

Chapter 2

Discrete Static Games

In an optimization problem, we have a single decision maker, his feasible decision alternative set, and an objective function depending on the selected alternative. In game theoretical models, we have several decision makers who are called the *players*, each of them has a feasible alternative set, which is called the player's *strategy set*, and each player has an objective function what is called the player's *payoff function*. The payoff of each player depends on the strategy selections of all players, so the outcome depends on his own decision as well as on the decisions of the other players. Let N be the number of players, S_k the strategy set of player k ($k = 1, 2, \dots, N$) and it is assumed that the payoff function ϕ_k of player k is defined on $S_1 \times S_2 \times \dots \times S_N$ and is real valued. That is, $\phi_k : S_1 \times S_2 \times \dots \times S_N \mapsto \mathbb{R}$. So if $s_1, s_2, \dots, s_N$ are the strategy selections of the players, $s_k \in S_k$ ($k = 1, 2, \dots, N$), then the payoff of player k is $\phi_k(s_1, s_2, \dots, s_N)$. The game can be denoted as $\Gamma(N; S_1, S_2, \dots, S_N; \phi_1, \phi_2, \dots, \phi_N)$ which is usually called the *normal form* representation of the game.

A game is called discrete, if the strategy sets are countable, in most cases only finite. The most simple discrete game has only two players, each of them has only two possible strategies to select from. Therefore, there are only four possible outcomes of the game.

2.1 Examples of Two-Person Finite Games

We start with the prisoner's dilemma game, which is the starting example in almost all game theory books and courses.

Example 2.1 (Prisoner's dilemma) Assume two criminals robbed a jewellery store for hire. After doing this job they escaped with a stolen can and delivered the stolen items to a mafia boss who hired them. After getting rid of the clear evidence the police

Table 2.1 Payoff table of Example 2.1

		2	
		<i>C</i>	<i>D</i>
1	<i>C</i>	(−2, −2)	(−10, −1)
	<i>D</i>	(−1, −10)	(−5, −5)

stopped them for a traffic violation and arrested them for using a stolen car. However, the police had a very strong suspicion that they robbed the jewellery store because the method they used was already known to the authorities, but there was no evidence for the serious crime, only for the minor offense of using a stolen car. In order to have evidence, the two prisoners were placed to separate cells from each other, so they could not communicate, and investigators told to each of them that his partner already admitted the robbery and encouraged him to do the same for a lighter sentence. In this situation the two criminals are the players, each of them has the choice from two alternatives: cooperate (*C*) with his partner by not confessing or defect (*D*) from his partner by confessing. So we have four possible states, (*C*, *C*), (*C*, *D*), (*D*, *C*), and (*D*, *D*) where the first (second) symbol shows the strategy of the first (second) player. The payoff values are the lengths of the prison sentences given to the two players. They are given in Table 2.1, where the first number is the payoff value of player 1 and the second number is that of player 2. The rows correspond to the strategies of player 1 and the columns to the strategies of player 2.

If both players cooperate, then they get only a light sentence because the police has no evidence for the robbery. If only one player defects, then he gets a very light sentence as the exchange for his testimony against his partner, who will receive a very harsh punishment. If both players confess, then they get stronger punishment then in the case of (*C*, *C*) but lighter than the cooperating player in the case when his partner defects.

In this situation the players can think in several different ways. They can look for a stable outcome or they can try to get as good as possible outcome under this condition.

The state (*C*, *C*) is not stable, since it is the interest of the first player to change his strategy from *C* to *D*, when his 2-year sentence would decrease to only 1 year. By this change the second player would get a very harsh 10-year sentence. The state (*C*, *D*) is not stable either, since if the first player would change his strategy to *D*, then his sentence would decrease in the expense of the second player. The state (*D*, *C*) is similar by interchanging the two players. The state (*D*, *D*) is stable in the sense that none of the players has the incentive to change strategy, that is, if any of the players changes strategy and the other player keeps his choice, then the strategy change can result in the same or worse payoff values. So the state (*D*, *D*) is the only stable state. It is usually called the *Nash equilibrium* (Nash 1950).

Definition 2.1 A Nash equilibrium gives a strategy choice for all players such that no player can increase his payoff by unilaterally changing strategy.

Another way of leading to the same solution is based on the notion of *best response*, which is the best strategy selection of each player given the strategy selection(s) of the other player(s). We can find the best response function of player 1 as follows. If player 2 selects C , then the payoff of player 1 is either -2 or -1 depending on his choice of C or D . Since -1 is more preferable than -2 , player 1 selects D in this case:

$$R_1(C) = D.$$

Similarly, if player 2 selects D , then the payoff of player 1 is either -10 or -5 , and again -5 is better with the strategy choice of D ,

$$R_1(D) = D.$$

We can see that strategy D is the best response of player 1 regardless of the strategy selection of the other player. Therefore, D is called a *dominant strategy*, so it is the players' optimal choice. Player 2 thinks in the same way, so his optimal choice is always D . So the players select the state (D, D) . ▼

Example 2.2 (Competition of gas stations) Two gas stations compete in an intersection of a city. They are the players, and for the sake of simplicity assume that they can select low (L) or high (H) selling price.

The payoff values are given in Table 2.2. If both charge high price, then they share the market and both enjoy high profit. If only one charges high price, then almost all customers select the station with low price, so its profit will be high by the high volume, while the other station will get only small profit by the very low volume. If both select low price, then they share the market with low profits. By using the same argument as in the previous example we can see that the only stable state is (L, L) , and L is dominant strategy for both players. The state (L, L) provides 20 units profit to each player. Notice that by cooperating with both selecting high price their profits would be 40 units. However, without cooperation such case cannot occur because of the usual lack of trust between the players. The same comment can be made in Example 2.1 as well, however, the illegality of price fixing also prohibits the players to cooperate. ▼

Example 2.3 (Game of privilege) Consider a house with two apartments and several common areas, such as laundry, storage, stairs, etc. The two families are supposed to take turns in cleaning and maintaining these common areas. In this situation the two families are the players, their possible strategies are participating (P) in the joint

Table 2.2 Payoff table of Example 2.2

		2	
		H	L
1	H	(40, 40)	(10, 50)
	L	(50, 10)	(20, 20)

Table 2.3 Payoff table of
Example 2.3

		2	
		<i>P</i>	<i>N</i>
1	<i>P</i>	(3, 3)	(1, 2)
	<i>N</i>	(2, 1)	(0, 0)

effort or not (*N*). The payoff table is given in Table 2.3. If both families participate, then the common areas are always nice and clean resulting in the highest payoff for both players. If only one participates, then the common areas are not as clean as in the previous case, and payoff of the participating player is even less than that of the other player because of its efforts.

If none of the players participate, then the common areas are not taken care resulting in the least payoffs. The best responses of the first player are as follows:

$$R_1(P) = P \text{ and } P_1(N) = P.$$

That is, *P* is dominant strategy. The same holds for player 2 as well, so the only Nash equilibrium is (*P*, *P*). ▼

Example 2.4 (Chicken game) Consider a very narrow street in which two teenagers stand against each other on motorbikes. For a signal they start driving toward each other. The one who gives way to the other is called the chicken. In this situation the teenagers want to show to their friends or to a gang that how determined they are. They are the two players with two possible strategies: becoming a chicken (*C*) or not (*N*). Table 2.4 shows the payoff values.

If both players are chickens, then their payoffs are higher than the payoff of a single chicken and lower than a nonchicken when the other player is a chicken. The worst possible outcome occurs with the state (*N*, *N*), when they collide and might suffer serious injuries. The best responses are the following:

$$R_k(C) = N \text{ and } R_k(N) = C (k = 1, 2).$$

Therefore, both states (*C*, *N*) and (*N*, *C*) are Nash equilibria, since in both cases the strategy choice of each player is its best response against the corresponding strategy of the other player. This result, however, does not help the players in their choices in a particular situation, since both strategies are equilibrium strategies and a choice among them requires the knowledge of the selected strategy of the other player. ▼

Table 2.4 Payoff table of
Example 2.4

		2	
		<i>C</i>	<i>N</i>
1	<i>C</i>	(3, 3)	(2, 4)
	<i>N</i>	(4, 2)	(1, 1)

Table 2.5 Payoff table of Example 2.5

		W	
		F	M
H	F	(2, 1)	(0, 0)
	M	(0, 0)	(1, 2)

Example 2.5 (Battle of sexes) A husband (H) and wife (W) want to spend an evening together. There are two possibilities, either they can go to a football game (F) or to a movie (M). The husband would prefer F , while the wife would like to go to M . They do not decide on the common choice in the morning and plan to call each other in the afternoon to finalize the evening program. However, they cannot communicate for some reason (unexpected meeting in work or power shortage), so each of them selects F or M independently of the other, travels there hoping to meet his/her spouse. The payoff values are given in Table 2.5.

If both players go to F , then they spend the evening together with positive payoff values, and since F is the preferred choice of the husband, his payoff is higher than that of his wife. The state (M, M) is similar in which case the wife gets a bit higher payoff. In the cases of (F, M) and (M, F) they cannot meet, no joint event occurs with zero payoff values. Clearly, for both players $k = H, W$,

$$R_k(F) = F \text{ and } R_k(M) = M$$

so both states (F, F) and (M, M) are equilibria. Similarly to the previous example, this solution does not give a clear choice in particular situations. ▼

Example 2.6 (Good citizens) Assume a robbery takes place in a dark alley and there are two witnesses of this crime. Both of them have a mobile phone, so they have the choice of either calling the police (C) or not (N). If at least one of them makes the call, then the criminal is arrested resulting in a positive payoff to the society including both witnesses. However, the caller will be used to testify in the trial against the criminal, which takes time and possible revenge from the criminal's partners. So the possible strategies of the witnesses are C and N , and the corresponding payoff values are given in Table 2.6.

The arrest of the criminal gives a 10 units benefit, however, making the call to the police decreases it by 3 units. If no phone call is made, then no benefit is obtained without any cost. In this case

$$R_k(C) = N \text{ and } R_k(N) = C (k = 1, 2)$$

resulting in two equilibria (C, N) and (N, C) . ▼

Table 2.6 Payoff table of Example 2.6

		2	
		C	N
1	C	(7, 7)	(7, 10)
	N	(10, 7)	(0, 0)

Table 2.7 Payoff table of
Example 2.7

		IRS	
		C	N
T	C	(-10, 9)	(0, 0)
	H	(-5, 4)	(-5, 5)

The previous examples show that equilibrium can be unique or multiple. In the following example, we will show case when no equilibrium exists.

Example 2.7 (Checking tax return) A tax payer (T) has to pay an income tax of 5,000 dollars, however, he has the option of not declaring his income and to avoid paying tax. However, in this second case he might get into trouble if IRS checks his tax return. In formulating this situation as a two-person game, player 1 is the taxpayer with two possible strategies: cheating (C) or being honest (H) with the tax return; and player 2 is the IRS who can check (C) the tax return or not (N). In determining the payoff values we notice that in the case of cheating the taxpayer has to pay his entire income tax of \$5,000 and a penalty \$5,000 as well if his tax return is checked. In checking a tax return the IRS has a cost of \$1,000. Table 2.7 shows the payoff values of the two players.

The best responses of the two players are as follows:

$$R_T(C) = H \text{ and } R_T(N) = C,$$

$$R_{IRS}(C) = C \text{ and } R_{IRS}(H) = N.$$

We can easily verify that there is no equilibrium, that is, no state is stable in the sense that in the cases of all states at least one player can increase its payoff by changing strategy.

In the case of state (C, C) player 1 has the incentive to change its strategy to H . In the case of (C, N) player 2 can increase its payoff by changing strategy to C . In the case of state (H, C) player 2 has again the incentive to change strategy to N , and finally, in the case of state (H, N) player 1 would want to change to C . ▼

Example 2.8 (Waste management) A waste management company plans to place dangerous waste on the border between two counties causing damages D_1 and D_2 units to them. In order to avoid these damages at least one county has to support intensive lobbying against the waste management company, which would cost them C_1 and C_2 units, respectively. Both counties have two possible strategies: supporting (S) the lobbying or not (N). So we have four possible states with payoff values given in Table 2.8.

If both support lobbying, then both counties face costs but there is no damage. If only one of them is supporter, then neither county faces damage but only one of them pays for lobbying. If none of them is supporter, then both face damages without any cost.

Table 2.8 Payoff table of Example 2.8

		2	
		S	N
1	S	$(-C_1, -C_2)$	$(-C_1, 0)$
	N	$(0, -C_2)$	$(-D_1, -D_2)$

We can easily check the conditions under which the different states provide equilibrium. State (S, S) is an equilibrium, if S is best response of both players against the strategy choice of S of the other player, which occurs when $-C_1 \geq 0$ and $-C_2 \geq 0$. This is impossible, so (S, S) cannot be an equilibrium. State (N, S) is an equilibrium if $0 \geq -C_1$ and $-C_2 \geq -D_2$, which can be rewritten as $C_2 \leq D_2$. State (S, N) is an equilibrium if $-C_1 \geq -D_1$ and $0 \geq -C_2$, that is, when $C_1 \leq D_1$. And finally, (N, N) is an equilibrium if $-D_1 \geq -C_1$, and $-D_2 \geq -C_2$, which can be rewritten as $D_1 \leq C_1$ and $D_2 \leq C_2$. Figure 2.1 shows these cases. Clearly, there is always an equilibrium, and it is not unique if $C_1 \leq D_1$ and $C_2 \leq D_2$. ▼

Example 2.9 (Advertisement game) Consider m markets of potential customers and assume that each of two agencies plans an intensive advertisement campaign on one of the markets. So they select a market and perform intensive advertisement there. If only one agency advertises on a market, then it will get all customers, however, if they select the same market, then they have to share the customers. So the set of strategies of both agencies is $\{1, 2, \dots, m\}$. Let $a_1 \geq a_2 \geq \dots \geq a_m$ denote the number of potential customers in the different markets. The payoff values ϕ_1 and ϕ_2 of the two agencies are given in Table 2.9, where $q_k = 1 - p_k$ for $k = 1, 2, \dots, m$. A strategy pair (i, j) is an equilibrium if strategy i is the best response of player 1 if player 2 selects strategy j , and also strategy j is best response of player 2 if player 1 chooses strategy i . That is, the $\phi_1(i, j)$ payoff value in the ϕ_1 table is the largest in its column, and $\phi_2(i, j)$ is the largest in its row in the ϕ_2 table. Notice first that in the ϕ_1 table the elements of the first row and the value at $(2, 1)$ can be the largest in their columns, so only these elements can provide equilibrium. In the ϕ_2 table only

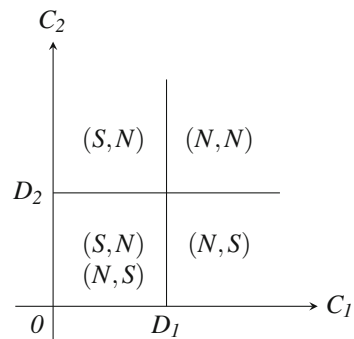
Fig. 2.1 Equilibria in Example 2.8

Table 2.9 Payoff tables of Example 2.9

1 \ 2	1	2	...	m
1	$p_1 a_1$	a_1	...	a_1
2	a_2	$p_2 a_2$	...	a_1
$\vdots$	$\vdots$	$\vdots$		$\vdots$
m	a_m	a_m	...	$p_m a_m$

ϕ_1

1 \ 2	1	2	...	m
1	$q_1 a_1$	a_2	...	a_m
2	a_1	$q_2 a_2$	...	a_m
$\vdots$	$\vdots$	$\vdots$		$\vdots$
m	a_1	a_2	...	$q_m a_m$

ϕ_2

the first column and element $\phi_2(1, 2)$ can be the largest in their rows. There are only three strategy pairs satisfying both row and column maximum conditions,

$$(2, 1), (1, 1) \text{ and } (1, 2).$$

The state $(2, 1)$ is equilibrium, if $a_2 \geq p_1 a_1$; the state $(1, 1)$ is equilibrium if $p_1 a_1 \geq a_2$ and $q_1 a_1 \geq a_2$, and similarly $(1, 2)$ is an equilibrium if $a_2 \geq q_1 a_1$. ▼

2.2 General Description of Two-Person Finite Games

Up to this point, we have introduced two-person finite games, when the players had only finitely many strategies to select from. Assume that player 1 has m strategies and player 2 has n strategies. Then the strategy sets are $S_1 = \{1, 2, \dots, m\}$ and $S_2 = \{1, 2, \dots, n\}$ for the two players. As we did it in the examples, the payoff values can be shown in the payoff tables, the general forms of which are given in Table 2.10.

A strategy pair (or state) (i, j) is an equilibrium, if the element a_{ij} is the largest in its column in the ϕ_1 table, and the element b_{ij} is the largest in its row in the ϕ_2

Table 2.10 Payoff tables of two-person finite games

1 \ 2	1	2	...	n
1	a_{11}	a_{12}	...	a_{1n}
2	a_{21}	a_{22}	...	a_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
m	a_{m1}	a_{m2}	...	a_{mn}

ϕ_1

1 \ 2	1	2	...	n
1	b_{11}	b_{12}	...	b_{1n}
2	b_{21}	b_{22}	...	b_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
m	b_{m1}	b_{m2}	...	b_{mn}

ϕ_2

table. As it was illustrated in the previous examples, there is no guarantee for the existence of an equilibrium, and even if it exists the uniqueness of the equilibrium is not guaranteed either.

A two-person game is called *zero-sum* if $\phi_1(i, j) + \phi_2(i, j) = 0$ with all strategy pairs (i, j) . That is, the gain of a player is the loss of the other. In this case $b_{ij} = -a_{ij}$, so there is no need to give the table for ϕ_2 , since its elements are the negatives of the corresponding elements of ϕ_1 . A strategy pair (i, j) is an equilibrium, if a_{ij} is the largest among the elements $a_{1j}, a_{2j}, \dots, a_{mj}$ and $-a_{ij}$ is the largest among the elements $-a_{i1}, -a_{i2}, \dots, -a_{in}$. The second condition can be rewritten as a_{ij} is the smallest among the numbers $a_{i1}, a_{i2}, \dots, a_{in}$. That is, a_{ij} is the largest in its column and also the smallest in its row. The equilibria of zero-sum games are often called the *saddle points* (think of a person sitting on a horse who is observed from the side and from the back of the horse). In general, zero-sum, two-person games do not necessarily have equilibrium, and if equilibrium exists, it is not necessarily unique. However, we can easily show that in the case of multiple equilibria the strategies are different but the corresponding payoff values are identical.

Lemma 2.1 *Let (i, j) and (k, l) be two equilibria of a two-person zero-sum game. Then $\phi_1(i, j) = \phi_1(k, l)$.*

Proof Let $\phi_1(i, j) = a_{ij}$ and $\phi_1(k, l) = a_{kl}$. Then

$$a_{ij} \geq a_{kj} \geq a_{kl}$$

since a_{ij} is the largest in its column and a_{kl} is the smallest in its row. Similarly,

$$a_{ij} \leq a_{il} \leq a_{kl}$$

since a_{ij} is the smallest in its row and a_{kl} is the largest in its column. These relations imply that $a_{ij} = a_{kl}$. ■

It is an interesting problem to find out the proportion of two-person zero-sum finite games which have at least one equilibrium. As the following theorem shows this ratio is getting smaller by increasing the size of the payoff table. Consider a two-person, zero-sum game in which the players have m and n strategies, respectively. Assume that the payoff values a_{ij} are independent, identically distributed random variables with a continuous cumulative distribution function. Then the following fact can be proved (Goldberg et al. 1968).

Theorem 2.1 *Under the above conditions the probability that the game has an equilibrium is*

$$P_{m,n} = \frac{m!n!}{(m+n-1)!}.$$

Proof Notice first that

- (i) the elements of the payoff table are different with probability one;
- (ii) all elements a_{ij} have the same probability to be equilibrium;
- (iii) the probability that there is an equilibrium is mn times the probability that a_{11} is equilibrium.

Fact (i) follows from the assumption that the distribution function is continuous and the table elements are independent, (ii) is implied by the assumption that the table elements are identically distributed. From (i), the probability that multiple equilibria exists is zero.

The element a_{11} is equilibrium if it is the largest in its column and the smallest in its row. So if we list the elements of the first row and column in increasing order, then all other elements of the first column should be before a_{11} and all other elements of the first row have to be after a_{11} . The $(m - 1)$ other elements of the first column can be permuted in $(m - 1)!$ different ways and the $(n - 1)$ other elements of the first row can be permuted in $(n - 1)!$ different ways, therefore there are $(m - 1)!(n - 1)!$ possible permutations in which a_{11} is in the equilibrium position. Since the $m + n - 1$ elements of the first row and column have altogether $(m + n - 1)!$ permutations, the probability that the element a_{11} is an equilibrium equals

$$\frac{(m - 1)!(n - 1)!}{(m + n - 1)!}.$$

Hence the probability that equilibrium exists is

$$mn \frac{(m - 1)!(n - 1)!}{(m + n - 1)!} = \frac{m!n!}{(m + n - 1)!}.$$

■

Notice that the value of P_{mn} does not depend on the distribution type of the elements, it depends on only the size of the payoff table.

In order to gain a feeling about this value let us consider some special cases and relations:

$$\begin{aligned} P_{2,2} &= \frac{2!2!}{3!} = \frac{2}{3} \\ P_{2,3} &= \frac{2!3!}{4!} = \frac{1}{2} \\ P_{2,4} &= \frac{2!4!}{5!} = \frac{2}{5}, \end{aligned}$$

from which we see that the probability value decreases if the size of the table becomes larger. This is true in general, since

$$\frac{P_{m,n+1}}{P_{m,n}} = \frac{m!(n+1)!}{(m+n)!} \cdot \frac{(m+n-1)!}{m!n!} = \frac{n+1}{m+n} < 1.$$

The same result is obtained if m increases, since $P_{m,n} = P_{n,m}$. Notice that

$$P_{2n} = \frac{2!n!}{(2+n-1)!} = \frac{2}{n+1} \rightarrow 0$$

as $n \rightarrow \infty$, therefore with any $m \geq 2$,

$$P_{mn} \leq P_{2n} \rightarrow 0$$

as $n \rightarrow \infty$. Therefore, P_{mn} is decreasing in m and n , furthermore it converges to zero if either m or n tends to infinity.

We can also show that the theorem does not hold necessarily if the distribution of table elements is discrete.

Example 2.10 Consider therefore the case of $m = n = 2$ and assume that the four elements of the table are randomly generated from a Bernoulli distribution such that

$$P(a_{ij} = 1) = p \text{ and } P(a_{ij} = 0) = q = 1 - p.$$

Since each element can take two possible values 0 and 1 furthermore there are four elements, we have $16(=2^4)$ possible payoff tables:

$$\begin{aligned} & \begin{pmatrix} \textcircled{0} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \textcircled{0} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \textcircled{0} & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & \textcircled{0} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \textcircled{0} & 1 \end{pmatrix} \\ & \begin{pmatrix} \textcircled{1} & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \textcircled{0} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \textcircled{0} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & \textcircled{1} \end{pmatrix} \\ & \begin{pmatrix} 1 & \textcircled{1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & \textcircled{1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \textcircled{1} & 1 \end{pmatrix} \begin{pmatrix} \textcircled{1} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \textcircled{1} \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

There are only two of them without an equilibrium. In the other tables an equilibrium element is circled. The probability of each table

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

equals p^2q^2 , so the probability that no equilibrium exists is $2p^2q^2$, and the probability that there is at least one equilibrium is $1 - 2p^2q^2$, which is not necessarily equal to $\frac{2}{3}$. ▼

As an example of two-person zero-sum finite games consider the following situation:

Fig. 2.2 Structure of the city

a_{11}	a_{12}	a_{13}		$a_{1,n-2}$	$a_{1,n-1}$	a_{1n}
a_{21}	a_{22}	a_{23}		$a_{2,n-2}$	$a_{2,n-1}$	a_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
a_{m1}	a_{m2}	a_{m3}		$a_{m,n-2}$	$a_{m,n-1}$	a_{mn}

Example 2.11 (Antiterrorism game) A rectangular-shaped city is divided into m block-rows and n block-columns by $E - W$ and $S - N$ streets as shown in Fig. 2.2. So there are mn blocks, and their values are listed in the figure.

Assume now that a terrorist group placed a bomb in one of the city blocks and demands a large amount of money as well as the release of prisoners from jail. The city administration clearly does not want to negotiate, they try to find the bomb and avoid damages. However, they have sufficient resources to check only one complete block-row or a complete block-column, so if the bomb is placed there, then it is certainly found. In this situation, the city and the terrorist group are the two players. The city can choose the block-row or the block-column which will be checked, the terrorists can select any block of the city. The payoff of the city is positive when they can find the bomb and save that block. The corresponding payoff of the terrorist group is the negative of that of the city, since they lose the damage opportunity. Table 2.11 shows the payoff table of the city.

If the city checks block-row i , then the bomb is found if it is placed in one of the blocks $(i, 1), (i, 2), \dots, (i, n)$ and if the city checks block-column j , then the bomb is found if it is in one of the blocks $(1, j), (2, j), \dots, (m, j)$. This is a zero-sum game, so an element of the table provides equilibrium if it is the largest in its column and the smallest in its row. Every column has positive element, so the largest element is always positive. Every row has zero elements, so the smallest element is always zero. Therefore, there is no element in the table which satisfies both conditions of an equilibrium. Consequently, the game has no equilibrium. ▼

Table 2.11 Payoff table of Example 2.11

1 \ 2	(1, 1) (1, 2) ... (1, n)				(2, 1) (2, 2) ... (2, n)				...	(m, 1) (m, 2) ... (m, n)			
	a_{11}	a_{12}	...	a_{1n}	a_{21}	a_{22}	...	a_{2n}		a_{m1}	a_{m2}	...	a_{mn}
1	a_{11}	a_{12}	...	a_{1n}	a_{21}	a_{22}	...	a_{2n}		a_{m1}	a_{m2}	...	a_{mn}
2													
$\vdots$													
m													
1	a_{11}				a_{21}					a_{m1}			
2		a_{12}				a_{22}					a_{m2}		
$\vdots$			$\ddots$				$\ddots$					$\ddots$	
n				a_{1n}				a_{2n}					a_{mn}

2.3 N -person Finite Games

Let N denote the number of players and assume that the players have finitely many strategies to select from. Assume that player k ($1 \leq k \leq N$) has m_k strategies which can be denoted by $1, 2, \dots, m_k$. So the set of strategies of player k is the finite set $S_k = \{1, 2, \dots, m_k\}$. If player 1 selects strategy i_1 , player 2 selects i_2 , and so on, player N selects i_N , then the N -tuple $\underline{s} = (i_1, i_2, \dots, i_N)$ is called a *simultaneous strategy* of the players. So $\underline{s} \in S_1 \times S_2 \times \dots \times S_N$, and the payoff function of each player k is a real valued function defined on $S = S_1 \times S_2 \times \dots \times S_N$ which can be denoted by $\phi_k(\underline{s})$. Similarly to the two-player case, a simultaneous strategy $\underline{s}^* = (i_1^*, i_2^*, \dots, i_N^*) \in S$ is an equilibrium, if i_k^* is the best response of all players k with given strategies $i_1^*, \dots, i_{k-1}^*, i_{k+1}^*, \dots, i_N^*$ of the other players.

Example 2.12 (Voting game) Consider a city with two candidates for an office, like to become the mayor. Let A and B denote the candidates. The potential voters are divided between the candidates. If N denotes the number of voter eligible individuals, then we can define an N -person game in the following way. The potential voters are the players. Each of them has two possible strategies voting or not. In defining the payoff functions two factors have to be taken into consideration. For any voter the benefit is 1 if his/her candidate is the winner, 0 in the case of a tie, and -1 if the other candidate wins. However, voting has some cost (time, car usage, etc.), which is assumed to be less than unity. In finding conditions for the existence of an equilibrium we have to consider the following simple facts:

- (i) There is no equilibrium when a candidate wins.
If at least one player votes in the losing group, then by not voting he/she would increase payoff by eliminating voting cost. If nobody votes in the losing group, then we have two subcases. If more than one person votes in the winning group, then one of them could change strategy to not voting and would increase payoff. If only one person is in the winning group, then any person in the losing group could make the election result a tie by going to vote, and in this way increase payoff.
- (ii) So the election result has to be a tie in any equilibrium, and everybody has to vote.
Assume that there is a person who does not vote. By going to vote he/she could make his/her group winner and so the payoff would increase.

In summary, the only possibility for an equilibrium is if N is even, equal number of people support the two candidates and everybody votes. This is really an equilibrium, since if any player changes strategy by not voting, then his/her group becomes the losing group and the payoffs decrease for its members. ▼

If at least one player has infinitely many strategies, then the payoff matrices become infinite. Nash equilibria are defined in the same way as in finite games, however the existence of best responses is not guaranteed in general.

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