

Chapter 2

Analytical Approximation Methods

2.1 Introduction

As we mentioned in the previous chapter, most of the nonlinear ODEs have no *explicit solutions*, i.e., solutions, which are expressible in finite terms. Even if an explicit solution can be determined, it is often too complicated to analyze the principal features of this solution. Due to such difficulties, the study of nonlinear mathematical problems is the most time-consuming and difficult task for researchers dealing with nonlinear models in the natural sciences, engineering, and scientific computing. With the increasing interest in the development of nonlinear models, a variety of analytical asymptotic and approximation techniques have been developed in recent years to determine *approximate solutions* of partial and ordinary differential equations. Some of these techniques are the perturbation method, the variational iteration method, the homotopy perturbation method, the energy balance method, the variational approach method, the parameter-expansion method, the amplitude-frequency formulation, the iteration perturbation method, and the Adomian decomposition method.

In this chapter, we present the variational iteration method and the Adomian decomposition method since these techniques have good convergence characteristics and can be used to treat strongly nonlinear ODEs.

2.2 The Variational Iteration Method

The variational iteration method (VIM) was first proposed by He (see e.g. [49, 50]) and systematically elucidated in [51, 54, 126]. The method treats partial and ordinary differential equations without any need to postulate restrictive assumptions that may change the physical structure of the solutions. It has been shown that the VIM solves effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions, see e.g. [127]. Examples

for such problems are the Fokker–Planck equation, the Lane–Emden equation, the Klein–Gordon equation, the Cauchy reaction–diffusion equation, and biological population models.

To illustrate the basic idea of the VIM, we consider, the ODE

$$Ly + N(y) = f(x), \quad x \in I, \quad (2.1)$$

where L and N are linear and nonlinear differential operators, respectively, and $f(x)$ is an given inhomogeneous term defined for all $x \in I$. In the VIM, a *correction functional* of the Eq. (2.1) is defined in the following form

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau) (Ly_n(\tau) + N(\tilde{y}_n(\tau)) - f(\tau)) d\tau, \quad (2.2)$$

where $\lambda(\tau)$ is a general Lagrange multiplier, which can be identified using the variational theory [38]. Furthermore, $y_n(x)$ is the n th approximation of $y(x)$ and $\tilde{y}_n(x)$ is considered as a restricted variation, i.e., $\delta \tilde{y}_n(x) = 0$.

By imposing the variation and by considering the restricted variation, Eq. (2.2) is reduced to

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) + \delta \left(\int_0^x \lambda(\tau) Ly_n(\tau) d\tau \right) \\ &= \delta y_n(x) + \left[\lambda(\tau) \left(\int_0^\tau L \delta y_n(\xi) d\xi \right) \right]_{\tau=0}^{\tau=x} \\ &\quad - \int_0^x \lambda'(\tau) \left(\int_0^\tau L \delta y_n(\xi) d\xi \right) d\tau. \end{aligned} \quad (2.3)$$

Obviously, in (2.3) we have used integration by parts, which is based on the following formula

$$\int \lambda(\tau) y_n'(\tau) d\tau = \lambda(\tau) y_n(\tau) - \int \lambda'(\tau) y_n(\tau) d\tau. \quad (2.4)$$

In the next sections, we will also use two other formulas for the integration by parts, namely

$$\int \lambda(\tau) y_n''(\tau) d\tau = \lambda(\tau) y_n'(\tau) - \lambda'(\tau) y_n(\tau) + \int \lambda''(\tau) y_n(\tau) d\tau, \quad (2.5)$$

and

$$\begin{aligned} \int \lambda(\tau) y_n'''(\tau) d\tau &= \lambda(\tau) y_n''(\tau) - \lambda'(\tau) y_n'(\tau) + \lambda''(\tau) y_n(\tau) \\ &\quad - \int \lambda'''(\tau) y_n(\tau) d\tau. \end{aligned} \quad (2.6)$$

Now, by applying the stationary conditions for (2.3), the optimal value of the Lagrange multiplier $\lambda(\tau)$ can be identified (see e.g. [50], formula (2.13) and the next section). Once $\lambda(\tau)$ is obtained, the solution of the Eq. (2.1) can be readily determined by calculating the successive approximations $y_n(x)$, $n = 0, 1, \dots$, using the formula (see Eq. (2.2))

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau) (Ly_n(\tau) + N(y_n(\tau)) - f(\tau)) d\tau, \quad (2.7)$$

where $y_0(x)$ is a starting function, which has to be prescribed by the user.

In the paper [50] it is shown, that the approximate solution $y_n(x)$ of the exact solution $y(x)$ can be achieved using any selected function $y_0(x)$. Consequently, the approximate solution is given as the limit $y(x) = \lim_{n \rightarrow \infty} y_n(x)$. In other words, the correction functional (2.2) will give a sequence of approximations and the exact solution is obtained at the limit of the successive approximations. In general, it is difficult to calculate this limit. Consequently, an accurate solution can be obtained by considering a large value for n . This value depends on the interval I where a good approximation of the solution is desired.

Let us consider, the following IVP

$$y'(x) + y(x)^2 = 0, \quad y(0) = 1. \quad (2.8)$$

The corresponding exact solution is

$$y(x) = (1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \dots \quad (2.9)$$

Here, we have

$$Ly = y', \quad N(y) = y^2, \quad f(x) \equiv 0.$$

To determine the Lagrange multiplier, we insert these expressions into (2.2) and obtain

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau) \left(\frac{dy_n(\tau)}{d\tau} + \tilde{y}_n(\tau)^2 \right) d\tau. \quad (2.10)$$

Making the above correction functional stationary w.r.t. y_n , noticing that $\delta \tilde{y}_n(x) = 0$ and $\delta y_n(0) = 0$, it follows with (2.3)

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) + \delta \left(\int_0^x \lambda(\tau) \frac{dy_n(\tau)}{d\tau} d\tau \right) \\ &= \delta y_n(x) + \left[\lambda(\tau) \int_0^\tau \frac{d \delta y_n(\xi)}{d\xi} d\xi \right]_{\tau=0}^{\tau=x} \\ &\quad - \int_0^x \lambda'(\tau) \left(\int_0^\tau \frac{d \delta y_n(\xi)}{d\xi} d\xi \right) d\tau \end{aligned}$$

$$\begin{aligned}
&= \delta y_n(x) + \lambda(\tau) \delta y_n(\tau)|_{\tau=x} - \int_0^x \lambda'(\tau) \delta y_n(\tau) d\tau \\
&\doteq 0.
\end{aligned}$$

Thus, we obtain the equations

$$1 + \lambda(x) = 0 \text{ and } \lambda'(x) = 0. \quad (2.11)$$

Now, we substitute the solution $\lambda = -1$ of (2.11) into (2.10). It results the successive iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(\frac{dy_n(\tau)}{d\tau} + y_n(\tau)^2 \right) d\tau. \quad (2.12)$$

We have to choose a starting function $y_0(x)$, which satisfies the given initial condition $y(0) = 1$. Starting with $y_0(x) \equiv 1$, we compute the following successive approximations

$$\begin{aligned}
y_0(x) &= 1, \\
y_1(x) &= 1 - x, \\
y_2(x) &= 1 - x + x^2 - \frac{1}{3}x^3, \\
y_3(x) &= 1 - x + x^2 - x^3 + \frac{2}{3}x^4 - \frac{1}{3}x^5 + \frac{1}{9}x^6 - \frac{1}{63}x^7, \\
y_4(x) &= 1 - x + x^2 - x^3 + x^4 - \frac{13}{15}x^5 + \dots - \frac{1}{59535}x^{15}, \\
y_5(x) &= 1 - x + x^2 - x^3 + x^4 - x^5 + \frac{43}{45}x^6 - \dots - \frac{1}{109876902975}x^{31}.
\end{aligned}$$

In Fig. 2.1 the first iterates $y_0(x), \dots, y_4(x)$ are plotted.

Comparing the iterates with the Taylor series of the exact solution (see (2.9)), we see that in $y_5(x)$ the first six terms are correct. The value of the exact solution at $x = 1$ is $y(1) = 1/2 = 0.5$. In Table 2.1, the corresponding value is given for the iterates $y_i(x)$, $i = 0, \dots, 10$.

In the above example, the linear operator is $L = \frac{d}{dx}$. More generally, let us assume that $L = \frac{d^m}{dx^m}$, $m \geq 1$.

In [86], the corresponding optimal values of the Lagrange multipliers are given. It holds

Fig. 2.1 The first successive iterates $y_i(x)$ for the IVP (2.8). The *solid line* represents the exact solution $y(x)$

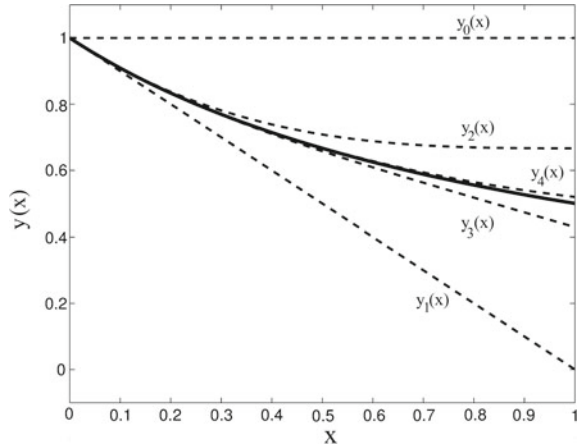


Table 2.1 The successive iterates at the right boundary for the IVP (2.8)

i	$y_i(1)$
0	1
1	0.0
2	0.66666667
3	0.42857143
4	0.51954733
5	0.49529971
6	0.50094000
7	0.49983557
8	0.50002547
9	0.49999645
10	0.50000045

$$\begin{aligned}
 \lambda &= -1, & \text{for } m &= 1, \\
 \lambda &= \tau - x, & \text{for } m &= 2, \\
 \lambda &= \frac{(-1)^m}{(m-1)!} (\tau - x)^{m-1}, & \text{for } m &\geq 1.
 \end{aligned} \tag{2.13}$$

Substituting (2.13) into the correction functional (2.2), we get the following iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(-1)^m}{(m-1)!} (\tau - x)^{m-1} (Ly_n(\tau) + N(y_n(\tau)) - f(\tau)) d\tau, \tag{2.14}$$

where $y_0(x)$ must be given by the user.

2.3 Application of the Variational Iteration Method

In this section, we will consider some nonlinear ODEs and show how the VIM can be used to approximate the exact solution of these problems.

Example 2.1 Solve the following IVP for the Riccati equation

$$y'(x) + \sin(x)y(x) = \cos(x) + y^2, \quad y(0) = 0.$$

Solution. In (2.1), we set

$$Ly \equiv \frac{dy}{dx}, \quad N(y) \equiv \sin(x)y - y^2, \quad f(x) \equiv \cos(x).$$

Thus, the correction functional (2.2) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau) \left(\frac{dy_n(\tau)}{d\tau} + \sin(\tau)y_n(\tau) - y_n(\tau)^2 - \cos(\tau) \right) d\tau.$$

Since L is the first derivative, i.e., $m = 1$, a look at formula (2.13) shows that $\lambda = -1$ is the optimal value of the Langrange multiplier. The resulting successive iteration formula is

$$y_{n+1}(x) = y_n(x) - \int_0^x (y_n'(\tau) + \sin(\tau)y_n(\tau) - y_n(\tau)^2 - \cos(\tau)) d\tau. \quad (2.15)$$

Let us choose $y_0(x) \equiv 0$ as starting function. Notice that $y_0(x)$ satisfies the given initial condition. Now, with (2.15) we obtain the following successive approximations

$$\begin{aligned} y_1(x) &= \sin(x), \\ y_2(x) &= \sin(x), \\ &\vdots \\ y_n(x) &= \sin(x). \end{aligned}$$

Obviously, it holds $\lim_{n \rightarrow \infty} y_n(x) = \sin(x)$. The exact solution is $y(x) = \sin(x)$. \square

Example 2.2 Determine with the VIM a solution of the following IVP for the second order ODE

$$y''(x) + \omega^2 y(x) = g(y(x)), \quad y(0) = a, \quad y'(0) = 0. \quad (2.16)$$

This problem is the prototype of nonlinear oscillator equations (see, e.g., [30, 99]). The real number ω is the angular frequency of the oscillator and must be determined in advance. Moreover, g is a known discontinuous function.

Solution. To apply the VIM, we set (see formula (2.1))

$$Ly \equiv y'' + \omega^2 y, \quad N(y) \equiv -g(y), \quad f(x) \equiv 0.$$

The corresponding correction functional is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau, x) \left(\frac{d^2 y_n(\tau)}{d\tau^2} + \omega^2 y_n(\tau) - g(\tilde{y}_n(\tau)) \right) d\tau.$$

Before we identify an optimal λ , we apply the formula (2.5) for the following integration by parts

$$\begin{aligned} \int_0^x \lambda(\tau, x) y_n''(\tau) d\tau &= \lambda(\tau, x) y_n'(\tau) \Big|_{\tau=0}^{\tau=x} - \frac{\partial \lambda(\tau, x)}{\partial \tau} y_n(\tau) \Big|_{\tau=0}^{\tau=x} \\ &\quad + \int_0^x \frac{\partial^2 \lambda(\tau, x)}{\partial \tau^2} y_n(\tau) d\tau. \end{aligned}$$

Using this relation in the correction functional, imposing the variation, and making the correction functional stationary, we obtain

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) + \lambda(\tau, x) \delta y_n'(\tau) \Big|_{\tau=x} - \frac{\partial \lambda(\tau, x)}{\partial \tau} \delta y_n(\tau) \Big|_{\tau=x} \\ &\quad + \int_0^x \left(\frac{\partial^2 \lambda(\tau, x)}{\partial \tau^2} + \omega^2 \lambda(\tau, x) \right) \delta y_n(\tau) d\tau \\ &\doteq 0. \end{aligned}$$

Thus, the stationary conditions are

$$\begin{aligned} \delta y_n : \quad & \frac{\partial^2 \lambda(\tau, x)}{\partial \tau^2} + \omega^2 \lambda(\tau, x) = 0, \\ \delta y_n' : \quad & \lambda(\tau, x) \Big|_{\tau=x} = 0, \\ \delta y_n : \quad & 1 - \frac{\partial \lambda(\tau, x)}{\partial \tau} \Big|_{\tau=x} = 0. \end{aligned} \tag{2.17}$$

The solution of the Eqs. in (2.17) is

$$\lambda(\tau, x) = \frac{1}{\omega} \sin(\omega(x - \tau)), \tag{2.18}$$

which leads to the following iteration formula

$$y_{n+1}(x) = y_n(x) + \frac{1}{\omega} \int_0^x \sin(\omega(x - \tau)) \left(\frac{d^2 y_n(\tau)}{d\tau^2} + \omega^2 y_n(\tau) - g(y_n(\tau)) \right) d\tau. \quad (2.19)$$

□

Example 2.3 Let us consider the following IVP of the Emden-Lane-Fowler equation (see e.g. [48])

$$y'' + \frac{2}{x} y' + x^k y^\mu = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

This ODE is used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules.

Solve this equation for $k = 0$ and $\mu = 5$, which has a closed form solution.

Solution. For the given parameters, the IVP to be solved is

$$y'' + \frac{2}{x} y' + y^5 = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Obviously, there is a singularity at $x = 0$. To overcome this singularity, we set $y \equiv z/x$. Then, we get

$$z'' + x^{-4} z^5 = 0, \quad z(0) = 0, \quad z'(0) = 1.$$

We set

$$Lz \equiv \frac{d^2 z}{dx^2}, \quad N(z) \equiv x^{-4} z^5, \quad f(x) \equiv 0.$$

Thus, the correction functional (2.2) is

$$z_{n+1}(x) = z_n(x) + \int_0^x \lambda(\tau) \left(\frac{d^2 z_n(\tau)}{d\tau^2} + \tau^{-4} z_n^5(\tau) \right) d\tau.$$

Looking at formula (2.13), we obtain for $m = 2$ the Lagrange multiplier $\lambda = \tau - x$. Therefore, the corresponding iteration formula (2.14) is

$$z_{n+1}(x) = z_n(x) + \int_0^x (\tau - x) \left(\frac{d^2 z_n(\tau)}{d\tau^2} + \tau^{-4} z_n^5(\tau) \right) d\tau.$$

Starting with $z_0(x) = x$, we obtain the following successive approximations

$$\begin{aligned} z_0(x) &= x, \\ z_1(x) &= x - \frac{x^3}{6}, \\ z_2(x) &= x - \frac{x^3}{6} + \frac{x^5}{24}, \\ z_3(x) &= x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{x^7}{432}. \end{aligned}$$

It is not difficult to show that

$$\lim_{n \rightarrow \infty} z_n(x) = z(x) = x \left(1 - \frac{x^2}{6} + \frac{x^4}{24} - \frac{5x^6}{432} + \cdots \right) = x \left(1 + \frac{x^2}{3} \right)^{-1/2}.$$

Thus

$$y(x) = \frac{z(x)}{x} = \left(1 + \frac{x^2}{3} \right)^{-1/2}$$

is the exact solution of the given IVP. \square

Example 2.4 One of the problems that has been studied by several authors is Bratu's BVP (see, e.g., [18, 71, 80, 87, 100, 108]), which is given in one-dimensional planar coordinates by

$$y'' = -\alpha e^y, \quad y(0) = 0, \quad y(1) = 0, \quad (2.20)$$

where $\alpha > 0$ is a real parameter. This BVP plays an important role in the theory of the electric charge around a hot wire and in certain problems of solid mechanics.

The exact solution of (2.20) is

$$y(x) = -2 \ln \left(\frac{\cosh(0.5(x - 0.5)\theta)}{\cosh(0.25\theta)} \right),$$

where θ satisfies

$$\theta = \sqrt{2\alpha} \cosh(0.25\theta).$$

Bratu's problem has zero, one or two solutions when $\alpha > \alpha_c$, $\alpha = \alpha_c$, and $\alpha < \alpha_c$, respectively, where the critical value α_c satisfies

$$1 = 0.25\sqrt{2\alpha_c} \sinh(0.25\theta).$$

In Chap. 5, the value α_c is determined as

$$\alpha_c = 3.51383071912.$$

Use the VIM to solve the BVP (2.20).

Solution. Let us expand e^y and use three terms of this expansion. We obtain

$$y'' + \alpha e^y = y'' + \alpha \sum_{i=0}^{\infty} \frac{y^i}{i!} \approx y'' + \alpha \left(1 + y + \frac{y^2}{2} \right).$$

Setting

$$Ly \equiv \frac{d^2 y}{dx^2}, \quad N(y) \equiv \alpha \left(1 + y + \frac{y^2}{2} \right), \quad f(x) \equiv 0,$$

the corresponding correction functional (2.2) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau) \left(\frac{d^2 y_n(\tau)}{d\tau^2} + \alpha \left(1 + \tilde{y}_n(\tau) + \frac{\tilde{y}_n(\tau)^2}{2} \right) \right) d\tau.$$

Looking at formula (2.13), we obtain for $m = 2$ the Lagrange multiplier $\lambda = \tau - x$. Therefore, the corresponding iteration formula (2.14) is

$$y_{n+1}(x) = y_n(x) + \int_0^x (\tau - x) \left(\frac{d^2 y_n(\tau)}{d\tau^2} + \alpha \left(1 + y_n(\tau) + \frac{y_n(\tau)^2}{2} \right) \right) d\tau. \quad (2.21)$$

Let us start with $y_0(x) = kx$, where k is a real number. The next iterate is

$$y_1(x) = kx + \alpha \int_0^x (\tau - x) \left(1 + k\tau + \frac{k^2 \tau^2}{2} \right) d\tau.$$

Integrating by parts leads to

$$y_1(x) = kx - \frac{\alpha x^2}{2!} - \frac{\alpha k x^3}{3!} - \frac{\lambda k^2 x^4}{4!}.$$

Substituting $y_1(x)$ into the right-hand side of (2.21), we obtain the next iterate

$$\begin{aligned} y_2(x) = & kx - \frac{\alpha x^2}{2!} - \frac{\alpha k x^3}{3!} - \frac{\alpha k^2 x^4}{4!} \\ & + \int_0^x (\tau - x) \left(-\frac{\alpha \tau^2}{2} - \frac{2\alpha k \tau^3}{3} + \frac{\alpha}{24} (3\alpha - 5k^2) \tau^4 + \frac{\alpha k}{24} (2\alpha - k^2) \tau^5 \right. \\ & \left. + \frac{5\alpha^2 k^2 \tau^6}{144} + \frac{\alpha^2 k^3 \tau^7}{144} + \frac{\alpha^2 k^4 \tau^8}{1152} \right) d\tau. \end{aligned}$$

Again, integration by parts yields

$$y_2(x) = kx - \frac{\alpha x^2}{2!} - \frac{\alpha k x^3}{3!} - \frac{\alpha(k^2 - \alpha)x^4}{4!} + \frac{4\alpha^2 k x^5}{5!} + \frac{\alpha^2(5k^2 - 3\alpha)x^6}{6!} \\ + \frac{5\alpha^2 k(k^2 - 2\alpha)x^7}{7!} - \frac{25\alpha^3 k^2 x^8}{8!} - \frac{35\alpha^3 k^3 x^9}{9!} - \frac{35\alpha^3 k^4 x^{10}}{10!}. \quad (2.22)$$

The function $y_2(x)$ must satisfy the initial conditions (see formula (2.20)). For a given α , the equation $y_2(1) = 0$ is a fourth degree polynomial in k . When an appropriate k is chosen from the corresponding four roots, the function $y_2(x)$ can be accepted as an approximation of the exact solution $y(x)$ for $x \in (0, 1)$.

Let us consider Bratu's problem with $\alpha = 1$. The polynomial in k is

$$35k^4 - 3250k^3 + 128250k^2 - 3137760k + 1678320 = 0.$$

Solving this algebraic equation by a numerical method, the following approximated roots are obtained:

$$k_1 = 0.546936690480377, \quad k_2 = 55.687874088793869, \\ k_{3,4} = 18.311166038934306 \pm 35.200557613929831 \cdot i.$$

When we substitute $k = k_1$ and $\alpha = 1$ into (2.22), the next iterate is determined. In Table 2.2, $y_2(x)$ is compared with $y(x)$ for $x = 0.1, 0.2, \dots, 0.9$.

Next, let us consider Bratu's problem with $\alpha = 2$. The polynomial in k is

$$7k^4 - 290k^3 + 5490k^2 - 71136k + 78624 = 0.$$

Solving this algebraic equation by a numerical method, the following approximated roots are obtained:

Table 2.2 Numerical results for Bratu's problem with $\alpha = 1$; $\theta = 1.5171645990507543685218444212962$

x	$y_2(x)$	$y(x)$	Relative error
0.1	0.049605613312791	0.049846791245413	0.004838384309118
0.2	0.088710501172383	0.089189934628823	0.005375421099202
0.3	0.116898896821669	0.117609095767941	0.006038639627618
0.4	0.133864761192820	0.134790253884190	0.006866169212538
0.5	0.139428101877514	0.140539214400472	0.007906067553444
0.6	0.133548005691170	0.134790253884190	0.009216157379500
0.7	0.116331618650518	0.117609095767941	0.010862060532661
0.8	0.088038206883444	0.089189934628823	0.012913203156517
0.9	0.049077324553953	0.049846791245413	0.015436634379762

Table 2.3 Numerical results for Bratu's problem with $\alpha = 2$; $\theta = 2.3575510538774020425939799885899$

x	$y_2(x)$	$y(x)$	Relative error
0.1	0.110752223723751	0.114410743267745	0.031977062988159
0.2	0.199192293844683	0.206419116487609	0.035010432976831
0.3	0.263300560244860	0.273879311825552	0.038625595742077
0.4	0.301549640325367	0.315089364225670	0.042971059761337
0.5	0.313080790744633	0.328952421341114	0.048249015866102
0.6	0.297850396206136	0.315089364225670	0.054711361209852
0.7	0.256726478126018	0.273879311825552	0.062629168976661
0.8	0.191509696354622	0.206419116487609	0.072228872919732
0.9	0.104847118151250	0.114410743267745	0.083590271711759

$$k_1 = 1.211500000137995, \quad k_2 = 25.631365803713045, \\ k_{3,4} = 7.292852812360195 \pm 17.564893217135829 \cdot i.$$

As before, when we substitute $k = k_1$ and $\alpha = 2$ into (2.22), the next iterate is determined. In Table 2.3, $y_2(x)$ is compared with $y(x)$ for $x = 0.1, 0.2, \dots, 0.9$.

The results in the Tables 2.2 and 2.3 show that the VIM is efficient and quite reliable. Only two iterations have lead to acceptable results. There is no doubt, if more terms of the expansion and/or more iterates are used, the VIM will generate far better results. \square

2.4 The Adomian Decomposition Method

The Adomian decomposition method (ADM) is a semi-analytical technique for solving ODEs and PDEs. The method was developed by the Armenian-American mathematician George Adomian [7, 8, 9]. The ADM is based on a decomposition of the solution of nonlinear operator equations in appropriate function spaces into a series of functions. The method, which accurately computes the series solution, is of great interest to the applied sciences. The method provides the solution in a rapidly convergent series with components that are computed elegantly. The convergence of this method is studied in [1, 2, 73].

Let the general form of an ODE be

$$F(y) = f,$$

where F is the nonlinear differential operator with linear and nonlinear terms. In the ADM, the linear term is decomposed as $L + R$, where L is an easily invertible operator and R is the remainder of the linear term. For convenience L is taken as the highest-order derivative. Thus the ODE may be written as

$$Ly + Ry + N(y) = f(x), \quad (2.23)$$

where $N(y)$ corresponds to the nonlinear terms.

Let L be a first-order differential operator defined by $L \equiv \frac{d}{dx}$. If L is invertible, then the inverse operator L^{-1} is given by

$$L^{-1}(\cdot) = \int_0^x (\cdot) d\tau.$$

Thus,

$$L^{-1}Ly = y(x) - y(0). \quad (2.24)$$

Similarly, if $L^2 \equiv \frac{d^2}{dx^2}$, then the inverse operator L^{-1} is regarded as a double integration operator given by

$$L^{-1}(\cdot) = \int_0^x \int_0^\tau (\cdot) dt d\tau.$$

It follows

$$L^{-1}Ly = y(x) - xy'(0). \quad (2.25)$$

We can use the same operations to find relations for higher-order differential operators. For example, if $L^3 \equiv \frac{d^3}{dx^3}$, then it is not difficult to show that

$$L^{-1}Ly = y(x) - y(0) - xy'(0) - \frac{1}{2!}x^2y''(0). \quad (2.26)$$

The basic idea of the ADM is to apply the operator L^{-1} formally to the expression

$$Ly(x) = f(x) - Ry(x) - N(y(x)).$$

This yields

$$y(x) = \Psi_0(x) + g(x) - L^{-1}Ry(x) - L^{-1}N(y(x)), \quad (2.27)$$

where the function $g(x)$ represents the terms, which result from the integration of $f(x)$, and

$$\Psi_0(x) \equiv \begin{cases} y(0), & \text{for } L = \frac{d}{dx}, \\ y(0) + xy'(0), & \text{for } L^2 = \frac{d^2}{dx^2}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2y''(0), & \text{for } L^3 = \frac{d^3}{dx^3}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2y''(0) + \frac{1}{3!}x^3y'''(0), & \text{for } L^4 = \frac{d^4}{dx^4}. \end{cases}$$

Now, we write

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \quad \text{and} \quad N(y(x)) = \sum_{n=0}^{\infty} A_n(x),$$

where

$$A_n(x) \equiv A_n(y_0(x), y_1(x), \dots, y_{n-1}(x))$$

are known as the *Adomian polynomials*. Substituting these two infinite series into (2.27), we obtain

$$\sum_{n=0}^{\infty} y_n(x) = \Psi_0(x) + g(x) - L^{-1}R \sum_{n=0}^{\infty} y_n(x) - L^{-1} \sum_{n=0}^{\infty} A_n(x). \quad (2.28)$$

Identifying the zeroth component $y_0(x)$ by $\Psi_0(x) + g(x)$, the remaining components $y_k(x)$, $k \geq 1$, can be determined by using the recurrence relation

$$\begin{aligned} y_0(x) &= \Psi_0(x) + g(x), \\ y_k(x) &= -L^{-1}R y_{k-1}(x) - L^{-1}A_{k-1}(x), \quad k = 1, 2, \dots \end{aligned} \quad (2.29)$$

Obviously, when some of the components $y_k(x)$ are determined, the solution $y(x)$ can be approximated in form of a series. Under appropriate assumptions, it holds

$$y(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n y_k(x).$$

The polynomials $A_k(x)$ are generated for each nonlinearity so that A_0 depends only on y_0 , A_1 depends only on y_0 and y_1 , A_2 depends on y_0 , y_1 , y_2 , etc. [7]. An appropriate strategy to determine the Adomian polynomials is

$$\begin{aligned} A_0 &= N(y_0), \\ A_1 &= y_1 N'(y_0), \\ A_2 &= y_2 N'(y_0) + \frac{1}{2!} y_1^2 N''(y_0), \\ A_3 &= y_3 N'(y_0) + y_1 y_2 N''(y_0) + \frac{1}{3!} y_1^3 N^{(3)}(y_0), \\ A_4 &= y_4 N'(y_0) + \left(\frac{1}{2!} y_2^2 + y_1 y_3 \right) N''(y_0) + \frac{1}{2!} y_1^2 y_2 N^{(3)}(y_0) \\ &\quad + \frac{1}{4!} y_1^4 N^{(4)}(y_0), \\ &\vdots \end{aligned}$$

where $N^{(k)}(y) \equiv \frac{d^k}{dy^k} N(y)$.

The general formula is

$$A_k = \frac{1}{k!} \frac{\partial^k}{\partial \lambda^k} \left[N \left(\sum_{n=0}^{\infty} y_n \lambda^n \right) \right]_{\lambda=0}, \quad k = 0, 1, 2, \dots \quad (2.30)$$

A MATHEMATICA program that generates the polynomials A_k automatically can be found in [16]. Moreover, in [20] a simple algorithm for calculating Adomian polynomials is presented. According to this algorithm the following formulas for the Adomian polynomials result:

$$\begin{aligned} A_0 &= N(y_0), \\ A_1 &= y_1 N'(y_0), \\ A_2 &= y_2 N'(y_0) + \frac{1}{2} y_1^2 N''(y_0), \\ A_3 &= y_3 N'(y_0) + y_1 y_2 N''(y_0) + \frac{1}{6} y_1^3 N^{(3)}(y_0), \\ A_4 &= y_4 N'(y_0) + \left(y_1 y_3 + \frac{1}{2} y_2^2 \right) N''(y_0) + \frac{1}{2} y_1^2 y_2 N^{(3)}(y_0) + \frac{1}{24} y_1^4 N^{(4)}(y_0). \end{aligned} \quad (2.31)$$

Before we highlight a few examples and show how the ADM can be used to solve concrete ODEs, let us list the Adomian polynomials for some classes of nonlinearity.

1. $N(y) = \exp(y)$:

$$\begin{aligned} A_0 &= \exp(y_0), & A_1 &= y_1 \exp(y_0), \\ A_2 &= \left(y_2 + \frac{1}{2!} y_1^2 \right) \exp(y_0), & A_3 &= \left(y_3 + y_1 y_2 + \frac{1}{3!} y_1^3 \right) \exp(y_0); \end{aligned} \quad (2.32)$$

2. $N(y) = \ln(y)$, $y > 0$:

$$\begin{aligned} A_0 &= \ln(y_0), & A_1 &= \frac{y_1}{y_0}, \\ A_2 &= \frac{y_2}{y_0} - \frac{1}{2} \frac{y_1^2}{y_0^2}, & A_3 &= \frac{y_3}{y_0} - \frac{y_1 y_2}{y_0^2} + \frac{1}{3} \frac{y_1^3}{y_0^3}; \end{aligned} \quad (2.33)$$

3. $N(y) = y^2$:

$$\begin{aligned} A_0 &= y_0^2, & A_1 &= 2y_0 y_1, \\ A_2 &= 2y_0 y_2 + y_1^2, & A_3 &= 2y_0 y_3 + 2y_1 y_2; \end{aligned} \quad (2.34)$$

4. $N(y) = y^3$:

$$\begin{aligned} A_0 &= y_0^3, & A_1 &= 3y_0^2 y_1, \\ A_2 &= 3y_0^2 y_2 + 3y_0 y_1^2, & A_3 &= 3y_0^2 y_3 + 6y_0 y_1 y_2 + y_1^3; \end{aligned} \quad (2.35)$$

5. $N(y) = yy'$:

$$\begin{aligned} A_0 &= y_0 y_0', & A_1 &= y_0' y_1 + y_0 y_1', \\ A_2 &= y_0' y_2 + y_1' y_1 + y_2' y_0, & A_3 &= y_0' y_3 + y_1' y_2 + y_2' y_1 + y_3' y_0; \end{aligned} \quad (2.36)$$

6. $N(y) = (y')^2$:

$$\begin{aligned} A_0 &= (y_0')^2, & A_1 &= 2y_0' y_1', \\ A_2 &= 2y_0' y_2' + (y_1')^2, & A_3 &= 2y_0' y_3' + 2y_1' y_2'; \end{aligned} \quad (2.37)$$

7. $N(y) = \cos(y)$:

$$\begin{aligned} A_0 &= \cos(y_0), & A_1 &= -y_1 \sin(y_0), \\ A_2 &= -y_2 \sin(y_0) - \frac{1}{2} y_1^2 \cos(y_0), & A_3 &= -y_3 \sin(y_0) - y_1 y_2 \cos(y_0) \\ & & & + \frac{1}{6} y_1^3 \sin(y_0); \end{aligned} \quad (2.38)$$

8. $N(y) = \sin(y)$:

$$\begin{aligned} A_0 &= \sin(y_0), & A_1 &= y_1 \cos(y_0), \\ A_2 &= y_2 \cos(y_0) - \frac{1}{2} y_1^2 \sin(y_0), & A_3 &= y_3 \cos(y_0) - y_1 y_2 \sin(y_0) \\ & & & - \frac{1}{6} y_1^3 \cos(y_0). \end{aligned} \quad (2.39)$$

2.5 Application of the Adomian Decomposition Method

In this section, we consider some IVPs for first-order and second-order ODEs.

Example 2.5 Solve the IVP

$$y'(x) = 1 - x^2 y(x) + y(x)^3, \quad y(0) = 0.$$

Solution. This is Abel's equation and its exact solution is $y(x) = x$. We apply the ADM to solve it. First, let us look at formula (2.23). We have

$$Ly \equiv y', \quad Ry \equiv x^2 y, \quad N(y) \equiv -y^3, \quad f(x) \equiv 1.$$

Using (2.27), we obtain

$$\begin{aligned} y(x) &= \Psi_0(x) + g(x) - L^{-1}Ry(x) - L^{-1}N(y(x)) \\ &= y(0) + x - \int_0^x \tau^2 y(\tau) d\tau + \int_0^x y(\tau)^3 d\tau. \end{aligned}$$

Now, applying formula (2.28), we get

$$\sum_{n=0}^{\infty} y_n(x) = x - \int_0^x \left(\tau^2 \sum_{n=0}^{\infty} y_n(\tau) \right) d\tau - \int_0^x \left(\sum_{n=0}^{\infty} A_n(\tau) \right) d\tau.$$

The Adomian polynomials, which belong to the nonlinearity $N(y) = y^3$, are given in (2.35). Setting

$$y_0(x) = \Psi_0(x) + g(x) = y(0) + x = x,$$

the recurrence relation (2.29) yields

$$\begin{aligned} y_1(x) &= - \int_0^x \tau^3 d\tau - \int_0^x A_0(\tau) d\tau = - \int_0^x \tau^3 d\tau + \int_0^x \tau^3 d\tau = 0, \\ y_2(x) &= - \int_0^x \tau^2 y_1(\tau) d\tau + 3 \int_0^x y_0(\tau)^2 y_1(\tau) d\tau = 0, \\ &\vdots \end{aligned}$$

Hence

$$y(x) = \sum_{k=0}^{\infty} y_k(x) = x + 0 + \cdots + 0 + \cdots = x.$$

□

Example 2.6 Solve the IVP

$$y'(x) + x e^{y(x)} = 0, \quad y(0) = 0.$$

Solution. Here, we have

$$Ly \equiv y', \quad Ry \equiv 0, \quad N(y) \equiv x e^y, \quad f(x) \equiv 0.$$

Using (2.27), we obtain

$$\begin{aligned} y(x) &= \Psi_0(x) + g(x) - L^{-1}Ry(x) - L^{-1}N(y(x)) \\ &= y(0) - \int_0^x \tau e^{y(\tau)} d\tau = - \int_0^x \tau e^{y(\tau)} d\tau. \end{aligned}$$

Now, applying formula (2.28), we get

$$\sum_{n=0}^{\infty} y_n(x) = - \int_0^x \tau \left(\sum_{n=0}^{\infty} A_n(\tau) \right) d\tau.$$

The Adomian polynomials, which belong to the nonlinearity $\exp(y)$, are given in (2.32). Setting

$$y_0(x) = \Psi_0(x) + g(x) = y(0) = 0,$$

the recurrence relation (2.29) yields

$$\begin{aligned} y_1(x) &= - \int_0^x \tau \cdot 1 d\tau = -\frac{x^2}{2}, \\ y_2(x) &= \int_0^x \tau \cdot \frac{\tau^2}{2} d\tau = \int_0^x \frac{\tau^3}{2} d\tau = \frac{x^4}{8}, \\ y_3(x) &= - \int_0^x \tau \left(\frac{\tau^4}{8} + \frac{1}{2} \left(-\frac{\tau^2}{2} \right)^2 \right) \cdot 1 d\tau = - \int_0^x \frac{\tau^5}{4} d\tau \\ &= -\frac{x^6}{24}, \\ &\vdots \end{aligned}$$

Thus,

$$y_n(x) = - \int_0^x \tau A_{n-1}(\tau) d\tau = \frac{1}{n} \left(-\frac{x^2}{2} \right)^n, \quad n = 1, 2, \dots,$$

and it holds

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = -\frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{24}x^6 + \dots = -\ln \left(1 + \frac{x^2}{2} \right).$$

□

Example 2.7 Solve the IVP

$$y''(x) + 2y(x)y'(x) = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution. Here, we have

$$Ly \equiv y'', \quad Ry \equiv 0, \quad N(y) \equiv 2yy', \quad f(x) \equiv 0.$$

Using (2.27), we obtain

$$\begin{aligned} y(x) &= \Psi_0(x) + g(x) - L^{-1}Ry(x) - L^{-1}N(y(x)) \\ &= y(0) + xy'(0) - 2 \int_0^x \int_0^\tau y(t)y'(t)dt d\tau. \end{aligned}$$

Now, applying formula (2.28), we get

$$\sum_{n=0}^{\infty} y_n(x) = x - 2 \int_0^x \int_0^\tau \left(\sum_{n=0}^{\infty} A_n(t) \right) dt d\tau.$$

The Adomian polynomials, which belong to the nonlinearity yy' , are given in (2.36). Setting

$$y_0(x) = \Psi_0(x) + g(x) = y(0) + xy'(0) = x,$$

the recurrence relation (2.29) yields

$$\begin{aligned} y_1(x) &= -2 \int_0^x \int_0^\tau (y_0(t)y_0'(t))dt d\tau = -2 \int_0^x \int_0^\tau t dt d\tau \\ &= -2 \int_0^x \frac{\tau^2}{2} d\tau = -\int_0^x \tau^2 d\tau = -\frac{x^3}{3}, \\ y_2(x) &= -2 \int_0^x \int_0^\tau (y_0'(t)y_1(t) + y_0(t)y_1'(t))dt d\tau \\ &= 2 \int_0^x \int_0^\tau \left(1 \cdot \frac{t^3}{3} + t \cdot t^2 \right) dt d\tau = \frac{4 \cdot 2}{3} \int_0^x \int_0^\tau t^3 dt d\tau \\ &= \frac{2}{3} \int_0^x \tau^4 d\tau = \frac{2}{15}x^5, \\ y_3(x) &= -2 \int_0^x \int_0^\tau (y_0'(t)y_2(t) + y_1'(t)y_1(t) + y_2'(t)y_0(t))dt d\tau \\ &= \int_0^x \int_0^\tau \left(1 \cdot \frac{2}{15}t^5 + t^2 \cdot \frac{t^3}{3} + \frac{2}{3}t^4 \cdot t \right) dt d\tau \\ &= -\frac{2 \cdot 17}{15} \int_0^x \int_0^\tau t^5 dt d\tau = -\frac{17}{15 \cdot 3} \int_0^x \tau^6 d\tau = -\frac{17}{315}x^7. \end{aligned}$$

Obviously, it holds

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$$

□

Example 2.8 Solve the IVP

$$y''(x) - y'(x)^2 + y(x)^2 = e^x, \quad y(0) = y'(0) = 1.$$

Solution. Here, we have

$$Ly \equiv y'', \quad Ry \equiv 0, \quad N(y) \equiv -(y')^2 + y^2, \quad f(x) \equiv e^x.$$

We write

$$N(y) = N_1(y) + N_2(y), \quad N_1(y) \equiv -(y')^2, \quad N_2(y) \equiv y^2.$$

Using (2.27), we obtain

$$\begin{aligned} y(x) &= \Psi_0(x) + g(x) - L^{-1}N_1(y(x)) - L^{-1}N_2(y(x)) \\ &= y(0) + xy'(0) + \int_0^x \int_0^\tau e^t dt d\tau + \int_0^x \int_0^\tau y'(t)^2 dt d\tau \\ &\quad - \int_0^x \int_0^\tau y(t)^2 dt d\tau \end{aligned}$$

Now, applying formula (2.28), we get

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) &= 1 + x + e^x - x - 1 + \int_0^x \int_0^\tau \left(\sum_{n=0}^{\infty} A_n(t) \right) dt d\tau \\ &\quad - \int_0^x \int_0^\tau \left(\sum_{n=0}^{\infty} B_n(t) \right) dt d\tau. \end{aligned}$$

The Adomian polynomials A_n , which belong to the nonlinearity $(y')^2$, are given in (2.37), and for the nonlinearity y^2 , the Adomian polynomials B_n are given in (2.34). Setting

$$y_0(x) = \Psi_0(x) + g(x) = e^x,$$

the recurrence relation (2.29) yields

$$\begin{aligned} y_1(x) &= \int_0^x \int_0^\tau A_0(t) dt d\tau - \int_0^x \int_0^\tau B_0(t) dt d\tau \\ &= \int_0^x \int_0^\tau (e^t)^2 dt d\tau - \int_0^x \int_0^\tau (e^t)^2 dt d\tau = 0. \end{aligned}$$

This implies $y_n(x) \equiv 0$, $n = 1, 2, \dots$, and we obtain the exact solution of the given problem:

$$y(x) = e^x + 0 + 0 + \dots = e^x.$$

□

The convergence of the ADM can be accelerated if the so-called *noise terms phenomenon* occurs in the given problem (see, e.g., [9, 10]). The noise terms are the identical terms with opposite sign that appear within the components $y_0(x)$ and $y_1(x)$. They only exist in specific types of nonhomogeneous equations. If noise terms indeed exist in the $y_0(x)$ and $y_1(x)$ components, then, in general, the solution can be obtained after two successive iterations.

By canceling the noise terms in $y_0(x)$ and $y_1(x)$, the remaining non-canceled terms of $y_0(x)$ give the exact solution. It has been proved that a necessary condition for the existence of noise terms is that the exact solution is part of $y_0(x)$.

Example 2.9 Solve the IVP

$$y''(x) - y'(x)^2 + y(x)^2 = 1, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution. As in the Example 2.8, we set

$$Ly \equiv y'', \quad Ry \equiv 0, \quad N_1(y) \equiv -(y')^2, \quad N_2(y) \equiv y^2, \quad f(x) \equiv 1.$$

Using (2.27), we obtain

$$y(x) = 1 + \frac{x^2}{2} + \int_0^x \int_0^\tau y'(t)^2 dt d\tau - \int_0^x \int_0^\tau y(t)^2 dt d\tau$$

Now, applying formula (2.28), we get

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) &= 1 + \frac{x^2}{2} + \int_0^x \int_0^\tau \left(\sum_{n=0}^{\infty} A_n(t) \right) dt d\tau \\ &\quad - \int_0^x \int_0^\tau \left(\sum_{n=0}^{\infty} B_n(t) \right) dt d\tau. \end{aligned}$$

The Adomian polynomials A_n , which belong to the nonlinearity $(y')^2$, are given in (2.37), and for the nonlinearity y^2 , the Adomian polynomials B_n are given in (2.34). Setting

$$y_0(x) = 1 + \frac{1}{2}x^2,$$

the recurrence relation (2.29) yields

$$\begin{aligned}
 y_1(x) &= \int_0^x \int_0^\tau t^2 dt d\tau - \int_0^x \int_0^\tau \left(1 + \frac{t^2}{2}\right)^2 dt d\tau \\
 &= \int_0^x \left[\frac{t^3}{3}\right]_0^\tau d\tau - \int_0^x \left[\frac{1}{20}t^5 + \frac{1}{3}t^3 + t\right]_0^\tau d\tau \\
 &= \int_0^x \frac{\tau^3}{3} d\tau - \int_0^x \left(\frac{1}{20}\tau^5 + \frac{1}{3}\tau^3 + \tau\right) d\tau \\
 &= \frac{1}{12}x^4 - \frac{1}{120}x^6 - \frac{1}{12}x^4 - \frac{1}{2}x^2 \\
 &= -\frac{1}{120}x^6 - \frac{1}{2}x^2.
 \end{aligned}$$

Comparing $y_0(x)$ with $y_1(x)$, we see that there is the noise term $\frac{1}{2}x^2$. Therefore, we can conclude that the solution of the given IVP is $y(x) = 1$. \square

Several authors have proposed a variety of modifications of the AMD (see, e.g., [12]) by which the convergence of the iteration (2.29) can be accelerated. Wazwaz [124, 126] suggests the following reliable modification which is based on the assumption that the function $h(x) \equiv \Psi_0(x) + g(x)$ in formula (2.27) can be divided into two parts, i.e.,

$$h(x) \equiv \Psi_0(x) + g(x) = h_0(x) + h_1(x).$$

The idea is that only the part $h_0(x)$ is assigned to the component $y_0(x)$, whereas the remaining part $h_1(x)$ is combined with other terms given in (2.29). It results the modified recurrence relation

$$\begin{aligned}
 y_0(x) &= h_0(x), \\
 y_1(x) &= h_1(x) - L^{-1}Ry_0(x) - L^{-1}A_0(x), \\
 y_k(x) &= -L^{-1}Ry_{k-1}(x) - L^{-1}A_{k-1}(x), \quad k = 2, 3, \dots
 \end{aligned} \tag{2.40}$$

Example 2.10 Solve the IVP

$$y''(x) - y(x)^2 = 2 - x^4, \quad y(0) = y'(0) = 0.$$

Solution. Here, we have

$$Ly \equiv y'', \quad Ry \equiv 0, \quad N(y) \equiv -y^2, \quad f(x) \equiv 2 - x^4.$$

Using (2.27), we obtain

$$\begin{aligned} y(x) &= \Psi_0(x) + g(x) - L^{-1}Ry(x) - L^{-1}N(y(x)) \\ &= x^2 - \frac{1}{30}x^6 + \int_0^x \int_0^\tau y(t)^2 dt d\tau. \end{aligned}$$

Now, applying formula (2.28), we get

$$\sum_{n=0}^{\infty} y_n(x) = x^2 - \frac{1}{30}x^6 + \int_0^x \int_0^\tau \left(\sum_{n=0}^{\infty} A_n(t) \right) dt d\tau.$$

The Adomian polynomials, which belong to the nonlinearity y^2 , are given in (2.34). Dividing

$$h(x) = x^2 - \frac{1}{30}x^6$$

into $h_0(x) \equiv x^2$ and $h_1(x) \equiv -\frac{1}{30}x^6$, and starting with $y_0(x) = x^2$, the recurrence relation (2.29) yields

$$\begin{aligned} y_1(x) &= -\frac{1}{30}x^6 + \int_0^x \int_0^\tau (t^2)^2 dt d\tau \\ &= -\frac{1}{30}x^6 + \int_0^x \frac{\tau^5}{5} d\tau \\ &= -\frac{1}{30}x^6 + \frac{1}{30}x^6 = 0. \end{aligned}$$

This implies

$$y_k(x) = 0, \quad k = 1, 2, \dots$$

Thus, we can conclude that the exact solution of the given IVP is $y(x) = x^2$.

Let us compare the modified ADM with the standard method. The ADM is based on the recurrence relation (2.29). Here, we have to set

$$y_0(x) = x^2 - \frac{1}{30}x^6.$$

Now, the recurrence relation (2.29) yields

$$\begin{aligned} y_1(x) &= \int_0^x \int_0^\tau \left(t^2 - \frac{1}{30}t^6 \right)^2 dt d\tau = \int_0^x \left(\frac{1}{11700}\tau^{13} - \frac{1}{135}\tau^9 - \frac{1}{5}\tau^5 \right) d\tau \\ &= \frac{1}{30}x^6 - \frac{1}{1350}x^{10} + \frac{1}{163800}x^{14}. \end{aligned}$$

$$\begin{aligned}
y_2(x) &= 2 \int_0^x \int_0^\tau \left(t^2 - \frac{1}{30} t^6 \right) \left(\frac{1}{163800} x^{14} - \frac{1}{1350} x^{10} + \frac{1}{30} x^6 \right) dt d\tau \\
&= \int_0^x \left(-\frac{1}{51597000} \tau^{21} + \frac{227}{62653500} \tau^{17} - \frac{1}{3510} \tau^{13} + \frac{1}{135} \tau^9 \right) d\tau \\
&= \frac{1}{1350} x^{10} - \frac{1}{49140} x^{14} + \frac{227}{1127763000} x^{18} - \frac{1}{1135134000} x^{22}.
\end{aligned}$$

Since

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots,$$

we see that the terms in x^6 and x^{10} cancel each other. The cancelation of terms is continued when further components y_k , $k \geq 3$, are added.

This is an impressive example of how fast the modified ADM generates the exact solution $y(x) = x^2$, compared with the standard method. \square

Many problems in the mathematical physics can be formulated as ODEs of *Emden-Fowler type* (see, e.g., [26, 30]) defined in the form

$$\begin{aligned}
y''(x) + \frac{\alpha}{x} y'(x) + \beta f(x) g(y) &= 0, \quad \alpha \geq 0, \\
y(0) = a, \quad y'(0) &= 0,
\end{aligned} \tag{2.41}$$

where f and g are given functions of x and y , respectively. The standard Emden-Lane-Fowler ODE results when we set $f(x) \equiv 1$ and $g(y) \equiv y''$.

Obviously, a difficulty in the analysis of (2.41) is the singularity behavior that occurs at $x = 0$. Before the ADM can be applied, a slight change of the problem is necessary to overcome this difficulty. The strategy is to define the differential operator L in terms of the two derivatives, $y'' + (\alpha/x)y'$, which are contained in the ODE. First, the ODE (2.41) is rewritten as

$$Ly(x) = -\beta f(x) g(y), \tag{2.42}$$

with

$$L \equiv x^{-\alpha} \frac{d}{dx} \left(x^\alpha \frac{d}{dx} \right).$$

The corresponding inverse operator L^{-1} is

$$L^{-1}(\cdot) = \int_0^x \tau^{-\alpha} \int_0^\tau t^\alpha (\cdot) dt d\tau.$$

Applying L^{-1} to the first two terms of (2.41), we obtain

$$\begin{aligned}
 & L^{-1} \left(y''(x) + \frac{\alpha}{x} y'(x) \right) \\
 &= \int_0^x \tau^{-\alpha} \int_0^\tau t^\alpha \left(y''(t) + \frac{\alpha}{t} y'(t) \right) dt d\tau \\
 &= \int_0^x \left[\tau^\alpha y'(\tau) - \int_0^\tau \alpha t^{\alpha-1} y'(t) dt + \int_0^\tau \alpha t^{\alpha-1} y'(t) dt \right] d\tau \\
 &= \int_0^x y'(\tau) d\tau = y(x) - y(0) = y(x) - a.
 \end{aligned}$$

Now, operating with L^{-1} on (2.41), we find

$$y(x) = a - \beta L^{-1}(f(x)g(y)). \quad (2.43)$$

It is interesting, that only the first initial condition is sufficient to represent the solution $y(x)$ in this form. The second initial condition can be used to show that the obtained solution satisfies this condition.

Let us come back to the Adomian decomposition method. As before, the solution $y(x)$ is represented by an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \quad (2.44)$$

In addition, the given nonlinear function $g(y)$ is represented by an infinite series of Adomian polynomials (as we have done it for $N(y)$)

$$g(y(x)) = \sum_{n=0}^{\infty} A_n(x), \quad (2.45)$$

where

$$A_n(x) \equiv A_n(y_0(x), y_1(x), \dots, y_{n-1}(x)).$$

Substituting (2.44) and (2.45) into (2.43) gives

$$\sum_{n=0}^{\infty} y_n(x) = a - \beta L^{-1} \left(f(x) \sum_{n=0}^{\infty} A_n(x) \right). \quad (2.46)$$

Now, the components $y_n(x)$ are determined recursively. The corresponding recurrence relation is

$$\begin{aligned}
 y_0(x) &= a, \\
 y_k(x) &= -\beta L^{-1}(f(x)A_{k-1}(x)), \quad k = 1, 2, \dots,
 \end{aligned}$$

or equivalently

$$\begin{aligned} y_0(x) &= a, \\ y_k(x) &= -\beta \left(\int_0^x \tau^{-\alpha} \int_0^\tau t^\alpha (f(t) A_{k-1}(x)) dt d\tau \right), \quad k = 1, 2, \dots \end{aligned} \quad (2.47)$$

Example 2.11 Solve the IVP

$$y''(x) + \frac{2}{x} y'(x) + e^{y(x)} = 0, \quad y(0) = y'(0) = 0.$$

Solution. This problem is a special case of (2.41), where $\alpha = 2$, $\beta = 1$, $f(x) \equiv 1$, $g(y) = \exp(y)$ and $a = 0$. A particular solution of the ODE is

$$y(x) = \ln \left(\frac{2}{x^2} \right).$$

Obviously, this solution does not satisfy the initial conditions. We will see that the ADM can be used to determine a solution which satisfies the ODE as well as the initial conditions.

For the nonlinearity $g(y) = \exp(y)$, the Adomian polynomials are given in Eq. (2.32). Using the recurrence relation (2.47), we obtain

$$\begin{aligned} y_0(x) &= 0, \\ y_1(x) &= - \int_0^x \tau^{-2} \int_0^\tau t^2 \cdot 1 dt d\tau = - \int_0^x \tau^{-2} \frac{\tau^3}{3} d\tau = - \int_0^x \frac{\tau}{3} d\tau \\ &= -\frac{1}{6} x^2, \\ y_2(x) &= - \int_0^x \tau^{-2} \int_0^\tau t^2 \cdot \left(\frac{-t^2}{6} \right) dt d\tau = \int_0^x \tau^{-2} \int_0^\tau \frac{t^4}{6} dt d\tau \\ &= \int_0^x \tau^{-2} \frac{\tau^5}{30} d\tau = \int_0^x \frac{\tau^3}{30} d\tau \\ &= \frac{1}{120} x^4, \\ y_3(x) &= - \int_0^x \tau^{-2} \int_0^\tau t^2 \cdot \left(\frac{t^4}{120} + \frac{t^4}{72} \right) dt d\tau = - \int_0^x \tau^{-2} \int_0^\tau \frac{t^6}{45} dt d\tau \\ &= - \int_0^x \tau^{-2} \frac{\tau^7}{315} d\tau = - \int_0^x \frac{\tau^5}{315} d\tau \\ &= -\frac{1}{1890} x^6, \end{aligned}$$

$$\begin{aligned}
y_4(x) &= - \int_0^x \tau^{-2} \int_0^\tau t^2 \left(-\frac{1}{1890} t^6 - \frac{1}{6} t^2 \cdot \frac{1}{120} t^4 - \frac{1}{6} \cdot \frac{1}{216} t^6 \right) dt d\tau \\
&= \int_0^x \tau^{-2} \left(\frac{61}{204120} \tau^9 \right) d\tau = \int_0^x \frac{61}{204120} \tau^7 d\tau \\
&= \frac{61}{1632960} x^8.
\end{aligned}$$

Thus, we have

$$y(x) = -\frac{1}{6} x^2 + \frac{1}{120} x^4 - \frac{1}{1890} x^6 + \frac{61}{1632960} x^8 + \dots$$

In [125] further variants of the general Emden-Fowler equation are discussed, and solved by the ADM. \square

2.6 Exercises

Exercise 2.1 Solve following ODEs by VIM or ADM:

- (1) $y'(x) - y(x) = -y(x)^2, \quad y(0) = 1,$
- (2) $y'(x) = 1 + \frac{y(x)}{x} + \left(\frac{y(x)}{x} \right)^2, \quad y(0) = 0,$
- (3) $y''(x) + \frac{5}{x} y'(x) + \exp(y(x)) + 2 \exp\left(\frac{y(x)}{2}\right) = 0, \quad y(0) = y'(0) = 0,$
- (4) $y''(x) + 2y(x)y'(x) - y(x) = \sinh(2x), \quad y(0) = 0, \quad y'(0) = 1,$
- (5) $y''(x) + \cos(y(x)) = 0, \quad y(0) = 0, \quad y'(0) = \frac{\pi}{2},$
- (6) $y''(x) + \frac{8}{x} y'(x) + 2y(x) = -4y(x) \ln(y(x)), \quad y(0) = 1, \quad y'(0) = 0,$
- (7) $y'''(x) + y''(x)^2 + y'(x)^2 = 1 - \cos(x), \quad y(0) = y''(0) = 0, \quad y'(0) = 1.$

Exercise 2.2 Given the following IVP for the Emden-Fowler ODE

$$y''(x) + \frac{2}{x} y'(x) + \alpha x^m y(x)^\mu = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Approximate the solution of this IVP by the VIM and the ADM.

Exercise 2.3 Given the following IVP for the Emden-Fowler ODE

$$y''(x) + \frac{2}{x} y'(x) + \alpha x^m e^{y(x)} = 0, \quad y(0) = y'(0) = 0.$$

Approximate the solution of this IVP by the VIM and the ADM.

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