Chapter 2
Uncertain Variable

Uncertainty theory was founded by Liu [36] in 2007 and perfected by Liu [38] in 2009 to deal with human's belief degree based on four axioms, and uncertain variable is the main tool to model a quantity with human uncertainty in uncertainty theory. The emphases of this chapter are on uncertain measure, uncertain variable, uncertainty distribution, inverse uncertainty distribution, operational law, expected value, and variance.

2.1 Uncertain Measure

Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. Then each element $\Lambda \in \mathcal{L}$ is called an event. Uncertain measure $\mathcal{M}$ is a function from $\mathcal{L}$ to [0, 1], that is, it assigns to each event Λ a number $\mathcal{M}\{\Lambda\}$ which indicates the belief degree that the event Λ will occur. According to the properties of belief degree, Liu [36] proposed the following three axioms that an uncertain measure is supposed to satisfy:

Axiom 1 (*Normality Axiom*) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 (*Duality Axiom*) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 (*Subadditivity Axiom*) For every countable sequence of events Λ_1, $\Lambda_2, \ldots,$ we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Definition 2.1 (*Liu [36]*) A set function $\mathcal{M}$ on a σ-algebra $\mathcal{L}$ of a nonempty set Γ is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms. In this case, the triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

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Example 2.1 Let $\mathcal{L}$ be the power set of $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. Define

$$\mathcal{M}\{\gamma_1\} = 0.5, \quad \mathcal{M}\{\gamma_2\} = 0.4, \quad \mathcal{M}\{\gamma_3\} = 0.3,$$

$$\mathcal{M}\{\gamma_1, \gamma_2\} = 0.7, \quad \mathcal{M}\{\gamma_1, \gamma_3\} = 0.6, \quad \mathcal{M}\{\gamma_2, \gamma_3\} = 0.5,$$

$$\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Gamma\} = 1.$$

Then $\mathcal{M}$ is an uncertain measure, and $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space.

Example 2.2 Let $\lambda(x)$ be a nonnegative function on $\Re$ satisfying

$$\sup_{x \neq y}(\lambda(x) + \lambda(y)) = 1,$$

and $\mathscr{B}$ be the Borel algebra of $\Re$. For each Borel set $B \in \mathscr{B}$, define

$$\mathcal{M}\{B\} = \begin{cases} \sup\limits_{x \in B} \lambda(x), & \text{if } \sup\limits_{x \in B} \lambda(x) < 0.5 \\ 1 - \sup\limits_{x \in B^c} \lambda(x), & \text{if } \sup\limits_{x \in B} \lambda(x) \geq 0.5. \end{cases}$$

Then $\mathcal{M}$ is an uncertain measure, and $(\Re, \mathscr{B}, \mathcal{M})$ is an uncertainty space.

Example 2.3 Let $\rho(x)$ be a nonnegative and integrable function on $\Re$ satisfying

$$\int_{\Re} \rho(x)\mathrm{d}x \geq 1,$$

and $\mathscr{B}$ be the Borel algebra of $\Re$. For each Borel set $B \in \mathscr{B}$, define

$$\mathcal{M}\{B\} = \begin{cases} \displaystyle\int_B \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_B \rho(x)\mathrm{d}x < 0.5 \\ 1 - \displaystyle\int_{B^c} \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_{B^c} \rho(x)\mathrm{d}x < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Then $\mathcal{M}$ is an uncertain measure, and $(\Re, \mathscr{B}, \mathcal{M})$ is an uncertainty space.

Theorem 2.1 (Liu [40], Monotonicity Theorem) *For any events $\Lambda_1 \subset \Lambda_2$, we have*

$$\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}. \tag{2.1}$$

Proof Since $\Lambda_1 \subset \Lambda_2$, we have $\Gamma = \Lambda_1^c \cup \Lambda_2$. By using the subadditivity and duality of uncertain measure, we obtain

$$1 = \mathcal{M}\{\Gamma\} \le \mathcal{M}\{\Lambda_1^c\} + \mathcal{M}\{\Lambda_2\} = 1 - \mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\}.$$

Hence $\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}$. The theorem is proved.

Theorem 2.2 (Liu [36]) *For any event* Λ, *we have*

$$0 \le \mathcal{M}\{\Lambda\} \le 1. \tag{2.2}$$

Proof Since $\emptyset \subset \Lambda \subset \Gamma$, $\mathcal{M}\{\Gamma\} = 1$, and $\mathcal{M}\{\emptyset\} = 1 - \mathcal{M}\{\Gamma\} = 0$, we have $0 \le \mathcal{M}\{\Lambda\} \le 1$ according to the monotonicity of uncertain measure. The theorem is proved.

Product Uncertain Measure

Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$, $k = 1, 2, \ldots$ be a sequence of uncertainty spaces. Write $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots$ as the universal set, and $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots$ as the product σ-algebra. In 2009, Liu [38] defined an uncertain measure $\mathcal{M}$ on $\mathcal{L}$, producing the fourth axiom of uncertain measure.

Axiom 4 (*Product Axiom*) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

For an arbitrary event $\Lambda \in \mathcal{L}$, its uncertain measure could be obtained via

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda} \min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda} \min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c} \min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c} \min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, \qquad \text{otherwise.} \end{cases}$$

Peng and Iwamura [58] showed that the triple $(\Gamma, \mathcal{L}, \mathcal{M})$ derived as above from the uncertainty spaces $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$, $k = 1, 2, \ldots$ is an uncertainty space. Interested readers may refer to Peng and Iwamura [58] or Liu [40] for details.

2.2 Uncertain Variable

Definition 2.2 (*Liu [36]*) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \left\{\gamma \in \Gamma \;\middle|\; \xi(\gamma) \in B\right\} \tag{2.3}$$

is an event.

Example 2.4 Consider an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ and $\mathcal{M}\{\gamma_1\} = 0.5$, $\mathcal{M}\{\gamma_2\} = 0.4$, $\mathcal{M}\{\gamma_3\} = 0.3$. The function ξ defined by

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \\ 1, & \text{if } \gamma = \gamma_3 \end{cases}$$

is an uncertain variable on $(\Gamma, \mathcal{L}, \mathcal{M})$.

Theorem 2.3 (Liu [36]) *Suppose f is a measurable function, and $\xi_1, \xi_2, \ldots, \xi_n$ are uncertain variables on* $(\Gamma, \mathcal{L}, \mathcal{M})$. *Then the function*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \tag{2.4}$$

defined by

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma \tag{2.5}$$

is an uncertain variable.

Proof Since $\xi_1, \xi_2, \ldots, \xi_n$ are measurable functions from $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, and f is a measurable function on the set of real numbers, the composite function $f(\xi_1, \xi_2, \ldots, \xi_n)$ is also a measurable function from $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. Hence ξ is an uncertain variable. The theorem is proved.

Example 2.5 Let ξ_1 and ξ_2 be two uncertain variables. Then the sum $\eta = \xi_1 + \xi_2$ defined by

$$\eta(\gamma) = \xi_1(\gamma) + \xi_2(\gamma), \quad \forall \gamma \in \Gamma$$

is an uncertain variable, and the product $\tau = \xi_1 \cdot \xi_2$ defined by

$$\tau(\gamma) = \xi_1(\gamma) \cdot \xi_2(\gamma), \quad \forall \gamma \in \Gamma$$

is also an uncertain variable.

Independence

Definition 2.3 (*Liu [38]*) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\} \tag{2.6}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Theorem 2.4 (Liu [38]) *The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigvee_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\} \tag{2.7}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Proof On the one hand, suppose that $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables. Then we have

$$\begin{aligned}\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\right\} &= 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i^c)\right\} \\ &= 1 - \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i^c\} = \bigvee_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}.\end{aligned}$$

Thus Eq. (2.7) holds. On the other hand, suppose that Eq. (2.7) holds. Then we have

$$\begin{aligned}\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} &= 1 - \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i^c)\right\} \\ &= 1 - \bigvee_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i^c\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}.\end{aligned}$$

Hence the uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent. The theorem is proved.

Uncertain Vector

Definition 2.4 (*Liu [36]*) Let $\xi_1, \xi_2, \ldots, \xi_m$ be some uncertain variables. Then the vector

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_m) \tag{2.8}$$

is called an m-dimensional uncertain vector.

The concept of independence for uncertain vectors is a generalization of that for uncertain variables.

Definition 2.5 (*Liu [45]*) The m-dimensional uncertain vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in \boldsymbol{B}_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in \boldsymbol{B}_i\} \tag{2.9}$$

for any Borel sets $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_n$ of m-dimensional real vectors.

2.3 Uncertainty Distribution

Definition 2.6 (*Liu [36]*) Let ξ be an uncertain variable. Then its uncertainty distribution Φ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}, \quad \forall x \in \Re. \tag{2.10}$$

Example 2.6 Consider an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ and $\mathcal{M}\{\gamma_1\} = 0.5, \mathcal{M}\{\gamma_2\} = 0.4, \mathcal{M}\{\gamma_3\} = 0.3$. The uncertain variable ξ defined by

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \\ 1, & \text{if } \gamma = \gamma_3 \end{cases}$$

has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < -1 \\ 0.5, & \text{if } -1 \le x < 0 \\ 0.7, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1. \end{cases}$$

Uncertain variables are said to be identically distributed if they share a common uncertainty distribution. Peng and Iwamura [57] showed that a function $\Phi : \Re \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$. Interested readers may refer to Peng and Iwamura [57] or Liu [40] for details.

Example 2.7 An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/(b-a), & \text{if } a \le x \le b \\ 1, & \text{if } x \ge b \end{cases}$$

where a and b are real numbers with $a < b$. For simplicity, this could be denoted by $\xi \sim \mathcal{L}(a, b)$.

Example 2.8 An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re$$

where e and σ are real numbers with $\sigma > 0$. For simplicity, this could be denoted by $\xi \sim \mathcal{N}(e, \sigma)$.

Example 2.9 An uncertain variable ξ is called lognormal if it has a lognormal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \geq 0$$

where e and σ are real numbers with $\sigma > 0$. For simplicity, this could be denoted by $\xi \sim \mathcal{LOGN}(e, \sigma)$.

Operational Law

Theorem 2.5 (Liu [40]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with continuous uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \tag{2.11}$$

has an uncertainty distribution

$$\Phi(x) = \sup_{f(x_1,x_2,\ldots,x_n)\leq x} \left(\min_{1\leq i\leq m} \Phi_i(x_i) \wedge \min_{m+1\leq i\leq n} (1 - \Phi_i(x_i))\right). \tag{2.12}$$

Proof For simplicity, we only prove the case of $m = 1$ and $n = 2$. Note that $f(x_1, x_2)$ is strictly increasing with respect to x_1 and strictly decreasing with respect to x_2, and ξ_1 and ξ_2 are independent uncertain variables. On the one hand, we have

$$\begin{aligned}
&\mathcal{M}\{f(\xi_1, \xi_2) \leq x\} = \mathcal{M}\left\{\bigcup_{f(x_1,x_2)\leq x} (\xi_1 \leq x_1) \cap (\xi_2 \geq x_2)\right\} \\
&\geq \sup_{f(x_1,x_2)\leq x} \mathcal{M}\{(\xi_1 \leq x_1) \cap (\xi_2 \geq x_2)\} = \sup_{f(x_1,x_2)\leq x} \Phi_1(x_1) \wedge (1 - \Phi_2(x_2)).
\end{aligned}$$

On the other hand, we have

$$\mathcal{M}\{f(\xi_1,\xi_2) > x\} = \mathcal{M}\left\{\bigcup_{f(x_1,x_2)>x} (\xi_1 \ge x_1) \cap (\xi_2 \le x_2)\right\}$$
$$\ge \sup_{f(x_1,x_2)>x} \mathcal{M}\{(\xi_1 \ge x_1) \cap (\xi_2 \le x_2)\} = \sup_{f(x_1,x_2)>x} (1-\Phi_1(x_1)) \wedge \Phi_2(x_2).$$

Since

$$\sup_{f(x_1,x_2)\le x} \Phi_1(x_1) \wedge (1-\Phi_2(x_2)) + \sup_{f(x_1,x_2)>x} (1-\Phi_1(x_1)) \wedge \Phi_2(x_2) = 1$$

and

$$\mathcal{M}\{f(\xi_1,\xi_2) \le x\} + \mathcal{M}\{f(\xi_1,\xi_2) > x\} = 1,$$

we have

$$\Phi(x) = \mathcal{M}\{f(\xi_1,\xi_2) \le x\} = \sup_{f(x_1,x_2)\le x} \Phi_1(x_1) \wedge (1-\Phi_2(x_2)).$$

The theorem is proved.

Remark 2.1 If f is a strictly increasing function, then $\xi = f(\xi_1,\xi_2,\ldots,\xi_n)$ has an uncertainty distribution

$$\Phi(x) = \sup_{f(x_1,x_2,\ldots,x_n)\le x} \min_{1\le i\le n} \Phi_i(x_i).$$

Remark 2.2 If f is a strictly decreasing function, then $\xi = f(\xi_1,\xi_2,\ldots,\xi_n)$ has an uncertainty distribution

$$\Phi(x) = \sup_{f(x_1,x_2,\ldots,x_n)\le x} \min_{1\le i\le n} (1-\Phi_i(x_i)).$$

Example 2.10 Assume that ξ_1 and ξ_2 are independent uncertain variables with continuous uncertainty distributions Φ_1 and Φ_2, respectively. Then $\xi_1 \vee \xi_2$ has an uncertainty distribution

$$\Psi(x) = \sup_{x_1 \vee x_2 \le x} \Phi_1(x_1) \wedge \Phi_2(x_2) = \Phi_1(x) \wedge \Phi_2(x),$$

and $\xi_1 \wedge \xi_2$ has an uncertainty distribution

$$\Upsilon(x) = \sup_{x_1 \wedge x_2 \le x} \Phi_1(x_1) \wedge \Phi_2(x_2) = \Phi_1(x) \vee \Phi_2(x).$$

Example 2.11 Assume that ξ_1 and ξ_2 are independent uncertain variables with continuous uncertainty distributions Φ_1 and Φ_2, respectively. Then $\xi_1+\xi_2$ has an uncertainty distribution

$$\Psi(x) = \sup_{x_1+x_2\le x} \Phi_1(x_1)\wedge\Phi_2(x_2) = \sup_{y\in\Re}\Phi_1(x-y)\wedge\Phi_2(y),$$

and $\xi_1-\xi_2$ has an uncertainty distribution

$$\Upsilon(x) = \sup_{x_1-x_2\le x} \Phi_1(x_1)\wedge(1-\Phi_2(x_2)) = \sup_{y\in\Re}\Phi_1(x+y)\wedge(1-\Phi_2(y)).$$

Example 2.12 Assume that ξ_1 and ξ_2 are independent and positive uncertain variables with continuous uncertainty distributions Φ_1 and Φ_2, respectively. Then $\xi_1\cdot\xi_2$ has an uncertainty distribution

$$\Psi(x) = \sup_{x_1\cdot x_2\le x} \Phi_1(x_1)\wedge\Phi_2(x_2) = \sup_{y>0}\Phi_1(x/y)\wedge\Phi_2(y),$$

and ξ_1/ξ_2 has an uncertainty distribution

$$\Upsilon(x) = \sup_{x_1/x_2\le x} \Phi_1(x_1)\wedge(1-\Phi_2(x_2)) = \sup_{y>0}\Phi_1(xy)\wedge(1-\Phi_2(y)).$$

2.4 Inverse Uncertainty Distribution

Definition 2.7 (*Liu [40]*) An uncertainty distribution Φ is called regular if it is a continuous and strictly increasing function, and

$$\lim_{x\to-\infty}\Phi(x)=0, \quad \lim_{x\to+\infty}\Phi(x)=1.$$

Note that a regular uncertainty distribution Φ has an inverse function Φ^{-1} on the open interval (0, 1). Besides, the domain of Φ^{-1} could be extended to [0, 1] via

$$\Phi^{-1}(0)=\lim_{\alpha\downarrow 0}\Phi^{-1}(\alpha), \quad \Phi^{-1}(1)=\lim_{\alpha\uparrow 1}\Phi^{-1}(\alpha)$$

provided that the limits exist.

Definition 2.8 (*Liu [40]*) Let ξ be an uncertain variable with a regular uncertainty distribution Φ. Then the inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ.

Liu [43] showed that a function $\Phi^{-1}:(0,1)\to\Re$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function.

Example 2.13 The inverse uncertainty distribution of a linear uncertain variable $\mathcal{L}(a, b)$ is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b.$$

Example 2.14 The inverse uncertainty distribution of a normal uncertain variable $\mathcal{N}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = e + \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha}{1 - \alpha}.$$

Example 2.15 The inverse uncertainty distribution of a lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = \exp(e) \left(\frac{\alpha}{1 - \alpha}\right)^{\sqrt{3}\sigma/\pi}.$$

Operational Law

Theorem 2.6 (Liu [40]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \tag{2.13}$$

has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1 - \alpha), \ldots, \Phi_n^{-1}(1 - \alpha)). \tag{2.14}$$

Proof For simplicity, we only prove the case of $m = 1$ and $n = 2$. Note that $f(x_1, x_2)$ is strictly increasing with respect to x_1 and strictly decreasing with respect to x_2, and ξ_1 and ξ_2 are independent uncertain variables. On the one hand, we have

$$\{f(\xi_1, \xi_2) \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1 - \alpha))\} \supset \{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \ge \Phi_2^{-1}(1 - \alpha)\}$$

and

$$\mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} \ge \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \wedge \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1 - \alpha)\} = \alpha \wedge \alpha = \alpha.$$

On the other hand, we have

$$\{f(\xi_1, \xi_2) \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1 - \alpha))\} \subset \{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \ge \Phi_2^{-1}(1 - \alpha)\}$$

and

$$\mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} \le \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \vee \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1 - \alpha)\} = \alpha \vee \alpha = \alpha.$$

Hence we have $\mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} = \alpha$, which implies Φ is just the uncertainty distribution of ξ. The theorem is proved.

Remark 2.3 If f is a strictly increasing function, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

Remark 2.4 If f is a strictly decreasing function, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Example 2.16 Assume that ξ_1 and ξ_2 are independent uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Then $\xi_1 \vee \xi_2$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \vee \Phi_2^{-1}(\alpha),$$

and $\xi_1 \wedge \xi_2$ has an inverse uncertainty distribution

$$\Upsilon^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \wedge \Phi_2^{-1}(\alpha).$$

Example 2.17 Assume that ξ_1 and ξ_2 are independent uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Then $\xi_1 + \xi_2$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha),$$

and $\xi_1 - \xi_2$ has an inverse uncertainty distribution

$$\Upsilon^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha).$$

Example 2.18 Assume that ξ_1 and ξ_2 are independent and positive uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Then $\xi_1 \cdot \xi_2$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \cdot \Phi_2^{-1}(\alpha),$$

and ξ_1/ξ_2 has an inverse uncertainty distribution

$$\Upsilon^{-1}(\alpha) = \Phi_1^{-1}(\alpha)/\Phi_2^{-1}(1-\alpha).$$

2.5 Expected Value

Definition 2.9 (*Liu [36]*) Let ξ be an uncertain variable. Then its expected value $E[\xi]$ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\}\mathrm{d}x \tag{2.15}$$

provided that at least one of the two integrals is finite.

Theorem 2.7 (Liu [36]) *Let ξ be an uncertain variable with an uncertainty distribution Φ. If the expected value $E[\xi]$ exists, then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \tag{2.16}$$

Proof Note that $\mathcal{M}\{\xi \geq x\} = 1 - \Phi(x)$ holds for almost every real number x and $\mathcal{M}\{\xi \leq x\}$ is just $\Phi(x)$. It follows from Definition 2.9 of expected value that

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\}\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \end{aligned}$$

The theorem is proved.

Theorem 2.8 (Liu [40]) *Let ξ be an uncertain variable with a regular uncertainty distribution Φ. If the expected value $E[\xi]$ exists, then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{2.17}$$

Proof By using the method of changing variables, it follows from Theorem 2.7 that

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x \\ &= \int_{\Phi(0)}^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_0^{\Phi(0)} \Phi^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \end{aligned}$$

The theorem is proved.

Example 2.19 The linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ has an expected value

$$E[\xi] = \frac{a + b}{2}.$$

Example 2.20 The normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has an expected value

$$E[\xi] = e.$$

Example 2.21 The lognormal uncertain variable $\xi \sim \mathcal{LOGN}(e, \sigma)$ has an expected value

$$E[\xi] = \begin{cases} \sqrt{3}\sigma \exp(e) \csc(\sqrt{3}\sigma), & \text{if } \sigma < \pi/\sqrt{3} \\ +\infty, & \text{if } \sigma \ge \pi/\sqrt{3}. \end{cases}$$

Interested readers may refer to Yao [71] for a detailed proof.

Theorem 2.9 (Liu and Ha [51]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \tag{2.18}$$

has an expected value

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha \tag{2.19}$$

provided that $E[\xi]$ exists.

Proof The theorem follows immediately from Theorems 2.6 and 2.8.

Example 2.22 Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Then

$$E[\xi\eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha)\mathrm{d}\alpha,$$

and

$$E\left[\frac{\xi}{\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1-\alpha)}\mathrm{d}\alpha.$$

Theorem 2.10 (Liu [36], Markov Inequality) *Let ξ be an uncertain variable. Then for any given real number $r > 0$, we have*

$$\mathcal{M}\{|\xi| \ge r\} \le \frac{E[|\xi|]}{r}. \tag{2.20}$$

Proof It follows from Definition 2.9 of expected value that

$$\begin{aligned}E[|\xi|] &= \int_0^{+\infty} \mathcal{M}\{|\xi| \geq x\}\mathrm{d}x \geq \int_0^{r} \mathcal{M}\{|\xi| \geq x\}\mathrm{d}x \\ &\geq \int_0^{r} \mathcal{M}\{|\xi| \geq r\}\mathrm{d}x = r \cdot \mathcal{M}\{|\xi| \geq r\}.\end{aligned}$$

Then we have

$$\mathcal{M}\{|\xi| \geq r\} \leq \frac{E[|\xi|]}{r}.$$

The theorem is proved.

2.6 Variance

Definition 2.10 (*Liu [36]*) Let ξ be an uncertain variable with a finite expected value e. Then its variance $V[\xi]$ is defined by

$$V[\xi] = E[(\xi - e)^2]. \tag{2.21}$$

Remark 2.5 Note that the variance of an uncertain variable is defined based on the expected value. If the expected value of an uncertain variable is infinite or does not exist at all, then the variance of the uncertain variable does not exist.

For an uncertain variable ξ with a finite expected value e, if we know only its uncertainty distribution Φ, then we can derive an upper bound of its variance $V[\xi]$ as follows:

$$\begin{aligned}V[\xi] &= \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \mathcal{M}\{(\xi \geq e + \sqrt{x}) \cup (\xi \leq e - \sqrt{x})\}\mathrm{d}x \\ &\leq \int_0^{+\infty} (\mathcal{M}\{\xi \geq e + \sqrt{x}\} + \mathcal{M}\{\xi \leq e - \sqrt{x}\})\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))\mathrm{d}x.\end{aligned}$$

In this case, we stipulate that

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))\mathrm{d}x. \tag{2.22}$$

Theorem 2.11 (Yao [84]) *Let ξ be an uncertain variable with a regular uncertainty distribution Φ. If its expected value e exists, then its variance*

$$V[\xi]=\int_0^1(\Phi^{-1}(\alpha)-e)^2\mathrm{d}\alpha. \tag{2.23}$$

Proof It follows from Stipulation (2.22) that

$$V[\xi]=\int_0^{+\infty}(1-\Phi(e+\sqrt{x}))\mathrm{d}x+\int_0^{+\infty}\Phi(e-\sqrt{x})\mathrm{d}x.$$

For the first term, substituting $\Phi(e+\sqrt{x})$ with α, and x with $\left(\Phi^{-1}(\alpha)-e\right)^2$ accordingly, we have

$$\begin{aligned}
&\int_0^{+\infty}(1-\Phi(e+\sqrt{x}))\mathrm{d}x\\
&=\int_{\Phi(e)}^1(1-\alpha)\mathrm{d}\left(\Phi^{-1}(\alpha)-e\right)^2\\
&=(1-\alpha)\left(\Phi^{-1}(\alpha)-e\right)^2\Big|_{\Phi(e)}^1-\int_{\Phi(e)}^1\left(\Phi^{-1}(\alpha)-e\right)^2\mathrm{d}(1-\alpha)\\
&=\int_{\Phi(e)}^1\left(\Phi^{-1}(\alpha)-e\right)^2\mathrm{d}\alpha.
\end{aligned}$$

For the second term, substituting $\Phi(e-\sqrt{x})$ with α, and x with $\left(e-\Phi^{-1}(\alpha)\right)^2$ accordingly, we have

$$\begin{aligned}
&\int_0^{+\infty}\Phi(e-\sqrt{x})\mathrm{d}x\\
&=\int_{\Phi(e)}^0\alpha\mathrm{d}\left(e-\Phi^{-1}(\alpha)\right)^2\\
&=\alpha\left(e-\Phi^{-1}(\alpha)\right)^2\Big|_{\Phi(e)}^0-\int_{\Phi(e)}^0\left(e-\Phi^{-1}(\alpha)\right)^2\mathrm{d}\alpha\\
&=\int_0^{\Phi(e)}\left(\Phi^{-1}(\alpha)-e\right)^2\mathrm{d}\alpha.
\end{aligned}$$

Hence

$$\begin{aligned}
V[\xi]&=\int_{\Phi(e)}^1\left(\Phi^{-1}(\alpha)-e\right)^2\mathrm{d}\alpha+\int_0^{\Phi(e)}\left(\Phi^{-1}(\alpha)-e\right)^2\mathrm{d}\alpha\\
&=\int_0^1\left(\Phi^{-1}(\alpha)-e\right)^2\mathrm{d}\alpha.
\end{aligned}$$

The theorem is proved.

Example 2.23 The linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ has a variance

$$V[\xi] = \frac{(b-a)^2}{12}.$$

Example 2.24 The normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has a variance

$$V[\xi] = \sigma^2.$$

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