

# An Optimal Partially Backlogged Policy of Deteriorating Items with Quadratic Demand

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**Abstract** An EOQ (Economic Order Quantity) model for a deteriorating item with quadratic demand pattern and quadratic holding cost and constant deterioration rate is considered in this paper. In addition, shortages and partial backlogging are allowed. It is assumed that the backlogging rate acts as not only a variable, but also depends on the length of the waiting time up to next replenishment during the stock out period. For this model, average total cost is derived. Finally, a numerical example for illustration is provided.

**Keywords** EOQ • Partial backlogging • Quadratic demand • Quadratic holding cost

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## 1 Introduction

In the past few decades, several mathematical techniques have been developed to control and maintain the inventories of deteriorating items. Generally, deterioration is defined as the change, damage, decay, dryness, spoilage, vaporization, etc., that cannot be used in original purposes. Researches began the inventory models by considering the time-proportional demand in their models. Bahari-Kasani [1] and Dave and Patel [3] studied their models by considering time-proportional demand. The literature survey by Raafat [10], Goyal and Giri [4] and Li et al. [8] discuss the up to date review on deteriorating inventory model. In the real market situation, the demand rate for any product cannot be constant. Many researchers have studied their models on linear and exponential time-dependent demand. A time-dependent linear demand leads to uniform change whereas time-dependent exponential demand seems unrealistic. Therefore, the more realistic approach is to consider time-dependent quadratic demand rate because it experiences the accelerated growth or decline in demand. Khanra and Chaudhuri [6], Khanra et al. [7] and Singh and Pattnayak [11] developed inventory models considering time-quadratic demand rate which is more realistic.

In real life situations, some but not all customers prefer to wait for backlogged items during the shortage period. Therefore, the longer the waiting time is, the smaller the backlogging rate will be. The opportunity cost due to lost sales should be considered in the inventory models. Researchers like Hollier and Mak [5], Chang and Dye [2] and Ouyang et al. [9] developed the inventory model considering partial backlogging. A general EOQ model for deteriorating items with time-dependent linear demand, time-proportional and partial backlogging have been developed by Singh and Pattnayak [12].

In this paper, an EOQ model for deteriorating items has been developed by considering constant deterioration rate. The demand rate and the holding cost are assumed to be quadratic function of time. The optimal shortage time, optimal cycle length, optimal order quantity and average total cost are derived. Finally, a numerical example for illustration of the model is provided.

## 2 Notations and Assumptions

The following assumptions and notations are used to develop the model.

- (i) A single item is considered over a prescribed time period.
- (ii) Shortages in the inventory system are allowed and partially backlogged.
- (iii) Replenishment rate is infinite.
- (iv)  $R(t)$ : The demand rate at any time  $t$  is given by  $R(t) = \begin{cases} a + bt + ct^2, & I(t) > 0, \\ K, & I(t) \leq 0, \end{cases}$  where  $a, b, c (a \geq 0, b \neq 0, c \neq 0) & K$  are positive constant.

- (v)  $\theta$  : The constant deterioration rate where  $0 < \theta < 1$ .
- (vi) The rate of backlogging during the stock out period is assumed as not only variable but also it depends on the length of the cycle time up to next replenishment. If the waiting time is longer, then the rate of backlogging will be smaller. Let  $B(t) = \frac{1}{1+\delta t}$  be the negative inventory rate of backlogging where  $\delta (> 0)$  is the backlogging parameter in the interval  $0 \leq t \leq T$ .
- (vii)  $h$  : The quadratic inventory holding cost nd is given by  $h(t) = h + \gamma t + \beta t^2$  where  $h > 0, \gamma \neq 0$  &  $\beta \neq 0$ .
- (viii)  $C_P$  : The purchase cost of the inventory item per unit.
- (ix)  $C_O$  : The ordering cost of the inventory per order.
- (x)  $C_S$  : The constant shortage cost per unit time unit.
- (xi)  $C_L$  : The cost of constant lost sales per unit.
- (xii)  $M$  : The length of time at which the shortage period starts.
- (xiii)  $T$  : The length of cycle time.
- (xiv)  $W_{Max}$  : The unit of items that arrive in the system at the starting of each cycle.
- (xv)  $S_{Max}$  : The maximum amount of demand backlogged per cycle.
- (xvi)  $Q_0$  : the order quantity per cycle.

### 3 Model Development

Considering the effect of quadratic demand and constant deterioration during  $[M, T]$ , the inventory level at any instant during the period  $[M, T]$  can be represented by the differential equation as:

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = -R(t), \quad 0 \leq t \leq M \quad (1)$$

where  $R(t) = a + bt + ct^2$ .

Further, the inventory level reaches zero at time  $t = M$ . Thus, using the boundary condition  $I_1(M) = 0$ , the solution of Eq. (1) becomes

$$I_1(t) = \left[ \frac{a + bM + cM^2}{\theta} - \frac{b + 2cM}{\theta^2} + \frac{2c}{\theta^3} \right] e^{\theta(M-t)} - \frac{a + bt + ct^2}{\theta} + \frac{b + 2ct}{\theta^2} - \frac{2c}{\theta^3}, \quad 0 \leq t \leq M. \quad (2)$$

With the help of  $I_1(0) = W$ , the inventory will be

$$W_{Max} = \left[ \frac{a + bM + cM^2}{\theta} - \frac{b + 2cM}{\theta^2} + \frac{2c}{\theta^3} \right] e^{\theta M} - \frac{a}{\theta} + \frac{b}{\theta^2} - \frac{2c}{\theta^3}. \quad (3)$$

During the stock out period  $[M, T]$ , a fraction of demand is backlogging. Thus, the state of inventory is governed by the differential equation as:

$$\frac{dI_2(t)}{dt} = R(t)B(t), \quad M \leq t \leq T. \quad (4)$$

where  $R(t) = K$  and  $B(T - M) = \frac{1}{1 + \delta(T - M)}$ .

The solution of Eq. (4) with  $I_2(M) = 0$  is

$$I_2(t) = \frac{K}{\delta} [\ln(1 + \delta(T - t)) - \ln(1 + \delta(T - M))], \quad M \leq t \leq T. \quad (5)$$

Maximum backordered units will be

$$S_{Max.} = -I(T) = \frac{K}{\delta} \ln[1 + \delta(T - M)]. \quad (6)$$

Thus, the order size during the time period  $[0, T]$  will be

$$Q_0 = W_{Max.} + S_{Max.} = \left[ \frac{a + bM + cM^2}{\theta} - \frac{b + 2cM}{\theta^2} + \frac{2c}{\theta^3} \right] e^{\theta M} - \frac{a}{\theta} + \frac{b}{\theta^2} - \frac{2c}{\theta^3} + \frac{K}{\delta} \ln[1 + \delta(T - M)]. \quad (7)$$

The following cost components are needed for the calculation of per cycle average total cost:

The per cycle ordering cost is

$$CO = C_O. \quad (8)$$

The per cycle cost of carrying is

$$\begin{aligned} CC &= \int_0^M h(t)I_1(t)dt = \int_0^M [(h + \gamma t + \beta t^2)I_1(t)]dt \\ &= \left[ \frac{a + bM + cM^2}{\theta} - \frac{b + 2cM}{\theta^2} + \frac{2c}{\theta^3} \right] \\ &\quad \times \left[ \frac{he^{\theta M} - h - \gamma M - \beta M^2}{\theta} + \frac{\gamma e^{\theta M} - \gamma - 2\beta M}{\theta^2} + \frac{2\beta e^{\theta M} - 2\beta}{\theta^3} \right] \\ &\quad - \frac{h}{\theta^3} \left[ \theta^2 \left( aM + \frac{bM^2}{2} + \frac{cM^3}{3} \right) - \theta(bM + cM^2) + 2cM \right] \\ &\quad - \frac{\gamma}{\theta^3} \left[ \theta^2 \left( \frac{aM^2}{2} + \frac{bM^3}{3} + \frac{cM^4}{4} \right) - \theta \left( \frac{bM^2}{2} + \frac{2cM^3}{3} \right) + cM^2 \right] \\ &\quad - \frac{\beta}{\theta^3} \left[ \theta^2 \left( \frac{aM^3}{3} + \frac{bM^4}{4} + \frac{cM^5}{5} \right) - \theta \left( \frac{bM^3}{3} + \frac{cM^4}{4} \right) + \frac{2cM^3}{3} \right]. \end{aligned} \quad (9)$$

The per cycle cost of deterioration is

$$\begin{aligned}
 CD &= C_P \left[ W_{Max} - \int_0^M (a + bt + ct^2) dt \right] \\
 &= C_P \left[ \left( \frac{a + bM + cM^2}{\theta} - \frac{b + 2cM}{\theta^2} + \frac{2c}{\theta^3} \right) e^{\theta M} \right] \\
 &\quad - C_P \left[ \frac{a}{\theta} - \frac{b}{\theta^2} + \frac{2c}{\theta^3} + aM + \frac{bM^2}{2} + \frac{cM^3}{3} \right].
 \end{aligned} \tag{10}$$

The per cycle cost of shortage is

$$CS = C_S \left[ - \int_M^T I_2(t) dt \right] = \frac{C_S K}{\delta} \left[ (T - M) - \frac{1}{\delta} \ln(1 + \delta(T - M)) \right]. \tag{11}$$

The per cycle cost of opportunity due to lost sales is

$$\begin{aligned}
 COLS &= C_L \left[ \int_M^T K \left( 1 - \frac{1}{1 + \delta(T - M)} \right) dt \right] \\
 &= C_L K \left[ (T - M) - \frac{1}{\delta} \ln(1 + \delta(T - M)) \right].
 \end{aligned} \tag{12}$$

Therefore, the per cycle average total cost is

$$CO + CC + CD + CS + COLS. \tag{13}$$

The per cycle average total cost per unit is

$$\begin{aligned}
 ATC(M, T) &= \frac{1}{T} [CO + CC + CD + CS + COLS] \\
 &= \frac{1}{T} \left[ \frac{a + bM + cM^2}{\theta} - \frac{b + 2cM}{\theta^2} + \frac{2c}{\theta^3} \right] \\
 &\quad \times \left[ \frac{he^{\theta M} - h - \gamma M - \beta M^2}{\theta} + \frac{\gamma e^{\theta M} - \gamma - 2\beta M}{\theta^2} + \frac{2\beta e^{\theta M} - 2\beta}{\theta^3} \right] \\
 &\quad - \frac{h}{\theta^3 T} \left[ \theta^2 \left( aM + \frac{bM^2}{2} + \frac{cM^3}{3} \right) - \theta(bM + cM^2) + 2cM \right] \\
 &\quad - \frac{\gamma}{\theta^3 T} \left[ \theta^2 \left( \frac{aM^2}{2} + \frac{bM^3}{3} + \frac{cM^4}{4} \right) - \theta \left( \frac{bM^2}{2} + \frac{2cM^3}{3} \right) + cM^2 \right] \\
 &\quad - \frac{\beta}{\theta^3 T} \left[ \theta^2 \left( \frac{aM^3}{3} + \frac{bM^4}{4} + \frac{cM^5}{5} \right) - \theta \left( \frac{bM^3}{3} + \frac{cM^4}{4} \right) + \frac{2cM^3}{3} \right] \\
 &\quad + \frac{C_P}{T} \left[ \left( \frac{a + bM + cM^2}{\theta} - \frac{b + 2cM}{\theta^2} + \frac{2c}{\theta^3} \right) e^{\theta M} \right] \\
 &\quad - \frac{C_P}{T} \left[ \frac{a}{\theta} - \frac{b}{\theta^2} + \frac{2c}{\theta^3} + aM + \frac{bM^2}{2} + \frac{cM^3}{3} \right] \\
 &\quad + \frac{C_O}{T} + \frac{C_S K}{\delta T} \left[ (T - M) - \frac{1}{\delta} \ln(1 + \delta(T - M)) \right].
 \end{aligned} \tag{14}$$

The necessary conditions for minimizing the average total cost per unit will be

$$\begin{aligned}
 \frac{\partial ATC(M, T)}{\partial M} = & \frac{1}{T} \left[ \frac{a + bM + cM^2}{\theta} - \frac{b + 2cM}{\theta^2} + \frac{2c}{\theta^3} \right] \\
 & \times \left[ \frac{h\theta e^{\theta M} - \gamma - 2\beta M}{\theta} + \frac{\alpha\theta e^{\theta M} - 2\beta}{\theta^2} + \frac{2\beta\theta e^{\theta M}}{\theta^3} \right] + \frac{1}{T} \left[ \frac{b + 2cM}{\theta} - \frac{2c}{\theta^2} \right] \\
 & \times \left[ \frac{he^{\theta M} - h - \gamma M - \beta M^2}{\theta} + \frac{\alpha e^{\theta M} - \gamma - 2\beta M}{\theta^2} + \frac{2\beta e^{\theta M} - 2\beta}{\theta^3} \right] \\
 & - \frac{h + \gamma M + \beta M^2}{\theta^3 T} [\theta^2 (a + bM + cM^2) - \theta(b + 2cM) + 2c] \\
 & + \frac{C_P}{T} [a + bM + cM^2] [e^{\theta M} - 1] - \frac{K(C_S + \delta C_L)(T - M)}{T(1 + \delta(T - M))} = 0.
 \end{aligned} \tag{15}$$

and

$$\frac{\partial ATC(M, T)}{\partial T} = \frac{1}{T} \left[ \frac{K(C_S + \delta C_L)(T - M)}{1 + \delta(T - M)} - (ATC(M, T)) \right] = 0 \tag{16}$$

The pair of  $[M, T]$  calculated from (15) and (16) subject to the conditions

$$\frac{\partial^2 ATC(M, T)}{\partial M^2} > 0, \frac{\partial^2 ATC(M, T)}{\partial T^2} > 0 \ \& \ \left( \frac{\partial^2 ATC(M, T)}{\partial M^2 \partial T} \right)^2 > 0 \text{ will minimize } ATC(M, T).$$

## 4 Numerical Example

*Example 1* Let us consider the following parametric values of the inventory system as:  $a = 12$ ,  $b = 8$ ,  $c = 2$ ,  $h = 4$ ,  $\gamma = 3$ ,  $\beta = 2$ ,  $C_P = 2$ ,  $C_O = 4$ ,  $C_S = 4$ ,  $C_L = 3$ ,  $K = 10$ ,  $\theta = 0.05$  and  $\delta = 3$  in appropriate units.

Solving the simultaneous Eqs. (15) and (16), the optimal shortage period and optimal cycle length are obtained as  $M^* = 0.262993$  unit time and  $T^* = 0.480607$  unit time respectively. Now substituting the pair  $(M^*, T^*)$  in Eqs. (3), (7) and (14), we get the optimal order quantity  $Q_0^* = 5.14309$  units, the optimal maximum inventory level  $W_{Max}^* = 3.4681$  units and the minimum average total cost per unit time  $ATC^*(M, T) = 17.1159$  respectively.

## 5 Conclusion

For the business scenario, a large amount of capital was invested for maintaining and controlling inventory. Therefore, the demand pattern and deterioration rate are two important factors for growing the business. In this paper, a deterministic

inventory model is considered with quadratic demand rate, quadratic carrying cost and constant deterioration rate. Shortages are allowed and partially backlogged in this model. The main reason of considering a quadratic demand instead of a linear demand or an exponential demand because of the accelerated growth and accelerated decline in demand. During the shortage period, the backlogging rate is inversely proportional to the waiting time for the next replenishment. Furthermore, the model has been solved with a numerical example by minimizing the total cost by simultaneously optimizing the shortage period and the length of the cycle.

There are many scopes of extending the present model as future work. Firstly, it can be extended by considering more realistic as stock dependent demand and stochastic demand. Secondly, it can be extended to variable deterioration like the generalized Weibull distribution deterioration rate.

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