Compactness in Infinitary Gödel Logics

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Abstract. We outline some model-building procedures for infinitary Gödel logics, including a suitable ultrapower construction. As an application, we provide two proofs of the fact that the usual characterizations of cardinals κ such that the Compactness and Weak Compactness Theorems hold for the infinitary language $\mathcal{L}_{\kappa,\kappa}$ are also valid for the corresponding Gödel logics.

Keywords: Gödel logic · Infinitary logic · Compactness

1 Introduction

Infinitary logics, or logics with infinitely long expressions, were first studied by Scott and Tarski [7,8]. Specifically, let κ and λ be cardinal numbers and consider a language $\mathcal{L}_{\kappa,\lambda}$ consisting of the following *non-logical symbols*:

1. finitary predicate symbols,
2. finitary function symbols,
3. constants,

and the following *logical symbols*:

4. a set of variables of size κ,
5. conjunctions $\bigwedge_{\iota<\delta} A_\iota$ and disjunctions $\bigvee_{\iota<\delta} A_\iota$ for $\delta < \kappa$,
6. implication and negation,
7. quantifier chains $\forall_{\iota<\delta} x_\iota$ and $\exists_{\iota<\delta} x_\iota$, for $\delta < \lambda$.

Note, in particular, that we do not necessarily include equality in the language. We give ourselves as much notational freedom as the context allows. For example, we might write $\forall \vec{x}$ or $\bigwedge A_\iota$ if the precise length of the connective is not important.

Infinitary languages quickly gathered interest due to their rich model-theoretic properties and expressive power. For example, the following formula separates the standard model of arithmetic from non-standard models:

$$\forall x \bigvee_{n<\omega} n > x.$$

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As is well known, the usual finitary logic ($\mathcal{L}_{\omega,\omega}$ in this notation) is compact. The natural question arose as to whether the languages $\mathcal{L}_{\kappa,\lambda}$ could satisfy suitable analogs of compactness. Recall that a cardinal κ is *weakly compact* if, and only if, it is inaccessible and satisfies the *tree property*, i.e., any tree of size κ such that every level has $< \kappa$ nodes has a branch B of length κ. If so, we say B is a branch *through* the tree. A filter[1] U on some set is *κ-complete* if the intersection of less than κ-many sets in U is also in U. A cardinal κ is *strongly compact* if any κ-complete filter on any set can be extended to a κ-complete ultrafilter. If U is an ultrafilter and $X \in U$, we say X has *measure one with respect to* U (and X has measure zero if $X \notin U$). Let

$$\mathcal{P}_\kappa A = \{S \subset A \colon |S| < \kappa\}.$$

We say an ultrafilter on $\mathcal{P}_\kappa A$ is a *fine measure* if it contains all sets of the form

$$A\uparrow := \{S \in \mathcal{P}_\kappa A \colon A \subset S\}.$$

It is well known (see, for example, [4,5]) that a cardinal κ is strongly compact if, and only if, for every cardinal λ, there exists a fine measure on $\mathcal{P}_\kappa\lambda$. By results of Keisler and Tarski [6] and Hanf [3], the languages $\mathcal{L}_{\kappa,\omega}$ and $\mathcal{L}_{\kappa,\kappa}$ satisfy a strong (resp. weak) analog of the usual compactness theory for classical logic if, and only if, κ is a strongly (resp. weakly) compact cardinal. Specifically, whenever Σ is an arbitrary set (resp. a set where at most κ-many non-logical symbols appear) of formulae such that every subset of Σ of cardinality $< \kappa$ has a model, then Σ has a model. We show that, in a sense made precise below, this is also true when the underlying logic is replaced by any first-order Gödel logic. As we will see, although the proofs are essentially as in the classical case, we need to circumvent a few minor technicalities that arise. In particular, we will need to introduce the notion of *coherent models* for Gödel logics and prove Łoś's Theorem for a suitable ultrapower construction. It has a similar flavor to the analog in continuous model theory (for example, see [2]). An important difference is that, of course, not all logical connectives in Gödel logics are continuous.

2 Gödel Logics

Definition 1. *Let U be a set and*[2] $V \subset [0,1]$ *be closed and containing* 0 *and* 1. *A* valuation $[\![\cdot]\!]$ *of $\mathcal{L}_{\kappa,\lambda}$ for U and V consists of*

1. *For each variable v, a value $[\![v]\!] \in U$;*
2. *For each function symbol f of arity n, a function $[\![f]\!] \colon U^n \to U$*
3. *Similarly, for each predicate symbol, a function $[\![P]\!] \colon U^n \to V$;*

A model *(or V-model, if we want to be precise) is a structure $(U, [\![\cdot]\!])$.*

[1] Recall that a (proper) filter $U \neq \wp(X)$ on a set X is a collection of subsets of X that is closed under binary intersections and supersets.

[2] As unfortunate as it is, 'V' is the usual notation for this.

In this paper, the term'model' is used both as in Definition 1 and in the classical sense. The meaning shall always be clear from the context. Also, V will always denote a closed subset of $[0,1]$ containing 0 and 1. Valuations are naturally extended to map any term t to an element $[\![t]\!] \in U$ and any $\mathcal{L}_{\kappa,\lambda}$-formula to a truth value $r \in V$:

$$[\![\bigwedge_{\iota<\delta} A_\iota]\!] = \inf\{[\![A_\iota]\!] \colon \iota < \delta\};$$

$$[\![\bigvee_{\iota<\delta} A_\iota]\!] = \sup\{[\![A_\iota]\!] \colon \iota < \delta\};$$

$$[\![A \to B]\!] = \begin{cases} [\![B]\!] & \text{if } [\![A]\!] > [\![B]\!], \\ 1 & \text{if } [\![A]\!] \leq [\![B]\!]; \end{cases}$$

$$[\![\forall_{\iota<\delta} x_\iota \, A(\vec{x})]\!] = \inf\{[\![A(\vec{u})]\!] \colon u_\iota \in U \text{ for each } \iota < \delta\};$$

$$[\![\exists_{\iota<\delta} x_\iota \, A(\vec{x})]\!] = \sup\{[\![A(\vec{u})]\!] \colon u_\iota \in U \text{ for each } \iota < \delta\}.$$

We will also sometimes abuse terminology by making statements about 'all $\vec{u} \subset U$,' when in reality we mean 'all $\vec{u} \subset U$ of the appropriate length.' Hence, the last line of the above definition could have been written as

$$[\![\exists_{\iota<\delta} x_\iota \, A(\vec{x})]\!] = \sup\{[\![A(\vec{u})]\!] \colon \vec{u} \subset U\}.$$

Negation is defined by $\neg A = A \to \bot$, so that

$$[\![\neg A]\!] = \begin{cases} 0 & \text{if } [\![A]\!] > 0, \\ 1 & \text{if } [\![A]\!] = 0; \end{cases} \tag{1}$$

in particular:

$$[\![\neg\neg A]\!] = \begin{cases} 0 & \text{if } [\![A]\!] = 0, \\ 1 & \text{if } [\![A]\!] > 0. \end{cases} \tag{2}$$

If Γ is a set of formulae, we define $[\![\Gamma]\!] = \inf\{[\![B]\!] \colon B \in \Gamma\}$. We say that a set Γ of $\mathcal{L}_{\kappa,\lambda}$-formulae 1-*entails* A, and write $\Gamma \models A$, if $1 = [\![\Gamma]\!]$ implies $1 = [\![A]\!]$ for any valuation $[\![\cdot]\!]$. Given a language $\mathcal{L}_{\kappa,\lambda}$ and a truth-value set V, we can formally define the *Gödel logic* G_V as the set of pairs (Γ, A) such that $\Gamma \models A$.

Indeed, a notion of entailment is usually taken as the central semantic notion for Gödel logics, instead of that of satisfiability. This is due to the fact that satisfiability can in general be defined from entailment, but not conversely (for a general treatment of first-order Gödel logics, see [1]).

Suppose Γ is a set of $\mathcal{L}_{\kappa,\lambda}$-sentences. We say that a set $S \subset \Gamma$ of cardinality $< \kappa$ is a κ-*reduction* for (Γ, A) if $\Gamma \models A$ implies $S \models A$. The following is the main definition:

Definition 2.

- *We say that* $\mathcal{L}_{\kappa,\lambda}$ satisfies the Weak Compactness Theorem for G_V *if every pair* (Γ, A) *where at most* κ*-many non-logical symbols appear has a* κ*-reduction.*

– *We say that* $\mathcal{L}_{\kappa,\lambda}$ satisfies the Compactness Theorem for G_V *if every pair* (Γ, A) *has a* κ*-reduction.*

The first-order language $\mathcal{L}$ under consideration is not important for the previous definition. It should rather be regarded as a statement about κ, λ, and/or V.

2.1 Models Coherent with an Enumeration

Note that our valuations include both interpretations and variable assignments. Hence, we might find two morally equal models that differ only in this regard. To remedy this, we consider the following notion:

Definition 3. *Let* $\mathfrak{U} = (U, [\![\cdot]\!])$ *and* $\mathfrak{W} = (U, (\!|\cdot|\!))$ *be models over the same language. We say* $\mathfrak{U}$ *and* $\mathfrak{W}$ *are* equivalent *if they coincide except perhaps for the values of variables, i.e.,* $[\![P(\vec{u})]\!] = (\!|P(\vec{u})|\!)$ *and* $[\![f(\vec{u})]\!] = (\!|f(\vec{u})|\!)$ *for each* $\vec{u} \subset U$*, each predicate symbol* P *and each function symbol* f*.*

We denote by $\mathcal{T}(\mathcal{L}_{\kappa,\kappa})$ the set of all terms in the language $\mathcal{L}_{\kappa,\kappa}$. In the future, we might be tempted to assume that the set of $\mathcal{L}_{\kappa,\kappa}$-formulae has cardinality κ. This occurs, e.g., if $\kappa = \kappa^{<\kappa}$ and only κ-many non-logical symbols appear in $\mathcal{L}_{\kappa,\kappa}$, as this implies that the set of $\mathcal{L}_{\kappa,\kappa}$-formulae has cardinality $\kappa^{<\kappa}$.

Under this assumption, we shall describe a procedure to replace a G_V-model by an equivalent one where quantified formulae are nicely witnessed. Although it is tailored for our purposes, it can easily be adapted to different contexts. This procedure and its kin will usually be used as Skolemnization supplements for Gödel logics. Let $\mathcal{F}(\mathcal{L}_{\kappa,\kappa}) = \{F_\iota \colon \iota < \kappa\}$ be an enumeration of all $\mathcal{L}_{\kappa,\kappa}$-formulae and $\{y^{\xi,i}_\iota \colon \xi, \iota < \kappa, i < \omega\}$ be a set of distinguished variables whose complement has size κ.

We say an occurrence of a formula F_ι in $\mathcal{F}(\mathcal{L}_{\kappa,\kappa})$ is *irregular* if ι is of the form $\gamma + k$ with γ limit, $0 < k < \omega$, $\vec{x}$ are free variables in F_ι and $F_\gamma = \forall \vec{x}\, F_\iota$ or $F_\gamma = \exists \vec{x}\, F_\iota$. We say an occurrence of a formula is *regular* if it is not irregular.

Lemma 4. *If* κ *is uncountable and the set of* $\mathcal{L}_{\kappa,\kappa}$*-formulae has cardinality* κ*, then there is an enumeration* $\mathcal{F}(\mathcal{L}_{\kappa,\kappa})$ *of* $\mathcal{L}_{\kappa,\kappa}$ *such that:*

1. *each formula appears unboundedly often;*
2. *each formula appears regularly at least once;*
3. $y^{\xi,i}_\iota$ *does not appear in* $\{F_\gamma \colon \gamma < \iota\}$ *for any* ι, ξ, i*;*
4. *whenever* $F_\iota = \forall_{\xi<\delta} x_\xi\, F(x_\xi)_{\xi<\delta}$ *or* $F_\iota = \exists_{\xi<\delta} x_\xi\, F(x_\xi)_{\xi<\delta}$ *appears regularly for the first time in the sequence, then* $F_{\iota+i} = F(y^{\xi,i}_\iota)_{\xi<\delta}$ *for each* $0 < i < \omega$*.*

Proof. Assign a formula to each limit ordinal $< \kappa$ in such a way that conditions 1 and 3 are verified. Condition 2 is verified automatically, as a formula can only be irregular at a successor stage. If F_ι is a regular-for-the-first-time occurrence of a formula whose outermost symbol is a chain of quantifiers, define $F_{\iota+i}$ for $i < \omega$ in such a way that condition 4 is witnessed to hold; otherwise, set $F_{\iota+i} = F_\iota$ for $i < \omega$. □

We say an enumeration $\mathcal{F}(\mathcal{L}_{\kappa,\kappa}) = \{\mathcal{F}_\iota \colon \iota < \kappa\}$ is *suitable* if either $\kappa = \aleph_0$ or $\mathcal{F}(\mathcal{L}_{\kappa,\kappa})$ satisfies conditions 1–4 in the statement of Lemma 4.

Definition 5. *We say a model* $(U, [\![\cdot]\!])$ *is* $\mathcal{F}(\mathcal{L}_{\kappa,\kappa})$-coherent *if* $\mathcal{F}(\mathcal{L}_{\kappa,\kappa})$ *is suitable and whenever a formula* $F_\iota = \forall_{\xi<\delta} x_\xi\, F(x_\xi)_{\xi<\delta}$ *or* $F_\iota = \exists_{\xi<\delta} x_\xi\, F(x_\xi)_{\xi<\delta}$ *appears regularly for the first time in the sequence, then*

$$[\![F_\iota]\!] = \lim_{i<\omega} [\![F_{\iota+i}]\!] \tag{3}$$

Proposition 6. *Suppose the set of* $\mathcal{L}_{\kappa,\kappa}$*-formulae has cardinality* κ*. Let* $\mathcal{F} = \mathcal{F}(\mathcal{L}_{\kappa,\kappa})$ *be a suitable enumeration and* $\mathfrak{U} = (U, [\![\cdot]\!])$*. Then, there exists an* $\mathcal{F}$*-coherent model* $\mathfrak{W} = (U, (\!|\cdot|\!))$ *equivalent to* $\mathfrak{U}$*.*

Proof. This is clear if $\kappa = \aleph_0$. Suppose $\aleph_1 \leq \kappa$ and partition the set of variables in the language into $Y = \{y_\iota^{\xi,i} \colon \xi, \iota < \kappa, i < \omega\}$ and its complement, Y' and fix a bijection g from Y' onto the set of all variables. We define the valuation $(\!|\cdot|\!)$ to be equal to $[\![\cdot]\!]$ except for the values of variables. Set $(\!|v|\!) = [\![g(v)]\!]$ whenever $v \in Y'$. It remains to define $(\!|\cdot|\!)$ at Y. Let u_0 be an arbitrary, fixed element of U such that $[\![v]\!] = u_0$ for some variable v.

Suppose A is a formula with a chain (or a block of chains) of quantifiers as outermost symbol, e.g., $A = \forall \vec{x}\, F(\vec{x})$. We have that $[\![\forall \vec{x}\, F(\vec{x})]\!] = \inf\{F(\vec{t}) \colon \vec{t} \subset U\}$. Let $\eta = lh(\vec{t})$. Fix an ω-sequence of η-sequences $\{\vec{t_i} \subset U \colon i < \omega\}$ such that $\lim_{i<\omega} F(\vec{t_i}) = [\![\forall \vec{x}\, F(\vec{x})]\!]$. Let F_ι be the first regular occurrence of $\forall \vec{x}\, F(\vec{x})$ in $\mathcal{F}$. We define

$$(\!|y_\iota^{\xi,i}|\!) = \begin{cases} (t_\xi)_i & \text{if } \xi < \eta \\ u_0 & \text{otherwise.} \end{cases}$$

By construction, clearly (3) holds whenever F_ι has a chain (or a block of chains) of quantifiers as outermost symbol and appears regularly for the first time. Moreover, $[\![B(\vec{u})]\!]$ and $(\!|B(\vec{u})|\!)$ coincide for every formula B and every $\vec{u} \subset U$. □

2.2 Ultraproducts

Let U be an ultrafilter on some set I and let $\{\mathfrak{U}_\iota \colon \iota \in I\}$ be a family of models in the language $\mathcal{L}_{\kappa,\lambda}$. We define the ultraproduct of $\{\mathfrak{U}_\iota \colon \iota \in I\}$ in the obvious way, namely, by setting $U = \prod_{\iota \in I} U_\iota / \equiv$, where

$$f \equiv g \text{ if, and only if, } \{\iota \colon f(\iota) = g(\iota)\} \in U.$$

For a function symbol F, we set

$$F[f] = [g] \text{ if, and only if, } \{\iota \colon F(f(\iota)) = g(\iota)\} \in U.$$

For a predicate symbol P, we define $[\![P[f]]\!] = r$ if, and only if,

$$\text{for every } \varepsilon > 0,\ \{\iota \colon |P(f(\iota)) - r| < \varepsilon\} \in U.$$

The ultraproduct is well-defined:

Lemma 7. *Assume P is atomic. Then $\{\iota\colon |P(f(\iota)) - r| < \varepsilon\} \in U$ for exactly one $r \in [0,1]$, so that the ultraproduct is well-defined. Moreover, if U is $(2^{\aleph_0})^+$-complete, then $[\![P[f]]\!] = r$ if, and only if, $\{\iota\colon P(f(\iota)) = r\} \in U$.*

Proof. Suppose that for no r is it the case that $\{\iota\colon |P(f(\iota)) - r| < \varepsilon\} \in U$ for every ε. For each r, choose $\varepsilon_r > 0$ witnessing this. By (topological) compactness of V, finitely-many intervals $(r-\varepsilon_r, r+\varepsilon_r)$ cover V. However, by finite additivity of the ultrafilter, not all of the sets

$$\{\iota\colon |P(f(\iota)) - r| < \varepsilon_r\}$$

can have measure zero—a contradiction. Similarly, let r_0 and r_1 be distinct and $\varepsilon < |r_2 - r_1|/2$. Then $A_i = \{\iota\colon |P(f(\iota)) - r_i| < \varepsilon\}$ cannot have measure one for both $i = 0$ and $i = 1$, as $A_0 \cap A_1 = \varnothing$. A similar argument shows that if U is $(2^{\aleph_0})^+$-complete, then

$$\{\iota\colon P(f(\iota)) = r\} \in U$$

for exactly one $r \in V$. □

We now show that Łoś's Theorem holds in most cases of interest:

Proposition 8. *Assume U is a $(\kappa + \aleph_1)$-complete ultrafilter on I. Let $\mathfrak{U} = (W, [\![\cdot]\!])$ be the ultraproduct of $\{\mathfrak{U}_\iota\colon \iota \in I\}$ by U. Then, Łoś's Theorem holds for $\mathcal{L}_{\kappa,\lambda}$, i.e., for any formula $\varphi \in \mathcal{L}_{\kappa,\lambda}$,*

$$[\![\varphi[f]_{\xi<\delta}]\!] = r \text{ if, and only if, for every } \varepsilon > 0,\ \{\iota\colon |[\![\varphi(f(\iota))_{\xi<\delta}]\!] - r| < \varepsilon\} \in U. \quad (4)$$

Moreover, if $2^{\aleph_0} < \kappa$, then

$$[\![\varphi[f]_{\xi<\delta}]\!] = r \text{ if, and only if, } \{\iota\colon [\![\varphi(f(\iota))_{\xi<\delta}]\!] = r\} \in U. \quad (5)$$

Proof. To spare the reader from an otherwise unreadable proof, we will sometimes identify formulae with their truth values and assume predicates are monadic. The proof is by a straightforward induction as usual.

($\bigwedge$) Let $\varphi[f] = \bigwedge_\gamma \varphi_\gamma[f]$. Write $r = [\![\bigwedge_\gamma \varphi_\gamma[f]]\!] = \inf_\gamma [\![\varphi_\gamma[f]]\!]$ and $[\![\varphi_\gamma[f]]\!] = r_\gamma$. Let $\varepsilon > 0$. The induction hypothesis gives that for every γ,

$$A_\gamma := \{\iota\colon |\varphi_\gamma(f(\iota)) - r_\gamma| < \varepsilon/3\} \in U.$$

By κ-completeness, $A := \bigcap_\gamma A_\gamma \in U$. Pick γ_0 such that $r_{\gamma_0} - r < \varepsilon/3$. Since

$$\begin{aligned}|\varphi(f(\iota)) - r| \leq\ & |\varphi(f(\iota)) - \varphi_{\gamma_0}(f(\iota))| \\ & + |\varphi_{\gamma_0}(f(\iota)) - r_{\gamma_0}| + |r_{\gamma_0} - r|,\end{aligned}$$

it suffices to show that $|\varphi(f(\iota)) - \varphi_{\gamma_0}(f(\iota))| \leq \varepsilon/3$ in some measure-one set. Suppose not, so that for every ι in some $A' \in U$, there is some $\gamma(\iota)$ such that $\varphi_{\gamma_0}(f(\iota)) > \varphi_{\gamma(\iota)}(f(\iota)) + \varepsilon/3$. Since U is κ-complete and the set of all possible γ has cardinality $< \kappa$, then $\gamma(\iota)$ must take a constant value, say γ^*, in a measure-one subset of A'. We apply the induction hypothesis to φ_{γ_0} to obtain a refinement A'' of A' such that

$$A'' := A' \cap \{\iota\colon |\varphi_{\gamma_0}(f(\iota)) - r_{\gamma_0}| < \varepsilon/6\} \in U, \tag{6}$$

and once more to obtain a further refinement of A'' that witnesses the analog of (6) for γ^*. From this follows that $\varphi_{\gamma^*}[f] \leq \varphi_{\gamma_0}[f] - \varepsilon/3$. Hence, $r = \inf_\gamma r_\gamma \leq r_{\gamma^*} \leq r_{\gamma_0} - \varepsilon/3$; a contradiction.
Conversely, if $\varphi[f] = r' \neq r$, then by the argument above,

$$\{\iota\colon |\varphi(f(\iota)) - r| < |r' - r|/2\} \notin U.$$

($\forall$) Let $r = [\![\forall_{\xi<\delta} x_\xi\, \varphi(x_\xi)_{\xi<\delta}]\!] = [\![\forall \vec{x}\, \varphi(\vec{x})]\!] = \inf_{\vec{f}}\{[\![\varphi[\vec{f}]]\!]\}$. Choose a sequence of (sequences of) terms $\{\vec{f}_i\colon i < \omega\}$ such that $\varphi(\vec{f}_i)$ converges to r and let $r_i = [\![\varphi[\vec{f}_i]]\!]$. From the induction hypothesis follows that for any $i < \omega$, and any $\varepsilon > 0$,

$$\{\iota\colon |\varphi(\vec{f}_i(\iota)) - r_i| < \varepsilon\} \in U.$$

In fact, $\aleph_1$-completeness gives that for any $\varepsilon > 0$,

$$A := \{\iota\colon |\varphi(\vec{f}_i(\iota)) - r_i| < \varepsilon \text{ for every } i\} \in U.$$

Hence, $\forall \vec{x}\, \varphi(\vec{x}) \leq r$ in a measure-one set. We show that for every $\varepsilon > 0$,

$$\{\iota\colon r - \forall \vec{x} \varphi \vec{x}(\iota) \leq \varepsilon\} \in U.$$

Suppose towards a contradiction that for some $0 < \varepsilon^* < 1/2$, we have $\forall \vec{x}\, \varphi(\vec{x}) + \varepsilon^* < r$ in a measure-one subset of A. Define $\vec{g} \in \left(\prod_{\iota\in I} U_\iota\right)^\delta$ by setting

$$\vec{g}(\iota) = \begin{cases} \text{some sequence of terms } \vec{t} \text{ such that } [\![\varphi(\vec{t})]\!]_\iota + \varepsilon^*/2 < r & \text{if it exists} \\ \text{some arbitrary term} & \text{otherwise.} \end{cases}$$

We claim that $\vec{g}(\iota)$ is defined using the first clause in a measure-one set. This follows from the fact that $A' := \{\iota\colon [\![\forall \vec{x}\, \varphi(\vec{x})]\!]_\iota + \varepsilon^* < r\} \in U$. Indeed, for each $\iota \in A'$, there must exist some sequence of terms $\vec{t_\iota}$ such that $0 \leq [\![\varphi(\vec{t_\iota})]\!]_\iota - [\![\forall \vec{x} \varphi(\vec{x})]\!]_\iota < \varepsilon^*/2$. But then $[\![\varphi(\vec{t_\iota})]\!]_\iota + \varepsilon^*/2 < r$. Hence the claim follows.
Let $r' < r$ be such that for all $\varepsilon > 0$, $\{\iota\colon |\varphi(\vec{g}(\iota)) - r'| < \varepsilon\} \in U$. We must necessarily have $r' + \varepsilon^*/2 \leq r$. We apply the induction hypothesis to obtain, say, $\varphi[\vec{g}] + \varepsilon^*/3 < r$, which contradicts $r = \inf_{\vec{f}}\{\varphi[\vec{f}]\}$.
To obtain the converse implication, we use the one we just proved as in the first case to show that if $\forall \vec{x}\, \varphi(\vec{x}) = r' \neq r$, then

$$\{\iota\colon |\forall \vec{x}\, \varphi(\vec{x}(\iota)) - r| < |r' - r|/2\} \notin U.$$

($\to$) Let $r = A[f] \to B[f]$, $s = A[f]$, and $t = B[f]$. First suppose $s \leq t$ so that $r = 1$, and let $0 < \varepsilon < (t-s)/2$. By induction hypothesis,

$$\{\iota\colon |A(f(\iota)) - s| < \varepsilon \text{ and } |B(f(\iota)) - t| < \varepsilon\} \in U, \tag{7}$$

so that $A(f(\iota)) \to B(f(\iota)) = 1$ on a measure-one set. Now suppose $s > t$, so that $r = t$ and $0 < \varepsilon < (t-s)/2$. As above, the induction hypothesis gives (7) and so $A(f(\iota)) \to B(f(\iota)) = t$ on a measure-one set. The converse is obtained as before.

The remaining cases are similar. Finally, if $2^{\aleph_0} < \kappa$, then (5) holds for atomic formulae by Lemma 7 and the same inductive argument goes through. □

Corollary 9. *Łoś's Theorem holds for the language $\mathcal{L}_{\omega,\omega}$ and the logic G_V for ultraproducts by countably complete ultrafilters.*

Also, from the proof of Proposition 8 follows that:

Corollary 10. *Łoś's Theorem holds for the language $\mathcal{L}_{\omega,\omega}$ and the logic G_V whenever V is finite.*

3 Compactness Theorems

3.1 Weak Compactness

Theorem 11. *Let κ be an uncountable cardinal.*

1. *If $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem for G_V, then κ is weakly compact;*
2. *If κ is weakly compact, then $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem for G_V.*

Proof. 1. We only need the seemingly weaker assumption that $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem. Assume $\mathcal{L}_{\kappa,\omega}$ contains a unary predicate symbol $\mathtt{P}$ and a set of constant symbols $\{c_\alpha\colon \alpha < \kappa\}$. To see that κ is inaccessible, note that if $\{\kappa_\alpha\colon \alpha < \lambda\}$ were a sequence of length $\lambda < \kappa$ cofinal in κ, then there would be no κ-reduction for $(\Gamma, \bot)$, where Γ is the set consisting of the sentences

- $\bigvee_{\alpha<\lambda} \bigvee_{\iota<\kappa_\alpha} \mathtt{P}(c_\iota)$,
- $\neg\mathtt{P}(c_\iota)$ for $\iota < \kappa$.

Clearly $S \not\models \bot$ for any proper subset S of Γ—a model witnessing this is provided by interpreting c_ι as ι and setting $[\![\mathtt{P}]\!](\iota) = 1$ for each $\iota \in \{\xi < \kappa\colon \neg\mathtt{P}(c_\xi) \notin \Gamma\}$ and $[\![\mathtt{P}]\!](\iota) = 0$ for all other ι (if it is in Γ, $\bigvee_{\alpha<\lambda} \bigvee_{\iota<\kappa_\alpha} \mathtt{P}(c_\iota)$ is witnessed to be true by any ι such that $\neg\mathtt{P}(c_\iota) \notin \Gamma$); while $\Gamma \models \bot$ vacuously. Hence, κ is regular.

If κ were not a strong limit, so that $2^\lambda \geq \kappa$ for some $\lambda < \kappa$, then there would be no κ-reduction for $(\Gamma, \bot)$ if Γ is the set consisting of the formulae

$$\neg \bigwedge_{\alpha<\lambda} \neg^{1+f(\alpha)}\mathtt{P}(c_\alpha), \text{ for } f\colon \lambda \to 2, \tag{8}$$

where $\neg^n$ has the obvious meaning. Indeed, if S is a proper subset of Γ, then let $g\colon \lambda \to 2$ be such that the corresponding instance of (8) does not belong to S. Interpret each c_α as α and set $[\![\mathtt{P}]\!](\alpha)$ to be 0 or 1 according as $g(\alpha)$ equals 0 or 1. Then $[\![\neg^{1+f(\alpha)}\mathtt{P}(c_\alpha)]\!] = 1$ for each $\alpha < \lambda$ if, and only if, $f = g$, and $[\![\neg^{1+f(\alpha)}\mathtt{P}(c_\alpha)]\!] = 0$ for some α otherwise (negated formulae only take values 0 and 1 by (1)), so that (8) takes value 1 if, and only if, $f \neq g$; in particular, $[\![S]\!] = 1$. However, $\Gamma \models \perp$ vacuously as $[\![\Gamma]\!] = 1$ is impossible, for the function g on λ defined by $g(\alpha) = [\![\neg\neg\mathtt{P}(c_\alpha)]\!]$ must be distinct from each $f\colon \lambda \to 2$. To see this, notice that for any such f, we must have by (8) that $[\![\neg^{1+f(\alpha)}\mathtt{P}(c_\alpha)]\!] = 0$ for some α, but

$$[\![\neg^{1+[\![\neg\neg\mathtt{P}(c_\alpha)]\!]}\mathtt{P}(c_\alpha)]\!] = 1$$

for each $\alpha < \lambda$. To see this, notice that it follows by (2) we have:

$$[\![\neg^{1+[\![\neg\neg\mathtt{P}(c_\alpha)]\!]}\mathtt{P}(c_\alpha)]\!] = \begin{cases} [\![\neg P(c_\alpha)]\!] & \text{if } [\![\mathtt{P}(c_\alpha)]\!] = 0, \\ [\![\neg\neg P(c_\alpha)]\!] & \text{if } [\![\mathtt{P}(c_\alpha)]\!] > 0. \end{cases}$$

The claim then follows by (1) and (2). Hence, κ is inaccessible.

It remains to show κ has the tree property. Let T be a tree of size κ such that each level has cardinality $< \kappa$. Denote by $l(\alpha)$ the αth level of T. We consider the set of sentences Γ consisting of

- $\neg(\mathtt{P}(c_\alpha) \wedge \mathtt{P}(c_\beta))$, for every α and β that are T-incomparable, and
- $\bigvee_{\xi \in l(\alpha)} \mathtt{P}(c_\xi)$, for every α.

For any subset S of Γ of cardinality $< \kappa$, there is a model witnessing $S \not\models \perp$; namely, choose a large-enough downwards-closed fragment of T as universe, assign α to the constant c_α and have $\mathtt{P}$ take value 1 along a sufficiently-large well-ordered set and 0 everywhere else. By the Weak Compactness Theorem, there is also a model witnessing $\Gamma \not\models \perp$. For each T-incomparable α and β,

$$[\![\neg(\mathtt{P}(c_\alpha) \wedge \mathtt{P}(c_\beta))]\!] = 1$$

i.e.,

$$\text{either } [\![\mathtt{P}(c_\alpha)]\!] = 0 \text{ or } [\![\mathtt{P}(c_\beta)]\!] = 0.$$

In particular, all points lying on the same level are incomparable, so that

$$[\![\bigvee_{\xi \in l(\alpha)} \mathtt{P}(c_\xi)]\!] = 1$$

implies that $\mathtt{P}$ must evaluate to 1 on one point in each level. The ordinals α such that $[\![\mathtt{P}(c_\alpha)]\!] = 1$ determine a branch through T. Therefore, T has the tree property.

2. Let Γ be a set of $\mathcal{L}_{\kappa,\kappa}$-formulae of cardinality κ. Suppose κ is a weakly compact cardinal and $S \not\models A$ for every $S \subset \Gamma$ of cardinality $< \kappa$. We will assume all symbols in $\mathcal{L}_{\kappa,\kappa}$ appear in Γ, so that there are only κ-many $\mathcal{L}_{\kappa,\kappa}$-formulae,

and construct a model $(U, [\![\cdot]\!])$ such that $[\![B]\!] = 1$ for each $B \in \Gamma$ and $[\![A]\!] < 1$. The assumption that all symbols in $\mathcal{L}_{\kappa,\kappa}$ appear in Γ results in no loss of generality, for any symbol not appearing in Γ can be evaluated arbitrarily by the model while preserving the conclusion. Fix some $\mathcal{T} = \mathcal{T}(\mathcal{L}_{\kappa,\kappa})$, and some suitable (in the sense of Sect. 2.1) enumeration $\mathcal{F} = \mathcal{F}(\mathcal{L}_{\kappa,\kappa})$.

Let T be the subtree of $V^{<\kappa}$ consisting of all $t\colon \gamma \to V$ such that $\gamma < \kappa$ and there exists an $\mathcal{F}$-coherent model $(W, (\!|\cdot|\!))$ fulfilling the following three conditions:

1. $(\!|F_\iota|\!) = t(\iota)$ for all $\iota < \gamma$;
2. $t(\iota) = 1$ if $F_\iota \in \Gamma$;
3. $t(\iota) < 1$ if $F_\iota = A$;

By hypothesis and Proposition 6, there is one such t for each subset of Γ of cardinality $< \kappa$. Additionally, each level of T has size $|V|$ and κ has the tree property, whereby there exists a branch $\mathcal{B}$ through T. This branch assigns a unique value in V to each formula in $\mathcal{F}$. For each initial segment t of $\mathcal{B}$, there exists a model agreeing with t on all valuations.

Define a relation $\equiv$ to hold between two terms $r, s \in \mathcal{T}$ whenever for each atomic $P(x)$, there exists $\iota < \kappa$ such that $P(r)$ and $P(s)$ appear before F_ι in $\mathcal{F}$ and are assigned the same value by the branch. We let the universe U of the model to be equal to $\mathcal{T}/\equiv$. For each atomic formula $F_\iota \in \mathcal{F}$, we set

$$[\![F_\iota]\!] = r \text{ if, and only if, } t(\iota) = r \text{ for some } t \in \mathcal{B}. \tag{9}$$

This is well-defined, by the definition of $\equiv$. In order to finish the proof, it remains to check that Eq. (9) holds true for arbitrary formulae. If so, then we will have a model where $[\![\Gamma]\!] = 1$ and $[\![A]\!] < 1$. We will check that the following properties hold:

1. $[\![B \to C]\!] = [\![C]\!]$ if $[\![B]\!] > [\![C]\!]$, and $[\![B \to C]\!] = 1$ otherwise.
2. $[\![\bigvee_{\iota<\delta} B_\iota]\!] = \sup\{[\![B]\!]_\iota \colon \iota < \delta\}$.
3. $[\![\bigwedge_{\iota<\delta} B_\iota]\!] = \inf\{[\![B]\!]_\iota \colon \iota < \delta\}$.
4. $[\![\forall_{\iota<\delta} x_\iota\, B(\vec{x})]\!] = \inf\{[\![B(\vec{u})]\!] \colon u_\iota \in \mathcal{T} \text{ for each } \iota < \delta\}$.
5. $[\![\exists_{\iota<\delta} x_\iota\, B(\vec{x})]\!] = \sup\{[\![B(\vec{u})]\!] \colon u_\iota \in \mathcal{T} \text{ for each } \iota < \delta\}$.

Notice that $\mathcal{B}$ evaluates all validities to 1 and respects entailment: if $t(\iota) = 1$ and $F_\iota \models F_\xi$, then $t(\xi) = 1$. The following observation will be used repeatedly: if $t(\iota) = 1$ and $F_\iota = B \to C$, then $[\![B]\!] \leq [\![C]\!]$. This follows from the fact that in every model where $(\!|B \to C|\!) = 1$, we must have $(\!|B|\!) \leq (\!|C|\!)$. This already gives one half of property (1). Conversely, assume $t(\iota) < 1$ and $F_\iota = B \to C$. Let ξ be large enough so that both B and C appear before F_ξ. In any model agreeing with $\mathcal{B}$ up to ξ, necessarily $(\!|B \to C|\!) < 1$, whence $(\!|B \to C|\!) = (\!|C|\!)$ and so $[\![B \to C]\!] = [\![C]\!]$.

For property (2), notice that $B_\xi \to \bigvee_{\iota<\delta} B_\iota$ is valid and thus $[\![\bigvee_{\iota<\delta} B_\iota]\!] \geq \sup\{[\![B]\!]_\iota \colon \iota < \delta\}$. Conversely, let ι^* be an ordinal such that all B_ι appear before F_{ι^*}. Since any model must evaluate $\bigvee_\iota B_\iota$ to the infimum of the values of the B_ι and there exists a model agreeing with $\mathcal{B}$ up to ι^*, it follows that $[\![\bigvee_{\iota<\delta} B_\iota]\!] = \sup\{[\![B]\!]_\iota \colon \iota < \delta\}$. Property (3) is proved analogously.

We show (4): clearly $[\![\forall_{\iota<\delta} x_\iota\, B(\vec{x})]\!] \leq [\![B(\vec{u})]\!]$ for any sequence of terms $\vec{u}$ in $\mathcal{T}$, as $\forall_{\iota<\delta} x_\iota\, B(\vec{x}) \to B(\vec{u})$ is valid. To see that equality holds, it suffices to notice that, if F_ι is the first regular occurrence of $\forall_{\iota<\delta} x_\iota\, B(\vec{x})$, since there exists an $\mathcal{F}$-coherent model agreeing with $\mathcal{B}$ up to level $\iota+\omega$, then $[\![F_\iota]\!] = \lim_{i<\omega} [\![F_{\iota+i}]\!]$.

As for property (5), we clearly have $[\![\exists_{\iota<\delta} x_\iota\, B(\vec{x})]\!] \geq [\![B(\vec{u})]\!]$ for any sequence of terms $\vec{u}$. Suppose $[\![D]\!] \geq [\![B(\vec{u})]\!]$ for any sequence of terms $\vec{u}$ and some formula D. Then we have $[\![B(\vec{u}) \to D]\!] = 1$ by property (1). This implies $[\![\forall \vec{x}(B(\vec{x}) \to D)]\!] = 1$ by property (4), whereby also $[\![\exists \vec{x}\, B(\vec{x}) \to D]\!] = 1$, for

$$\forall \vec{x}(B(\vec{x}) \to D) \models \exists \vec{x}\, B(\vec{x}) \to D.$$

This yields $[\![\exists \vec{x}\, B(\vec{x})]\!] \leq [\![D]\!]$ as desired and finishes the proof. $\square$

3.2 Strong Compactness

Theorem 12. *Let κ be a cardinal.*

1. *If $\mathcal{L}_{\kappa,\kappa}$ satisfies the Compactness Theorem for G_V, then κ is strongly compact;*
2. *If κ is strongly compact, then $\mathcal{L}_{\kappa,\kappa}$ satisfies the Compactness Theorem for G_V.*

Proof. 1. The classical proof goes through. As before, we only suppose for the first claim that $\mathcal{L}_{\kappa,\omega}$ satisfies the Compactness Theorem for G_V. Let F be a κ-complete filter on some set I. Assume $\mathcal{L}_{\kappa,\omega}$ contains a unary predicate $\mathtt{S}$ for every subset S of I and a constant c. Let Γ be the set of

- (extension) sentences $\mathtt{S}(c)$ for every $S \in F$;
- all sentences true in the (classical) structure $(I, \{S\}_{S\subset I})$, in particular:
- (monotonicity) $\mathtt{S}(c) \to \mathtt{S}'(c)$ for every $S \subset S' \subset I$,
- (κ-completeness) $\bigwedge_{\iota<\delta} \mathtt{S}_\iota(c) \to \mathtt{S}(c)$, for $\delta < \kappa$ and $S = \bigcup_{\iota<\delta} S_\iota$,
- (maximality) $\mathtt{S}(c) \vee \neg\mathtt{S}(c)$ for every $S \subset I$.

For every subset Δ of Γ of cardinality $< \kappa$, there is a model witnessing $\Delta \not\models \bot$. In fact, there is a model that takes only values 0 and 1 obtained by taking I as universe and interpreting $\mathtt{S}$ as S for each predicate $\mathtt{S}$ appearing in Δ and c as some element belonging to $\bigcap_{\mathtt{S}\in\Delta} S$, which exists by κ-completeness. By the Compactness Theorem, there is a model $(U, [\![\cdot]\!], \{S^*\}_{S\subset I}, c)$ witnessing $\Gamma \not\models \bot$. Define

$$S \in F^* \text{ if, and only if, } [\![\mathtt{S}(c)]\!] = 1.$$

Clearly, F^* extends F, as $\mathtt{S}(c) \in \Gamma$ for every $S \in F$, whence $[\![\mathtt{S}(c)]\!] = 1$. Also, F^* is a κ-complete filter: suppose $S \in F^*$, so that $[\![\mathtt{S}(c)]\!] = 1$, and $S' \supset S$. Since $\mathtt{S}(c) \to \mathtt{S}'(c) \in \Gamma$, then $[\![\mathtt{S}(c) \to \mathtt{S}'(c)]\!] = 1$, which implies $[\![\mathtt{S}(c)]\!] = [\![\mathtt{S}'(c)]\!] = 1$.

Suppose $S_\iota \in F^*$ for every $\iota < \delta$ and $\delta < \kappa$. It follows that $[\![\mathtt{S}_\iota(c)]\!] = 1$ for each $\iota < \delta$. Letting $S = \bigcap_{\iota<\delta} S_\iota$, we have that $\bigwedge_{\iota<\delta} \mathtt{S}_\iota(c) \to \mathtt{S}(c) \in \Gamma$, whence $[\![\mathtt{S}(c)]\!] = 1$. Hence, F^* is a κ-complete filter. In fact, F^* is an ultrafilter, for $\mathtt{S}(c) \vee \neg\mathtt{S}(c) \in \Gamma$, so that if $S \notin F^*$, then $[\![\mathtt{S}(c)]\!] < 1$, and so the fact that $[\![\mathtt{S}(c) \vee \neg\mathtt{S}(c)]\!] = 1$ implies that $[\![\neg\mathtt{S}(c)]\!] = 1$.

2. Conversely, suppose that κ is a strongly compact cardinal and that for any $S \subset \Gamma$ of cardinality Γ, we have $S \not\models A$, as witnessed by a model $\mathfrak{U}_S = (U_S, [\![\cdot]\!]_S)$. Consider the ultraproduct $\mathfrak{U} = (U, [\![\cdot]\!])$ by a fine measure on $\mathcal{P}_\kappa \Gamma$. By Proposition 8 and the fact that $\kappa > (2^{\aleph_0})^+$ (κ is inaccessible), the ultraproduct satisfies (5), i.e., for any formula $\varphi(x_\xi)_{\xi<\delta}$,

$$[\![\varphi[f]_{\xi<\delta}]\!] = r \text{ if, and only if, } \{S \in \mathcal{P}_\kappa \Gamma\colon [\![\varphi(f(S))_{\xi<\delta}]\!] = r\} \text{ has measure one.}$$

Fineness of the measure implies that $\{\varphi\}\uparrow = \{S \in \mathcal{P}_\kappa \Gamma\colon \{\varphi\} \subset S\}$ has measure one for any $\varphi \in \Gamma$. Moreover, $[\![\varphi]\!]_S = 1$ for any $S \in \{\varphi\}\uparrow$, and so $[\![\varphi]\!] = 1$. Similarly, $[\![A]\!] < 1$, because $[\![A]\!]_S < 1$ for all $S \in \mathcal{P}_\kappa \Gamma$. □

4 An Alternative Proof

(The proofs of) Theorems 11 and 12 are evidence that, sufficiently high up Cantor's realm, the influence of logics' size on their behavior becomes progressively more prevalent, and, that of other traits, progressively less. An example of this is the fact that, for Gödel logics, the truth-value set V seems to play no role whatsoever, in clear contrast to usual finitary first-order logics.

This should not be surprising. Indeed, large cardinalities allow us to diffuse otherwise-characteristic properties of logics by means of codings. Herein, a key ingredient is the regularity of the models Proposition 6 yields. This provides us with alternative proofs of 11.2 and 12.2. These proofs are somewhat more extensive than the ones provided originally, although they do have the clear advantage that with little or no effort, they can be adapted into other contexts. For definiteness, we focus on weak compactness in the following.

Another proof of 11.2 Suppose κ is a weakly compact cardinal and $S \not\models A$ for every $S \subset \Gamma$ of cardinality $< \kappa$. As before, without loss of generality, we assume all symbols in $\mathcal{L}_{\kappa,\kappa}$ appear in Γ. Define a first-order infinitary language $\mathcal{L}'_{\kappa,\kappa}$ consisting of

- the same set of variables `Var` as $\mathcal{L}_{\kappa,\kappa}$,
- the same set of function and constant symbols as $\mathcal{L}_{\kappa,\kappa}$,
- a predicate $P^C_r(\vec{x})$ for every $r \in V$ whenever $C(\vec{x})$ is a $\mathcal{L}_{\kappa,\kappa}$-formula,
- predicates $S^{C,B}(\vec{x}, \vec{y})$ and $W^{C,B}(\vec{x}, \vec{y})$ whenever $C(\vec{x})$ and $B(\vec{y})$ are $\mathcal{L}_{\kappa,\kappa}$-formulae.

Only κ-many non-logical symbols appear in $\mathcal{L}_{\kappa,\kappa}$; thus, the set of $\mathcal{L}_{\kappa,\kappa}$-formulae has cardinality κ. Consequently, only κ-many non-logical symbols appear in $\mathcal{L}'_{\kappa,\kappa}$. We will interpret the infinitary G_V-logic over $\mathcal{L}_{\kappa,\kappa}$ in classical logic. The intended interpretation of $P^C_r(\vec{x})$ is '$C(\vec{x})$ has truth value r.' Similarly, the intended interpretations of $S^{C,B}(\vec{x}, \vec{y})$ and $W^{C,B}(\vec{x}, \vec{y})$ are, respectively, '$C(\vec{x})$ has a (strictly) smaller truth value than $B(\vec{y})$.' Let $\mathcal{F}(\mathcal{L}_{\kappa,\kappa}) = \{F_\iota\colon \iota < \kappa\}$ be a suitable enumeration of all $\mathcal{L}_{\kappa,\kappa}$-formulae with distinguished set of variables $\{y^{\xi,i}_\iota\colon \xi, \iota < \kappa, i < \omega\}$. If $C = \forall_{\xi<\delta} x_\xi F(x_\xi)_{\xi<\delta}$ or $C = \exists_{\xi<\delta} x_\xi F(x_\xi)_{\xi<\delta}$ and F_ι

is the first regular appearance of C in $\mathcal{F}$, we denote by $\mathtt{Var}(F,(x_\xi)_{\xi<\delta})$ the set $\{y_\iota^{\xi,i} : \xi<\delta, i<\omega\}$.

We use the fact that if κ is weakly compact, then $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem for classical logic, as recalled in Sect. 1. Let Σ consist of all sentences of one of the following forms:

1. $\bigvee_{r\in V} P_r^C(\vec{x})$, for each $C(\vec{x})\in\mathcal{L}_{\kappa,\kappa}$;
2. $P_r^C(\vec{x})\to\neg P_s^C(\vec{x})$, for each $C(\vec{x})\in\mathcal{L}_{\kappa,\kappa}$ and each $r\neq s$ in V;
3. $S^{C,B}(\vec{x},\vec{y}) \leftrightarrow \bigvee_{r\in V}\bigvee_{s\in V\cap(r,1]}\big(P_r^C(\vec{x})\wedge P_s^B(\vec{y})\big)$, for each $C(\vec{x})$, $B(\vec{y})\in\mathcal{L}_{\kappa,\kappa}$;
4. $W^{C,B}(\vec{x},\vec{y}) \leftrightarrow \bigvee_{r\in V}\bigvee_{s\in V\cap[r,1]}\big(P_r^C(\vec{x})\wedge P_s^B(\vec{y})\big)$, for each $C(\vec{x})$, $B(\vec{y})\in\mathcal{L}_{\kappa,\kappa}$;
5. $W^{\bigwedge_{\iota<\delta}C_\iota,C_\xi}((\vec{x}_\iota)_{\iota<\delta},\vec{x}_\xi)$ for each $\xi<\delta<\kappa$ and each $C(\vec{x})\in\mathcal{L}_{\kappa,\kappa}$;
6. $P_r^{\bigwedge_{\iota<\delta}C_\iota}(\vec{y}_\iota)_{\iota<\delta}\leftrightarrow\bigwedge_{\epsilon>0}\bigvee_{\xi<\delta}\bigvee_{t\in V\cap[r,r+\epsilon]}P_t^{C_\xi}(\vec{y}_\xi)$, for each sequence of formulae $C_\iota(\vec{y}_\iota)\in\mathcal{L}_{\kappa,\kappa}$;
7. $W^{C_\xi,\bigvee_{\iota<\delta}C_\iota}(\vec{x}_\xi,(\vec{x}_\iota)_{\iota<\delta})$ for each $\xi<\delta<\kappa$ and each $C(\vec{x})\in\mathcal{L}_{\kappa,\kappa}$;
8. $P_r^{\bigvee_{\iota<\delta}C_\iota}(\vec{y}_\iota)_{\iota<\delta}\leftrightarrow\bigwedge_{\epsilon>0}\bigvee_{\xi<\delta}\bigvee_{t\in V\cap[r-\epsilon,r]}P_t^{C_\xi}(\vec{y}_\xi)$, for each sequence of formulae $C_\iota(\vec{y}_\iota)\in\mathcal{L}_{\kappa,\kappa}$;
9. $W^{\forall\vec{x}C,C}(\vec{y})$ for each $C(\vec{y})\in\mathcal{L}_{\kappa,\kappa}$;
10. $P_r^{\forall\vec{x}C(\vec{x})}(\vec{y}) \leftrightarrow \bigwedge_{\epsilon>0}\bigvee_{\vec{z}\in\mathtt{Var}(C,\vec{x})}\bigvee_{t\in V\cap[r,r+\epsilon]}P_t^C(\vec{z},\vec{y})$, for each $C(\vec{z},\vec{y})\in\mathcal{L}_{\kappa,\kappa}$;
11. $W^{C,\exists\vec{x}C}(\vec{y})$ for each $C(\vec{y})\in\mathcal{L}_{\kappa,\kappa}$;
12. $P_r^{\exists\vec{x}C(\vec{x})}(\vec{y}) \leftrightarrow \bigwedge_{\epsilon>0}\bigvee_{\vec{z}\in\mathtt{Var}(C,\vec{x})}\bigvee_{t\in V\cap[r-\epsilon,r]}P_t^C(\vec{z},\vec{y})$, for each $C(\vec{z},\vec{y})\in\mathcal{L}_{\kappa,\kappa}$;
13. $W^{C,B}(\vec{x},\vec{y})\to P_1^{C\to B}(\vec{x},\vec{y})$, for every $C(\vec{x}),B(\vec{y})\in\mathcal{L}_{\kappa,\kappa}$;
14. $\big(S^{B,C}(\vec{x},\vec{y})\wedge P_r^B(\vec{x})\big)\to P_r^{C\to B}(\vec{x},\vec{y})$, for every $C(\vec{x}),B(\vec{y})\in\mathcal{L}_{\kappa,\kappa}$;
15. $\bigwedge_{r\in V\cap[0,1)}P_r^A$;
16. $P_1^B(\vec{x})$, for every $B(\vec{x})\in\Gamma$.

The first two conditions above state that each formula has exactly one truth value. Conditions 3 and 4 define the predicates $S^{C,B}(\vec{x},\vec{y})$ and $W^{C,B}(\vec{x},\vec{y})$. Conditions 5–14 define how truth values should behave in nonatomic formulae. Specifically, conditions 5–8 define conjunctions and disjunctions, conditions 9–12 define quantifiers, and conditions 13 and 14 define implication.

The restriction of the domain of the conjunction in conditions 10 and 12 is necessary in order to avoid a conjunction of length κ. The last two conditions state that any formula in Γ must have truth value 1 and A must not.

Let Δ be a subset of Σ of cardinality $<\kappa$ and

$$\Delta_0=\{B\in\mathcal{L}_{\kappa,\kappa} : P_1^B\in\Delta\}.$$

By hypothesis, there is a G_V-model witnessing $\Delta_0\not\models A$. The key point is that, by Proposition 6, we can find an $\mathcal{F}$-coherent model $\mathfrak{W}=(U,[\![\cdot]\!])$ witnessing $\Delta_0\not\models A$. We define a (classical) model $\mathfrak{U}$ for Δ with the same universe:

– For any function symbol f, we set

$$\mathfrak{U}\models f(\vec{x})=y \text{ if, and only if, } [\![f]\!](\vec{x})=y. \tag{10}$$

– For any atomic formula C, we set

$$\mathfrak{U} \models P_r^C(\vec{a}) \text{ if, and only if, } [\![C(\vec{a})]\!] = r. \tag{11}$$

Any $F \in \Delta$ of the form 1–16 is satisfied: one verifies by induction that (11) holds for arbitrary formulae $C(\vec{a})$. For example, if F is of the form 10 or 12, then $\mathfrak{U} \models F$ because $\mathfrak{W}$ is $\mathcal{F}$-coherent.

Hence, any subset of Σ of cardinality $< \kappa$ has a classical model, whereby the Weak Compactness Theorem for classical logic yields a model of Σ, say, $\mathfrak{U}$. Let U be the universe of this model. We define a G_V-model with universe U via (10) and (11).

The classical model $\mathfrak{U}$ satisfies sentences 1–16. Since it satisfies 1 and 2, each formula is assigned exactly one truth value in the G_V-model. One verifies—once more by induction—that (11) holds for arbitrary formulae $C(\vec{a})$. For example, suppose $C = \forall x F(x)$. Let $r = [\![\forall x F(x)]\!]$ and $r_u = [\![F(u)]\!]$, so that $r = \inf_{u \in \mathcal{T}} r_u$. By induction hypothesis, $\mathfrak{U} \models P_{r_u}^{F(u)}$ for each u. By Eqs. 1, 2, 4, and 9, $\mathfrak{U} \models P_s^{\forall x F(x)}$ for some $s \leq r$. But necessarily $s = r$, for $\{r_u \colon u \in \mathtt{Var}(C, \vec{x})\}$ converges to r by 10. The other cases are treated similarly. Finally, the model witnesses $\Gamma \not\models A$ by 15 and 16. □

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