of constitutive relation encodes the nature of material response. The constitutive equations of thermoelasticity and thermoviscoelasticity will be introduced. Restrictions imposed by the Second Law of thermodynamics, the principle of material frame indifference, and material symmetry will be discussed.

2.1 Kinematics

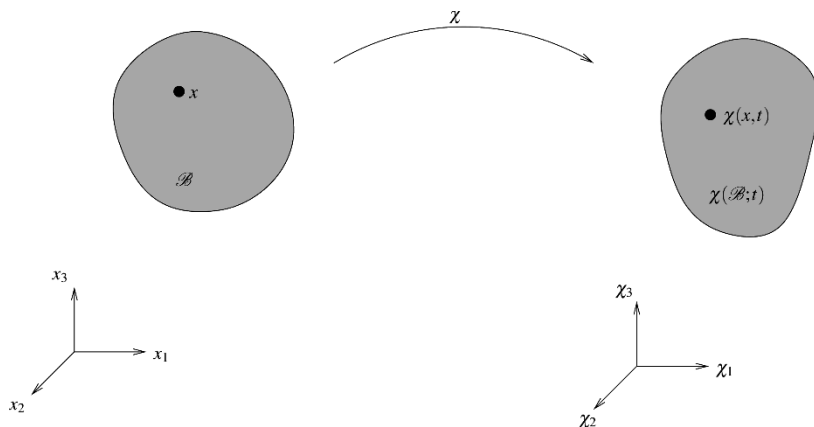


Fig. 2.1.1

The ambient space is \mathbb{R}^m , of dimension one, two or three. Two copies of \mathbb{R}^m shall be employed, one for the *reference space*, the other for the *physical space*. A *body*

is identified by a *reference configuration*, namely an open subset \mathcal{B} of the reference space. Points of \mathcal{B} will be called *particles*. The typical particle will be denoted by x and time will be denoted by t .

A *placement* of the body is a bilipschitz homeomorphism of its reference configuration \mathcal{B} to some open subset of the physical space. A *motion* of the body over the time interval (t_1, t_2) is a Lipschitz map χ of $\mathcal{B} \times (t_1, t_2)$ to \mathbb{R}^m whose restriction to each fixed t in (t_1, t_2) is a placement. Thus, for fixed $x \in \mathcal{B}$ and $t \in (t_1, t_2)$, $\chi(x, t)$ specifies the position in physical space of the particle x at time t ; for fixed $t \in (t_1, t_2)$, the map $\chi(\cdot, t) : \mathcal{B} \rightarrow \mathbb{R}^m$ yields the placement of the body at time t ; finally, for fixed $x \in \mathcal{B}$, the curve $\chi(x, \cdot) : (t_1, t_2) \rightarrow \mathbb{R}^m$ describes the trajectory of the particle x in physical space. See Fig. 2.1.1.

The reference configuration generally renders an abstract representation of the body. In practice, however, one often identifies the reference space with the physical space and employs as reference configuration an actual placement of the body, by identifying material particles with the point in physical space that they happen to occupy in that particular placement.

The aim of continuum physics is to monitor the evolution of various fields associated with the body, such as density, stress, temperature, etc. In the *referential* or *Lagrangian* description, one follows the evolution of fields along particle trajectories, while in the *spatial* or *Eulerian* description one monitors the evolution of fields at fixed position in space. The motion allows us to pass from one formulation to the other. For example, considering some illustrative field w , we write $w = f(x, t)$ for its referential description and $w = \phi(\chi, t)$ for its spatial description. The motion relates f and ϕ by $\phi(\chi(x, t), t) = f(x, t)$, for $x \in \mathcal{B}$, $t \in (t_1, t_2)$.

Either formulation has its relative merits, so both will be used here. Thus, in order to keep proper accounting, three symbols would be needed for each field, one to identify it, one for its referential description, and one for its spatial description (w , f , and ϕ in the example, above). However, in order to control the proliferation of symbols and make the physical interpretation of the equations transparent, the standard notational convention is to employ the single identifying symbol of the field for all three purposes. To prevent ambiguity in the notation of derivatives, the following rules will apply: Partial differentiation with respect to t will be denoted by an overdot in the referential description and by a t -subscript in the spatial description. Gradient, differential and divergence¹ will be denoted by Grad, ∇ and Div, with respect to the material variable x , and by grad, d and div, with respect to the spatial variable χ . Thus, referring again to the typical field w with referential description $w = f(x, t)$ and spatial description $w = \phi(\chi, t)$, \dot{w} will denote $\partial f / \partial t$, w_t will denote $\partial \phi / \partial t$, $\text{Grad} w$ will denote $\text{grad}_x f$, and $\text{grad} w$ will denote $\text{grad}_\chi \phi$. This notation may appear confusing at first but the student of the subject soon learns to use it efficiently and correctly.

The motion χ induces two important kinematical fields, namely the *velocity*

¹ For consistency with matrix notations, gradients will be realized as m -column vectors and differentials will be m -row vectors, namely the transpose of gradients. As in Chapter I, the divergence operator will be acting on row vectors.

$$(2.1.1) \quad v = \dot{\chi},$$

in $L^\infty(\mathcal{B} \times (t_1, t_2); \mathbb{R}^m)$, and the *deformation gradient*, which, its name notwithstanding, is the differential of the motion:

$$(2.1.2) \quad F = \nabla \chi,$$

in $L^\infty(\mathcal{B} \times (t_1, t_2); \mathbb{M}^{m \times m})$. In accordance with the definition of placement, we shall be assuming

$$(2.1.3) \quad \det F \geq a > 0 \quad \text{a.e.}$$

These fields allow us to pass from spatial to material derivatives; for example, assuming w is a Lipschitz field,

$$(2.1.4) \quad \dot{w} = w_t + (dw)v,$$

$$(2.1.5) \quad \text{Grad } w = F^\top \text{grad } w, \quad \nabla w = (dw)F.$$

By virtue of the polar decomposition theorem, the local deformation of the medium, expressed by the deformation gradient F , may be realized as the composition of a pure stretching and a rotation:

$$(2.1.6) \quad F = RU,$$

where the symmetric, positive definite matrix

$$(2.1.7) \quad U = (F^\top F)^{1/2}$$

is called the *right stretch tensor* and the proper orthogonal matrix R is called the *rotation tensor*.

Turning to the rate of change of deformation, we introduce the referential and spatial *velocity gradients* (which are actually differentials):

$$(2.1.8) \quad \dot{F} = \nabla v, \quad L = dv.$$

L is decomposed into the sum of the symmetric *stretching tensor* D and the skew-symmetric *spin tensor* W :

$$(2.1.9) \quad L = D + W, \quad D = \frac{1}{2}(L + L^\top), \quad W = \frac{1}{2}(L - L^\top).$$

The spin tensor is just a representation of the *vorticity* vector $\omega = \text{curl } v$ as a skew symmetric matrix.

The class of Lipschitz continuous motions allows for shocks but is not sufficiently broad to also encompass motions involving cavitation in elasticity, vortices in hydrodynamics, vacuum in gas dynamics, etc. Even so, we shall continue to develop the theory under the assumption that motions are Lipschitz continuous, deferring considerations of generalization until such need arises.

2.2 Balance Laws in Continuum Physics

Consider a motion χ of a body with reference configuration $\mathcal{B} \subset \mathbb{R}^m$, over a time interval (t_1, t_2) . The typical balance law of continuum physics postulates that the change over any time interval in the amount of a certain extensive quantity stored in any part of the body is balanced by a flux through the boundary and a production in the interior during that time interval. With space and time fused into space-time, the above statement yields a balance law of the type considered in Chapter I, ultimately reducing to a field equation of the form (1.2.3).

To adapt to the present setting the notation of Chapter I, we take space-time \mathbb{R}^{m+1} as the ambient space \mathbb{R}^k , and set $\mathcal{X} = \mathcal{B} \times (t_1, t_2)$, $X = (x, t)$. With reference to (1.4.1), we partition the flux density field A into a $n \times m$ matrix-valued spatial part Ψ and a \mathbb{R}^n -valued temporal part Θ , namely $A = [-\Psi \mid \Theta]$. In the notation introduced in the previous section, (1.4.1) now takes the form

$$(2.2.1) \quad \dot{\Theta} = \text{Div } \Psi + P.$$

This is the referential field equation for the typical balance law of continuum physics. The field Θ is the density of the balanced quantity; Ψ is the flux density field through material surfaces; and P is the production density.

The corresponding spatial field equation may be derived by appealing to Theorem 1.3.1. The map X^* that carries (x, t) to $(\chi(x, t), t)$ is a bilipschitz homeomorphism of \mathcal{X} to some subset \mathcal{X}^* of \mathbb{R}^{m+1} , with Jacobian matrix (cf. (1.3.2), (2.1.1), and (2.1.2)):

$$(2.2.2) \quad J = \begin{bmatrix} F & v \\ 0 & 1 \end{bmatrix}.$$

Notice that (1.3.3) is satisfied by virtue of (2.1.3). Theorem 1.3.1 and Remark 1.3.2 now imply that if $\Theta \in L_{loc}^1(\mathcal{X}; \mathbb{R}^n)$, $\Psi \in L_{loc}^1(\mathcal{X}; \mathbb{M}^{n \times k})$ and $P \in L_{loc}^1(\mathcal{X}; \mathbb{R}^n)$, then (2.2.1) holds in the sense of distributions on \mathcal{X} if and only if

$$(2.2.3) \quad \Theta_t^* + \text{div}(\Theta^* v^\top) = \text{div } \Psi^* + P^*$$

holds in the sense of distributions on \mathcal{X}^* , where the fields $\Theta^* \in L_{loc}^1(\mathcal{X}^*; \mathbb{R}^n)$, $\Psi^* \in L_{loc}^1(\mathcal{X}^*; \mathbb{M}^{n \times m})$ and $P^* \in L_{loc}^1(\mathcal{X}^*; \mathbb{R}^n)$ are defined by

$$(2.2.4) \quad \Theta^* = (\det F)^{-1} \Theta, \quad \Psi^* = (\det F)^{-1} \Psi F^\top, \quad P^* = (\det F)^{-1} P.$$

It has thus been established that the referential (Lagrangian) field equations (2.2.1) and the spatial (Eulerian) field equations (2.2.3) of the balance laws of continuum physics are related by (2.2.4) and are equivalent within the function class of fields considered here.

As we have seen, in order to pass from Lagrangian to Eulerian coordinates, and vice versa, one has to apply Theorem 1.3.1 for a bilipschitz homeomorphism that is not given in advance, but is generated by the motion itself, which also affects the balanced fields. This coupling, which has no bearing on whether the referential and

the spatial formulations of balance laws are equivalent, has nonetheless fostered an unwarranted aura of mystery about the issue.

In anticipation of the forthcoming discussion of material symmetry, it is useful to investigate how the fields Θ, Ψ, P and Θ^*, Ψ^*, P^* transform under *isochoric* changes of the reference configuration of the body, induced by a bilipschitz homeomorphism \bar{x} of \mathcal{B} to some subset $\bar{\mathcal{B}}$ of another reference space \mathbb{R}^m , with Jacobian matrix

$$(2.2.5) \quad H = \frac{\partial \bar{x}}{\partial x}, \quad \det H = 1,$$

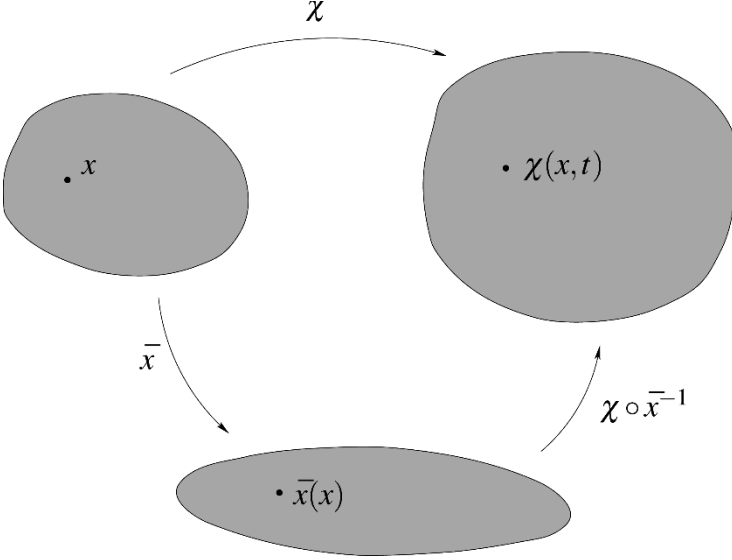


Fig. 2.2.1

see Figure 2.2.1. By virtue of Theorem 1.3.1, the Lagrangian field equation (2.2.1) on \mathcal{B} will transform into an equation of exactly the same form on $\bar{\mathcal{B}}$, with fields $\bar{\Theta}$, $\bar{\Psi}$ and \bar{P} related to Θ, Ψ and P by

$$(2.2.6) \quad \bar{\Theta} = \Theta, \quad \bar{\Psi} = \Psi H^\top, \quad \bar{P} = P.$$

In the corresponding Eulerian field equations, the fields $\bar{\Theta}^*, \bar{\Psi}^*$ and \bar{P}^* are obtained through (2.2.4): $\bar{\Theta}^* = (\det \bar{F})^{-1} \bar{\Theta}$, $\bar{\Psi}^* = (\det \bar{F})^{-1} \bar{\Psi} \bar{F}^\top$ and $\bar{P}^* = (\det \bar{F})^{-1} \bar{P}$, where \bar{F} denotes the deformation gradient relative to the new reference configuration $\bar{\mathcal{B}}$. By the chain rule, $\bar{F} = FH^{-1}$ and so

$$(2.2.7) \quad \bar{\Theta}^* = \Theta^*, \quad \bar{\Psi}^* = \Psi^*, \quad \bar{P}^* = P^*,$$

i.e., as was to be expected, the spatial fields are not affected by changing the reference configuration of the body.

In continuum physics, theories are identified by means of the list of balance laws that apply in their context. The illustrative example of thermomechanics will be presented in the next section. It should be noted, however, that in addition to balance laws with physical content there are others that simply express useful, purely kinematic properties. Equation (2.1.8), $\dot{F} = \nabla v$, which expresses the compatibility between the fields F and v , provides an example in that direction.

At first reading, one may skip the remainder of this section, which deals with a special topic for future use, and pass directly to the next Section 2.3.

In what follows, we derive, for $m = 3$, a set of kinematic balance laws whose referential form is quite complicated and yet whose spatial form is very simple or even trivial. This will demonstrate the usefulness of switching from the Lagrangian to the Eulerian formulation and vice versa.

A smooth function ϕ on the set of $F \in \mathbb{M}^{3 \times 3}$ with $\det F > 0$ is called a *null Lagrangian* if the Euler-Lagrange equation

$$(2.2.8) \quad \text{Div} [\partial_F \phi(F)] = 0,$$

associated with the functional $\int \phi(F) dx$, holds for every smooth deformation gradient field F . Any null Lagrangian ϕ admits a representation as an affine function

$$(2.2.9) \quad \phi(F) = \text{tr}(AF) + \text{tr}(BF^*) + \alpha \det F + \beta$$

of F , its determinant $\det F$, and its adjugate matrix $F^* = (\det F)F^{-1} = (\partial_F \det F)^\top$.

By combining (2.2.8) with $\dot{F} = \nabla v$, one deduces that if ϕ is any null Lagrangian (2.2.9), then the conservation law

$$(2.2.10) \quad \dot{\phi}(F) = \text{Div} [v^\top \partial_F \phi(F)]$$

holds for any smooth motion with deformation gradient F and velocity v .

The aim here is to show that, for any null Lagrangian (2.2.9), the “quasi-static” conservation law (2.2.8) as well as the “kinematic” conservation law (2.2.10) actually hold even for motions that are merely Lipschitz continuous, i.e.,

$$(2.2.11) \quad \text{Div} (\partial_F F) = 0,$$

$$(2.2.12) \quad \text{Div} (\partial_F F^*) = 0,$$

$$(2.2.13) \quad \text{Div} (\partial_F \det F) = 0,$$

$$(2.2.14) \quad \dot{F} = \text{Div} (v^\top \partial_F F),$$

$$(2.2.15) \quad \dot{F}^* = \text{Div} (v^\top \partial_F F^*),$$

$$(2.2.16) \quad \overline{\dot{\det F}} = \text{Div} (v^\top \partial_F \det F),$$

for any bounded measurable deformation gradient field F and velocity field v .

Clearly, (2.2.11) is obvious and (2.2.14) is just an alternative way of writing the familiar $\dot{F} = \nabla v$. Furthermore, since

$$(2.2.17) \quad \frac{\partial F_{\alpha i}^*}{\partial F_{j\beta}} = \sum_{k=1}^3 \sum_{\gamma=1}^3 \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} F_{k\gamma},$$

where ε_{ijk} and $\varepsilon_{\alpha\beta\gamma}$ are the standard permutation symbols, (2.2.12) follows from the observation that $\partial F_{k\gamma}/\partial x_\beta = \partial^2 \chi_k / \partial x_\beta \partial x_\gamma$ is symmetric in (β, γ) while $\varepsilon_{\alpha\beta\gamma}$ is skew-symmetric in (β, γ) .

To see (2.2.13), consider the trivial balance law (2.2.3), with $\Theta^* = 0$, $\Psi^* = I$, $P^* = 0$, and write its Lagrangian form (2.2.1), where on account of (2.2.4), $\Theta = 0$, $\Psi = (\det F)(F^\top)^{-1} = (F^*)^\top = \partial \det F / \partial F$, $P = 0$. Similarly, (2.2.16) is the Lagrangian form (2.2.1) of the trivial balance law (2.2.3), with $\Theta^* = 1$, $\Psi^* = v^\top$, and $P^* = 0$. Indeed, in that case, by virtue of (2.2.4) we deduce that $\Theta = \det F$, $\Psi = (\det F)(F^{-1}v)^\top = (F^*v)^\top = v^\top (\partial \det F / \partial F)$, and $P = 0$.

It remains to verify (2.2.15). We begin with the simple conservation law

$$(2.2.18) \quad (F^{-1})_t = (dx)_t = dx_t = -d(F^{-1}v),$$

in Eulerian coordinates, and derive its Lagrangian form (2.2.1), through (2.2.4). Thus $\Theta = (\det F)F^{-1} = F^*$, while the flux Ψ , in components form, reads

$$(2.2.19) \quad \Psi_{\alpha i \beta} = \sum_{j=1}^3 (\det F) \left[F_{\beta j}^{-1} F_{\alpha i}^{-1} - F_{\alpha j}^{-1} F_{\beta i}^{-1} \right] v_j.$$

The quantity in brackets vanishes when $\alpha = \beta$ and/or $i = j$; otherwise, it represents a minor of the matrix F^{-1} and thus is equal to $\det F^{-1}$ multiplied by the corresponding entry of the matrix $(F^{-1})^{-1} = F$. Hence, recalling (2.2.17),

$$(2.2.20) \quad (\det F) \left[F_{\beta j}^{-1} F_{\alpha i}^{-1} - F_{\alpha j}^{-1} F_{\beta i}^{-1} \right] = \sum_{k=1}^3 \sum_{\gamma=1}^3 \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma} F_{k\gamma} = \frac{\partial F_{\alpha i}^*}{\partial F_{j\beta}},$$

and this establishes (2.2.15).

2.3 The Balance Laws of Continuum Thermomechanics

Continuum thermomechanics, which will serve as a representative model throughout this work, is demarcated by the balance laws of mass, linear momentum, angular momentum, energy, and entropy whose referential and spatial field equations will now be introduced.

In the *balance law of mass*, there is neither flux nor production so the referential and spatial field equations read

$$(2.3.1) \quad \dot{\rho}_0 = 0,$$

$$(2.3.2) \quad \rho_t + \operatorname{div}(\rho v^\top) = 0,$$

where ρ_0 is the *reference density* and ρ is the *density* associated with the motion, related through

$$(2.3.3) \quad \rho = \rho_0(\det F)^{-1}.$$

Note that (2.3.1) implies that the value of the reference density associated with a particle does not vary with time: $\rho_0 = \rho_0(x)$. (2.3.2) is also referred to as the *equation of continuity*.

In the *balance law of linear momentum*, the production is induced by the *body force* (per unit mass) vector b , with values in \mathbb{R}^m , while the flux is represented by a *stress tensor* taking values in $\mathbb{M}^{m \times m}$. The referential and spatial field equations read

$$(2.3.4) \quad (\rho_0 v)^\cdot = \operatorname{Div} S + \rho_0 b,$$

$$(2.3.5) \quad (\rho v)_t + \operatorname{div}(\rho v v^\top) = \operatorname{div} T + \rho b,$$

where S denotes the *Piola-Kirchhoff stress* and T denotes the *Cauchy stress*, related by

$$(2.3.6) \quad T = (\det F)^{-1} S F^\top.$$

For any unit vector v , the value of Sv at (x, t) yields the stress (force per unit area) vector transmitted at the particle x and time t across a material surface with normal v ; while the value of Tv at (χ, t) gives the stress vector transmitted at the point χ in space and time t across a spatial surface with normal v .

In the *balance law of angular momentum*, production and flux are the moments about the origin of the production and flux involved in the balance of linear momentum. Consequently, the referential field equation is

$$(2.3.7) \quad (\chi \wedge \rho_0 v)^\cdot = \operatorname{Div}(\chi \wedge S) + \chi \wedge \rho_0 b,$$

where \wedge denotes cross product. Under the assumption that $\rho_0 v$, S and $\rho_0 b$ are in L^1_{loc} while the motion χ is Lipschitz continuous, we may use (2.3.4), (2.1.1) and (2.1.2) to reduce (2.3.7) into

$$(2.3.8) \quad S F^\top = F S^\top.$$

Similarly, the spatial field equation of the balance of angular momentum reduces, by virtue of (2.3.5), to the statement that the Cauchy stress tensor is symmetric:

$$(2.3.9) \quad T^\top = T.$$

There is no need to perform that calculation since (2.3.9) also follows directly from (2.3.6) and (2.3.8).

In the *balance law of energy*, the energy density is the sum of the (specific) *internal energy* (per unit mass) ε and kinetic energy. The production is the sum of the rate of work of the body force and the *heat supply* (per unit mass) r . Finally, the flux is the sum of the rate of work of the stress tensor and the *heat flux*. The referential and spatial field equations thus read

$$(2.3.10) \quad (\rho_0 \varepsilon + \tfrac{1}{2} \rho_0 |v|^2)^\cdot = \text{Div} (v^\top S + Q^\top) + \rho_0 v^\top b + \rho_0 r,$$

$$(2.3.11) \quad (\rho \varepsilon + \tfrac{1}{2} \rho |v|^2)_t + \text{div} [(\rho \varepsilon + \tfrac{1}{2} \rho |v|^2) v^\top] = \text{div} (v^\top T + q^\top) + \rho v^\top b + \rho r,$$

where the referential and spatial heat flux vectors Q and q , with values in \mathbb{R}^m , are related by

$$(2.3.12) \quad q = (\det F)^{-1} F Q.$$

Finally, the *balance law of entropy* is expressed by the *Clausius-Duhem inequality*

$$(2.3.13) \quad (\rho_0 s)^\cdot \geq \text{Div} \left(\frac{1}{\theta} Q^\top \right) + \rho_0 \frac{r}{\theta},$$

$$(2.3.14) \quad (\rho s)_t + \text{div} (\rho s v^\top) \geq \text{div} \left(\frac{1}{\theta} q^\top \right) + \rho \frac{r}{\theta},$$

in its referential and spatial form, respectively. The symbol s stands for (specific) *entropy* and θ denotes the (absolute) *temperature*. Thus, the *entropy flux* is just the heat flux divided by temperature. The term $\frac{r}{\theta}$ represents the external entropy supply (per unit mass), induced by the heat supply r . However, the fact that (2.3.13) and (2.3.14) are mere inequalities rather than equalities signifies that there may be additional *internal entropy production*, which is not specified a priori in the context of this theory, apart from being constrained to be nonnegative. This last condition is dictated by (and in fact expresses) the *Second Law of thermodynamics*. As a nonnegative distribution, the internal entropy production is necessarily a measure \mathcal{N} . Adding \mathcal{N} to the right-hand side turns the Clausius-Duhem inequality into an equality which, by virtue of Theorem 1.3.3, is the field equation of a balance law. In particular, this demonstrates that the referential form (2.3.13) and the spatial form (2.3.14) are equivalent even when the fields are merely locally integrable.

The motion and the entropy (or temperature) field together constitute a *thermodynamic process*. The fields of internal energy, stress, heat flux, and temperature (or entropy) are determined from the thermodynamic process by means of constitutive relations that characterize the material response of the body. In particular, the constitutive equation for the stress is required to satisfy identically the balance law of angular momentum as expressed by (2.3.8) or (2.3.9). Representative material classes will be introduced in the following Sections, 2.5 and 2.6.

The field equations of the balance laws of mass, linear momentum and energy, coupled with the constitutive relations, render a closed system of evolution equations

that should determine the thermodynamic process from assigned body force field b , heat supply field r , boundary conditions, and initial conditions.

The remaining balance law of entropy plays a markedly different role. The Clausius-Duhem inequality (2.3.13) or (2.3.14) is regarded as a criterion of *thermodynamic admissibility* for thermodynamic processes that already comply with the balance laws of mass, momentum and energy. In this regard, smooth thermodynamic processes are treated differently from thermodynamic processes with discontinuities.

It is a tenet of continuum thermodynamics that the constitutive relations should be constrained by the requirement that any smooth thermodynamic process that balances mass, momentum and energy must be automatically thermodynamically admissible. To implement this requisite, the first step is to derive from the Clausius-Duhem inequality the *dissipation inequality*

$$(2.3.15) \quad \rho_0 \dot{\epsilon} - \rho_0 \theta \dot{s} - \text{tr}(S \dot{F}^\top) - \frac{1}{\theta} Q \cdot G \leq 0,$$

$$(2.3.16) \quad \rho \dot{\epsilon} - \rho \theta \dot{s} - \text{tr}(T D) - \frac{1}{\theta} q \cdot g \leq 0,$$

in Lagrangian or Eulerian form, respectively, which does not involve the extraneously assigned body force and heat supply. The new symbols G and g appearing in (2.3.15) and (2.3.16) denote the *temperature gradient*:

$$(2.3.17) \quad G = \text{Grad } \theta, \quad g = \text{grad } \theta, \quad G = F^\top g.$$

To establish (2.3.15), one first eliminates the body force b between the field equations (2.3.1), (2.3.4) and (2.3.10) of the balance laws of mass, linear momentum and energy to get

$$(2.3.18) \quad \rho_0 \dot{\epsilon} = \text{tr}(S \dot{F}^\top) + \text{Div } Q^\top + \rho_0 r,$$

and then eliminates the heat supply r between the above equation and the Clausius-Duhem inequality (2.3.13). Similarly, (2.3.16) is obtained by combining (2.3.2), (2.3.5) and (2.3.11) with (2.3.14) in order to eliminate b and r . Of course, (2.3.15) and (2.3.16) are equivalent: either one implies the other by virtue of (2.3.3), (2.3.6), (2.3.17), (2.1.9) and (2.3.9). In the above calculations it is crucial that the underlying thermodynamic process is assumed smooth, because this allows us to apply the classical product rule of differentiation on terms like $|v|^2$, $v^\top S$, $\theta^{-1} Q$ etc., which induces substantial cancellation. It should be emphasized that the dissipation inequalities (2.3.15) and (2.3.16) are generally meaningless for thermodynamic processes with discontinuities.

The constitutive equations are required to satisfy identically the dissipation inequality (2.3.15) or (2.3.16), which will guarantee that any smooth thermodynamic process that balances mass, momentum and energy is automatically thermodynamically admissible. The implementation of this requisite for specific material classes will be demonstrated in the following Sections 2.5 and 2.6.

Beyond taking care of smooth thermodynamic processes, as above, the Clausius-Duhem inequality is charged with the additional responsibility of certifying the thermodynamic admissibility of discontinuous processes. This is a central issue, with many facets, which will surface repeatedly in the remainder of the book.

When dealing with continuous media with complex structure, e.g., mixtures of different materials, it becomes necessary to replace the Clausius-Duhem inequality with a more general entropy inequality in which the entropy flux is no longer taken a priori as heat flux divided by temperature but is instead specified by an individual constitutive relation. It turns out, however, that in the context of thermoelastic or thermoviscoelastic media, which are the main concern of this work, the requirement that such an inequality must hold identically for any smooth thermodynamic process that balances mass, momentum and energy implies in particular that entropy flux is necessarily heat flux divided by temperature, so that we fall back to the classical Clausius-Duhem inequality.

To prepare the ground for the forthcoming investigation of material symmetry, it is necessary to discuss the law of transformation of the fields involved in the balance laws when the reference configuration undergoes a change induced by an isochoric bilipshitz homeomorphism \bar{x} , with unimodular Jacobian matrix H (2.2.5); see Fig. 2.2.1. The deformation gradient F and the stretching tensor D (cf. (2.1.9)) will transform into new fields \bar{F} and \bar{D} :

$$(2.3.19) \quad \bar{F} = FH^{-1}, \quad \bar{D} = D.$$

The reference density ρ_0 , internal energy ε , Piola-Kirchhoff stress S , entropy s , temperature θ , referential heat flux vector Q , density ρ , Cauchy stress T , and spatial heat flux vector q , involved in the balance laws, will also transform into new fields $\bar{\rho}_0, \bar{\varepsilon}, \bar{S}, \bar{s}, \bar{\theta}, \bar{Q}, \bar{\rho}, \bar{T}$, and \bar{q} according to the rule (2.2.6) or (2.2.7), namely,

$$(2.3.20) \quad \bar{\rho}_0 = \rho_0, \quad \bar{\varepsilon} = \varepsilon, \quad \bar{S} = SH^\top, \quad \bar{s} = s, \quad \bar{\theta} = \theta, \quad \bar{Q} = HQ,$$

$$(2.3.21) \quad \bar{\rho} = \rho, \quad \bar{T} = T, \quad \bar{q} = q.$$

Also the referential and spatial temperature gradients G and g will transform into \bar{G} and \bar{g} with

$$(2.3.22) \quad \bar{G} = (H^{-1})^\top G, \quad \bar{g} = g.$$

2.4 Material Frame Indifference

The body force and heat supply are usually induced by external factors and are assigned in advance, while the fields of internal energy, stress, entropy and heat flux are determined by the thermodynamic process. Motions may influence these fields inasmuch as they deform the body: rigid motions, which do not change the distance between particles, should have no effect on internal energy, temperature or referential

heat flux and should affect the stress tensor in such a manner that the resulting stress vector, observed from a frame attached to the moving body, looks fixed. This requirement is postulated by the fundamental *principle of material frame indifference* which will now be stated with precision

Consider any two thermodynamic processes (χ, s) and $(\chi^\#, s^\#)$ of the body such that the entropy fields coincide, $s^\# = s$, while the motions differ by a rigid (time dependent) rotation²:

$$(2.4.1) \quad \chi^\#(x, t) = O(t)\chi(x, t), \quad x \in \mathcal{B}, \quad t \in (t_1, t_2),$$

$$(2.4.2) \quad O^\top(t)O(t) = O(t)O^\top(t) = I, \quad \det O(t) = 1, \quad t \in (t_1, t_2).$$

Note that the fields of deformation gradient $F, F^\#$, spatial velocity gradient $L, L^\#$ and stretching tensor $D, D^\#$ (cf. (2.1.8), (2.1.9)) of the two processes (χ, s) , $(\chi^\#, s^\#)$ are related by

$$(2.4.3) \quad F^\# = OF, \quad L^\# = OLO^\top + \dot{O}O^\top, \quad D^\# = ODO^\top.$$

Let $(\varepsilon, S, \theta, Q)$ and $(\varepsilon^\#, S^\#, \theta^\#, Q^\#)$ denote the fields for internal energy, Piola-Kirchhoff stress, temperature and referential heat flux associated with the processes (χ, s) and $(\chi^\#, s^\#)$. The principle of material frame indifference postulates:

$$(2.4.4) \quad \varepsilon^\# = \varepsilon, \quad S^\# = OS, \quad \theta^\# = \theta, \quad Q^\# = Q.$$

From (2.4.4), (2.3.17) and (2.4.3) it follows that the referential and spatial temperature gradients $G, G^\#$ and $g, g^\#$ of the two processes are related by

$$(2.4.5) \quad G^\# = G, \quad g^\# = Og.$$

Furthermore, from (2.3.6), (2.3.12) and (2.4.3) we deduce the following relations between the Cauchy stress tensors $T, T^\#$ and the spatial heat flux vectors $q, q^\#$ of the two processes:

$$(2.4.6) \quad T^\# = OTOT^\top, \quad q^\# = Oq.$$

The principle of material frame indifference should be reflected in the constitutive relations of continuous media, irrespectively of the nature of material response. Illustrative examples will be considered in the following two sections.

2.5 Thermoelasticity

In the framework of continuum thermomechanics, a *thermoelastic* medium is identified by the constitutive assumption that, for any fixed particle x and any motion, the

² An alternative, albeit equivalent, realization of this setting is to visualize a single thermodynamic process monitored by two observers attached to individual coordinate frames that rotate relative to each other. When adopting that approach, certain authors are allowing for reflections, in addition to proper rotations.

value of the internal energy ε , the Piola-Kirchhoff stress S , the temperature θ , and the referential heat flux vector Q , at x and time t , is determined solely by the value at (x, t) of the deformation gradient F , the entropy s , and the temperature gradient G , through constitutive equations

$$(2.5.1) \quad \begin{cases} \varepsilon = \hat{\varepsilon}(F, s, G), \\ S = \hat{S}(F, s, G), \\ \theta = \hat{\theta}(F, s, G), \\ Q = \hat{Q}(F, s, G), \end{cases}$$

where $\hat{\varepsilon}$, \hat{S} , $\hat{\theta}$ and \hat{Q} are smooth functions defined on the subset of $\mathbb{M}^{m \times m} \times \mathbb{R} \times \mathbb{R}^m$ with $\det F > 0$. Moreover, $\hat{\theta}(F, s, G) > 0$. When the thermoelastic medium is *homogeneous*, the same functions $\hat{\varepsilon}$, \hat{S} , $\hat{\theta}$ and \hat{Q} and the same value ρ_0 of the reference density apply to all particles $x \in \mathcal{B}$.

The Cauchy stress T and the spatial heat flux q are also determined by constitutive equations of the same form, which may be derived from (2.5.1) and (2.3.6), (2.3.12). When employing the spatial description of the motion, it is natural to substitute on the list (2.5.1) the constitutive equations of T and q for the constitutive equations of S and Q ; also on the list (F, s, G) of the state variables to replace the referential temperature gradient G with the spatial temperature gradient g (cf. (2.3.17)).

The above constitutive equations will have to comply with the conditions stipulated earlier. To begin with, as postulated in Section 2.3, every smooth thermodynamic process that balances mass, momentum and energy must satisfy identically the Clausius-Duhem inequality (2.3.13) or, equivalently, the dissipation inequality (2.3.15). Substituting from (2.5.1) into (2.3.15) yields

$$(2.5.2) \quad \text{tr}[(\rho_0 \partial_F \hat{\varepsilon} - \hat{S}) \dot{F}^\top] + \rho_0 (\partial_s \hat{\varepsilon} - \hat{\theta}) \dot{s} + \rho_0 \partial_G \hat{\varepsilon} \dot{G} - \hat{\theta}^{-1} \hat{Q} \cdot G \leq 0.$$

It is clear that by suitably controlling the body force b and the heat supply r one may construct smooth processes that balance mass, momentum and energy and attain at some point (x, t) arbitrarily prescribed values for F , s , G , \dot{F} , \dot{s} and \dot{G} , subject only to the constraint $\det F > 0$. Hence (2.5.2) cannot hold identically unless the constitutive relations (2.5.1) are of the following special form:

$$(2.5.3) \quad \begin{cases} \varepsilon = \hat{\varepsilon}(F, s), \\ S = \rho_0 \partial_F \hat{\varepsilon}(F, s), \\ \theta = \partial_s \hat{\varepsilon}(F, s), \\ Q = \hat{Q}(F, s, G), \end{cases}$$

$$(2.5.4) \quad \hat{Q}(F, s, G) \cdot G \geq 0.$$

Thus the internal energy may depend on the deformation gradient and on the entropy but not on the temperature gradient. The constitutive equations for stress and temperature are induced by the constitutive equation of internal energy, through caloric relations, and are likewise independent of the temperature gradient. Only the heat flux may depend on the temperature gradient, subject to the condition (2.5.4) which implies that heat always flows from the hotter to the colder part of the body.

Another requirement on constitutive relations is that they observe the principle of material frame indifference, formulated in Section 2.4. By combining (2.4.4) and (2.4.3)₁ with (2.5.3), we deduce that the functions $\hat{\varepsilon}$ and \hat{Q} must satisfy the conditions

$$(2.5.5) \quad \hat{\varepsilon}(OF, s) = \hat{\varepsilon}(F, s), \quad \hat{Q}(OF, s, G) = \hat{Q}(F, s, G),$$

for all proper orthogonal matrices O . A simple calculation verifies that when (2.5.5) hold, then the remaining conditions in (2.4.4) will be automatically satisfied, by virtue of (2.5.3)₂ and (2.5.3)₃.

To see the implications of (2.5.5), we apply it with $O = R^\top$, where R is the rotation tensor in (2.1.6), to deduce

$$(2.5.6) \quad \hat{\varepsilon}(F, s) = \hat{\varepsilon}(U, s), \quad \hat{Q}(F, s, G) = \hat{Q}(U, s, G).$$

It is clear that, conversely, if (2.5.6) hold then (2.5.5) will be satisfied for any proper orthogonal matrix O . Consequently, the principle of material frame indifference is completely encoded in the statement (2.5.6) that the internal energy and the referential heat flux vector may depend on the deformation gradient F solely through the right stretch tensor U .

When the spatial description of motion is to be employed, the constitutive equation for the Cauchy stress

$$(2.5.7) \quad T = \rho \partial_F \hat{\varepsilon}(F, s) F^\top,$$

which follows from (2.3.6), (2.3.3) and (2.5.3)₂, will satisfy the principle of material frame indifference (2.4.6)₁ so long as (2.5.6) hold. For the constitutive equation of the spatial heat flux vector

$$(2.5.8) \quad q = \hat{q}(F, s, g),$$

the principle of material frame indifference requires (recall (2.4.6)₂, (2.4.3)₁ and (2.4.5)₂):

$$(2.5.9) \quad \hat{q}(OF, s, Og) = O \hat{q}(F, s, g),$$

for all proper orthogonal matrices O .

The final general requirement for constitutive relations is that the Piola-Kirchhoff stress satisfy (2.3.8), for the balance of angular momentum. This imposes no additional restrictions, however, because a simple calculation reveals that once (2.5.5)₁ holds, S computed through (2.5.3)₂ will automatically satisfy (2.3.8). Thus in thermoelasticity, material frame indifference implies balance of angular momentum.

The constitutive equations undergo further reduction when the medium is endowed with *material symmetry*. Recall from Section 2.3 that when the reference configuration of the body is changed by means of an isochoric bilipschitz homeomorphism \bar{x} with unimodular Jacobian matrix H (2.2.5), then the fields transform according to the rules (2.3.19), (2.3.20), (2.3.21) and (2.3.22). It follows, in particular, that any medium that is thermoelastic relative to the original reference configuration will stay so relative to the new one, as well, even though the constitutive functions will generally change. Any isochoric transformation of the reference configuration that leaves invariant the constitutive functions for ε, T and θ manifests material symmetry of the medium. Consider any such transformation and let H be its Jacobian matrix. By virtue of (2.3.19)₁, (2.3.20)₂ and (2.5.3)₁, the constitutive function $\hat{\varepsilon}$ of the internal energy will remain invariant, provided

$$(2.5.10) \quad \hat{\varepsilon}(FH^{-1}, s) = \hat{\varepsilon}(F, s).$$

A simple calculation verifies that when (2.5.10) holds, the constitutive functions for T and θ , determined through (2.5.7) and (2.5.3), are automatically invariant under that H . On account of (2.3.19)₁ and (2.3.22)₂, the constitutive function \hat{q} of the heat flux will be invariant under H if

$$(2.5.11) \quad \hat{q}(FH^{-1}, s, g) = \hat{q}(F, s, g).$$

It is clear that the set of matrices H with determinant one for which (2.5.10) and (2.5.11) hold forms a subgroup \mathcal{G} of the special linear group $SL(m)$, called the *symmetry group* of the medium. In certain media, \mathcal{G} may contain only the identity matrix I in which case material symmetry is minimal. When \mathcal{G} is nontrivial, it dictates through (2.5.10) and (2.5.11) conditions on the constitutive functions of the medium.

Maximal material symmetry is attained when $\mathcal{G} \equiv SL(m)$. In that case the medium is a *thermoelastic fluid*. Applying (2.5.10) and (2.5.11) with selected matrix $H = (\det F)^{-1/m} F \in SL(m)$, we deduce that $\hat{\varepsilon}$ and \hat{q} may depend on F solely through its determinant or, equivalently by virtue of (2.3.3), through the density ρ :

$$(2.5.12) \quad \varepsilon = \tilde{\varepsilon}(\rho, s), \quad q = \tilde{q}(\rho, s, g).$$

The Cauchy stress may then be obtained from (2.5.7) and the temperature from (2.5.3)₃. The calculation gives

$$(2.5.13) \quad T = -pI,$$

$$(2.5.14) \quad p = \rho^2 \partial_\rho \tilde{\varepsilon}(\rho, s), \quad \theta = \partial_s \tilde{\varepsilon}(\rho, s).$$

In the standard texts on thermodynamics, (2.5.14) are usually presented in the guise of the *Gibbs relation*:

$$(2.5.15) \quad \theta ds = d\varepsilon + pd\left(\frac{1}{\rho}\right).$$

The constitutive function \tilde{q} in (2.5.12) must also satisfy the requirement (2.5.9) of material frame indifference which now assumes the simple form

$$(2.5.16) \quad \tilde{q}(\rho, s, Og) = O\tilde{q}(\rho, s, g),$$

for all proper orthogonal matrices O . The final reduction of \tilde{q} that satisfies (2.5.16) is

$$(2.5.17) \quad q = \kappa(\rho, s, |g|)g,$$

where κ is a scalar-valued function. We have thus shown that in a thermoelastic fluid the internal energy depends solely on density and entropy. The Cauchy stress is a *hydrostatic pressure*, likewise depending only on density and entropy. The heat flux obeys Fourier's law with *thermal conductivity* κ which may vary with density, entropy and the magnitude of the heat flux.

The simplest classical example of a thermoelastic fluid is the *ideal gas*, which is identified by Boyle's law

$$(2.5.18) \quad p = R\rho\theta,$$

combined with the constitutive assumption that internal energy is proportional to temperature:

$$(2.5.19) \quad \varepsilon = c\theta.$$

In (2.5.18), R is the *universal gas constant* divided by the molecular weight of the gas, and c in (2.5.19) is the *specific heat*. The constant $\gamma = 1 + R/c$ is the *adiabatic exponent*. The classical kinetic theory predicts $\gamma = 1 + 2/n$, where n is the number of degrees of freedom of the gas molecule. The maximum value $\gamma = 5/3$ is attained when the gas is monatomic.

Combining (2.5.18) and (2.5.19) with (2.5.13) and (2.5.14), one easily deduces that the constitutive relations for the ideal gas, in normalized units, read

$$(2.5.20) \quad \varepsilon = c\rho^{\gamma-1}e^{\frac{s}{c}}, \quad p = R\rho^{\gamma}e^{\frac{s}{c}}, \quad \theta = \rho^{\gamma-1}e^{\frac{s}{c}}.$$

The ideal gas model provides a satisfactory description of the behavior of ordinary gases, over a wide range of density and temperature, but it becomes less reliable at extreme values of the state variables, especially near the point of transition to the liquid phase. Accordingly, a large number of equations have been proposed, with theoretical or empirical provenances, that would apply to "real gases". The most classical example is the *van der Waals gas*, in which (2.5.18) is replaced by

$$(2.5.21) \quad (p + a\rho^2)(1 - b\rho) = R\rho\theta,$$

where a and b are positive parameters. It corresponds to constitutive relations

$$(2.5.22) \quad \varepsilon = c \left(\frac{\rho}{1-b\rho} \right)^{\gamma-1} e^{\frac{s}{c}} + a\rho, \quad p = R \left(\frac{\rho}{1-b\rho} \right)^{\gamma} e^{\frac{s}{c}}, \quad \theta = \left(\frac{\rho}{1-b\rho} \right)^{\gamma-1} e^{\frac{s}{c}}.$$

A more exotic model is the *Chaplygin gas*, with equations of state in the form

$$(2.5.23) \quad \varepsilon = \frac{1}{2\rho^2} f(s) - \frac{1}{\rho} g(s) + h(s), \quad p = g(s) - \frac{1}{\rho} f(s), \quad \theta = \frac{1}{2\rho^2} f'(s) - \frac{1}{\rho} g'(s) + h'(s).$$

Notice that at low density the pressure becomes negative, which runs counter to conventional wisdom. However, it is this feature that renders the Chaplygin gas attractive to cosmologists, as they are relating it to “dark matter”.

An *isotropic thermoelastic solid* is a thermoelastic material with symmetry group \mathcal{G} the proper orthogonal group $SO(m)$. In that case, to obtain the reduced form of the internal energy function $\hat{\varepsilon}$ we combine (2.5.10) with (2.5.6)₁. Recalling (2.1.7) we conclude that

$$(2.5.24) \quad \hat{\varepsilon}(OUO^\top, s) = \hat{\varepsilon}(U, s),$$

for any proper orthogonal matrix O . In particular, we apply (2.5.24) for the proper orthogonal matrices O that diagonalize the symmetric matrix $U : OUO^\top = \Lambda$. This establishes that, in consequence of material frame indifference and material symmetry, the internal energy of an isotropic thermoelastic solid may depend on F solely as a symmetric function of the eigenvalues of the right stretch tensor U . Equivalently,

$$(2.5.25) \quad \varepsilon = \tilde{\varepsilon}(J_1, \dots, J_m, s),$$

where (J_1, \dots, J_m) are invariants of U . In particular, when $m = 3$, one may employ $J_1 = |F|^2$, $J_2 = |F^*|^2$ and $J_3 = \det F$, where F^* is the adjugate matrix of F . The reduced form of the Cauchy stress for the isotropic thermoelastic solid, computed from (2.5.25) and (2.5.7), is recorded in the references cited in Section 2.9. The reader may also find there explicit examples of constitutive functions for specific compressible or incompressible isotropic elastic solids.

In an alternative, albeit equivalent, formulation of thermoelasticity, one regards the temperature θ , rather than the entropy s , as a state variable and writes a constitutive equation for s rather than for θ . In that case it is also expedient to monitor the *Helmholtz free energy*

$$(2.5.26) \quad \psi = \varepsilon - \theta s$$

in the place of the internal energy ε . One thus starts out with constitutive equations

$$(2.5.27) \quad \begin{cases} \psi = \bar{\psi}(F, \theta, G), \\ S = \bar{S}(F, \theta, G), \\ s = \bar{s}(F, \theta, G), \\ Q = \bar{Q}(F, \theta, G), \end{cases}$$

in the place of (2.5.1). The requirement that all smooth thermodynamic processes that balance mass, momentum and energy must satisfy identically the dissipation inequality (2.3.15) reduces (2.5.27) to

$$(2.5.28) \quad \begin{cases} \psi = \bar{\psi}(F, \theta), \\ S = \rho_0 \partial_F \bar{\psi}(F, \theta), \\ s = -\partial_\theta \bar{\psi}(F, \theta), \\ Q = \bar{Q}(F, \theta, G), \end{cases}$$

$$(2.5.29) \quad \bar{Q}(F, \theta, G) \cdot G \geq 0,$$

which are the analogs ³ of (2.5.3), (2.5.4). The principle of material frame indifference and the presence of material symmetry further reduce the above constitutive equations. In particular, $\bar{\psi}$ satisfies the same conditions as $\hat{\varepsilon}$, above.

We conclude the discussion of thermoelasticity with remarks on special thermodynamic processes. A process is called *adiabatic* if the heat flux Q vanishes identically; it is called *isothermal* when the temperature field θ is constant; and it is called *isentropic* if the entropy field s is constant. Note that (2.5.29) implies $\bar{Q}(F, \theta, 0) = 0$ so, in particular, all isothermal processes are adiabatic. Materials that are poor conductors of heat are commonly modeled as *nonconductors of heat*, characterized by the constitutive assumption $\hat{Q} \equiv 0$. Thus every thermodynamic process of a nonconductor is adiabatic.

In an isentropic process, the entropy is set equal to a constant, $s \equiv \bar{s}$; the constitutive relations for the temperature and the heat flux are discarded and those for the internal energy and the stress are restricted to $s = \bar{s}$:

$$(2.5.30) \quad \begin{cases} \varepsilon = \hat{\varepsilon}(F, \bar{s}), \\ S = \rho_0 \partial_F \hat{\varepsilon}(F, \bar{s}). \end{cases}$$

In particular, for an ideal gas, on account of (2.5.20),

$$(2.5.31) \quad \varepsilon = \frac{\kappa}{\gamma - 1} \rho^{\gamma-1}, \quad p = \kappa \rho^\gamma,$$

where $\kappa = R \exp(\bar{s}/c)$.

In an isentropic process, the motion is determined solely by the balance laws of mass and momentum, in conjunction with the constitutive relations (2.5.30). This may create the impression that isentropic thermoelasticity is isomorphic to the purely mechanical theory of *hyperelasticity*. However, this is not entirely accurate, because

³ The constitutive equations in the form (2.5.3) are called *caloric* and in the form (2.5.28) are called *thermal*.

isentropic thermoelasticity inherits from thermodynamics the Second Law in the following guise: To sustain an isentropic process, one must control the heat supply r in such a manner that the ensuing motion, under the constant entropy field, satisfies the energy balance law (2.3.10). When the process is also adiabatic, $Q = 0$, the Clausius-Duhem inequality (2.3.13) reduces to $r \leq 0$, in which case (2.3.10) implies

$$(2.5.32) \quad (\rho_0 \epsilon + \tfrac{1}{2} \rho_0 |v|^2)^\cdot \leq \text{Div}(v^\top S) + \rho_0 v^\top b.$$

The Eulerian form of this inequality is

$$(2.5.33) \quad (\rho \epsilon + \tfrac{1}{2} \rho |v|^2)_t + \text{div}[(\rho \epsilon + \tfrac{1}{2} \rho |v|^2)v^\top] \leq \text{div}(v^\top T) + \rho v^\top b.$$

The above inequalities play in isentropic thermoelasticity the role played by the Clausius-Duhem inequality (2.3.13), (2.3.14) in general thermoelasticity: for smooth motions, they hold identically, as equalities, by virtue of (2.3.4) and (2.5.30). By contrast, in the context of motions that are merely Lipschitz continuous, they are extra conditions serving as the test of *thermodynamic admissibility* of the motion.

In practice, the isentropic theory is employed when it is judged that the effect of entropy fluctuation is negligible. This is not an uncommon situation, for the following reason. In smooth adiabatic processes, $Q = 0$, and if in addition $r = 0$, (2.3.18) in conjunction with (2.5.3) yields $\dot{s} = 0$. Thus, in the absence of heat supply, adiabatic processes starting out isentropically stay isentropic for as long as they are smooth. The smoothness requirements are met when F, v and s are merely Lipschitz continuous, which allows for processes with weak fronts, though not with shocks. As we shall see later, even after shocks develop, so long as they remain weak, entropy fluctuation is small (of third order) in comparison to the fluctuation of density and velocity, and may thus be neglected.

In isothermal thermoelasticity, θ is set equal to a constant $\hat{\theta}$, the heat supply r is regulated to balance the energy equation, and the motion is determined solely by the balance laws of mass and momentum. The only constitutive equations needed are

$$(2.5.34) \quad \begin{cases} \psi = \bar{\psi}(F, \hat{\theta}), \\ S = \rho_0 \partial_F \bar{\psi}(F, \hat{\theta}), \end{cases}$$

namely the analogs of (2.5.30). The implications of the Second Law of thermodynamics are seen, as before, by combining (2.3.10) with (2.3.13), assuming now $\theta = \hat{\theta} = \text{constant}$. This yields

$$(2.5.35) \quad (\rho_0 \psi + \tfrac{1}{2} \rho_0 |v|^2)^\cdot \leq \text{Div}(v^\top S) + \rho_0 v^\top b,$$

with Eulerian form

$$(2.5.36) \quad (\rho \psi + \tfrac{1}{2} \rho |v|^2)_t + \text{div}[(\rho \psi + \tfrac{1}{2} \rho |v|^2)v^\top] \leq \text{div}(v^\top T) + \rho v^\top b,$$

which should be compared to (2.5.32) and (2.5.33). We conclude that isothermal and isentropic thermoelasticity are essentially isomorphic, with the Helmholtz free

energy, at constant temperature, in the former, playing the role of internal energy, at constant entropy, in the latter.

In an isothermal process $\theta = \hat{\theta}$ for an ideal gas,

$$(2.5.37) \quad \psi = k \log \rho, \quad p = k\rho,$$

where $k = R\hat{\theta}$.

2.6 Thermoviscoelasticity

We now consider an extension of thermoelasticity that encompasses materials with *internal dissipation* induced by *viscosity of the rate type*. The internal energy ε , the Piola-Kirchhoff stress S , the temperature θ , and the referential heat flux vector Q may now depend not only on the deformation gradient F , the entropy s and the temperature gradient G , as in (2.5.1), but also on the time rate \dot{F} of the deformation gradient:

$$(2.6.1) \quad \begin{cases} \varepsilon = \hat{\varepsilon}(F, \dot{F}, s, G), \\ S = \hat{S}(F, \dot{F}, s, G), \\ \theta = \hat{\theta}(F, \dot{F}, s, G), \\ Q = \hat{Q}(F, \dot{F}, s, G). \end{cases}$$

As stipulated in Section 2.3, every smooth thermodynamic process that balances mass, momentum and energy must satisfy identically the dissipation inequality (2.3.15). Substituting from (2.6.1) into (2.3.15) yields

$$(2.6.2) \quad \text{tr}[(\rho_0 \partial_F \hat{\varepsilon} - \hat{S}) \dot{F}^\top] + \text{tr}(\rho_0 \partial_{\dot{F}} \hat{\varepsilon} \ddot{F}^\top) + \rho_0 (\partial_s \hat{\varepsilon} - \hat{\theta}) \dot{s} + \rho_0 \partial_G \hat{\varepsilon} \dot{G} - \hat{\theta}^{-1} \hat{Q} \cdot G \leq 0.$$

By suitably controlling the body force b and heat supply r , one may construct smooth processes that balance mass, momentum and energy and attain at some point (x, t) arbitrarily prescribed values for $F, \dot{F}, s, G, \ddot{F}, \dot{s}$ and \dot{G} , subject only to the constraint $\det F > 0$. Consequently, the inequality (2.6.2) cannot hold identically unless the constitutive functions in (2.6.1) have the following special form:

$$(2.6.3) \quad \begin{cases} \varepsilon = \hat{\varepsilon}(F, s), \\ S = \rho_0 \partial_F \hat{\varepsilon}(F, s) + Z(F, \dot{F}, s, G), \\ \theta = \partial_s \hat{\varepsilon}(F, s), \\ Q = \hat{Q}(F, \dot{F}, s, G), \end{cases}$$

$$(2.6.4) \quad \text{tr}[Z(F, \dot{F}, s, G)\dot{F}^\top] + \frac{1}{\hat{\theta}(F, s)} \hat{Q}(F, \dot{F}, s, G) \cdot G \geq 0.$$

Comparing (2.6.3) with (2.5.3) we observe that, again, the internal energy, which may depend solely on the deformation gradient and the entropy, determines the constitutive equation for the temperature by the same caloric equation of state. On the other hand, the constitutive equation for the stress now includes the additional term Z which contributes the viscous effect and induces internal dissipation manifested in (2.6.4).

The constitutive functions must be reduced further to comply with the principle of material frame indifference, postulated in Section 3.4. In particular, frame indifference imposes on internal energy the same condition (2.5.5)₁ as in thermoelasticity, and the resulting reduction is, of course, the same:

$$(2.6.5) \quad \hat{\varepsilon}(F, s) = \hat{\varepsilon}(U, s),$$

where U denotes the right stretch tensor (2.1.7). Furthermore, when (2.6.5) holds, the constitutive equation for the temperature, derived through (2.6.3)₃, and the term $\rho_0 \partial_F \hat{\varepsilon}(F, s)$, in the constitutive equation for the stress, will be automatically frame indifferent. It remains to investigate the implications of frame indifference on Z and on the heat flux. Since the analysis will focus eventually on thermoviscoelastic fluids, it will be expedient to switch at this point from S and Q to T and q ; also to replace, on the list (F, \dot{F}, s, G) of state variables, \dot{F} with L (cf. (2.1.8)) and G with g (cf. (2.3.17)). We thus write

$$(2.6.6) \quad T = \rho \partial_F \hat{\varepsilon}(F, s) F^\top + \hat{Z}(F, L, s, g),$$

$$(2.6.7) \quad q = \hat{q}(F, L, s, g).$$

Recalling (2.4.3) and (2.4.5), we deduce that the principle of material frame indifference requires

$$(2.6.8) \quad \begin{cases} \hat{Z}(OF, OLO^\top + \dot{O}O^\top, s, Og) = O\hat{Z}(F, L, s, g)O^\top \\ \hat{q}(OF, OLO^\top + \dot{O}O^\top, s, Og) = O\hat{q}(F, L, s, g), \end{cases}$$

for any proper orthogonal matrix O . In particular, for any fixed state (F, L, s, g) with spin W (cf. (2.1.9)), we may pick $O(t) = \exp(-tW)$, in which case $O(0) = I$, $\dot{O}(0) = -W$. It then follows from (2.6.8) that \hat{Z} and \hat{q} may depend on L solely through its symmetric part D and hence (2.6.6) and (2.6.7) may be written as

$$(2.6.9) \quad T = \rho \partial_F \hat{\varepsilon}(F, s) F^\top + \hat{Z}(F, D, s, g),$$

$$(2.6.10) \quad q = \hat{q}(F, D, s, g),$$

with \hat{Z} and \hat{q} such that

$$(2.6.11) \quad \begin{cases} \hat{Z}(OF, ODO^\top, s, Og) = O\hat{Z}(F, D, s, g)O^\top \\ \hat{q}(OF, ODO^\top, s, Og) = O\hat{q}(F, D, s, g), \end{cases}$$

for all proper orthogonal matrices O .

For the balance law of angular momentum (2.3.9) to be satisfied, \hat{Z} must also be symmetric: $\hat{Z}^\top = \hat{Z}$. Notice that in that case the dissipation inequality (2.6.4) may be rewritten in the form

$$(2.6.12) \quad \text{tr}[\hat{Z}(F, D, s, g)D] + \frac{1}{\hat{\theta}(F, s)}\hat{q}(F, D, s, g) \cdot g \geq 0.$$

Further reduction of the constitutive functions results when the medium is endowed with material symmetry. The rules of transformation of the fields under isochoric change of the reference configuration are recorded in (2.3.19), (2.3.20), (2.3.21) and (2.3.22). As in Section 2.5, we introduce here the *symmetry group* \mathcal{G} of the material, namely the subgroup of $\text{SL}(m)$ formed by the Jacobian matrices H of those isochoric transformations \bar{x} of the reference configuration that leave the constitutive functions for ε , T , θ and q invariant. Thus, \mathcal{G} is the set of all $H \in \text{SL}(m)$ with the property

$$(2.6.13) \quad \begin{cases} \hat{\varepsilon}(FH^{-1}, s) = \hat{\varepsilon}(F, s), \\ \hat{Z}(FH^{-1}, D, s, g) = \hat{Z}(F, D, s, g), \\ \hat{q}(FH^{-1}, D, s, g) = \hat{q}(F, D, s, g). \end{cases}$$

The material will be called a *thermoviscoelastic fluid* when $\mathcal{G} \equiv \text{SL}(m)$. In that case, applying (2.6.13) with $H = (\det F)^{-1/m}F \in \text{SL}(m)$, we conclude that $\hat{\varepsilon}$, \hat{Z} and \hat{q} may depend on F solely through its determinant or, equivalently, through the density ρ . Therefore, the constitutive equations of the thermoviscoelastic fluid reduce to

$$(2.6.14) \quad \begin{cases} \varepsilon = \tilde{\varepsilon}(\rho, s), \\ T = -pI + \tilde{Z}(\rho, D, s, g), \\ p = \rho^2 \partial_\rho \tilde{\varepsilon}(\rho, s), \quad \theta = \partial_s \tilde{\varepsilon}(\rho, s), \\ q = \tilde{q}(\rho, D, s, g). \end{cases}$$

For frame indifference, \tilde{Z} and \tilde{q} should still satisfy, for any proper orthogonal matrix O , the conditions

$$(2.6.15) \quad \begin{cases} \tilde{Z}(\rho, ODO^\top, s, Og) = O\tilde{Z}(\rho, D, s, g)O^\top, \\ \tilde{q}(\rho, ODO^\top, s, Og) = O\tilde{q}(\rho, D, s, g), \end{cases}$$

which follow from (2.6.11). It is possible to write down explicitly the form of the most general functions \tilde{Z} and \tilde{q} that conform with (2.6.15). Here, it will suffice to

record the most general constitutive relations, for $m=3$, that are compatible with (2.6.15) and are linear in (D, g) , namely

$$(2.6.16) \quad T = -p(\rho, s)I + \lambda(\rho, s)(\text{tr} D)I + 2\mu(\rho, s)D,$$

$$(2.6.17) \quad q = \kappa(\rho, s)g,$$

which identify the compressible, heat-conducting *Newtonian fluid*.

The *bulk viscosity* $\lambda + \frac{2}{3}\mu$, *shear viscosity* μ and *thermal conductivity* κ of a Newtonian fluid are constrained by the inequality (2.6.12), which here reduces to

$$(2.6.18) \quad \lambda(\rho, s)(\text{tr} D)^2 + 2\mu(\rho, s)\text{tr} D^2 + \frac{\kappa(\rho, s)}{\theta(\rho, s)}|g|^2 \geq 0.$$

This inequality will hold for arbitrary D and g if and only if

$$(2.6.19) \quad \mu(\rho, s) \geq 0, \quad 3\lambda(\rho, s) + 2\mu(\rho, s) \geq 0, \quad \kappa(\rho, s) \geq 0.$$

For actual dissipation, at least one of μ , $3\lambda + 2\mu$ and κ should be strictly positive.

2.7 Incompressibility

Many fluids, and even certain solids, such as rubber, may be stretched or sheared with relative ease, while exhibiting disproportionately high stiffness when subjected to deformations that would change their volume. Continuum physics treats such materials as incapable of sustaining any volume change, so that the density ρ stays constant along particle trajectories. The incompressibility condition

$$(2.7.1) \quad \det F = 1, \quad \text{tr} D = \text{div } v^\top = 0,$$

in Lagrangian or Eulerian coordinates, is then appended to the system of balance laws, as a kinematic constraint. In return, the stress tensor is decomposed into two parts:

$$(2.7.2) \quad S = -p(F^{-1})^\top + \hat{S}, \quad T = -pI + \hat{T},$$

where \hat{S} or \hat{T} , called the *extra stress*, is determined, as before, by the thermodynamic process, through constitutive equations, while the other term, which represents a *hydrostatic pressure*, is not specified by a constitutive relation but is to be determined, together with the thermodynamic process, by solving the system of balance laws of mass, momentum and energy, subject to the kinematic constraint (2.7.1).

The salient property of the hydrostatic pressure is that it produces no work under isochoric deformations. To motivate (2.7.2) by means of the Second Law of thermodynamics, let us consider an incompressible thermoelastic material with constitutive equations for ε, θ and Q as in (2.5.1), but only defined for F with $\det F = 1$, and S unspecified. The dissipation inequality again implies (2.5.2) with \hat{S} replaced by

S , $\partial_F \hat{\varepsilon}$ replaced by the tangential derivative $\partial_F^\tau \hat{\varepsilon}$ on the manifold $\det F = 1$, and \dot{F} constrained to lie on the subspace

$$(2.7.3) \quad \text{tr}[(F^{-1})^\top \dot{F}^\top] = \text{tr}[(F^*)^\top \dot{F}^\top] = \text{tr}[(\partial_F \det F) \dot{F}^\top] = \overline{\det F} = 0.$$

Therefore, $\text{tr}[(\rho_0 \partial_F^\tau \hat{\varepsilon} - S) \dot{F}^\top] \leq 0$ for all \dot{F} satisfying (2.7.3) if and only if

$$(2.7.4) \quad S = -p(F^{-1})^\top + \rho_0 \partial_F^\tau \hat{\varepsilon}(F, s),$$

for some scalar p .

In incompressible Newtonian fluids, the stress is still given by (2.6.16), where, however, ρ is constant and $p(\rho, s)$ is replaced by the undetermined hydrostatic pressure p . When the incompressible fluid is inviscid, the entire stress tensor is subsumed by the undetermined hydrostatic pressure.

2.8 Relaxation

The state variables of continuum physics, introduced in the previous sections, represent statistical averages of certain physical quantities, such as velocity, translational kinetic energy, rotational kinetic energy, chemical energy etc., associated with the molecules of the material. These quantities evolve and eventually settle, or “relax”, to states in local equilibrium, characterized by equipartition of energy and other conditions dictated by the laws of statistical physics. The constitutive relations of thermoelasticity, considered in earlier sections, are relevant so long as local equilibrium is attained in a time scale much shorter than the time scale of the gross motion of the material body. In the opposite case, where the relaxation time is of the same order of magnitude as the time scale of the motion, relaxation mechanisms must be accounted for even within the framework of continuum physics. This is done by introducing additional, *internal state variables*, measuring the deviation from local equilibrium. The states in local equilibrium span a manifold embedded in the extended state space. The internal state variables satisfy special constitutive relations, in the form of balance laws with dissipative source terms that act to drive the state vector towards local equilibrium.

An enormous variety of relaxation theories are discussed in the literature; the reader may catch a glimpse of their common underlying structure through the following example.

We consider a continuous medium that does not conduct heat and whose isentropic response is governed by constitutive relations

$$(2.8.1) \quad \varepsilon = \hat{\varepsilon}(F, \Sigma),$$

$$(2.8.2) \quad S = P(F) + \rho_0 \Sigma,$$

for the internal energy and the Piola-Kirchhoff stress, where Σ is an internal variable taking values in $\mathbb{M}^{m \times m}$ and satisfying a balance law of the form

$$(2.8.3) \quad \rho_0 \dot{\Sigma} = \frac{1}{\tau} [\Pi(\Sigma) - F].$$

Thus, the material exhibits instantaneous elastic response, embodied in the term $P(F)$, combined with viscous response induced by relaxation of Σ . The positive constant τ is called the *relaxation time*.

The postulate that any smooth motion of the medium that balances linear momentum (2.3.4) must satisfy identically the entropy inequality (2.5.32) yields

$$(2.8.4) \quad S = \rho_0 \partial_F \hat{e}(F, \Sigma),$$

$$(2.8.5) \quad \text{tr} [\partial_\Sigma \hat{e}(F, \Sigma) \dot{\Sigma}^\top] \leq 0.$$

Upon combining (2.8.4) and (2.8.5) with (2.8.2) and (2.8.3), we deduce

$$(2.8.6) \quad \varepsilon = \sigma(F) + \text{tr}(\Sigma F^\top) + h(\Sigma),$$

$$(2.8.7) \quad P(F) = \rho_0 \partial_F \sigma(F), \quad \Pi(\Sigma) = -\partial_\Sigma h(\Sigma).$$

When h is strictly convex, the source term in (2.8.3) is dissipative and acts to drive Σ towards local equilibrium $\Sigma = H(F)$, where H is the inverse function of Π . Π^{-1} exists since $-\Pi$ is strictly monotone, namely,

$$(2.8.8) \quad \text{tr} \{ [\Pi(\Sigma) - \Pi(\bar{\Sigma})][\Sigma - \bar{\Sigma}]^\top \} < 0, \quad \text{for any } \Sigma \neq \bar{\Sigma}.$$

In local equilibrium the medium responds like an elastic material with internal energy

$$(2.8.9) \quad \varepsilon = \tilde{\varepsilon}(F) = \sigma(F) + \text{tr}[H(F)F^\top] + h(H(F))$$

and Piola-Kirchhoff stress

$$(2.8.10) \quad S = P(F) + \rho_0 H(F) = \rho_0 \partial_F \tilde{\varepsilon}(F).$$

2.9 Notes

The venerable field of continuum physics has been enjoying a resurgence, concomitant with the rise of interest in the behavior of materials with nonlinear response. The encyclopedic works of Truesdell and Toupin [1] and Truesdell and Noll [1] contain reliable historical information as well as massive bibliographies and may serve as excellent guides for following the development of the subject from its inception, in the 18th century, to the mid 1960's. The text by Gurtin [1] provides a clear, elementary introduction to the area. A more advanced treatment, with copious references, is found in the book of Silhavy [1]. The text by Müller [2] is an excellent presentation of thermodynamics from the perspective of modern continuum physics. Other good sources, emphasizing elasticity theory, are the books of Ciarlet [1], Hanyga [1],

Marsden and Hughes [1] and Wang and Truesdell [1]. The monograph by Antman [3] contains a wealth of material on the theory of elastic strings, rods, shells and three-dimensional bodies, with emphasis on the qualitative analysis of the governing balance laws.

The referential description of motion was conceived by Euler, and was eventually named Lagrangian so as to highlight the analogy with the formulation of Analytical Dynamics by Lagrange. On the other hand, the spatial, or Eulerian, description, which was effectively employed by Euler, was introduced by Daniel Benoulli and by D'Alembert.

On the equivalence of the referential (Lagrangian) and spatial (Eulerian) description of the field equations for the balance laws of continuum physics, see Dafermos [17] and Wagner [2,3]. It would be useful to know whether this holds under more general assumptions on the motion than Lipschitz continuity. For instance, when the medium is a thermoelastic gas, it is natural to allow regions of vacuum in the placement of the body. In such a region the density vanishes and the specific volume (determinant of the deformation gradient) becomes infinitely large. For other examples in which the equations get simpler as one passes from Eulerian to Lagrangian coordinates, see Peng [2].

The kinematic balance laws (2.2.15) and (2.2.16) were first derived by Qin[1], in the context of smooth motions, by direct calculation. It is interesting that, as we see here, they are valid when the motions are merely Lipschitz continuous and in fact, as shown by Demoulini, Stuart and Tzavaras [2], even under slightly weaker hypotheses. The connection to null Lagrangians was first pointed out in this last reference. For a detailed treatment of null Lagrangians, see Ball, Currie and Olver [1]. For the differential geometric interpretation of the kinematic balance laws, see Wagner [3,4].

The field equations for the balance laws considered here were originally derived by Euler [1,2], for mass, Cauchy [3,4], for linear and angular momentum, and Kirchhoff [1], for energy. The Clausius-Duhem inequality was postulated by Clausius [1], for the adiabatic case; the entropy flux term was introduced by Duhem [1] and the entropy production term was added by Truesdell and Toupin [1]. More general entropy inequalities were first considered by Müller [1].

The use of frame indifference and material symmetry to reduce constitutive equations originated in the works of Cauchy [4] and Poisson [2]. In the ensuing century, this program was implemented (mostly correctly but occasionally incorrectly) by many authors, for a host of special constitutive equations. In particular, the work of the Cosserats [1], Rivlin and Ericksen [1], and others in the 1940's and 1950's contributed to the clarification of the concepts. The principle of material frame indifference and the definition of the symmetry group were ultimately postulated with generality and mathematical precision by Noll [1].

The postulate that constitutive equations should be reduced so that the Clausius-Duhem inequality will be satisfied automatically by smooth thermodynamic processes that balance mass, momentum and energy was first stated as a general principle by Coleman and Noll [1]. The examples presented here were adapted from Coleman and Noll [1], for thermoelasticity, and Coleman and Mizel [1], for thermoviscoelasticity.

In his doctoral dissertation (1873), van der Waals introduced the equation of state that now bears his name, in order to account for the volume of gas molecules and for intermolecular forces. Gallavotti [1] discusses its interpretation from the standpoint of statistical physics. The van der Waals gas has served over the years as a simple model for phase transitions.

As we saw in the historical introduction, the special features of the Chaplygin gas were first noticed by Earnshaw [1]. The classical contributions of Chaplygin [1] are expounded in the text by von Mises [1]. The Chaplygin gas is currently finding new applications in cosmology. For a surprising application of the equation of state of this gas to differential geometry, see Section 18.7.

Coleman and Gurtin [1] have developed a general theory of thermoviscoelastic materials with internal state variables, of which the example presented in Section 2.8 is a special case. Constitutive relations of this type were first considered by Maxwell [1]. A detailed discussion of relaxation phenomena in gas dynamics is found in the book by Vincenti and Kruger [1].

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