

Chapter 1

Introduction—Dynamics of Relevant Variables

In physical modeling one faces almost contradictory requirements. We want to catch up with reality, but by definite notions and mathematical relations. We like to describe fast dynamical and slow—or even static—situations. We try to understand the sub-microscopic down to 10^{-34} m and the universe up to 10^{26} m. We want to know the properties of single elementary particles and of complex systems coupled to environments. The complexity of modeling is drastically reduced if one can find a set of few **relevant variables** appropriate for the specific questions to be answered. It turns out that often relevant variables emerge within a more detailed theory with elementary variables. Relevant variables are typically the slow variables in a system. When dynamically time scales begin to separate, relevant variables emerge. The best known example is thermodynamics where internal microscopic variables are summed up in a partition sum and only environmental conditions of equilibrium like temperature, chemical potential, pressure and applied magnetic field are known to calculate system properties like average energy, particle density and magnetization or secondary quantities like specific heat and magnetic susceptibility.

Despite the diversification of physics into many disciplines like atomic physics, optics, nuclear physics, particle physics, astrophysics and condensed matter physics it turned out in the last decades that a common methodological frame for theoretical physics exists, which allows an overall view on basic concepts in theoretical physics. This overall view can briefly be characterized as follows.

1. Theoretical physics is about models for probabilities of varying properties in stochastic processes. These probabilities are empirically controlled by counting documentable facts. In the modeling one tries to use as few as possible relevant variables. This is the most creative job in physical modeling as the finding of relevant variables is not automatic, yet.
2. A dynamical description of a system is given in terms of a step by step evolution, which Mathematicians call a (semi-)group.
3. When **fluctuations** in the relevant variables are inessential, a time evolution can be formulated directly for the relevant variables. The process is then called a deterministic process and it is described locally in time by differential equations

which solutions become determined paths starting at some initial value. When fluctuations are essential, the time evolution is searched for the **probability** distribution of relevant variables. Probability conservation relates the probability distribution to a probability current.

4. On each level of description one can however distinguish between two fundamental different classes of dynamics: **reversible and irreversible**.
5. Stationary states of a dynamic always play an important role, e.g. as asymptotic situations for scattering processes or as limiting situations of processes or as building blocks in analyzing the dynamics.
6. Irreversible processes with step by step evolution can be described in terms of a Chapman-Kolmogorov equation for the probability distribution and are called Markov processes. The corresponding local in time equation is called Master equation.
7. To reach reversible step by step evolution equations for processes with essential fluctuations, it turns out that it cannot be formulated directly for the probability distribution. It leads by its close relation to the current density generically to relaxation and irreversibility. Fortunately, it can instead be formulated for a two-component quantity called **pre-probability**. The components (put together as a complex number with modulus and phase) allow to calculate probability density and the current density as independent quantities, except for the probability conservation fulfilled automatically by construction. This theory of reversible step by step stochastic processes is just abstract quantum theory without explicit use of the quantum of action. The corresponding local in time equation is called Schrödinger equation. Fluctuating reversible step by step processes are called quantum processes.
8. Both, Markov and quantum processes can be solved formally by elegant and powerful methods exploiting the step by step character leading to **generating functionals** for all quantities of interest. Analyzing symmetries and topological constraints helps in moving from formal solutions by clear strategies of approximations to explicit approximate solutions.

To imagine the difference between reversible and irreversible processes consider watching a movie. A movie of reversible processes can be played in reverse and you will not take notice of this fact, while in situations of irreversible processes you will find the movie “funny”. The funny things are seemingly impossible motions. In reality they could only happen after an extraordinary sophisticated rearrangement of environment and initial conditions.

In reversible dynamics the notion of **energy** plays a key role. It is the generator of motion and serves as a conserved quantity that separates possible motions from impossible motions (see Fig. 1.1).

In irreversible systems the notion of **current** through the system and the notion of **entropy** play key roles (see Fig. 1.2). The entropy measures the dispersion of states in the space of possible states. In irreversible systems this dispersion increases unless it has reached a stationary state. In regions of currents between reservoirs new

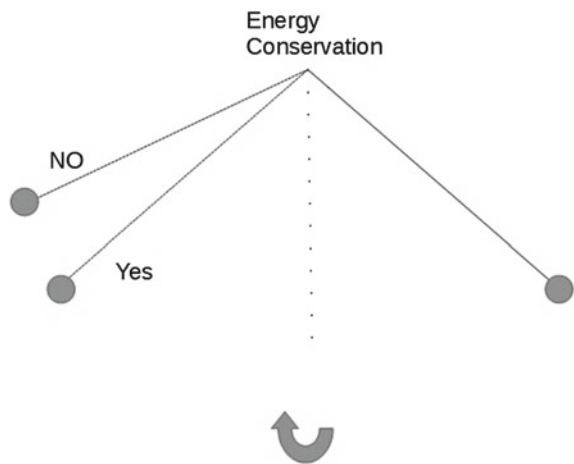


Fig. 1.1 Energy conservation

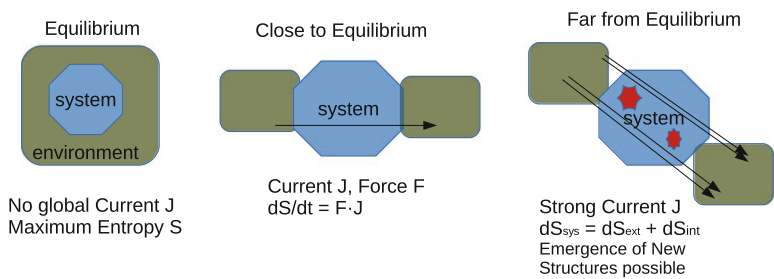


Fig. 1.2 Equilibrium and non-equilibrium

structures can emerge, while in global equilibrium global currents vanish. Therefore, entropy indicates which of several energetically allowed motions will probably take place.

In this book a physical system is defined by a set of relevant variables which form the **configuration manifold**. This manifold can be enlarged or (more often) reduced, the variables can be transformed and they have a time evolution. **Properties** of a physical system are functions on the configuration tangent bundle or cotangent bundle. The tangent bundle contains the configuration point x and local tangential vectors to describe velocities $\dot{x}$. Instead of vectors one can take a **dual** description in terms of linear forms on vectors. Actually, we usually gain information about vectors by linear forms. A basis of the vector space at a given point x is the Gaussian basis ∂_{x_k} and the dual basis is dx_j with $dx_j(\partial_{x_k}) = \delta_{jk}$. Discrete variables and their (perhaps discrete) time evolution are meant to be included in this notion; tangential objects are deviations in short time.

The methods presented here are thought to provide an organized toolbox for modeling quite general physical systems. The general strategy is: we like to

calculate **expectation values** and **correlations** of properties of relevant variables evolving in time, usually under some pre-described conditions. We use models where dynamic and stationary expectation values and correlations can be calculated from generating functionals.¹ The functionals are often called **effective action**. To reach such generating functionals one exploits duality whenever possible. Duality means that variables often have dual partners with respect to functionals, e.g. coordinate x and wave number k are dual by the function e^{ikx} . Fourier-Laplace and Legendre transformations are typical examples of changing from variables to dual variables. The underlying dynamics is modeled by Markov or quantum processes and limiting behavior of such processes. Such limits can be stationary equilibrium, stationary non-equilibrium, scattering situations, decoherent situations (crossover from quantum to Markov behavior) and so called classical deterministic behavior (negligible fluctuations for relevant variables). In finding relevant variables (usually the slow ones in a system) symmetries and topology of the configuration manifold are most helpful. Topology can lead to distinguished classes on the configuration manifold which can—to a large extent—be treated separately and symmetries help optimizing the choice of coordinates for configurations. A crucial indicator for being on the right track is: a **stability analysis** based on an expansion beyond a so called Gaussian approximation (quadratic approximation around a characteristic point) in relevant variables shows the right qualitative phase structure. Such stability analysis often runs under the name of renormalization group analysis. This modeling strategy is displayed in Fig. 1.3. To illustrate the methods we try to keep the formal and calculation effort low and try to use significant but simple examples instead of trying to reach full generality. We will consider simple abstract systems, simple toy systems and some model systems for real phenomena, e.g. ink in water, Laser light, the quantum Hall effect and superconductors.

The book is organized as follows. In Chap. 2 we introduce the dynamics with semi-group or group character. States evolve step by step from an initial state. For reversible systems the group has inverse elements as part of the dynamics. For semi-groups an inverse does not necessarily exist within the semi-group. The short time steps are captured by a time homogeneous generator resulting in differential equations in continuous time. States can be configurations, probabilities or pre-probabilities for configurations. In Chap. 3 formal methods of solving the dynamics are discussed. These are of algebraic or analytic type which both have some advantages as a starting point for approximate methods. In rare cases they even allow for explicit solutions. The analytic formal solutions by generating functionals (path integrals, partition sums) are most convenient as starting points for modeling systems by appropriate relevant variables, as displayed in Fig. 1.3. To enrich the toolbox in modeling it is helpful to know exactly solvable models and a variety of methods of finding them. This is the subject of Chap. 4. Then we broaden the view in Chap. 5 and work out a theory where properties and states are generalized as to have compact and flexible ways of calculation. We consider limits of stationarity and cross-over between different types of dynamics by taking system environments into account. In each type of

¹Functionals stand for functions in perhaps infinitely many degrees of freedom.

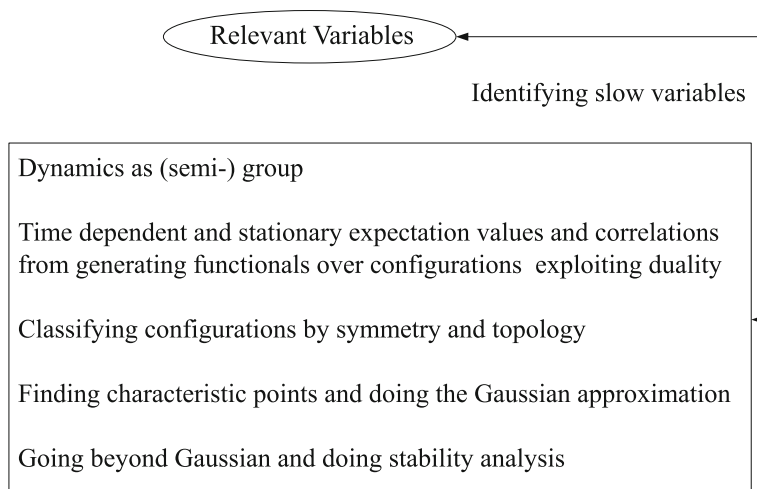


Fig. 1.3 Modeling strategy

dynamics or stationary limit, generating functionals appear as the unifying structure. When fields form the configuration space for infinitely many degrees of freedom one deals with stochastic field theories. In the case of quantum processes these are so called quantum field theories. They are met commonly in particle physics and condensed matter physics. As stressed before, one should study the topology and symmetry of the starting configuration space with respect to the generating functional to identify relevant variables. Thus, symmetry and topology are the subjects of Chaps. 6 and 7. Finally we use the toolbox in Chap. 8 for some selected applications of the author's choice to demonstrate their richness. Three appendices may help filling gaps for readers not yet familiar with probability theory (Appendix A), the method of characteristics (Appendix B) and many-body terminology (Appendix C).

Generated Dynamics of Markov and Quantum
Processes

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2016, XVII, 226 p. 36 illus., 1 illus. in color., Hardcover

ISBN: 978-3-662-49694-7