Preface

Imagine a large statistical ensemble of Young diagrams and pick up one. We would like to say something about the typical shape, if any, of a Young diagram we get. Mathematically, let $\mathbb{Y}_n$ be the set of Young diagrams of size n and introduce a probability $\mathbb{M}^{(n)}$ on $\mathbb{Y}_n$. We discuss probabilistic limit theorems, especially the law of large numbers, as $n \to \infty$ on the quantities describing the shape of a Young diagram. While a Young diagram grows with n, let us rescale it horizontally and vertically by $1/\sqrt{n}$ to keep its area, which enables us to recognize the visible limit shape. Among others, the Plancherel measure is the most important from the point of view of symmetry or group-theoretical meaning. It describes the relative size of each irreducible component in the bi-regular representation of a symmetric group. Moreover, because the Plancherel measure is defined also on the path space of the Young graph, we can discuss the limit shape of Young diagrams as a strong law of large numbers.

Such a limit shape problem for Young diagrams was first shown and solved by Vershik–Kerov [29] and Logan–Shepp [21]. Afterwards, Biane [1, 2] extended this problem to a wide range of group-theoretical ensembles and brought in new insights of Voiculescu's free probability theory. Analysis of Young diagram ensembles and random permutations has made great progress, strongly influenced by an explosive development of random matrix theory. Beyond the law of large numbers, the central limit theorem (fluctuation of the shape) and other limit theorems have been studied extensively. References would be too huge to mention here (Kerov's book [19] is the one I always cite as a rich source of ideas from asymptotic representation theory). Readers can search through keywords and researchers according to their tastes.

This book is intended to serve as an introduction to the limit shape problem for Young diagrams as sketched above. It does not cover a broad range but stays near the classical results of Vershik–Kerov and Logan–Shepp. However, we bring a contemporary point of view for methods of proofs and some approaches. A key ingredient will be the algebra of polynomial functions in several coordinates of Young diagrams, which was introduced by Kerov–Olshanski [20]. In this book, we call it the Kerov–Olshanski algebra (KO algebra) after [20]. We give complete and self-contained proofs to the main results within the framework of representations of symmetric groups, not relying on random matrix theory or representations of unitary groups. Another point put anew is to mention a dynamical model for the time evolution of profiles of random Young diagrams. Although we focus mostly on the representation–theoretical aspect of the model in this book, analysis of the time evolution of profiles will be a promising topic with relation to geometric partial differential equations.

It is essential to investigate in detail the relations between various generating systems of the KO algebra, which was performed by Ivanov–Olshanski [16]. Notions of free probability theory are brought into this algebra with the help of Kerov's transition measure, and Biane's method plays an active part therein. Actually, it may be an exaggeration that we bring in the KO algebra to show the classical result of Vershik–Kerov and Logan–Shepp on the limit shape with respect to the Plancherel measure. However, once we know some structure of this algebra, the rest will be reduced to a pleasant application of simple weight counting argument. The KO algebra is a very nice device having rich applications in asymptotic representation theory for symmetric groups, especially in that it enables us to proceed along an exact or non-asymptotic way up to certain stages. We willingly include some materials about the KO algebra in reasonable depth. Such being the case, this book owes much to the works of [2, 3, 16].

Because the scope of this book is kept rather limited, we let quite many materials drop out of the content which could be appropriately included as interesting related topics by a more skillful author; for example,

- the philosophical and phenomenological analogy between random permutations and random matrices
- exact and asymptotic analysis of random Young diagrams as a point process
- the nature of fluctuations for ensembles of Young diagrams
- harmonic and stochastic analysis on infinite-dimensional dual objects, e.g., the Martin boundary of a branching graph
- asymptotic representation theory in frameworks beyond group actions, e.g., an extension from Plancherel to Jack, and so on.

Let us briefly give the organization of the following chapters. Because Chap. 1 is nothing but a casual description of preliminaries, readers should look into appropriate references according to their backgrounds. Speaking of representations of the symmetric group, one can go ahead with little trouble by accepting the hook formula and Frobenius's character formula. Chapter 2 is devoted to analysis of the KO algebra, which makes a technical prop. Chapter 3 contains analytic descriptions of continuous diagrams, or continuous limits of Young diagrams. Solutions of the limit shape problem for the Plancherel ensemble are given in Chap. 4. We give the proofs not only by an application of the KO algebra but also through what is called a continuous hook. The latter is of interest leading to the large deviation principle. While the results in Chap. 4 are of static nature, Chap. 5 includes a dynamical model. Funaki–Sasada [11] treated hydrodynamic limit for evolution of the profiles of Young diagrams. Chapter 5 is based on [12], which was greatly inspired by [11].

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