

Chapter 2

Analysis of the Kerov–Olshanski Algebra

Abstract In this chapter, we investigate the algebra of polynomial functions in coordinates of Young diagrams as a nice framework in which various quantities on Young diagrams can be efficiently computed. This algebra was introduced by Kerov–Olshanski [20], analysis of which is substantially due to Ivanov–Olshanski [16]. Several systems of generators and associated generating functions are considered. It is important to understand the concrete transition rules between these generating systems, one of which is the Kerov polynomial.

2.1 Coordinates of a Young Diagram

In this section, we consider two kinds of coordinates encoding a Young diagram: the Frobenius coordinates and the min-max coordinates.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) \in \mathbb{Y}$ be a Young diagram having d boxes along the main diagonal. We call

$$a_i = a_i(\lambda) = \lambda_i - i + \frac{1}{2}, \quad b_i = b_i(\lambda) = \lambda'_i - i + \frac{1}{2}, \quad i \in \{1, 2, \dots, d\}$$

the Frobenius coordinates of λ and write as $\lambda = (a_1, \dots, a_d \mid b_1, \dots, b_d)$. The Frobenius coordinates of $\lambda \in \mathbb{Y}$ satisfy

$$-b_1 < -b_2 < \dots < -b_d < 0 < a_d < \dots < a_2 < a_1, \quad |\lambda| = \sum_{i=1}^d (a_i + b_i).$$

Let us display a Young diagram in the upper half of the xy -plane as in Fig. 2.1, where $\lambda = (4, 2, 2, 1)$ of the French style in Fig. 1.1 is rotated by 45° and put in such a way that the main diagonal boxes lie along the y -axis. The piecewise linear border indicated by bold lines in Fig. 2.1 is called the profile of a Young diagram. Since it is preferable that the corners of any profile have integral xy -coordinates, we always assume that the edge length of each box is $\sqrt{2}$ in the display as in Fig. 2.1.

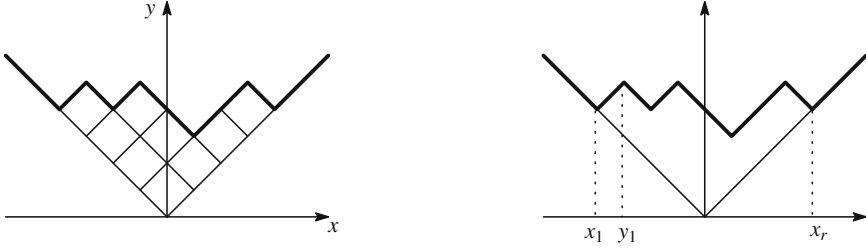


Fig. 2.1 (left) profile of $\lambda = (4, 2, 2, 1)$; (right) its min-max coordinates

For $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) \in \mathbb{Y}$, the subset of $\mathbb{Z} + \frac{1}{2}$ defined by $M(\lambda) = \{\lambda_i - i + \frac{1}{2}\}_{i \in \mathbb{N}}$ is called the Maya diagram of λ . It is easy to see

$$\begin{aligned} \{a_1, \dots, a_d\} &= M(\lambda) \cap (\mathbb{N} - \tfrac{1}{2}), & \{-b_1, \dots, -b_d\} &= (-M(\lambda')) \cap (-\mathbb{N} + \tfrac{1}{2}), \\ M(\lambda) \sqcup (-M(\lambda')) &= \mathbb{Z} + \tfrac{1}{2} \end{aligned}$$

for $\lambda = (a_1, \dots, a_d \mid b_1, \dots, b_d) \in \mathbb{Y}$. The set $\{b : \text{box} \mid b \in \lambda\}$ is bijective to $\{(s, t) \in M(\lambda) \times (-M(\lambda')) \mid s > t\}$. We have $h_\lambda(b) = s - t$ as the hook length under this bijective correspondence $b \leftrightarrow (s, t)$ and hence

$$\log \prod_{b \in \lambda} h_\lambda(b) = \sum_{(s, t) \in M(\lambda) \times (-M(\lambda')) : s > t} \log(s - t). \quad (2.1)$$

Through the hook formula (Proposition 1.1) and (2.1), maximizing $\dim \lambda$ in $\mathbb{Y}_n$ is equivalent to minimizing the RHS of (2.1).

Given $\lambda = (a_1, \dots, a_d \mid b_1, \dots, b_d) \in \mathbb{Y}$, we consider a polynomial of degree k in the Frobenius coordinates:

$$p_k(\lambda) = \sum_{i=1}^d (a_i^k + (-1)^{k-1} b_i^k), \quad k \in \mathbb{N}, \quad (2.2)$$

and a rational function with a_i and $-b_i$ as its pole and zero respectively:

$$\Phi(z; \lambda) = \prod_{i=1}^d \frac{z + b_i}{z - a_i}, \quad z \in \mathbb{C}. \quad (2.3)$$

We may set $\Phi(z; \emptyset) = 1$ though we do not consider the Frobenius coordinates of the empty diagram $\emptyset$. In a sufficiently large annulus $1 \ll |z| < \infty$, the Laurent expansion of Φ gives

$$\Phi(z; \lambda) = \prod_{i=1}^d \frac{1 + (b_i/z)}{1 - (a_i/z)} = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(\lambda)}{k} z^{-k}\right). \quad (2.4)$$

The x -coordinates of the interlacing valleys (=local minima) and peaks (=local maxima) of the profile of $\lambda \in \mathbb{Y}$ yields an integer sequence

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{r-1} < y_{r-1} < x_r, \quad r \in \mathbb{N}, \quad (2.5)$$

which is called the min-max coordinates of λ . Clearly, the last x_r is determined from $x_1, \dots, y_{r-1}$. It is not difficult to see the following characterization.

Lemma 2.1 *An interlacing real sequence of (2.5) forms the min-max coordinates of some $\lambda \in \mathbb{Y}$ if and only if*

$$\sum_{i=1}^r x_i - \sum_{i=1}^{r-1} y_i = 0 \quad \text{and} \quad x_1, \dots, x_r, y_1, \dots, y_{r-1} \in \mathbb{Z}.$$

We consider a rational function with min coordinate x_i and max coordinate y_i as its pole and zero respectively:

$$G(z; \lambda) = \frac{(z - y_1) \cdots (z - y_{r-1})}{(z - x_1) \cdots (z - x_r)}, \quad z \in \mathbb{C}. \quad (2.6)$$

In particular, $G(z; \emptyset) = 1/z$ for the empty diagram.

Transposing λ to λ' in (2.3) and (2.6), we readily have

$$\Phi(z; \lambda') = \Phi(-z; \lambda)^{-1}, \quad G(z; \lambda') = -G(-z; \lambda), \quad \lambda \in \mathbb{Y}, \quad z \in \mathbb{C}.$$

Proposition 2.1 *The rational functions Φ of (2.3) and G of (2.6) for*

$$\lambda = (a_1, \dots, a_d \mid b_1, \dots, b_d) = (x_1 < y_1 < \cdots < y_{r-1} < x_r) \in \mathbb{Y}$$

are connected as

$$\frac{\Phi(z - \frac{1}{2}; \lambda)}{\Phi(z + \frac{1}{2}; \lambda)} = z G(z; \lambda), \quad z \in \mathbb{C}. \quad (2.7)$$

Proof When we rewrite $\Phi(z; \lambda)$, which is expressed by the Frobenius coordinates of λ , in terms of the min-max coordinates, we have only to be careful about how the profile of λ traverses the y -axis. Consider the situations case by case.

2.2 Transition Measure I

In this section, we translate encoding of a Young diagram by its coordinates into two atomic measures on $\mathbb{R}$; one called Kerov’s transition measure and the other the Rayleigh measure. Such embedding into the space of measures enables us to develop asymptotic theory in a flexible framework.

We begin with a bit wider class than Young diagrams. A function $\lambda : \mathbb{R} \rightarrow \mathbb{R}$, or the graph $y = \lambda(x)$, satisfying the following conditions is called a (centered) rectangular diagram:

- (i) continuous and piecewise linear
- (ii) $\lambda'(x) = \pm 1$ except finite x ’s
- (iii) $\lambda(x) = |x|$ for $|x|$ large enough.

The set of rectangular diagrams is denoted by $\mathbb{D}_0$. A rectangular diagram is (the profile of) a Young diagram if and only if the exceptional x ’s in (ii) are all integers. This yields the natural inclusion $\mathbb{Y} \subset \mathbb{D}_0$. The definitions of the min-max coordinates and the rational function G , (2.5) and (2.6) respectively, are immediately extended from $\mathbb{Y}$ to $\mathbb{D}_0$.

Lemma 2.2 *An interlacing real sequence of (2.5) forms the min-max coordinates of some $\lambda \in \mathbb{D}_0$ if and only if*

$$\sum_{i=1}^r x_i - \sum_{i=1}^{r-1} y_i = 0.$$

To $\lambda = (x_1 < y_1 < \cdots < y_{r-1} < x_r) \in \mathbb{D}_0$ we assign an $\mathbb{R}$ -valued (probability) measure on $\mathbb{R}$ as

$$\tau_\lambda = \sum_{i=1}^r \delta_{x_i} - \sum_{i=1}^{r-1} \delta_{y_i} \quad (2.8)$$

and call it the Rayleigh measure of $\lambda \in \mathbb{D}_0$. Clearly, $\lambda \mapsto \tau_\lambda$ is injective. Under derivatives of Schwartz’ distributions we have

$$\tau_\lambda = \left(\frac{\lambda(x) - |x|}{2} \right)'' + \delta_0. \quad (2.9)$$

Let us use the notation of the k th moment $M_k(\cdot)$ for an $\mathbb{R}$ -valued measure on $\mathbb{R}$ also. Then (2.8) and (2.9) yield

$$M_k(\tau_\lambda) = \sum_{i=1}^r x_i^k - \sum_{i=1}^{r-1} y_i^k = \int_{\mathbb{R}} x^k \left(\frac{\lambda(x) - |x|}{2} \right)'' dx + \delta_{0k}, \quad k \in \mathbb{N} \cup \{0\}. \quad (2.10)$$

In particular, we have through integration by parts

$$M_2(\tau_\lambda) = \int_{\mathbb{R}} (\lambda(x) - |x|) dx = 2|\lambda|. \quad (2.11)$$

Lemma 2.3 *We can reconstruct $\lambda \in \mathbb{D}_0$ from its Rayleigh measure τ_λ by*

$$\lambda(u) = \int_{\mathbb{R}} |u - x| \tau_\lambda(dx), \quad u \in \mathbb{R}.$$

Proof We use (2.9), but note that $|u - x|$ is not differentiable. Take $a > 0$ such that $\text{supp}(\lambda(x) - |x|) \subset (-a, a)$. The function $(\lambda(x) - |x|)'$ is of bounded variation and $(\lambda(x) - |x|)''$ is an $\mathbb{R}$ -valued measure, both supported in $(-a, a)$. For $u \in (-a, a)$

$$\begin{aligned} & \int_{(-a, a)} |u - x| \left(\frac{\lambda(x) - |x|}{2} \right)'' dx \\ &= \int_{(-a, u)} (u - x) \left(\frac{\lambda(x) - |x|}{2} \right)'' dx + \int_{(u, a)} (x - u) \left(\frac{\lambda(x) - |x|}{2} \right)'' dx. \end{aligned} \quad (2.12)$$

The first term of the RHS of (2.12) is

$$\begin{aligned} \int_{(-a, u)} \left(\int_x^u dy \right) \left(\frac{\lambda(x) - |x|}{2} \right)'' dx &= \int_{(-a, u)} \left(\int_{(-a, y)} \left(\frac{\lambda(x) - |x|}{2} \right)'' dx \right) dy \\ &= \int_{(-a, u)} \left(\frac{\lambda(y) - |y|}{2} \right)' dy = \frac{\lambda(u) - |u|}{2}, \end{aligned}$$

and so is the second term. We thus have (2.12) to be $\lambda(u) - |u|$. The cases of $u \geq a$ and $u \leq -a$ are easier to see. Combine this with $\int_{-\infty}^{\infty} |u - x| \delta_0(dx) = |u|$.

In order to define the transition measure of a rectangular diagram, we consider the partial fraction expansion of (2.6) for $\lambda = (x_1 < y_1 < \dots < y_{r-1} < x_r) \in \mathbb{D}_0$:

$$G(z; \lambda) = \frac{(z - y_1) \cdots (z - y_{r-1})}{(z - x_1) \cdots (z - x_r)} = \frac{\mu_1}{z - x_1} + \dots + \frac{\mu_r}{z - x_r}, \quad (2.13)$$

$$\mu_i = \frac{(x_i - y_1) \cdots (x_i - y_{r-1})}{(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_r)}, \quad i \in \{1, \dots, r\}. \quad (2.14)$$

The interlacing property (2.5) assures $\mu_i > 0$ in (2.14). Multiplying (2.13) by z and letting $z \rightarrow \infty$ yield $\sum_{i=1}^r \mu_i = 1$. We thus have an atomic probability on $\mathbb{R}$

$$\mathfrak{m}_\lambda = \sum_{i=1}^r \mu_i \delta_{x_i} \in \mathcal{P}(\mathbb{R}), \quad \text{supp } \mathfrak{m}_\lambda = \{x_1, \dots, x_r\} \quad (2.15)$$

called (Kerov's) transition measure of $\lambda \in \mathbb{D}_0$. Note that (2.13) is the Stieltjes transform of $\mathfrak{m}_\lambda$:

$$G_{\mathfrak{m}_\lambda}(z) = \int_{\mathbb{R}} \frac{1}{z - x} \mathfrak{m}_\lambda(dx) = G(z; \lambda), \quad z \in \mathbb{C}. \quad (2.16)$$

Proposition 2.2 *Given $\lambda \in \mathbb{D}_0$, the two moment sequences $\{M_n(\mathfrak{m}_\lambda)\}_{n \in \mathbb{N}}$ and $\{M_k(\tau_\lambda)\}_{k \in \mathbb{N}}$ are connected to each other by*

$$\sum_{n=0}^{\infty} M_n(\mathfrak{m}_\lambda) z^{-n} = \exp\left(\sum_{k=1}^{\infty} \frac{M_k(\tau_\lambda)}{k} z^{-k}\right). \quad (2.17)$$

Hence $\{M_n(\mathfrak{m}_\lambda)\}$ and $\{M_k(\tau_\lambda)\}$ are expressed by polynomials in each other.

Proof Setting $\mu = \sum_{i=1}^r \mu_i \delta_{x_i}$ in (2.13) for interlacing $x_1 < y_1 < \cdots < y_{r-1} < x_r$ and μ_i of (2.14), we get for $|z| \gg 1$

$$\begin{aligned} \sum_{n=0}^{\infty} M_n(\mu) z^{-n} &= z G_\mu(z) = \frac{z(z - y_1) \cdots (z - y_{r-1})}{(z - x_1) \cdots (z - x_r)} \\ &= \exp\left\{\sum_{i=1}^{r-1} \log\left(1 - \frac{y_i}{z}\right) - \sum_{i=1}^r \log\left(1 - \frac{x_i}{z}\right)\right\} = \exp\left\{\sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{i=1}^r x_i^k - \sum_{i=1}^{r-1} y_i^k\right) z^{-k}\right\}. \end{aligned} \quad (2.18)$$

Specialization to the min-max coordinates of $\lambda \in \mathbb{D}_0$ yields (2.17).

As the terms of z^{-1} and z^{-2} in (2.17), we have

$$M_1(\mathfrak{m}_\lambda) = M_1(\tau_\lambda) = 0, \quad M_2(\mathfrak{m}_\lambda) = \frac{1}{2} M_2(\tau_\lambda). \quad (2.19)$$

Proposition 2.3 *The map $\lambda \mapsto \mathfrak{m}_\lambda$ gives a bijection of $\mathbb{D}_0$ to the set of probabilities on $\mathbb{R}$ with mean 0 and finite supports.*

Proof Since the injectivity is immediate from (2.13), we verify the surjectivity. Take any

$$\mu = \sum_{i=1}^r \mu_i \delta_{x_i}, \quad x_1 < \cdots < x_r, \quad \mu_i > 0, \quad \sum_{i=1}^r \mu_i = 1, \quad \sum_{i=1}^r x_i \mu_i = 0.$$

Determine a monic real polynomial $f(z)$ of degree $r - 1$ by

$$\frac{\mu_1}{z - x_1} + \cdots + \frac{\mu_r}{z - x_r} = \frac{f(z)}{(z - x_1) \cdots (z - x_r)}.$$

Since $f(x_1), f(x_2), \dots, f(x_r)$ have alternating sign changes, f has $r - 1$ zeros y_i satisfying $x_1 < y_1 < x_2 < \cdots < x_{r-1} < y_{r-1} < x_r$. We hence have the same equality as (2.13) and then (2.18), in particular $M_1(\mu) = \sum_{i=1}^r x_i - \sum_{i=1}^{r-1} y_i$ as the coefficient of z^{-1} . Lemma 2.2 assures the existence of $\lambda \in \mathbb{D}_0$ such that $\mathfrak{m}_\lambda = \mu$.

While the Rayleigh measure τ_λ reflects the shape of $\lambda \in \mathbb{Y}_n$ more or less directly, the transition measure m_λ gives us information about the irreducible representation of $\mathfrak{S}_n$ labeled by λ . Let us see a few instances.

The Plancherel measure M_{Pl} on the path space $\mathfrak{T}$ defined by (1.17) induces a Markov chain on $\mathbb{Y}$. In fact, assuming $\lambda^0 = \emptyset \nearrow \lambda^1 \nearrow \dots \nearrow \lambda^{n-1} \nearrow \lambda (\in \mathbb{Y}_n)$ forms a path in $\mathfrak{T}_n$, we have the conditional probability

$$\begin{aligned} M_{\text{Pl}}(t(n+1) = \mu \mid t(0) = \lambda^0, \dots, t(n) = \lambda) \\ = \begin{cases} \frac{M_{\text{Pl}}(C_\lambda^0 \nearrow \dots \nearrow \lambda \nearrow \mu)}{M_{\text{Pl}}(C_\lambda^0 \nearrow \dots \nearrow \lambda)} = \frac{\dim \mu}{(n+1) \dim \lambda}, & \lambda \nearrow \mu, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This chain is often called the Plancherel growth process. Let $(x_1 < y_1 < x_2 < \dots < y_{r-1} < x_r)$ be the min-max coordinates of $\lambda \in \mathbb{Y}_n$ and $\mu^{(i)} \in \mathbb{Y}_{n+1}$ denote the Young diagram obtained by putting a box at the i th valley (of the x -coordinate x_i) of λ . The following fact gives a good reason for m_λ to be called the transition measure.

Lemma 2.4 *Under the above notations,*

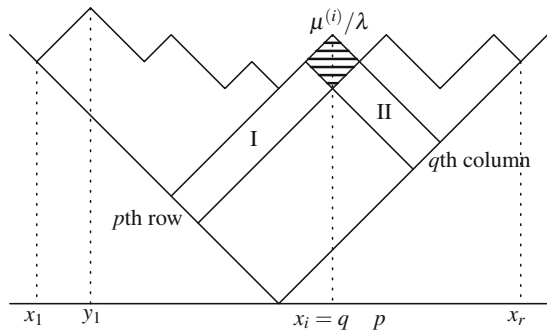
$$m_\lambda(\{x_i\}) = \frac{\dim \mu^{(i)}}{(n+1) \dim \lambda}, \quad i \in \{1, \dots, r\}. \quad (2.20)$$

Proof The hook formula (Proposition 1.1) implies that the RHS of (2.20) is

$$\prod_{b \in \lambda} h_\lambda(b) / \prod_{b \in \mu^{(i)}} h_{\mu^{(i)}}(b).$$

When we rewrite this quantity in terms of the min-max coordinates, we have only to focus on the boxes lying in zone I and zone II in Fig. 2.2, where $\mu^{(i)}/\lambda$ is the (p, q) box in $\mu^{(i)}$. The hook length at $(p, 1)$ box in zone I is $h_{\mu^{(i)}}(p, 1) = x_i - x_1$, and so on. Successive cancellations yield (2.14) and hence $m_\lambda(\{x_i\})$.

Fig. 2.2 Min-max coordinates and hook length ratio



Theorem 1.2 tells the irreducible character value at a cycle, where (1.8) is expressed in terms of row lengths of a Young diagram. We now rewrite this formula by using the Frobenius coordinates and the min-max coordinates, and connect it with the transition measure. In order to regard the irreducible character values at a cycle as a function on $\mathbb{Y}$, set

$$\Sigma_k(\lambda) = \begin{cases} |\lambda|^{\downarrow k} \tilde{\chi}_{(k, 1^{|\lambda|-k})}^\lambda, & |\lambda| \geq k, \\ 0, & |\lambda| < k \end{cases} \quad (2.21)$$

for $k \in \mathbb{N}$ and $\lambda \in \mathbb{Y}$. In particular, $\Sigma_1(\lambda) = |\lambda|$.

Theorem 2.1 For $k \in \mathbb{N}$ and $\lambda \in \mathbb{Y}$,

$$\Sigma_k(\lambda) = -\frac{1}{k} [z^{-1}] \left\{ z^{\downarrow k} \frac{\Phi(z + \frac{1}{2}; \lambda)}{\Phi(z - k + \frac{1}{2}; \lambda)} \right\} \quad (2.22)$$

$$= -\frac{1}{k} [z^{-1}] \left\{ \frac{1}{G_{m_\lambda}(z) G_{m_\lambda}(z-1) \cdots G_{m_\lambda}(z-k+1)} \right\}. \quad (2.23)$$

Proof First we verify that the RHS of (2.22) is 0 if $|\lambda| < k$. In terms of the Frobenius coordinates $\lambda = (a_1, \dots, a_d \mid b_1, \dots, b_d)$,

$$z^{\downarrow k} \frac{\Phi(z + \frac{1}{2}; \lambda)}{\Phi(z - k + \frac{1}{2}; \lambda)} = z^{\downarrow k} \prod_{i=1}^d \frac{(z + \frac{1}{2} + b_i)(z - k + \frac{1}{2} - a_i)}{(z + \frac{1}{2} - a_i)(z - k + \frac{1}{2} + b_i)}. \quad (2.24)$$

The poles of (2.24) are all integers and satisfy

$$0 \leq a_d - \frac{1}{2} < \cdots < a_1 - \frac{1}{2} < -b_1 + k - \frac{1}{2} < \cdots < -b_d + k - \frac{1}{2} \leq k - 1$$

since $a_1 + b_1 \leq |\lambda| < k$. Multiplied by $z^{\downarrow k}$, the denominator is then canceled. Hence (2.24) proves to be a polynomial in z .

Let us assume $|\lambda| \geq k$. We show (2.22). By (1.8) and (2.21),

$$\Sigma_k(\lambda) = -\frac{1}{k} [z^{-1}] \left\{ z^{\downarrow k} \prod_{i=1}^n \frac{z - k - (\lambda_i + n - i)}{z - (\lambda_i + n - i)} \right\} \quad (2.25)$$

where $\lambda = (\lambda_i \geq \lambda_2 \geq \cdots)$ and $n = |\lambda|$. We note the equality

$$\Phi(z; \lambda) = \prod_{i=1}^{\infty} \frac{z - (-i + \frac{1}{2})}{z - (\lambda_i - i + \frac{1}{2})}. \quad (2.26)$$

In fact, multiplying both the numerator and the denominator in the RHS of (2.3) by $\prod_{c \in M(\lambda) \cap (-\mathbb{N} + \frac{1}{2})} (z - c)$, we get $\prod_{i=1}^{\infty} z - (-i + \frac{1}{2})$ as the new denominator. Now that $\lambda_{n+1} = 0$, (2.26) yields

$$\begin{aligned}\Phi(z - n + \tfrac{1}{2}; \lambda) &= \prod_{i=1}^n \frac{z - n + i}{z - n - \lambda_i + i}, \\ \Phi(z - n - k + \tfrac{1}{2}; \lambda) &= \prod_{i=1}^n \frac{z - n - k + i}{z - n - k - \lambda_i + i},\end{aligned}$$

and hence

$$\frac{\Phi(z - n + \tfrac{1}{2}; \lambda)}{\Phi(z - n - k + \tfrac{1}{2}; \lambda)} = \prod_{i=1}^n \frac{z - n + i}{z - n - k + i} \prod_{i=1}^n \frac{z - k - \lambda_i - n + i}{z - \lambda_i - n + i}.$$

Noting that the first product of the RHS is $z^{\downarrow k} / (z - n)^{\downarrow k}$, we have from (2.25)

$$\Sigma_k(\lambda) = -\frac{1}{k} [z^{-1}] \left\{ (z - n)^{\downarrow k} \frac{\Phi(z - n + \tfrac{1}{2}; \lambda)}{\Phi(z - n - k + \tfrac{1}{2}; \lambda)} \right\}. \quad (2.27)$$

For given λ and k , we can take a sufficiently large annulus $1 \ll |z| < \infty$ in which changing the contours $C \leftrightarrow C - n$ in the integral expressions is valid. Therefore, (2.22) follows from (2.27).

Finally, we verify the equality in (2.23). However, that is immediate from (2.7) and (2.16).

2.3 The Kerov–Olshanski Algebra

In this section, we focus on the algebra of polynomial functions in the coordinates of Young diagrams. Analysis of its structure in particular yields the Kerov polynomial and an asymptotic formula for irreducible characters of the symmetric groups.

We know two kinds of polynomials of ‘degree’ k as functions on $\mathbb{Y}$; one being $p_k(\lambda)$ of (2.2) in the Frobenius coordinates and the other $M_k(\tau_\lambda)$ of (2.10) in the min-max coordinates. Their generating functions of exponential type appear in (2.4) and (2.16)–(2.17) respectively. Since they are connected as (2.7), we can get the following relation between $\{p_k(\lambda)\}$ and $\{M_k(\tau_\lambda)\}$.

Proposition 2.4 *There exists an infinite matrix A satisfying*

- *A is upper-triangular*
- *All entries of A are nonnegative and rational*
- *All diagonal entries of A are equal to 1*

and

$$[M_2(\tau_\lambda) M_3(\tau_\lambda) M_4(\tau_\lambda) \cdots] = [2p_1(\lambda) 3p_2(\lambda) 4p_3(\lambda) \cdots]A. \quad (2.28)$$

Proof We begin with (2.7) and use (2.3) and (2.6):

$$\frac{(1 - \frac{y_1}{z}) \cdots (1 - \frac{y_{r-1}}{z})}{(1 - \frac{x_1}{z}) \cdots (1 - \frac{x_r}{z})} = \prod_{i=1}^d \frac{(1 - \frac{-b_i}{z-(1/2)})(1 - \frac{a_i}{z+(1/2)})}{(1 - \frac{a_i}{z-(1/2)})(1 - \frac{-b_i}{z+(1/2)})} \quad (2.29)$$

where $\lambda \in \mathbb{Y}$ has the Frobenius coordinates $(a_1, \dots, a_d \mid b_1, \dots, b_d)$ and the min-max coordinates $(x_1 < y_1 < \cdots < y_{r-1} < x_r)$. Expand logarithms of the both sides of (2.29) in $|z| \gg 1$. The LHS yields by (2.10) $\sum_{n=1}^{\infty} (M_n(\tau_\lambda)/n)z^{-n}$, while the RHS proceeds by (2.2) to

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{p_k(\lambda)}{k} \left\{ \left(z - \frac{1}{2}\right)^{-k} - \left(z + \frac{1}{2}\right)^{-k} \right\} &= \sum_{k=1}^{\infty} \frac{p_k(\lambda)}{k} z^{-k} \left\{ \left(1 - \frac{1}{2z}\right)^{-k} - \left(1 + \frac{1}{2z}\right)^{-k} \right\} \\ &= \sum_{k=1}^{\infty} \frac{p_k(\lambda)}{k} z^{-k} \sum_{j=0}^{\infty} \binom{-k}{2j+1} \frac{-1}{2^{2j}} z^{-(2j+1)} \\ &= \sum_{k=1}^{\infty} \frac{p_k(\lambda)}{k} z^{-k} \sum_{j=0}^{\infty} \frac{k^{\uparrow(2j+1)}}{(2j+1)!2^{2j}} z^{-(2j+1)} \\ &= \sum_{n=2}^{\infty} z^{-n} \sum_{0 \leq j \leq (n/2)-1} p_{n-2j-1}(\lambda) \frac{(n-1)^{\downarrow(2j)}}{(2j+1)!2^{2j}}. \end{aligned}$$

Hence we have

$$M_n(\tau_\lambda) = \sum_{0 \leq j \leq (n/2)-1} \binom{n}{2j+1} \frac{1}{2^{2j}} p_{n-2j-1}(\lambda), \quad n \in \{2, 3, \dots\},$$

which gives (2.28) and the other conditions for A .

Proposition 2.5 *Both $\{p_n(\lambda)\}_{n \in \mathbb{N}}$ and $\{M_n(\tau_\lambda)\}_{n \in \{2,3,\dots\}}$ are algebraically independent.*

Proof ¹We show algebraic independence of $\{p_n(\lambda)\}_{n \in \mathbb{N}}$. Provided that

$$f(p_1(\lambda), \dots, p_n(\lambda)) = \sum_{k_1, \dots, k_n} \alpha_{k_1 \dots k_n} p_1(\lambda)^{k_1} \cdots p_n(\lambda)^{k_n} = 0 \quad (2.30)$$

holds for a polynomial f in n variables, let us show $f = 0$. In (2.30), the partial sum of the terms in which $k = k_1 + 2k_2 + \cdots + nk_n$ is maximal is denoted by $f^{\natural}$.

¹The argument follows Proposition 1.5 in [16].

It suffices to verify that any coefficient $\alpha_{k_1 \dots k_n}$ in $f^{\natural}$ vanishes because it then proves to be the case for all k 's inductively. Let $x = (x_1, \dots, x_l) \in \mathbb{R}^l$, $x_1 \geq \dots \geq x_l > 0$, $l \geq k$, take $m \in \mathbb{N}$ and set $\lambda_i = \lfloor mx_i \rfloor$ for $i \in \{1, \dots, l\}$, $\lambda = (\lambda_1 \geq \dots \geq \lambda_l) \in \mathbb{Y}$. Putting this λ into (2.30), dividing the expression by the highest power of m and letting $m \rightarrow \infty$, we get

$$f^{\natural}(p_1(x), p_2(x), \dots, p_n(x)) = 0 \quad (2.31)$$

where $p_j(x) = p_j(x_1, \dots, x_l) = x_1^j + \dots + x_l^j$ is the power sum in l variables. In fact, the effect of b_i 's in the Frobenius coordinates and the other terms than $f^{\natural}$ tend to 0 as $m \rightarrow \infty$. Since $\{p_1(x_1, \dots, x_l)^{k_1} \dots p_n(x_1, \dots, x_l)^{k_n} \mid k_1 + \dots + nk_n = k (\leq l)\}$ is linearly independent by Proposition 1.4 (or a version of finite variables suffices), all coefficients in (2.31) is 0. This yields $f = 0$ and thus algebraic independence of $\{p_n(\lambda)\}_{n \in \mathbb{N}}$.

Provided that there exists an algebraic relation $g(M_2(\tau_\lambda), \dots, M_{n+1}(\tau_\lambda)) = 0$ between $\{M_n(\tau_\lambda)\}_{n \in \{2,3,\dots\}}$, rewrite it by using Proposition 2.4 as

$$g(2p_1(\lambda), \dots, (n+1)p_n(\lambda)) + h(2p_1(\lambda), \dots, (n+1)p_n(\lambda)) = 0.$$

By upper triangularity of A in (2.28), we get $g^{\natural}(2p_1(\lambda), \dots, (n+1)p_n(\lambda)) = 0$ similarly to (2.31). Again through an inductive argument, we are led to $g = 0$. This completes the proof of algebraic independence of $\{M_n(\tau_\lambda)\}_{n \in \{2,3,\dots\}}$.

The algebra $\mathbb{A}$ of functions on $\mathbb{Y}$ generated by $\{p_n(\lambda)\}_{n \in \mathbb{N}}$, or equivalently by $\{M_n(\tau_\lambda)\}_{n \in \{2,3,\dots\}}$, is isomorphic to Λ of the symmetric functions. We call $\mathbb{A}$ the Kerov–Olshanski algebra after [20]. The two kinds of generators above induce the degrees of an element of $\mathbb{A}$. The canonical degree in $\mathbb{A}$ is defined by regarding $p_n(\lambda)$ as a homogeneous element of degree n . This is clearly the one inherited from Λ . On the other hand, the weight degree in $\mathbb{A}$ is defined by regarding $M_n(\tau_\lambda)$ as a homogeneous element of degree n . These degrees are denoted by $\deg$ and wt respectively: $\deg p_n(\lambda) = n$, $\text{wt } M_n(\tau_\lambda) = n$. If $f \in \mathbb{A}$ is not homogeneous, $\deg f$ and $\text{wt } f$ indicate the degrees of the respective top homogeneous terms of f . For example, $\text{wt } p_n(\lambda) = n + 1$.

Recall that $\{M_n(\tau_\lambda)\}_{n \in \{2,3,\dots\}}$ and $\{M_n(\mathbf{m}_\lambda)\}_{n \in \{2,3,\dots\}}$ are in polynomial relations to each other through (2.17) as was seen in Proposition 2.2. Actually, the relation is (a specialization of) the one between the power sums and the complete symmetric functions in Λ . Furthermore, moments of a probability on $\mathbb{R}$ are in polynomial relations to three kinds of cumulants, classical, free and Boolean, through the cumulant-moment formulas. In particular, we can take $\{M_n(\mathbf{m}_\lambda)\}_{n \in \{2,3,\dots\}}$ or $\{R_n(\mathbf{m}_\lambda)\}_{n \in \{2,3,\dots\}}$ as generators of $\mathbb{A}$. As is seen in the sequel, $\{\Sigma_k(\lambda)\}_{k \in \mathbb{N}}$ also generates $\mathbb{A}$. A key observation might be a resemblance between the two expressions (1.32) and (2.23). In the beginning, we have

$$\Sigma_1(\lambda) = R_2(\mathbf{m}_\lambda) (= |\lambda|), \quad \Sigma_2(\lambda) = R_3(\mathbf{m}_\lambda). \quad (2.32)$$

Indeed, (2.11) and (2.19) yield

$$\Sigma_1(\lambda) = |\lambda| = \frac{1}{2} M_2(\tau_\lambda) = M_2(\mathfrak{m}_\lambda) = R_2(\mathfrak{m}_\lambda).$$

Moreover, (2.23) and (1.27) yield

$$\begin{aligned} \Sigma_2(\lambda) &= -\frac{1}{2} [z^{-1}] \left\{ \left(z - \sum_{k=1}^{\infty} \frac{B_k(\mathfrak{m}_\lambda)}{z^{k-1}} \right) \left(z - 1 - \sum_{k=1}^{\infty} \frac{B_k(\mathfrak{m}_\lambda)}{(z-1)^{k-1}} \right) \right\} \\ &= -\frac{1}{2} (-B_2(\mathfrak{m}_\lambda) - B_3(\mathfrak{m}_\lambda) + B_2(\mathfrak{m}_\lambda) - B_3(\mathfrak{m}_\lambda)) \\ &= B_3(\mathfrak{m}_\lambda) = M_3(\mathfrak{m}_\lambda) = R_3(\mathfrak{m}_\lambda) \end{aligned} \quad (2.33)$$

by noting $B_1(\mathfrak{m}_\lambda) = R_1(\mathfrak{m}_\lambda) = M_1(\mathfrak{m}_\lambda) = 0$.

Theorem 2.2 *For any $k \in \mathbb{N}$, $k \geq 3$, there exists a polynomial $P_k(x_2, \dots, x_{k-1})$ in $k-2$ variables satisfying*

$$\Sigma_k(\lambda) = R_{k+1}(\mathfrak{m}_\lambda) + P_k(R_2(\mathfrak{m}_\lambda), \dots, R_{k-1}(\mathfrak{m}_\lambda)) \quad (2.34)$$

where a possible value of the weight degree of each term in the lower part

$$P_k(R_2(\mathfrak{m}_\lambda), \dots, R_{k-1}(\mathfrak{m}_\lambda))$$

belongs to $\{k-1, k-3, \dots\}$ (every other integer) $\subset \mathbb{N}$.

Proof Let us write $G_\lambda = G_{\mathfrak{m}_\lambda}$, $M_k(\lambda) = M_k(\mathfrak{m}_\lambda)$, $R_k(\lambda) = R_k(\mathfrak{m}_\lambda)$, $B_k(\lambda) = B_k(\mathfrak{m}_\lambda)$ for short.

[Step 1] We will have an expression of $G_\lambda(z)^{-1} \dots G_\lambda(z-k+1)^{-1}$ in (2.23) in terms of the Laurent series in z in a similar way as (2.33). The expansion (1.27) of Boolean cumulant coefficients yields

$$\frac{1}{G_\lambda(z-r)} = z - r - \sum_{j=1}^{\infty} \frac{B_j(\lambda)}{(z-r)^{j-1}} = z - r - \sum_{j=1}^{\infty} \frac{B_j(\lambda)}{z^{j-1}} \left(\sum_{l=0}^{\infty} \frac{r^l}{z^l} \right)^{j-1} \quad (2.35)$$

for $r \in \{1, \dots, k-1\}$. Putting

$$\begin{aligned} \left(\sum_{l=0}^{\infty} t^l \right)^{j-1} &= \sum_{l_1, \dots, l_{j-1}=0}^{\infty} t^{l_1 + \dots + l_{j-1}} = \sum_{i=0}^{\infty} \alpha_{i,j-1} t^i, \\ \alpha_{i,j-1} &= \left| \{(l_1, \dots, l_{j-1}) \in (\mathbb{N} \cup \{0\})^{j-1} \mid l_1 + \dots + l_{j-1} = i\} \right| \end{aligned}$$

into (2.35), we continue (2.35) as

$$\begin{aligned}
&= z - r - \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} \frac{\alpha_{i,j-1} r^i B_j(\lambda)}{z^{i+j-1}} = z - r - \sum_{p=1}^{\infty} \frac{1}{z^{p-1}} \left(\sum_{j=1}^p \alpha_{p-j,j-1} r^{p-j} B_j(\lambda) \right) \\
&= z - \sum_{p=1}^{\infty} A_{p,r}(\lambda) \frac{1}{z^{p-1}} \tag{2.36}
\end{aligned}$$

where

$$A_{p,r}(\lambda) = \begin{cases} \sum_{j=1}^p \alpha_{p-j,j-1} r^{p-j} B_j(\lambda), & p \geq 2, \\ r + B_1(\lambda), & p = 1. \end{cases}$$

Since $\text{wt } B_j(\lambda) = \text{wt } M_j(\lambda) = \text{wt } M_j(\tau_\lambda) = j$ holds, we have $\text{wt } A_{p,r}(\lambda) = p$ and

$$A_{p,r}(\lambda) = B_p(\lambda) + (\text{wt-lower terms}), \quad p \in \mathbb{N}. \tag{2.37}$$

[Step 2] Put (2.36) and (2.37) into $G_\lambda(z-r)^{-1}$ of (2.23):

$$\begin{aligned}
&(G_\lambda(z)G_\lambda(z-1) \cdots G_\lambda(z-k+1))^{-1} \\
&= \left(z - \sum_{p=1}^{\infty} \frac{B_p(\lambda)}{z^{p-1}} \right) \left(z - \sum_{p=1}^{\infty} \frac{B_p(\lambda)}{z^{p-1}} + \sum_{p=1}^{\infty} \frac{*_1^p}{z^{p-1}} \right) \cdots \left(z - \sum_{p=1}^{\infty} \frac{B_p(\lambda)}{z^{p-1}} + \sum_{p=1}^{\infty} \frac{*_{k-1}^p}{z^{p-1}} \right)
\end{aligned}$$

where $*_1^p, \dots, *_{k-1}^p$ are terms of weight degree $\leq p-1$. Continue as

$$\begin{aligned}
&= \left(z - \sum_{p=1}^{\infty} \frac{B_p(\lambda)}{z^{p-1}} \right)^k + \sum_{j=1}^{k-1} \left(z - \sum_{p=1}^{\infty} \frac{B_p(\lambda)}{z^{p-1}} \right)^j \underbrace{\left\{ \sum_{\sharp} \left(\sum_{p=1}^{\infty} \frac{*}{z^{p-1}} \right) \cdots \left(\sum_{p=1}^{\infty} \frac{*}{z^{p-1}} \right) \right\}}_{(k-j) \text{ product}} \\
&= G_\lambda(z)^{-k} + (\star) \tag{2.38}
\end{aligned}$$

where $(k-j)*$'s are of weight degree $\leq p-1$ though they are not identical. Moreover, $\sum_{\sharp}$ indicates a finite sum with the number depending k and j . Each j -term of $(\star)$ in

(2.38) has such an expression as

$$z^i \frac{B_{p_1}(\lambda)}{z^{p_1-1}} \cdots \frac{B_{p_{j-i}}(\lambda)}{z^{p_{j-i}-1}} \frac{[\text{wt} \leq q_1-1]}{z^{q_1-1}} \cdots \frac{[\text{wt} \leq q_{k-j}-1]}{z^{q_{k-j}-1}}, \quad i \in \{0, 1, \dots, j\}.$$

To pick up the term of z^{-1} , the requirement for the index is

$$i - \{(p_1-1) + \cdots + (p_{j-i}-1) + (q_1-1) + \cdots + (q_{k-j}-1)\} = -1.$$

Then the weight degree of the coefficient is bounded by

$$p_1 + \cdots + p_{j-i} + (q_1 - 1) + \cdots + (q_{k-j} - 1) = j + 1 \leq k.$$

We have thus $\text{wt}([z^{-1}](\star)) \leq k$. Combining this with (1.32), we get

$$\Sigma_k(\lambda) = -\frac{1}{k} [z^{-1}] \left\{ \frac{1}{G_\lambda(z)^k} + (\star) \right\} = R_{k+1}(\lambda) + F(\lambda) \quad (2.39)$$

with $F \in \mathbb{A}$, $\text{wt } F \leq k$.

[Step 3] Since $\{R_k(\lambda)\}$ generates $\mathbb{A}$, (2.39) yields existence of a polynomial P_k such that

$$\Sigma_k(\lambda) = R_{k+1}(\lambda) + P_k(R_2(\lambda), \dots, R_k(\lambda)), \quad \text{wt } P_k(R_2(\lambda), \dots, R_k(\lambda)) \leq k, \quad (2.40)$$

where it clearly suffices to take generators up to $R_k(\lambda)$ from the relations between generators of $\mathbb{A}$. Let us consider the involution

$$\text{inv}(f)(\lambda) = f(\lambda'), \quad f \in \mathbb{A}$$

induced by the transposition $\lambda \mapsto \lambda'$. Taking the character values at a k -cycle of $\lambda' \cong \lambda \otimes \text{sgn}$, we have

$$\text{inv}(\Sigma_k)(\lambda) = \Sigma_k(\lambda') = (-1)^{k-1} \Sigma_k(\lambda).$$

On the other hand, since the transition measure obeys $\mathfrak{m}_{\lambda'}(A) = \mathfrak{m}_\lambda(-A)$ for any Borel set A of $\mathbb{R}$, its moment satisfies $\text{inv}(M_k)(\lambda) = M_k(\lambda') = (-1)^k M_k(\lambda)$. Then, (1.23) yields also for its free cumulant

$$\text{inv}(R_k)(\lambda) = R_k(\lambda') = (-1)^k R_k(\lambda).$$

Taking inv of (2.40):

$$(-1)^{k-1} \Sigma_k(\lambda) = (-1)^{k+1} R_{k+1}(\lambda) + P_k(R_2(\lambda), \dots, (-1)^k R_k(\lambda))$$

and comparing it with (2.40), we have

$$P_k(R_2(\lambda), -R_3(\lambda), \dots, (-1)^k R_k(\lambda)) = (-1)^{k-1} P_k(R_2(\lambda), R_3(\lambda), \dots, R_k(\lambda)). \quad (2.41)$$

When k is even, (2.41) implies that the sum of the terms of even weight degree in $P_k(R_2(\lambda), \dots, R_k(\lambda))$ vanishes. Similarly, when k is odd, that of odd weight degree in $P_k(R_2(\lambda), \dots, R_k(\lambda))$ vanishes. Hence we conclude that possible weight degrees for the terms in $P_k(R_2(\lambda), \dots, R_k(\lambda))$ of (2.40) belong to $\{k-1, k-3, \dots\}$. In particular, $R_k(\lambda)$ does not appear since there are no terms of weight degree k .

Seen from the viewpoint of the canonical degree, the following holds instead of Theorem 2.2.

Theorem 2.3 *For any $k \in \mathbb{N}$, there hold*

$$\Sigma_k(\lambda) = M_{k+1}(\mathfrak{m}_\lambda) + (\text{deg-lower terms}) \quad (2.42)$$

$$= p_k(\lambda) + (\text{deg-lower terms}). \quad (2.43)$$

Proof We first verify that the RHSs of (2.42) and (2.43) agree. Note $\deg M_k(\tau_\lambda) = k - 1$ by (2.28). The relation between $M_n(\mathfrak{m}_\lambda)$'s and $M_k(\tau_\lambda)$'s yield

$$M_{k+1}(\mathfrak{m}_\lambda) = \frac{1}{k+1} M_{k+1}(\tau_\lambda) + (\text{terms of } \deg \leq k-1), \quad (2.44)$$

which together with (2.28) implies (2.42) agrees with (2.43). Needless to say, the terms of lower canonical degrees in both equations are not identical.

Next we show the equality of (2.42). In (2.34), the lower terms in the RHS satisfy $\text{wt} \leq k-1$ and $\deg \leq k-2$. In the free cumulant-moment formula

$$R_{k+1}(\mathfrak{m}_\lambda) = \sum_{\pi \in \text{NC}(k+1)} m_{\text{NC}(k+1)}(\pi, 1_{k+1}) M_\pi(\mathfrak{m}_\lambda), \quad (2.45)$$

we have $\deg M_\pi(\mathfrak{m}_\lambda) = k+1 - b(\pi)$ where $b(\pi)$ denotes the number of blocks of $\pi \in \text{NC}(k+1)$. Indeed, (2.44) gives $\deg M_n(\mathfrak{m}_\lambda) = \deg M_n(\tau_\lambda) = n-1$. Hence the term of the highest canonical degree in the RHS of (2.45) is the one of $b(\pi) = 1$, namely $M_{k+1}(\mathfrak{m}_\lambda)$. This completes the proof of (2.42).

Corollary 2.1 *Both $\{\Sigma_k(\lambda)\}_{k \in \mathbb{N}}$ and $\{R_k(\mathfrak{m}_\lambda)\}_{k \in \{2,3,\dots\}}$ are algebraically independent.*

Corollary 2.2 *In Theorem 2.2, uniqueness of the polynomial P_k holds also. To be precise, the expression of (2.34) is unique without mentioning the weight degree of $P_k(R_2(\mathfrak{m}_\lambda), \dots, R_{k-1}(\mathfrak{m}_\lambda))$.*

Proof This follows from Corollary 2.1.

Definition 2.1 Theorem 2.2, Corollary 2.2 and (2.32) determine the following sequence of polynomials:

$$\begin{aligned} K_2(x_2) &= x_2, & K_3(x_2, x_3) &= x_3, \\ K_{k+1}(x_2, \dots, x_{k+1}) &= x_{k+1} + P_k(x_2, \dots, x_{k-1}), & k &\geq 3. \end{aligned}$$

The polynomial K_k is called the Kerov polynomial.

Remark 2.1 The derivation of the Kerov polynomials based on comparing (1.32) and (2.23) is due to Okounkov as suggested in [3]. Carrying out Step 2 of the proof of Theorem 2.2, one has

$$K_4(x_2, x_3, x_4) = x_4 + x_2, \quad K_5(x_2, x_3, x_4, x_5) = x_5 + 5x_3, \quad \dots$$

The fact that all coefficients of the Kerov polynomials are positive integers is conjectured by Kerov and proved first by Féray [8]. Explicit forms of the first several Kerov polynomials are presented in [3].

Remark 2.2 Along the above discussion, Theorem 2.3 was proved by using the Kerov polynomials (Theorem 2.2), which does not seem to be optimal as readers might notice. It would be more natural to deduce (2.43) directly from (2.22) for a proof of Theorem 2.3.

Extending (2.21) to a general conjugacy class, set for $\rho \in \mathbb{Y}$

$$\Sigma_\rho(\lambda) = \begin{cases} |\lambda|^{\downarrow|\rho|} \tilde{\chi}_{(\rho, 1^{|\lambda|-|\rho|})}^\lambda, & |\lambda| \geq |\rho|, \\ 0, & |\lambda| < |\rho|, \end{cases} \quad \lambda \in \mathbb{Y}.$$

In particular, $\Sigma_\emptyset(\lambda) = 1$. As a linearizing formula, the following holds.

Proposition 2.6 For $\rho, \sigma \in \mathbb{Y}$,

$$\Sigma_\rho \Sigma_\sigma = \Sigma_{\rho \sqcup \sigma} + \sum_{\tau \in \mathbb{Y}: |\rho| \vee |\sigma| \leq |\tau| < |\rho| + |\sigma|} a_\tau \Sigma_\tau, \quad a_\tau \in \mathbb{Q}_{\geq 0}.$$

Proposition 2.6 yields that $\Sigma_\rho \in \mathbb{A}$ and $\deg \Sigma_\rho = |\rho|$. Hence we have

$$\Sigma_\rho = \Sigma_{\rho_1} \cdots \Sigma_{\rho_{l(\rho)}} + (\text{deg-lower terms}). \quad (2.46)$$

Similarly for the weight degree also, the following holds.

Proposition 2.7 For $\rho, \sigma \in \mathbb{Y}$,

$$\Sigma_\rho \Sigma_\sigma = \Sigma_{\rho \sqcup \sigma} + \sum_{\tau \in \mathbb{Y}: |\tau| + l(\tau) \leq |\rho| + l(\rho) + |\sigma| + l(\sigma) - 2} a_\tau \Sigma_\tau.$$

Hence we see $\text{wt } \Sigma_\rho = |\rho| + l(\rho)$ and

$$\Sigma_\rho = \Sigma_{\rho_1} \cdots \Sigma_{\rho_{l(\rho)}} + (\text{lower terms with weight degree} \leq \text{wt } \Sigma_\rho - 2). \quad (2.47)$$

The expression (2.46) or (2.47) tells that $\{\Sigma_\rho\}_{\rho \in \mathbb{Y}}$ forms a basis of $\mathbb{A}$.

See [16] for the proofs of Proposition 2.6 and Proposition 2.7. In [13], we included their proofs based on partial permutations developed in [15].

The Limit Shape Problem for Ensembles of Young
Diagrams

Hora, A.

2016, IX, 73 p. 9 illus., Softcover

ISBN: 978-4-431-56485-0