Mathematics and Reality

Mathematics as a Tool to Understand and Act on Reality

Sylvestre Huet: To start this chapter on the nature of mathematics and of its relationship with reality, I would like to quote you two extracts from interviews I carried out for *Libération*,[1] one of a mathematician you are familiar with—Alain Connes—and the other of a Belgian theoretical physicist and epistemologist—Dominique Lambert. Here is what Alain Connes said:

> There are two opposing extreme viewpoints about mathematical activity. The first one, which I am entirely in agreement with, follows the Platonists: it states that there is a raw, primitive mathematical reality which predates its discovery. A world whose exploration requires the creation of tools, in the same way as ships had to be invented to cross oceans. Mathematicians will, therefore, invent, create theories whose purpose is to shed some light on this preexisting reality. The second viewpoint is that of formalists; they deny mathematics any preexistence, considering that it is a formal game, founded on axioms and logical deductions, hence a purely human creation. This viewpoint seems more natural to non-mathematicians, who are reluctant to assume an unknown world which they do not perceive. People understand that mathematics is a language, but not that it is an external reality outside the human mind. Yet, the great discoveries of the 20th century, in particular Godel's work, have shown that the formalist viewpoint is unsustainable. Whatever be the means of exploration, the formal system used, there will always be mathematical truths beyond it, and mathematical reality cannot be reduced to the logical consequences of a formal system.

In contrast to this vision of mathematics, Dominique Lambert expressed the following point of view in another interview published in *Libération*, attempting to resolve the difficult issue of the relationship between mathematics and reality by a historical approach:

> "A mathematician seemingly invents with imagination as his only guide and mathematical rules as the only rules. Thirty years later, his invention helps to describe a particle or space-time. Why? Mathematics are efficient. Very much so. To the point of causing turmoil among physicists, the biggest "clients" of mathematics. Princeton physicist Eugene Wigner's

[1] A French daily, translator's note.

P. Cartier et al., *Freedom in Mathematics*,
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famous article is an example. He called it: "*The Unreasonable Effectiveness of Mathematics in the Natural Sciences*".[2] A compelling formula, but an old problem. As far back as Antiquity, numbers have been assigned some sort of power. Pythagoreans saw the meaning of the world, its true reality, in numbers to which they assigned a metaphysical status. With Plato, mathematics became a world apart, separated from the world of appearances, which provides access to that of ideas. As for Aristotle, he saw mathematics as a mere abstraction based on reality. And the debate is unable to find a way out of this brutal dichotomy. Each school accumulates examples supporting its case without being able to oust its opponent who opposes as many counterexamples. To break this deadlock, one needs to observe the history of the relationship between mathematics and physics. It is then possible to justify both these conceptions and to nuance their element of truth. The great rise of the Platonic conception goes back to the birth of modern physics. When Galileo wrote: "The great book of Nature is written in mathematical language", he was not surprised by the efficiency of mathematics, he assumes it. According to him, it is necessary to understand and to express the world through mathematics because mathematics has something to do with the structure, the essence of the world. And he is believable because his approach is efficient. Forgetting that he carefully chose basic physical systems (fall of a marble on the floor) that allow for approximations and simplifications. We forget that the mathematical notions he used—the point (without extension), the line (without depth)—have no physical counterparts. However, there was subsequently a tremendous upsurge in the development of mathematics. It often appeared to be independent of all else. This gradually reinforced this conception, which reached its peak with Cantor's sentence: "The essence of mathematics lies in its freedom". And with Bourbaki—un group of French mathematicians formed in the 1930s —for whom mathematics is a self-sufficient whole. This conception gives a "meta-physical" dimension to mathematics and separates them from empirical science and the world. In this context, efficiency can only be unreasonable, even miraculous. The "result of a pre-established harmony", as stated by Leibniz. This position is all the more powerful as the efficiency of mathematics in natural sciences, biology, and sometimes in economy cannot be denied. But is this efficiency connected to a "metaphysical" structure of the world? Or to some other reason, which is in no way miraculous or mysterious? To answer these questions, it is necessary to to delve into the history of mathematics. We then find that it looks like a co-evolution with natural sciences and especially with physics. Think about the co-evolution of flowers and pollen-gathering insects. Except that there are phases of relatively independent development as well as periods of strong interactions with empirical domains. The latter bring new problems, new information to mathematics which takes hold of them to initiate a new development of its own. Some sort of bilateral self-stimulating interaction, with phases of independent developments. But a co-evolution where the main stages of each protagonist do not necessarily coincide, contrary to the biological image. There are decades, if not centuries without any interaction. After Riemann developed curved spaces around 1850—this is pure geometry—six decades elapsed before physics grabbed hold of it with Einstein's general relativity.

Because of this time lag, isolating one of these sentences, you find a mathematician who asks himself questions for their own sake, and very rich domains appear to develop by themselves. Alain Connes described them as "generative domains", Jean Dieudonné of "fertile problems", which seems to agree with Platonism. But if we broaden our outlook, we see that, historically, mathematics would not have alone reached its present development. How can the development of integral calculus or of analysis be imagined without mechanics? It is because Werner Heisenberg strove to "glue" to the spectrum of the hydrogen atom that he managed to build the first mathematical apparatus of quantum theory. Later, Richard Feynman's work in quantum mechanic led to a whole series of mathematical developments. In the 1970 and 1980s, chaos theory and dynamical systems were stimulated by meteorol-

[2] E.P. Wigner, *Communications on Pure and Applied Mathematics*, XIII (1960), 1–14.

ogy or the desire to understand turbulence. If mathematicians were left alone too long, they would not go very far. Platonism does not account for this. Aristotle's shortcoming was to believe that mathematics is merely derived from reality. Plato's shortcoming was to believe that all mathematics exists in a universe autonomous from reality.

The connection between mathematics and reality proves to be more subtle. Let us start from vision. I see a cup and consider it real because when I look at it from different angles, its perception remains the same. If changing angles suffices to change the cup, it is an illusion. Behind the idea of invariance is the idea of reality. The physicist does the same. He recuperates sets of objects, defines operations on these objects, then, physical dimensions and measures are associated to concrete invariants: momentum, energy, charge. However, is this not precisely what mathematicians do: define sets of objects on which operations are carried out and where invariants are searched for? Mathematics is some sort of extension of the process of perception.

This close proximity between mathematics, physics and perception, explains why the formal nets woven by mathematicians can catch bits of reality. Conversely, the relative independence of mathematics explains why some of these nets do not catch anything. But also why several different nets may describe the same reality. Moreover, that the same equation can describe—as shown by Jean-Marc Lévy-Leblond—different natural phenomena shows that it does not touch its essence, even if this equation is often essential to think about them. The equation of the electron is not the electron. As the great mathematician René Thom warned us 'describing is not understanding'.

Gerhard Heinzmann: These interviews show that the attempt to describe the relationship between mathematics and reality by having recourse to the Platonic or the formalist position has shortcomings and leads to philosophical deadlocks which, firstly, do not have easy answers and secondly, cannot be broken by merely observing the history of the relationship between mathematics and physics. So as not to misuse the time allocated to me, I will omit the formalism (which, contrary to what Alain Connes said, is always sustainable[3] and I will confine myself to a brief discussion of Platonism: it is one thing to assume a primitive mathematical reality preexisting its discovery (every mathematician is free to believe in it and he may find it quite useful to do so) and another to give rational arguments explaining how these abstract entities can be grasped (these arguments are the task of the philosopher) without falling into the trap of an analogy with the perception of concrete objects. Thus, to avoid presupposing the mysterious capacity that gives access to Platonic entities, following the Duhem–Quine thesis,[4] philosophers have defended a moderate Platonism: quantifiers of a theory relate to sets of numbers, to functions as well as particles, and fields. It, therefore, seems reasonable to believe in the existence of theoretical elements in the same way as we believe in empirical elements as long as the theory resists critical tests. Nonetheless, this standpoint leads to a number of questions[5] and

[3]See for example Michel Detlefsen (2004), Formalism, in: Stewart Shapiro (ed.), *The Oxford Handbook of Philosophy of mathematics and Logic*, Oxford, Oxford University Press, pp. 236–317.

[4]This thesis states that it is impossible to test empirically a mathematical or physical hypothesis in isolation, but a bundle of physical and mathematical hypotheses can be tested.

[5]What are the mathematical entities truly vital for a given scientific theory? If it could be shown that these are numbers, the Platonic answer could reduce to a constructivist position! Which principles concerning these entities are necessary for the required mathematics?.

requires paying a heavy price: a clear semantic separation between abstract objects and concrete objects must be refuted.

Defining like Dominique Lambert mathematics as "some sort of extension of the process of perception," nothing is gained philosophically, and one remains stuck in the analogy.

So I propose another solution. It consists in saying that mathematics is a language used in the sciences to *decomplexify* reality—which may seem surprising to someone who has not done mathematics, and who finds it extremely complex. But in the end it is reality that is complex. And mathematicians, from the beginning, have been inventing a language to measure it, to simplify it, to decomplexify it. This is what Euclid did in geometry. In arithmetic, one needs to wait till Peano[6] for structures to be highlighted and axiomatic systems elaborated. Hence, arithmetic is seemingly more complex than the structure of geometry, which is also paradoxical. Moving away from numbers accessible to daily experience and writing down their structure was harder than moving away from concrete drawings.

This is what is essential in mathematics: it is a science which reflects on necessary structures to express the complex reality whose objects are "invented" as a position within a structure.[7] It does not describe reality, but serves as a tool to express it. Mathematics is our tool to access a reality, generally speaking, more complex than the one accessible to our senses and our experience.

Cédric Villani: I go along with what Gerhard just said, especially about the fact that mathematics initially arises from a desire to simplify. Indeed, what surrounds us is complicated, incomprehensible, and besides, what is reality? Even the simplest things…. Phase transition is a perfect illustration of this: take a pan, boil water, and what happens? Water goes through a phase change… and nobody has ever understood why! This is one of the still open problems in mathematical physics. The complexity that surrounds us is absolutely terrifying if you try to imagine it. The mathematical approach consists in extracting guiding principles that will enable us to describe the world around us, then to understand this world, and finally to act upon it. To understand and to act. These are the two basic motivations of the mathematical approach: the one is inseparable from the other—and conversely. To this end, all lines, all paths, are simplified and replaced by a line, some sort of extremely simple representation; for somewhat more complex forms, there is the

[6]Giuseppe Peano (1858–1932), Italian mathematician and linguist. A pioneer in the formalist approach to mathematics, he contributed to the development of an axiomatization of arithmetic.

[7]By using the term "invented", I obviously do not mean that the mathematician invents material objects, but that the mathematical conceptualization of reality is not a description, but preserves a certain degree of freedom. To illustrate what I want to say, it is convenient to take the traditional example of geometry before 1905 (space-time); I can either take Euclidean geometry or one of the non-Euclidean geometries to measure street angles; none of the descriptions obtained is more true than the others and for this reason none is a *description* in the sense of the expression of a biunivocal relation, but is an "invention". Distinct theories are, therefore, empirically equivalent or, as formulated by American philosopher Willard Ban Orman Quine: theories are sub-determined by experience.

triangle… And gradually, one comes closer to the complexity of the world around us, by attempting to rely on a set of simple axioms, simple rules, logical rules that enable us to somewhat find our bearings, and put some order in a world so indecipherable that it is absolutely brutal.

And coming back to the initial question on abstract/concrete aspects, it is a dialectic approach from the outset: going toward the world, returning, looking, reflecting. This differs from an experimental process (reflecting on an experiment is followed by testing it against the world, and then returning), but still it is a bit like that: a theory is elaborated, observation shows whether it provides a good description, viewpoints are changed, and so on. Take the parabola, which is one of Galileo's greatest successes: he made a flagrant mistake, and what is remarkable is to see him finally reach a correct result (in other words, the trajectory of a parabola) by accumulating mathematical mistakes. It may seem simple today, but at the time finding the trajectory of a projectile was extremely complicated. People continued to get excited by this for a long time,

Pierre Cartier: Moreover, for Galileo, the continuous motion of a solid (according to the inertia principle he himself defined) is not a motion along a straight line at constant velocity, but a circular motion at constant velocity! Nonetheless, if the parabola is identified to its osculating circle,[8] in the conditions of the experience, we recover something coherent.

Cédric Villani: This example is a good illustration of the extent to which what surrounds us is complicated. In Galileo's times, the arts were flourishing, there were already masterpieces in literature, painting, sculpture… but in terms of the description of the familiar world around us by natural laws, reflection still centered around trivial issues such as: "If I throw this object in the air, how is it going to fall down?"

Jean Dhombres: Let us then play an intellectual game to place our discussion in the thinking of an earlier period… I am professor of Aristotelian physics in Padua around 1604, and so am a colleague of Galileo and I also teach about falling bodies—for such is required by the program which enforces a commentary of "*Mechanical Questions*", a book then attributed to Aristotle and dealing with heavy bodies thrown through the air. Now, my colleague Galileo uses simplification of a mathematical nature (the fall of bodies in the void is seemingly founded on a new law expressed mathematically), whereas this is a real phenomenon. Galileo can only be wrong for scrimping on Aristotelian thought like a bad student and by denying all efforts to interpret the texts of so many past professors. In the name of academic dignity itself, I propose a compromise theory, which includes various positions, one of them being taken from a book written more that 70 years ago by a mathematician named

[8] The osculating circle of a curve at a point is the circle best approximating the curve passing through the point, keeping its direction and curvature.

Tartaglia,[9] who introduced so many novelties in the algebraic domain of equations. As an accredited professor, I dogmatically stipulate that when I throw a stone in the air at an angle, the stone first goes straight because when throwing it, I gave it this *impetus* which I refrain from defining or quantifying; but this *impetus* runs out and when it has run out, at the end of the straight line, the stone no longer knows where to go. Not in mere rhetoric can I say that not knowing where to go, it goes in a circle. When it returns to the same height, the stone lets itself fall straight down. For those of us reading today, it is clear that there is not *one* trajectory, but a variety of motions indicated by different forms, curves or lines. There is a rectilinear motion, then something emotional, or psychological: the stone thinks, it thinks it does not know where to go, hence it goes in a circle. Then it falls straight down because it tries to reach the center of the world to which it aspires, since it has taken a fancy for this imagined center.

Cédric Villani: One could play devil's advocate and say that these arguments appear to be much richer and more interesting that Galileo's spartan arguments.

Jean Dhombres: Precisely. There is something rude in the mathematics presented by Galileo. Does not the latter say one should forget "fancies"? When his frequently polemical texts are read honestly, despite the experimental and deep side about the parabola which is the only curve he finds for the trajectory of the stone thrown,—we have drawings and accounts of fairly brilliant experiments by Galileo concerning the parabola—, we basically get the impression that the experiments come later. After coming up with the idea that there is no dissipated *impetus*, the unique trajectory stems from the composition of two totally distinct effects and that are both standpoints on our perception of the world. "Normally" the trajectory should be "a straight line with uniform velocity". This uniformity, which is the exact opposite of the dissipation of the *impetus* because the fact that Galileo enforces the conservation of velocity means that he represents time on a line, and that he makes equal amounts of distance covered correspond to equal amounts of time. But at every position that the stone should occupy on the line followed when thrown, the stone falls, this is the second effect. Galileo checked that the length of vertical fall is proportional to the square of the time elapsed since the start of the throw—such is the aim of experiments on the motion of a heavy ball on an inclined plane representing an artificial decrease in velocity of the vertical fall giving rise to observation by bringing into play the slope parameter of the inclined plane. Which means, and here he uses mathematics that then seemed sophisticated, that the length of the vertical fall at each point is proportional to the square of the inclined distance. As any high school student today would recognize, in one go I obtain the Cartesian equation of the parabola, with respect to non-orthogonal axes, with on the one hand the direction of the throw and

[9] Niccolo Tartaglia (around 1500–1557) was an Italian mathematician, who gave away his technique for finding the roots of polynomial equations of the third degree to his fellow countryman Girolamo Cardano (1501–1571), Cardan in French, translated Euclid into Italian, and among other domains worked in ballistics.

on the other the vertical at the point of throw. This amounts to analytic geometry. On no account is it a proof in the physical sense, and Galileo presents arguments in which the terms used fit a mathematically formulated definition, contrary to what was done with the *impetus*. The role of experimentation remains modest here; it is a question of constructing mathematical facts, uniform velocity and vertical fall; simplification lies in the exclusion of all the rest, that is to say of the form of the stone thrown, air resistance, the place where the stone is thrown, etc. Today we would say that Galileo has eliminated all “noise” around the phenomenon. And two physical concepts then arise: initial velocity, which happily replaces *impetus*, and acceleration, which we call gravity. Economy, simplification and indeed, one can speak of an abstraction of reality, and even of a rift, in the sense “I kill reality” to conceptualize it. Mathematics plays with reality, with the purpose both of describing in the simplest manner and of being able to act, in the manner of an experiment. Because I am not at all sure that the purpose is—at least for Galileo—to understand a priori. To use scholastic language, Galileo did not search for the substance of the fall, but its form, and mathematical abstraction enabled him to succeed physically, and led him naturally to ballistics and water jet techniques.

Pierre Cartier: Indeed Galileo was foremost an engineer; he built fountains…

Jean Dhombres: I was almost going to say that his a priori mathematical understanding of the fall of bodies is indissolubly linked to his physical sense. Let me add that since Galileo, no one has seen a water jet coming out at an angle from a fountain in some park in the world without detecting a parabola. This realization of mathematics by a curve invented so many centuries ago is for me one of the most enlightening signs of modernity. Because, it indicates a modification of Greek mathematics, in the sense that one recovers geometric properties of the parabola (such as the existence of an axis of symmetry, whose perception is not obvious at the outset when one looks at a water jet) only because one gives an analytic description of it which here takes the form of a trajectory—what specialists call a parametrization. The object “parabola” has, therefore, changed even in the eyes of mathematicians: it has become enriched with an interpretative network. Similarly human consciousness, which out of the parabolic form of water jets created symmetry in French style gardens, Versailles being one of the most successful examples.

Cédric Villani: There are two trends: the “we want to understand” one and the “engineer” one. And both of these trends co-exist. We find them in Galileo and others. Let us mention Euler, who started fluid mechanics because he too had fountains to build and all sorts of things.

Pierre Cartier: Moreover, the first prize obtained by Euler from the French Academy of Sciences concerns boats and sails, when he had never seen a single sailing boat—not many could be seen in Basel!

Cédric Villani: Let us also mention Newton,[10] who wanted to understand the order governing the planets. Here we have a desire to understand, with also a rather surprising relationship between reality and theory. It is said that Newton somewhat altered the results of his observations so as to make them fit well with his theory.

Pierre Cartier: Copernicus and Kepler as well. They all cheated a bit!

Jean Dhombres: Certainly! But did they cheat more that the Padua professor in my story, or more than those who wanted to keep the Ptolemaic system?

Mathematics as a Self-Sufficient Whole

Multiple Realities, One Language

Sylvestre Huet: In other words, mathematics is developed primarily to describe the world, then to understand it, and as the world is complex, it is simplified. Moreover, at the beginning, the most simplifiable natural phenomena were chosen so as to be able to use mathematical objects and mathematical methods available in earlier times. Besides, from then on, the mathematical part of all this becomes self-contained in the sense that there is already a reflection about the most basic objects intending to go beyond a simple abstraction of reality in order to simplify it, describe it, understand it, even act on it. Why, how, and what are, in your opinion. the consequences of this mathematical development which, starting from this period, desired to cut the link with the reality of the physical world—even agree with the Platonic vision: the meaning of reality would seem to be in this abstract definition and not in the reality itself. It is Plato's cave: the idea of the table is better than the table, the latter being only a more or less perfect realization of this idea. Is it still thought today that nature is written in mathematical language (Galileo) or that there is a "pre-established harmony" (Leibniz) between mathematics and reality?

Jean Dhombres: I will go back to Galileo's example. Basic objects held a fascination for the mathematician Galileo. And when he discovered with the help of his parabola, which is not a basic object, that he can say that the distances covered in equal amounts of time are like odd integers $(1, 3, 5, 7, 9\ldots)$, he was fascinated by these ancient objects that are not arbitrary numbers. A study that may be considered a study of reality, even if many things have been set aside since only the stone's motion is being observed,

[10]Isaac Newton (1642–1727), English mathematician, physicist, philosopher and astronomer. This emblematic scientific figure is especially known for having founded classical mechanics, elaborated the theory of universal gravitation and, along with Leibniz, developed infinitesimal calculus.

now included objects, odd numbers, that have been already studied and that have well-known properties…

Pierre Cartier: To understand the sudden appearance of odd integers here, the *gnomon* needs to be mentioned. This plane figure known since time immemorial is a square made up of small squares and cut into strips. The figure readily shows that the strips are successively made up of 1, 3, 5… small squares, thus implying relations:

$1^2 = 1$
$2^2 = 1 + 3$
$3^2 = 1 + 3 + 5 \ldots$

which are the ones that fascinated Galileo.

Jean Dhombres: Exactly, and when Galileo found this correspondence, he marveled that seemingly independent things, namely number games, appear at the core of something real. To recover something which seems to be from "elsewhere" starting from what can be labeled some form of reality is almost a proof of truth for Galileo. It cannot be a matter of chance. However, the proof is not of mathematical type. I should, therefore, like to stress that Galileo's famous 1623 book "The Assayer" does not mention mathematics as the language of nature, which at the time meant the language of physics in the old sense, but as the language of the universe. Probably not so much to describe all there is, and refer to reality, but to encompass all that has a structure, and in particular he could only be thinking of the Solar System.

Gerhard Heinzmann: I think you will find an answer to your question if you change a word in your description. It suffices to say that mathematics helps to *articulate* reality, instead of saying that it *describes* reality. If mathematics is considered a language, namely that of the most general relations, the mathematician takes "his" language as the object of his study, just like the grammarian looks at the structure of the natural language. However, mathematics can be reduced to logic and to set theory (belonging is then the only relation related to a content). This leads to more or less two possibilities: either you consider sets as primitive abstract objects (in a Platonic perspective or an instrumentalist perspective) forming the positions of your mathematical structure (=language), or you consider the structure of the sets themselves as a primitive object. It is then a useful tool but in principle open to review.

I think it is better to get free of the idea of Platonism with respect to objects—there are mathematical objects in a self-contained universe with respect to reality… I am not a historian and I do not consider the real development of mathematics, but I am interested in your question on the language aspect of mathematics from a logical point of view. And from a logical point of view, there is no difficulty in explaining that a mathematician is someone who develops a language he understands. We all speak a language, we all make abstractions using plain language: for example, if I show you a table saying "table", it is an abstraction for you, you cannot even really see this table as it can be seen from so many possible angles… And yet, the word "table" is as a general rule sufficient to very sensibly indicate the object singled out, though the object is not strictly speaking defined. Primitive mathematical objects (like structures or sets) are obviously very abstract objects but sufficiently sensible to be used as the foundation of a theory. The language of mathematics is "in harmony" with reality because it is our tool for getting to know (using appropriate scientific definitions) reality and not because nature is written in mathematical language or because there is any preestablished harmony.

Mathematical Tools to Articulate Reality

Pierre Cartier: I am going to incorporate a second keyword in our discussion. It is *tool*. On the one hand, language and on the other, the tool. I am extremely struck by the mastery of the mathematical tool acquired by ordinary people in the last 50 years. Allow me to mention an anecdotal example: when I was 14, at the end of the war, a mutualist school was founded in my hometown on Proudhon or Bertrand Russell model. The purpose was to educate young workers to enable them to obtain an advanced specialized technical degree or its equivalent. Facing me were 18-year-old "kids"—when I was only 14—and I used to teach them mathematics. The lesson on negative numbers set me the greatest difficulty. It was really stupendously difficult to explain that minus multiplied by minus gives a plus, etc. Sixty years later, I had left my car in an underground parking and asked my 4-year-old granddaughter whether she remembered where it was and she confidently replied; "in -7". At 4, my granddaughter knows what a negative number is, she feels it, the tool has been incorporated.

Let me take another similar example: my son-in-law teaches in a technical high school. One assumes that in a technical high school, the level of mathematics is surely not very high. Well this is wrong! His mathematics class is more difficult than what he would teach in a general high school, but it does not focus on the same things. For instance, he teaches complex numbers[11] in his class on electricity in conjunction with experiments. Yet, I can assure you that in my final high school year, our mathematics teacher did not know what a complex number was. Our physics teacher was slightly more knowledgeable. Today, it is part of standard skills, of our background. To some extent I think of the progress of civilization as the gradual expansion of the notion of a number, and more precisely of the notion of numbers that can be manipulated. Obviously today, the notation 10^x has come into common use: there are megas (10^6), gigas (10^9), teras (19^{12})..., which was not the case forty years ago.

[11] See note p. 52.

On the other hand, I also insist on the fact that a new tool is widely used since the 20th century (invented in the previous century): the *matrix*.[12] When Heisenberg[13] gave his matrix formulation of quantum mechanics, he went to see Born[14]—who was his boss—and the latter said: ‘Wait,this reminds me of something! There is a construction resembling this in a former mechanics course. Lets see this together”! Hence in the 1920–1930s, the best minds discovered matrices—I must admit that this is not quite true, some algebraists (Jordan, Minkowski, Artin…) had already manipulated them, but more confidentially. However, matrices have now become a basic tool, in particular in statistics. Likewise, even at a basic level, electrical (impedance, etc.) and linear optical laws are no longer taught without mentioning that there are matrices behind. A good course must mention them. Furthermore, it is a usable tool, you can compute eigenvalues on your calculator, etc. Now, let us not forget that most of Le Verrier’s[15] work on the discovery of Neptune is based on the diagonalization of 4 by 4 symmetric matrices. Today with a calculator he would get the result instantaneously.

Mathematics thus create tools, and society’s appropriation of these tools is part of mathematical progress. It can, therefore, be said that in some simplified way, there is an investment starting from reality, an abstraction, and a return to reality.

Cédric Villani: The notion of tool described by Pierre is central, and to come back to the question about why one studies mathematics “in themselves”, sometimes in a disconnected manner from the “real world”, I would say that in mankind’s history, each time a new tool appears, in any field, there is always someone to rush in and study for its own sake. There are specialists in political law for the sake of political

[12] A matrix is a square or rectangular array of numbers, such as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 3 & 4 \\ 9 & 8 & 7 \end{pmatrix}.$$

Two matrices of the same size (2×2 for example) can be added by adding the corresponding entries together:

$$\begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 8 \\ 9 & 11 \end{pmatrix} = \begin{pmatrix} 7+2 & 5+8 \\ 4+9 & 3+11 \end{pmatrix}$$

There is also a multiplication rule. Let’s just say that:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ so that the matrix } \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

is a faithful representation of the complex number $a + bi$. As a result, calculation rules for complex numbers are justified by matrix calculus. In geometry, a rotation on the plane (or space) is represented by a 2×2 (or 3×3) matrix and matrix multiplication corresponds to the composition of rotations.

[13] Werner Heisenberg (1901–1976), German physicist, who received the Nobel prize in physics in 1932, famous for his uncertainty principle.

[14] Max Born (1882–1970) German theoretical physicist, who received the physics Nobel prize in 1954 for his remarkable work in quantum theory.

[15] Urbain Le Verrier (1811–1877), French astronomer and mathematician specialized in celestial mechanics, discoverer of Neptune and founder of modern French meteorology.

law, specialists in economy for the sake of economy, specialists in the dialect of the Isle of Man or in I do not what, who are in this field *for its own sake*, who go more thoroughly into it *for its own sake* before interacting with others. Likewise, we mathematicians have an extraordinary tool, which we study for its own sake, in all its aspects, which we take to greater depth and so on.

I would like to add two remarks on the question of status. At first, we talked about simplification starting from the real world, in which the dual motion of enrichment and explanation which exists between reality and mathematics is already underway. From time to time, the need to explain something arises and enriches the existing corpus with a new mathematical concept which takes its place as a new tool. For personal reasons, I found one of these concepts fascinating. It is the notion of *entropy*,[16] introduced in statistical physics by physicists Boltzmann[17] and Maxwell.[18] I want to emphasize that Boltzmann created a mathematical concept to understand the physical world. Once this concept became an object, a mathematical object, this tool was further developed and used as mathematics in numerous fields. In a single motion "from the concrete to the abstract and then back to the concrete", entropy can be found everywhere, in every form, in problems concerning the study of gases as well as information exchanges between mobile phones, or even the study of living phenomena… We see how a new tool is shaped: we have a new concept, a new mathematical tool, which is first studied as such—entropy for entropy's sake—, then come those who make the link, the bridge. There is a constant dialectic, concrete/abstract, mathematics/return to the real world.

Complex Numbers: A Tool Then a Reality

Jean Dhombres: In my view, this idea of construction based on a back-and-forth approach between phenomena and concepts is essential to understand the role of mathematical tools even nowadays and since the times of Galileo, Kepler, and Descartes. So I would l like to give another historical example, which always perturbed me, that of complex numbers.[19] They are known to have been developed in

[16]Entropy is a quantity characterizing the disorder of a system.

[17]Ludwig Boltzmann (1844–1906), Austrian physicist, founder of statistical mechanics.

[18]James Clerk Maxwell (1831–1879), Scottish physicist and mathematician, famous for his equations unifying electricity and magnetism, as well as for his work on the kinetic theory of gases.

[19]A complex number: it is known that the square of an arbitrary number (integer or decimal) is positive, so -1 cannot be a square. However, if we wish all second-degree equations, without exception, to have two roots, then admittedly, the equation $x^2 + 1$ has two roots, i and $-i$, using conventional notation. If we combine the new number i with the old ones, and want to perform all the usual operations (addition, subtraction, multiplication, division), all these numbers can be reduced to the normal form $a + bi$, which defines complex numbers (as opposed to usual numbers such as a and b said to be "real"). Gauss, Argand, and Cauchy's great discovery around 1800 was to give a geometric representation of these numbers: $a + bi$ corresponds to a point on the plane with coordinates $(a; b)$. As Euler found out, trigonometry can be very much simplified using complex numbers.

the 16th century as a pure mathematicians' object to solve polynomial equations of the third and fourth degrees—one hardly used to go beyond. These numbers play no role in what can be alled the scientific revolution, even if Descartes devised the far more general idea of imaginary numbers. And then, almost one and half century later, after some calculatory type work like that of Viète on trigonometric formulas expressing the cosine or the sine of the multiple of an angle as a polynomial function of the same cosine and sine of this angle, complex numbers appeared in a completely different form. As the complex exponential presented by Euler in 1748 in a book with a fascinating title, *Introduction to Analysis of the Infinite*—a title unfortunately reduced in French to "infinitely small". This second rebirth is the real birth of complex numbers, though still called "imaginary quantities", following Descartes. Euler obtained his famous relation $e^{ix} = \cos x + i \sin x$, which connects complex numbers to reality in that the geometrical angle, here, x is henceforth precisely represented by a purely analytic formula, and in which what is denoted by i here is represented by Euler as an expression whose square is -1.

Pierre Cartier: Indeed, this is how the theory of angles was really founded.

Jean Dhombres: And Euler realized this immediately and formulated it as: the complex exponential underpins the angle.

Pierre Cartier: The same holds for trigonometry, which had no logical foundation before.

Cédric Villani: It also underpins the construction of π. Until the 20th century, the definition of π remained a matter of controversy, to such an extent that the German mathematician, Edmund Landau, defined π as follows: after constructing the complex exponential, π is defined as the smallest root of sine. Unfortunately for him, in his days, his work was considered anti-Aryan or at least degenerate, because having such a twisted mind… Euler's relation was still controversial two centuries after its discovery.

Jean Dhombres: That is quite understandable: it was an analytic construction (let us say of an argument not focused on a figure) which shed light on (I do not say for a moment contradicted!) what the geometry resulting from Euclid had not been able to fully master. But once Euler obtained this construction, a change becomes visible in the mentality of mathematicians: they then tried to find properties of complex numbers for their own sake. We would like to say that they identified two structures: a two-dimensional normed real vector space structure (what is called the topological plane), and a field structure, to use terminology in vogue for almost two centuries, even that they obtained the only commutative field structure not consisting of real numbers, but this will only be proved by the Russian school in the 20th century, in the theory of Banach algebras. To put it differently, we would like these 17th century mathematicians to have interpreted as necessary the operations of addition and multiplication of complex numbers regarded earlier as mere calculation methods.

However, the addition of two complex numbers can be seen in the form of a parallelogram whose construction is based on two of these numbers starting from the origin and whose diagonal is the sum.

The second structure, that of a field, reduces the question to that of the angle of geometry, understood as similitude: it suffices to see that multiplying by Euler's e^{ix} is the same as performing a rotation of angle x. Taking the norm of a vector, also called the absolute value of the complex number, multiplication will then give a homothety. Homothety and rotation, by composition we get all planar similitudes. But this is not what happened: even among brilliant mathematicians, the idea spread that Euler's formula justifies the unproblematic generalization of all mathematical formulas to complex numbers. It is Cauchy who, through his 1821 course, managed to put an end to blindly optimistic analytic practices, which in effect were destroying the entire mathematical tradition.

In my view, what is surprising is that 50 years earlier, mathematicians, in particular D'Alembert, had decided to study axiomatically matters concerning the angle, especially to find the minimal assumptions on which it is possible to found addition of unnamed objects that had become familiar in the form of force because of analytic mechanics and that today could be called vectors. Remarkable work followed in so far as addition which we call vectorial was found to be unique. But they do not put forward the vector space structure. The interpretation of the tool "complex numbers as elements of a two-dimensional normed vector space" had to go through another event: the geometric interpretation of complex numbers. This is a major reform of Euclid's geometry because it provides a means to compute punctual planar transformations. The introduction of this fundamental representation—far from minor—is mostly due to Argand,[20] considered a "minor author". He understood something and successfully imposed it (but how slowly…) because with his representation, he could prove—in less than a page—that every polynomial has at least a complex root. Even though a topological property was missing, it was clean, new, in line with the rigorous spirit of the times. The advantage of Argand's approach is that it grasps what it is that makes the difference between real numbers, over which it is only possible to go straight in one or the other direction around a point, whereas for complex numbers, one can go round a number, the operation being performed algebraically by multiplication, and hence by operations well adapted to polynomials.

Cédric Villani: Is it then Argand who is at the origin of the fundamental theorem of Algebra?

Jean Dhombres: Yes, in the sense that he was the first to give the simplest proof, which is today adopted by every textbook, but no one wants to recognize this! In France it is in general attributed to Cauchy, who only copied Argand, hiding the fact he had borrowed it.

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[20] Jean-Robert Argand (1768–1822), Swiss mathematician, who worked in Paris and was known for having introduced the planar representation of complex numbers in 1806.

Pierre Cartier: Gauss, like Cauchy, also copied, but these are illustrious names remembered by the history of mathematics, contrary to that of Argand. No one reproduces the frightfully complicated proof given by Gauss in his thesis in 1799.

Cédric Villani: This theorem is even called "D'Alembert's fundamental theorem of Algebra".

Jean Dhombres: Except in one German tradition, where it is still called Argand's theorem—and clearly this is a tradition that goes against Gauss! My purpose was not to take a historical standpoint, but to underline that a purely constructivist theory—"I take elements and then I build"—does not always work.

Cédric Villani: You are a bit harsh when you say that the constructivist theory does not always work, but let us say that indeed it does not lead to everything. It is a part of the structure and then there are also bridges, motivations…

Jean Dhombres: Exactly. To be precise, I think there is in Argand what I would call intuition with all due precaution, as it is the growing awareness of the reality of what "going around" means, which can be made algebraic in that it can be expressed in terms of polynomials. The precise statement of Argand's theorem that "in the neighbourhood of a complex value which is not the root of a polynomial there is another value for which the absolute value of the polynomial is strictly less than the value of the polynomial for the initial complex number". It is, therefore, a simple formulation of the extreme value theorem for differentiable functions of a complex variable, an idea due to Legendre. But I do not intend to turn Argand into a prophet. By understanding the perfect match existing between complex numbers and polynomials, by building as Cédric just said a bridge, Argand did what was appropriate, which corresponds to what the Greeks called "suitability" or "symmetry"; in those times, it was in no way included in the "constructed" definition of a polynomial, but it has become natural.

Pierre Cartier: I would like to add a historical detail about complex numbers. In the 18th century, in an article on winds,[21] D'Alembert introduced complex numbers in a very explicit manner: he introduced two coordinates x and y in the plane, he did $x + iy$, and he expressed in fluid dynamics what later came to be known as Cauchy-Riemann equations.

Cédric Villani: He already saw the connection with complex analysis?

Pierre Cartier: Yes, in effect he uses a method which will be later taken up in aerodynamics in the 20th century. It uses (holomorphic) functions of a complex

[21] *Reflections on the General Cause of Winds*, 1745.

variable and thus performs a conformal transformation.[22] This enabled D'Alembert to study the distribution of winds on a large scale.

Jean Dhombres: In his study of winds, D'Alembert defined complex exponentials as though they were actions, and to explain this, I am going to use "I", thus reproducing a frequent habit among mathematicians (remember our school years: "I draw the line from A to B", "I set the operation…"). So a priori, I set $e^{ix} = a + ib$. I now make the two coefficients a and b depend on x by considering them as functions. I accept that the fundamental property of the exponential is to equal its derivative. I thus obtain $ie^{ix}dx = da + idb$, and as the left hand side is just $i(a + ib)dx$, separating the real term from the imaginary one, I get two equations from which it is easy to deduce that the function b/a is the solution of the differential equation of a tangent ($d(b/a) = (1+(b/a)^2)dx$). This introduces the angle and then gives Euler's formula $e^{ix} = \cos x + i \sin x$. It is a heavy-handed proof, which is so inductive that it does not seem to satisfy the axiomatic methodology. We agree to say it lacks rigor, when what is troublesome is not this but that the existence of the exponential as a complex number is presumed, and that this presumption leads to a unique object. The method, none other than that of Descartes and called the method of undetermined coefficients by D'Alembert, is probably the place where one best sees the mathematician's freedom of imagination, but also its limits, as though the real world were holding out against it. Something similar to the gesture of the fisherman in Indian thought who throws his net into the unknown, rather than logical thinking, led D'Alembert to set $e^{ix} = a+ib$ (who was not familiar with the then unpublished work of Euler). D'Alembert can be said to have brought back in his net the angle, a notion that is part of our everyday life, our experiments, our experience, albeit in a different guise, but not distorted, included in the construction of the field of complex numbers.

Pierre Cartier: Two small historical additions. We have talked of Euclid's rigor, and rightly so. But it is important to know that before the 17th century, and essentially before Euler and D'Alembert's work, the theory of angles did not exist. What is an angle of 1 °? To answer this question you need to know how to construct the regular polygon with 360 sides, which, according to Gauss, is impossible. Hence talking of an angle of 1° amounts to talking of something that does not exist. And in this context Euclid's first proof (the construction of the equilateral triangle) is wrong: I have a base, I draw a circular arc, another circular, they intersect, and I have won! But why do these arcs intersect? In Euclid's first proof, nothing in the axioms guarantees that the two circular arcs intersect. In fact, it is only at the of the 19th century that Pasch[23] and others made sure this works—which was later summarized in Hilbert's[24] geometry book on the foundations of geometry.

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[22] Such as when drawing the map of some region of the Earth, angles are respected, but not lengths…

[23] Moritz Pasch (1843–1930), German mathematician, who wrote one of the first books axiomatizing geometry (1882).

[24] David Hilbert, *Foundations of Geometry* (1899), ed. Jacques Gabay, 1997.

Cédric Villani: Euclid supposed the two circular arcs intersect: it is intuitively obvious, but in fact it does not follow from the axioms of geometry… So how can the gap in the proof be filled? An analysis argument is needed, a study of the distance function…?

Pierre Cartier: Absolutely, one has to use analysis.

Cédric Villani: How sad…

Gerhard Heinzmann: The purity of methods has to be abandoned. Aristotle used to forbid going from "one genus to another" in a proof, "for instance, proving geometrical truths by arithmetic".[25] And though Descartes and others later applied algebraic methods to geometry, this ideal of purity remained largely untouched.[26] Among those who gave up the epistemological value of the purity of methods there is Henri Poincaré, for whom analytic methods are not only more concise but enable us to obtain new results in "geometry" (=topology) through association:

> "Indeed, the sole purpose of geometry is not to provide immediate descriptions of bodies falling within the scope of our senses: it is foremost an analytic study of a group; as a consequence, nothing prevents us to deal with analogous and more general groups. It will be said, why should we not preserve the analytic language and replace it by a geometric language, which loses all its benefits as soon as our senses can no longer be involved. The fact is this new language is concise; next that the similarity with ordinary geometry can create fertile associations of ideas and suggest useful generalizations".

Pierre Cartier: To found geometry, analysis cannot be dispensed with. Hilbert's book on the foundations of geometry—and even better, Artin's *Geometric Algebra*—does a superb job of showing that a whole geometry can be founded over an ordered field (algebraic numbers, rational numbers, etc.), but that to obtain unsophisticated, yet fundamental, results on angles, intersecting curves, etc., at some point, recourse to analysis cannot be avoided, in other words, the whole field of real numbers needs to be generated. And the latter cannot be defined without extensive recourse to infinity.

Jean Dhombres: In support of all that Pierre said about the unavoidable aspect of infinity when it is reworked by analysis, and this time going from geometry to algebra, I would like to return to Argand and his proof of the fundamental theorem. On the one hand, because he lacked a property ensuring that the extremum of the absolute value of a polynomial in a disk is effectively reached for some value of the variable (Cauchy too lacked this). Topology got really started in the last third of the 19th century when the lack of a correct proof for such a property was realized; on the other hand, because the manner Argand studied inequalities consisted in cuttings, now ordinary in analysis, by which we ensure that the omission of certain terms is

[25] *Posterior Analytics*, Book I, Part 7.

[26] Michael Detlefsen, Andrew Arana (2011), "Purity of Methods", *Philosophers imprint* 11 (2), 1–20, p. 5.

safe. In my view, the game of epsilons and deltas, which makes up the anthropology of the analyst, begins with Argand, even if it is with Weierstrass[27] that this became a general rule.

Gerhard Heinzmann: Historically, this change in attitude is remarkable. From the Euclidean model one passes to an analytic model and the tool changes the mathematical problems.

Pierre Cartier: This is what the whole changeover of the 19th century is about.

Cédric Villani: A changeover which is found in embryonic form in Euler's history: analysis is put back at the core of the entire structure.

The Nature of Mathematical Objects

Intuition: To Discover or to Invent?

Cédric Villani: In my view, the issue of the status of mathematics is not all that important. Is it Platonic (mathematical objects preexisting in a world of ideas), Aristotelian (mathematical objects used to describe the *real* world), etc.? What is important is not the tool, it is the way it is used… Of course, philosophical conceptions can influence the manner research is conducted in mathematics, in the sense that someone convinced there is something intrinsic waiting to be discovered will not pursue research in the same way as someone persuaded it is a manmade construction movement. Personally, I belong to the school of thought that posits a preexisting harmony and which, for each given problem, seeks for the gem, being convinced that it exists. I am one of the miracle seekers, but not one who creates it or seeks for something very clever in his own resources.

Gerhard Heinzmann: But what are the conditions for acceding to this truth with no clausal link with us? How can it be affirmed there exists something abstract? Either we allow ourselves an abstraction procedure, or we are forced to say "it is by a particular ability", some sort of unexplained intuition.

Cédric Villani: Yes, an unexplained intuition, a personal and almost religious conviction.

Gerhard Heinzmann: This is the weak point of this conception. The others, the constructivists, are more rational But perhaps they are less successful.

[27] Karl Weierstrass (1815–1897), German mathematician considered to be the founder of modern analysis.

Cédric Villani: Constructivists, who have less faith in miracles, may not rush in so many wrong directions, but this is to be seen... Anyhow, my way of proceeding, which always consists in searching for bridges between already existing things is more that of someone persuaded of a preexisting harmony.

Sylvestre Huet: Henri Poincaré said: "Logic ... is the instrument of demonstration; intuition is the instrument of invention".[28] Is this remark directly connected to your exchange?

Gerhard Heinzmann: Yes and no! Yes, because the order of invention may proceed without logic. Associations, intuition, imagination, dreams, fictitious entities: everything can be useful and justified as an invention tool. However, to justify, in other words to prove, for example the hypothesis that the study of some transformation groups leads to geometries with constant curvature, correct "logical" arguments are needed. Now, the difficulty resides in determining what is meant by "logically correct". Contrary to what the above quoted extract from Poincaré may lead to believe, for Poincaré logic was neither the only means to reach certainty, nor external to the realm of intuition. On the contrary, *pure* intuition also gives certainty and enables us not only to invent but also to *demonstrate*.[1] Poincaré was opposed to the logicians' thesis who claim to be able to demonstrate all mathematical truths without having recourse to intuition, once the principles of logic are admitted. Everything then depends on what is admitted as "logic". Since, with the extensive development of mathematical logic, the answer to this question is now more balanced than in Poincaré's days, Poincaré's interesting and modern argument against logicism is not so much the conjecture that there does not exist any purely logical transposition for each mathematical reasoning, but the affirmation that this transposition would lack epistemic values necessary to understand mathematical reasoning—and I think he is right!

Pierre Cartier: I largely subscribe to the thesis developed by Gerhard. I would happily use a metaphor, that of the motion of a pendulum. Watching it gives the impression of cogs and wheels fitted together in a complicated and precise manner, which enables it to start once the initial impulsion is given. This is equivalent to a logical system, and a very complicated computer program is an object of the same type. But the conception of the motion of a pendulum—or of a computer program, or even of the world's great clock according to Kepler and Newton—is the result of a different process, which cannot be reduced to a game of cogs and wheels. A long and complicated mathematical proof also requires preplanning, and hiding this from the reader may jeopardize his understanding. Is this not some sort of intuition? Does intention the same as intuition?

Jean Dhombres: At least in mathematics, constructivism flourished for a long time! This is what I was emphasizing by recalling the success, but also the weight, of

[28] Henri Poincaré, *The Value of Science*, p. 23.

Euclid's *Elements* as a structure. Can it be said that it is through intuition that Euler obtained his formula? We saw that the same formula follows from an impressive feat in D'Alembert! Anyhow, because of this formula, Viète's trigonometric formulas become trivial, or rather they can thereby be explained, and connections are forged between algebra and trigonometry coming from geometry. In return, is it not possible to think that Viète's formulas, which made him burst with joy, because of their very complexity have pushed mathematicians to find the hidden reasons? The motivation to simplify ties up with the idea that what is complicated is deficient. In my view, this motivation paradoxically justifies what Gerhard said about the role of logic. To take just one example: around 1807, after many pages of calculations, Fourier found what today is known as the Fourier series of a sawtooth function. The coefficients he obtains can be expressed simply, but their computation is awful. It is not logical. He, therefore, felt there must be a deep reason for this simplicity. As testified by his manuscript, he then immediately discovered the orthogonality relations: the whole theory could then develop analytically and he formulated in his "Analytic Theory of Heat". In order not to fail historical truth as well as not to prevent the understanding of what Karl Popper calls "the logic of discovery", he decided to include the computational stage.

Infinity: A Mathematical Necessity, a Physical Abnormality

Sylvestre Huet: Could you try to summarize this whole issue of the relationship between mathematical objects and reality, and why the ancient quarrel Plato/Aristotle is still going on? For example, Alain Connes made this unequivocal statement: "In my view, mathematics represents the only coherent strategy to understand and specify the external material reality unambiguously". In other words, he admits being at one extreme philosophically. In my professional experience, I noticed that this extreme position is shared by most "great" mathematicians. It is probably no coincidence. Anyhow, what is somewhat fascinating in the way in which you are presenting this relationship is, on the one hand, that notions of wonder and fascination stem both from discovering that a material reality can be mathematized in order to specify it better, possibly to understand it and *in fine*, to act; and on the other hand, the fact that self-contained developments become tools to specify this reality, as for example complex numbers that were in no way initially conceived to strike a significant chord with the real world.

In the evolution of mathematics and of their relationship with other empirical sciences, it is rather extraordinary that more and more complicated objects can be gradually mathematized, in particular coarse objects such as fractals used to describe the shape of clouds made of water vapor, snowflakes, or the coastline of Brittany... All this proves the awesome power of mathematics in the construction of other sciences (natural sciences, physical sciences, geography, etc.). How do you see this tremendous extension of mathematical tools? And, according to you, you who seem not very Platonic, is the ancient quarrel Plato/Arstotle still relevant?

Cédric Villani: In fact, I consider myself to be a Platonic mathematician, but I think this quarrel should not be continued, that it should be overcome. Mathematics should be addressed not in terms of conceptions, but in terms of approach, psychologically speaking. Everyone has different aptitudes, some are good at something, others at something else...

Gerhard Heinzmann: Platonism is a philosophical position which says that "mathematical entities exist autonomously, independently of the mathematician, and that the latter can reliably and indisputably access them". As I mentioned above, as a founding position, this position is controversial and, like all philosophical positions, has advantages and disadvantages. However, I get the impression that Cédric does not at all wish to defend Platonism as a philosophical position justifying his mathematical activities, but rather as a heuristic position to *do* mathematics. He is free to do so and a majority of mathematicians are probably Platonists in this sense, though they do not defend Platonism in a philosophical sense.

Sylvestre Huet: Let us take a concrete example... Pierre Cartier said we need infinity, which is a purely mathematical object, to construct basic mathematical tools such as real numbers. Now, the theoretical physicist's great fear is to find an infinity in his equations for he knows that at that "place", he will no longer be able to apply the laws of physics: in the real world, nothing is infinite. The obsession of the physicist will, therefore, be to get rid of it, even though he previously needed it in his arguments.

Cédric Villani: How can infinity be imagined? There are many ways of imagining infinity. In fact there are researchers who try to do without infinity—and it is far harder to work without infinity that with it. Infinity is an idea, an abstraction and a shortcut for a very large number, a number as large as wanted, which simplifies things. Historically, immense progress has been achieved by trying to do without infinity, by replacing it with numbers "as large as wanted" having quantitatively measurable, precise bounds; in contrast, immense progress have been achieved using infinity. Instead of being afraid, a physicist encountering a singularity (something discontinuous, a punctual change of the model's properties, somewhere density becomes infinite... a black hole, the big bang, or anything else) thinks: "This is where the thing is hiding". For example, Landau[29] discovered "Landau damping" by studying the singularities of the Vlasov equation in plasma physics, etc. I am sure that it is possible to find numerous examples. Thus, for those with a certain mind set, who are very pragmatic and want to build tools, doing without infinity becomes a major challenge; and for others who are seeking the key to the mystery, the idea that there are phenomena in the real world governed by some sort of "infinite idea" supposedly existing in a certain universe is also very important. In short, I think that it is necessary to move beyond the discussion "is mathematics like this or like that"?

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[29]Lex Davidovitch Landau (1908–1968), Russian theoretical physicist, who received the Nobel prize in physics in 1962 "for his pioneering theories on the condensed state of matter, in particular liquid helium".

Sylvestre Huet: Are we to understand that the Platonic/Aristotelian position depends on the way mathematicians work? In other words, when Alain Connes says "I explore an external world…", the important word is not his Platonic standpoint about the fact that there are supposedly mathematical objects existing independently of man or of material reality, but the verb "explore" which describes his way of working.

Cédric Villani: Absolutely.

Pierre Cartier: Besides, this can be illustrated using fractals. Even if I have some reservations about Mandelbrot,[30] his stroke of genius should be recognized: He *saw* what was before everyone's eyes and what no one had seen before him. Yes, these wretched fractals were there in front of us! They are everywhere, in painting, in clouds represented on the photo hanging on this wall…

Now, mathematics can sometimes give rise to a certain myopia if one is focussed on something too precise, forgetting what is beside it. It is well-known that great discoveries—not only in mathematics—are often things that are looking us right in the eye, but that nobody sees, and all of a sudden: the king is naked! What was visible to everyone is discovered and explained to others.

Cédric Villani: It is like this idea which lasted for a long time that all functions are differentiable; even Cauchy at some point though that continuous functions were always differentiable...

Pierre Cartier: Galois also used to believe this.

Cédric Villani: Galois too? I did not know. And then one day non-differentiable functions were constructed. Initially, everyone said: "What is this, it is not possible?", then finally everyone changed their mind: "But yes, not all functions are differentiable, we have been using them for centuries and we never realized!".

Pierre Cartier: This hesitation is understandable because we are always a bit frightened in front of infinity. Now, when you want to construct a non-differentiable function or fractals, you have to iterate infinitely many times. If you stop halfway, you're done for. On the other hand, I gave a student from Strasbourg a master's thesis subject based on Linnik's[31] and Erdös'[32] work on stochastic properties of the distribution of prime numbers: I asked him to simulate Brownian motion using prime numbers. The student showed me his result, a curve, and I immediately replied, shocked: "No, it is not a Brownian motion". I could not readily explain why—it greatly swerved

[30]Benoît Mandelbrot (1924–2010), French-American mathematician, famous for having developed a new class of mathematical objects: fractal objects (or fractals).

[31]Yuri Linnik (1915–1972), Russian mathematician whose work was mainly in number theory, probability theory and statistics.

[32]Paul Erdös (1913–1996), Hungarian mathematician who largely contributed to the development of number theory and combinatorics.

and then returned too slowly. We then analyzed his result and understood where the mistake came from because we are rational beings: he had simply wrongly copied a formula. With a minor correction, it worked. I want to show through this example that it is possible to get the intuition of what a Brownian motion is, of what a fractal is. In the case at hand, my intuition resulted in a shock: the student's result did not correspond to what I was expecting. This is both one of the difficulties and one of the beauties of mathematics: to know whether there are mathematical ideas in some empyrean or other, but anyway there is a certain mathematical reality with different realizations—either in the calculations performed, or in the images under study and which strike you, raise questions, and force you to answer them. Still there is something in front of us. Even if I do not believe in the ideas of the cave which would have us restricting ourselves to observing the shadows of reality of another world beyond us, I am sure that a mathematician does not act in a void. Though I am not a formalist—anyhow not too much of a formalist, just what is needed...–, I recognize the power of formalist reasoning, of the formalization of language and of its extensive logical contributions in the 20th century. But formalism does not exhaust everything because on occasion there is nevertheless this element of surprise or shock which expresses something deep, and which is not easy to analyze. A rational analysis is always possible after the fact, but it does not tell the whole story.

Cédric Villani: This is why Mandelbrot is nonetheless an important figure: he put back intuition at the heart of the debate, its sensual and visual aspect, at a time when this was badly regarded.

Pierre Cartier: I spoke just after Mandelbrot at the yearly meeting of the (French) Association of Mathematics Professors in Public Education, and in fact, I was surprised that we both said the same thing: "Do not be afraid, use drawings". It was quite daring to say this in the early 1980s in front of 2000 professors, in the temple of the Association, at a time when using drawings was not rather unacceptable!

Gerhard Heinzmann: But obviously, intuition should not be only seen in drawings, that is to say in its sensitive form. It is also present in a non-sensitive form in the understanding of a proof. It is legitimate to ask ourselves whether all can be formalized? And, indeed, the understanding of a proof cannot be formalized, under penalty of a regression to infinity. It requires some sort of intuition, understood to be the overall view of a "know how". Of course, it a step-by-step proof is possible, but this does not amount to an understanding of a proof.

Cédric Villani: In a proof, we usually mention: "This is important, this and this". We don't only give the logical design which solves the problem. For the listener to understand, we extract the key points that will enable him to convince himself of the truth of the proof and to recreate it, when required. This separation between what is "important" and what is "secondary" is not logical (for logic dictates that all is important!), but intuitive, calling upon usual diagrams, reflexes, analogies, summaries...

Gerhard Heinzmann: Precisely, this understanding forms a broad field in the hermeneutics and analytics of philosophical approaches.

Jean Dhombres: As is very often the case, history teaches us modesty in our epistemological conclusions, by not taking sides, leaving all their importance to the answers that have been brought above. The first time infinity played a role as such in a mathematical proof is in a 1795 paper of Laplace. By skillfully introducing a parameter that Gauss will take up again in 1815—forgetting to say where he had found it!—, Laplace proved the existence of a function from the reals to a finite set (in fact pairs of roots of a given polynomial equation). He did not at all know how to compute this function, and could not in any way draw it, and equally had no intuition about this function: yet the infinite characteristic of real numbers asserts that the said function will take the same value for at least two distinct real values. This is sufficient for Laplace to finish his proof of the fundamental theorem of algebra, without having any means to effectively construct roots.

Aesthetic Criteria in Mathematics

Cédric Villani: The issue of aesthetics in mathematics, which in fact was fundamental in the exhibition *Mathematics, a sudden change of scenery*,[33] is equally essential. Visitors to the exhibition were struck, but also very surprised by the beauty of the exhibits. The general public indeed finds mentions of the aesthetic aspect of mathematics surprising, whereas for mathematicians, this is trivial…—they are even surprised that they are asked about it!

Jean Dhombres: An elegant proof or a beautiful proof…, these words have meaning in mathematics, but which practitioners keep deep inside them.

Gerhard Heinzmann: It is nevertheless necessary to explain why a proof is beautiful. On this point, there is something in Poincaré that sets him apart from tradition: he regarded as aesthetic what has greater possibilities of applications, in other words of becoming possible models:

> "the mathematician must have worked as artist.
>
> What we ask of him is to help us to see […]. Now, he sees best who stands highest".[34]
>
> "For a construction to …serve as stepping-stone to one wishing to mount, it must first of all possess a sort of unity enabling us to see in it something besides the juxtaposition of its elements. …A construction, therefore, becomes interesting only when it can be ranged beside other analogous constructions, forming species of the same genus".[35]

[33] Exhibition *Mathematics, a sudden change of scenery* (Cartier Foundation) welcomed 80 000 visitors between October 21, 2011 and March 18, 2012.

[34] *The Value of Science*, pp. 77–78.

[35] *Science and Hypothesis*, p. 17.

In his opinion, topology (*Analysis Situs*) is the most aesthetic subject because it has applications everywhere:

> "[in] the study of curves defined by differential equations and to generalize them to higher order differential equations and in particular in the study of the three-body problem. [He] needed it to study non-uniform functions of two variables, to study periods of multiple integrals and to apply this study to the development of the perturbation function. In short, [for him], *Analysis Situs* was a way to approach an important problem in group theory, namely the search for discrete groups or finite groups contained in a given continuous group".[36]

This vision of aesthetics has a modern sense which stands out from the traditional science of beauty or from the criticism of taste; it includes a cognitive function: a proof is all the more aesthetic when it can be used in other fields.

Jean Dhombres: It is a beautiful answer in tune with its times, a time when philosophical utilitarianism had the upper hand; one and half century earlier, Newton would have probably mentioned the aesthetics involved in the practice of a unique method leading to the unification, if not of science, at least of one of its sectors, for what will become celestial mechanics. As for the eccentric English mathematician Godfrey H. Hardy, he mentioned the aesthetics of all mathematical theories having no applications! And for André Weil, aesthetics resided in a kind of metaphysical shiver that takes hold of the inventor, in a hurry to get rid of this beautiful weight by formalizing it for common consumption.

Pierre Cartier: The Hungarian mathematician, Paul Erdös,[37] a highly colorful character, used to talk of the proofs "in the Book". He meant those that have such a resonance, or such an element of surprise that nothing can surpass them and that they need to be included in an anthology [such an anthology exists, published by Springer]. Such a proof needs to go right to the point, while revealing some unexpected aspects. The law of quadratic reciprocity provides an example; Gauss gave six different proofs for it; the simple character of each is stunning, and so is the diversity: a book of hours!

Grothendieck has a somewhat different idea: to develop the natural foundations in such a way that difficulties dissolve, like the sea flooding the beach and becoming "etale" (one of his fetish words). In this sense, the most beautiful proof is the one about which one can simply say: "See!", which harmoniously fits in the structure. Grothendieck started his scientific work with "Functional Analysis", a field where Gerfand[38] had showed the way. The discovery after 1945 of Gelfand's thesis on "normed rings" gave rise to a great sense of wonder. It contained difficult and deep theorems—N. Wiener's Tauberian theorem, G. Birkoff's ergodic theorem—like so

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[36] See Henri Poincaré, Analyse des travaux scientifiques de Henri Poincaré faite par lui-même, *Acta Mathematica*, 38, 1921, pp. 1–135, p. 101.

[37] See his biography in *Erdös, the man who loved only numbers*, Berlin, 2000.

[38] Israel Gelfand (1913–2009), Russian mathematician, student of Andrei Kolmogorov, whose work is related to every mathematical field. He was, however, mostly known for his contributions to functional analysis and their impact in quantum mechanics.

many ripe fruits of a new theory harmoniously unraveling while retaining a high degree of abstraction: the eagle that descends from the sky upon its prey.

Aestheticism has pitfalls. In mathematics, a proof that is too beautiful is not often very *pliable*; it does not lend itself to modifications, it can close a path.

Sylvestre Huet: And is this aestheticism the same for all mathematicians?

Pierre Cartier: Mostly.

Gerhard Heinzmann: It is not about value, but about cognitive quality. A proof using many *ad hoc* elements and only usable in the field considered is less aesthetic than a proof applicable to several fields, for example like proofs in model theory.

Cédric Villani: Poincaré talked about the value of aestheticism as something that enables it to become useful, harmonious, but also as something the enables to guide mathematicians. These feelings are widely shared.

Sylvestre Huet: Are there other approaches of the aesthetic aspect of mathematics apart from that of Poincaré? In particular, it is not unusual to hear that a proof is elegant because it is short, clear, and uses very few axioms... Are there rules that need to be upheld, criteria like Ockham's razor?

Gerhard Heinzmann: The symptoms you mention must indeed be present to be able to talk of a mathematical aestheticism, independently of its theoretical conception. In this regard, in her book "Aestheticism and Mathematics",[39] Caroline Julien distinguishes between the Poincaré solution and the Platonic solution: according to the latter, the idea of beauty merges with the truth of mathematical facts that exist independently of the mind. On the contrary, for Poincaré, "beauty is not an idea in the Platonic sense, but a category that enables us to explain the pleasure of the senses". Nevertheless, American philosopher Nelson Goodman (1906–1998) is the one who profoundly influenced the history of aestheticism in the 20th century by replacing the question "what is art" by the question "when is art?". His answer thus contains technical and precise symptoms which answer your question: "simplicity", "saturation", and "exemplification" are symptoms (too technical to be clarified here) interpreted in the case of mathematics by Caroline Julien in her book.

Jean Dhombres: The "aestheticism of simplicity" may be classical, but, in mathematics, one also finds an aestheticism that could be qualified as baroque. The simplest example I can think of is found in John von Neumann's approach, which introduced the notion of a Hilbert space, concerning the mathematical foundations of quantum mechanics. Two models of quantum mechanics were available to him in his days, both were satisfactory realizations of experimental data. One of them (that of

[39]Caroline Julien, *Esthétique et mathématiques. Une exploration goodmanienne*, Presses universitaires de Rennes, 2008.

Heisenberg) used matrices, and was, therefore, a matter of the discrete character of numerical algorithms. The other one (that of Schrodinger) used wave functions, and fell under continuity and differential analysis. Von Neumann's viewpoint, which I would call aesthetic and baroque because he brings order into objective disorder, was to say that these two theories, of very different type, were both a matter of the same structure... which still needed to be found by suitable selection. And this is where construction intervenes. In this matter, what is it that is said to be beautiful: the idea of a common structure, or the determination of this structure?

Determinism, Chaos, and Predictability

Sylvestre Huet: How do you feel about the current state of discussions started 30 years ago on determinism and indeterminism, non-linear dynamic systems and deterministic chaos? Has this approach been fruitful for the sciences in general and mathematics in particular?

Pierre Cartier: There is no doubt that for celestial mechanics this was a major step forward, confirming Poincaré's intuition and paving the way for a new methodology to study the long-term evolution and stability of the Solar System. Many conventional ideas were knocked down. For the rest of mathematics, there is a small domain named "Study of dynamical systems" where ideas of this type are developed, but I do not think it has really irrigated the entire field of mathematics—contrary to that of physics.

Generally speaking, the discussion on determinism and indeterminism is multi-faceted and difficult. In Lapalace's perspective, an "omniscient" mind could clearly foresee the future—and recover the past. More realistically, information comes at a cost; the economic assessment has to be borne in mind: cost of the information to be collected versus benefits of forecasts made by means of this information.

A particularly striking example was discovered by J. Moser in the 1970s. It relates to an astronomical system consisting of two massive stars, rotating around a common center in accordance with Kepler's laws. A low-mass comet is traveling along the perpendicular axis to the plane of the two orbits, passing through the center of the system. Let us take the revolution period of the two stars as our year. Question: for one thousand consecutive years, it has been observed that the comet passes through the center. Will it come back or not the following year? Answer: no answer can be given, the reason being that the answer may depend on the 50,000th decimal place of the number giving the initial velocity of the comet, and that computing this number to this decimal place would require incredibly long observations. I don't have enough space here to develop all new viewpoints about Darwin's theory of evolution, about the foundations of thermodynamics and of the theory of gases,[40] about teleology, and finally about the supposed indeterminism of quantum mechanics. It is a very open

[40] Maxwell imagined a demon opening a door for fast molecules and closing it for slow ones. Precise experiments simulate such a demon.

field, and the debate between chance and necessity has become far more positive and fruitful.

Cédric Villani: Nowadays these concepts have become part of our common culture. They are the means of interpretation of numerous phenomena... Nonetheless, as Poincaré and then Lorenz fully realized, this theory is based on two pillars: the first pillar, exponentially growing uncertainty which makes all precise long-term forecasts impossible, is remembered, but the second one is, on the contrary, often forgotten: statistical predictability, enabling us to assert that this or that will happen with a certain chance of success... The greater the chaos, the better statistical predictability.

Jean Dhombres: I would like to add an anecdote, just to show that mathematicians are truly at the origin of the debate "determinism and indeterminism" when probability was seriously implemented in natural philosophy. In the 1758 second edition of his *Traité de dynamique* D'Alembert gave "proofs" for the principle of inertia, which is a form of simplified and absolute determinism. Eleven years later, the very young Laplace, a student with a barely one-month old masters degree from the University of Caen came to Paris; as he failed to be received by the academician, he wrote him a letter to tell him he had made a double mistake: the determinism of the principle of inertia cannot be proved by mathematics, nor by metaphysics.

But determinism had to be set out as a condition in order to do science, in particular to be able to turn probability into a tool to search for the causes of certain phenomena. His expressly unsubmissive manner won him D'alembert's trust, who found a professorship for Laplace in the *École militaire* of Paris... but did not change his dogmatic standpoint. Laplace never mentioned it again, but gave determinism his most famous statement, while insisting on the "delicate" handling of probability. It was no longer a question of "given that" which may characterize the Euclidean world, but of "do as if".

Sylvestre Huet: Besides this debate took on a very societal dimension, especially fed by a total distortion of Isabelle Stengers'[41] and Ilya Prigogine's[42] message. Actually the latter proved that Laplacian determinism—the demon that would be capable of forecasting the evolution of any system by knowing its laws and initial positions—cannot exist in a great many dynamical systems for they are not linear. Moreover, The French astronomer, Jacques laskars' work showed that this extends to the celestial mechanics of the Solar System, and that it is not possible to compute the future precise position of the Earth beyond some hundreds of millions of years. Nevertheless, for both researchers, this in no way amounts to an abdication of the human mind, on the contrary it represents a progress since it is now possible to compute forecast

[41] Isabelle Stengers, born in 1949, Belgian philosopher and historian of science.

[42] Ilya Prigogine, born in 1949, Belgian philosopher and historian of science of Russian origin, who received the Nobel Prize in chemistry in 1977 for his contribution to the thermodynamics of irreversible processes and to the theory of dissipative structures. In particular, he showed that when matter moves away from its equilibrium state, self-organization occurs—a phenomenon that can be observed in physics, biology and climatology.

horizons of systems whereas before the illusion of infinite predictability prevailed. This thought has been distorted and is often presented to the public in a caricatural form: “Laplacian determinism is a mistake, hence nothing can be predicted, and if nothing can be predicted, it is not possible to set out a political program for the transformation of society”.

Jean Dhombres: This was also Bruno Latour’s Nietzschean position, that of absolute relativism, all knowledge being merely a pose, for it hides the nature of social forces and the interests of all sides that it may impose it. But the tide has been turning, and the Sokal case, as well as Jean Bricmont’s comments, have had an impact; he has come back to feelings said to be of empathy with scientific practice.

Cédric Villani: It’s a dreadful discourse, which is still very much part of social sciences —and also other fields—, and which consists in saying: “Since it is very complex and we understand nothing, we should do nothing, or else we should change everything”. But when you ask those who say that “we should change everything” what it is that it should be replaced by, the answer is never very clear.

Jean Dhombres: One could also turn the argument around: since you got it wrong once—and one always gets in wrong once—, you are always going to get it wrong. This is a perfect sophist’s argument; and it is hard to counter without sinking into complacency, which is a bit the hallmark of scientism. This complacency sometimes lies in wait for mathematics, a science that has historically recognized insufficiencies, but never mistakes which would amount to contradictions and would have been accepted by a generation of mathematicians. It, therefore, also lies in wait for logic since the latter is largely mathematized. It is actually in the name of this sophism that Galileo ended up being condemned: by pulling down even a part of the Aristotelian construction, Galileo was seemingly undermining all truths, and thus creating chaos.

Gerhard Heinzmann: Indeed, going from sophism to scientism would be as absurd as giving up perception because of illusions, paradoxes or physiological anomalies, or as considering intuition inadequate because of its fallibility.

Cédric Villani: Ultimately, this is an issue in the moral sphere. A politician may make mistakes, but it is his duty to elaborate a project, it is his mandate. He has a vision, he tries, he adapts, but if for any reason he resigns then he no longer does his job. This goes beyond the issue of science.

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