

# Contents

<b>1</b>	<b>Prerequisite Concepts and Notations</b>	<b>1</b>
1.1	Set Theory	1
1.2	Groups and Fundamental Homomorphism Theorem	4
1.3	Group Representations, Free Groups, and Relations	7
1.3.1	Linear Representation of a Group	7
1.3.2	Free Groups and Relations	8
1.3.3	Betti Number and Structure Theorem for Finite Abelian Group	11
1.4	Exact Sequence of Groups	11
1.5	Free Product and Tensor Product of Groups	13
1.5.1	Free Product of Groups	13
1.5.2	Tensor Product of Groups	14
1.6	Torsion Group	14
1.7	Actions of Groups	15
1.8	Modules and Vector Spaces	16
1.8.1	Modules	16
1.8.2	Direct Sum of Modules	17
1.8.3	Tensor Product of Modules	17
1.8.4	Vector Spaces	20
1.9	Euclidean Spaces and Some Standard Notations	22
1.10	Set Topology	22
1.10.1	Topological Spaces: Introductory Concepts	23
1.10.2	Homeomorphic Spaces	25
1.10.3	Metric Spaces	27
1.10.4	Connectedness and Locally Connectedness	28
1.10.5	Compactness and Paracompactness	29
1.10.6	Weak Topology	30
1.11	Partition of Unity and Lebesgue Lemma	30
1.11.1	Lebesgue Lemma and Lebesgue Number	31

1.12	Separation Axioms, Urysohn Lemma, and Tietze Extension Theorem. . . . .	31
1.13	Identification Maps, Quotient Spaces, and Geometrical Construction . . . . .	32
1.14	Function Spaces . . . . .	37
1.15	Manifolds . . . . .	38
1.16	Exercises . . . . .	40
1.17	Additional Reading. . . . .	43
	References . . . . .	43
<b>2</b>	<b>Homotopy Theory: Elementary Basic Concepts</b> . . . . .	<b>45</b>
2.1	Homotopy: Introductory Concepts and Examples . . . . .	47
2.1.1	Concept of Homotopy . . . . .	47
2.1.2	Functorial Representation. . . . .	57
2.2	Homotopy Equivalence . . . . .	58
2.3	Homotopy Classes of Maps . . . . .	62
2.4	$H$ -Groups and $H$ -Cogroups . . . . .	64
2.4.1	$H$ -Groups and Loop Spaces . . . . .	65
2.4.2	$H$ -Cogroups and Suspension Spaces . . . . .	75
2.5	Adjoint Functors . . . . .	79
2.6	Contractible Spaces . . . . .	83
2.6.1	Introductory Concepts . . . . .	83
2.6.2	Infinite-Dimensional Sphere and Comb Space . . . . .	85
2.7	Retraction and Deformation . . . . .	88
2.8	NDR and DR Pairs. . . . .	95
2.9	Homotopy Properties of Infinite Symmetric Product Spaces. . . . .	96
2.10	Applications . . . . .	97
2.10.1	Extension Problems . . . . .	97
2.10.2	Fundamental Theorem of Algebra. . . . .	99
2.11	Exercises. . . . .	100
2.12	Additional Reading. . . . .	104
	References . . . . .	105
<b>3</b>	<b>The Fundamental Groups</b> . . . . .	<b>107</b>
3.1	Fundamental Groups: Introductory Concepts . . . . .	108
3.1.1	Basic Motivation . . . . .	108
3.1.2	Introductory Concepts . . . . .	109
3.1.3	Functorial Property of $\pi_1$ . . . . .	117
3.1.4	Some Other Properties of $\pi_1$ . . . . .	119
3.2	Alternative Definition of Fundamental Groups . . . . .	124
3.3	Degree Function and the Fundamental Group of the Circle . . . . .	126
3.4	The Fundamental Group of the Punctured Plane. . . . .	130
3.5	Fundamental Groups of the Torus . . . . .	131

3.6	Vector Fields and Fixed Points . . . . .	132
3.7	Knot and Knot Groups . . . . .	133
3.8	Applications . . . . .	136
3.8.1	Fundamental Theorem of Algebra. . . . .	136
3.8.2	An Alternative Proof of Brouwer Fixed Point Theorem . . . . .	137
3.8.3	Borsuk–Ulam Theorem . . . . .	139
3.8.4	Cauchy’s Integral Theorem of Complex Analysis . . . . .	140
3.9	Exercises. . . . .	141
3.10	Additional Reading. . . . .	144
	References . . . . .	145
<b>4</b>	<b>Covering Spaces . . . . .</b>	<b>147</b>
4.1	Covering Spaces: Introductory Concepts and Examples. . . . .	148
4.1.1	Introductory Concepts . . . . .	148
4.1.2	Some Interesting Properties of Covering Spaces . . . . .	151
4.1.3	Covering Spaces of $\mathbf{R}P^n$ . . . . .	152
4.2	Computing Fundamental Groups of Figure-Eight and Double Torus. . . . .	153
4.3	Path Lifting and Homotopy Lifting Properties . . . . .	155
4.4	Lifting Problems of Arbitrary Continuous Maps. . . . .	158
4.5	Covering Homomorphisms: Their Classifications and Galois Correspondence . . . . .	161
4.5.1	Covering Homomorphisms and Deck Transformations . . . . .	161
4.5.2	Classification of Covering Spaces by Using Group Theory. . . . .	163
4.5.3	Classification of Covering Spaces and Galois Correspondence . . . . .	167
4.6	Universal Covering Spaces and Computing $\pi_1(\mathbf{R}P^n)$ . . . . .	170
4.6.1	Universal Covering Spaces . . . . .	170
4.6.2	Computing $\pi_1(\mathbf{R}P^n)$ . . . . .	172
4.7	Fibrations and Cofibrations . . . . .	174
4.7.1	Homotopy Lifting Problems. . . . .	174
4.7.2	Fibration: Introductory Concepts. . . . .	176
4.7.3	Cofibration: Introductory Concepts . . . . .	179
4.8	Hurewicz Theorem for Fibration and Characterization of Fibrations . . . . .	182
4.9	Homotopy Liftings and Monodromy Theorem . . . . .	184
4.9.1	Path Liftings and Homotopy Liftings . . . . .	185
4.9.2	Monodromy Theorem . . . . .	185
4.10	Applications and Computations . . . . .	186
4.10.1	Actions of Fundamental Groups . . . . .	186
4.10.2	Fundamental Groups of Orbit Spaces . . . . .	187

4.10.3	Fundamental Group of the Real Projective Space $\mathbf{RP}^n$ . . . . .	189
4.10.4	The Fundamental Group of Klein's Bottle . . . . .	189
4.10.5	The Fundamental Groups of Lens Spaces . . . . .	190
4.10.6	Computing Fundamental Group of Figure-Eight by Graph-theoretic Method . . . . .	191
4.10.7	Application of Galois Correspondence. . . . .	191
4.11	Exercises. . . . .	192
4.12	Additional Reading. . . . .	195
	References . . . . .	196
<b>5</b>	<b>Fiber Bundles, Vector Bundles and <math>K</math>-Theory</b> . . . . .	<b>197</b>
5.1	Bundles, Cross Sections, and Examples. . . . .	198
5.1.1	Bundles. . . . .	199
5.1.2	Cross Sections . . . . .	200
5.1.3	Morphisms of Bundles . . . . .	201
5.1.4	Examples. . . . .	204
5.2	Fiber Bundles: Introductory Concepts . . . . .	206
5.3	Hopf and Hurewicz Fiberings . . . . .	210
5.3.1	Hopf Fiberings of Spheres. . . . .	210
5.3.2	Hurewicz Fiberings . . . . .	212
5.4	$G$ -Bundles and Principal $G$ -Bundles . . . . .	213
5.5	Homotopy Properties of Numerable Principal $G$ -Bundles. . . . .	218
5.6	Classifying Spaces: The Milnor Construction. . . . .	220
5.7	Vector Bundles: Introductory Concepts . . . . .	223
5.8	Charts and Transition Functions of Bundles. . . . .	229
5.9	Homotopy Classification of Vector Bundles. . . . .	233
5.10	$K$ -Theory: Introductory Concepts . . . . .	236
5.11	Principal $G$ -Bundles for Lie Groups $G$ . . . . .	240
5.12	Applications . . . . .	241
5.13	Exercises. . . . .	241
5.14	Additional Reading. . . . .	245
	References . . . . .	246
<b>6</b>	<b>Geometry of Simplicial Complexes and Fundamental Groups of Polyhedra</b> . . . . .	<b>249</b>
6.1	Geometry of Finite Simplicial Complexes . . . . .	250
6.2	Triangulations and Polyhedra. . . . .	253
6.3	Simplicial Maps . . . . .	257
6.4	Barycentric Subdivisions . . . . .	258
6.5	Simplicial Approximation . . . . .	261
6.6	Computing Fundamental Groups of Polyhedra . . . . .	264
6.7	Applications . . . . .	265

6.7.1	Application to Extension Problem . . . . .	265
6.7.2	Application to Graph Theory . . . . .	266
6.7.3	van Kampen Theorem . . . . .	267
6.8	Exercises . . . . .	268
6.9	Additional Reading . . . . .	270
	References . . . . .	271
<b>7</b>	<b>Higher Homotopy Groups . . . . .</b>	<b>273</b>
7.1	Absolute Homotopy Groups: Introductory Concept . . . . .	274
7.2	Absolute Homotopy Groups Defined by Hurewicz . . . . .	277
7.3	Functorial Properties of Absolute Homotopy Groups . . . . .	278
7.4	The Relative Homotopy Groups: Introductory Concepts . . . . .	281
7.5	The Boundary Operator and Induced Transformation . . . . .	282
7.5.1	Boundary Operator . . . . .	282
7.5.2	Induced Transformations . . . . .	283
7.6	Functorial Property of the Relative Homotopy Groups . . . . .	284
7.7	Homotopy Sequence and Its Exactness . . . . .	285
7.7.1	Homotopy Sequence and Its Exactness . . . . .	285
7.7.2	Some Consequences of the Exactness of the Homotopy Sequence . . . . .	287
7.8	Homotopy Sequence of Fiberings and Hopf Fiberings . . . . .	289
7.8.1	Homotopy Sequence of Fiberings . . . . .	289
7.8.2	Hopf Fiberings of Spheres . . . . .	290
7.8.3	Problems of Computing $\pi_m(S^n)$ . . . . .	290
7.9	More on Hopf Maps . . . . .	291
7.10	Freudenthal Suspension Theorem and Table of $\pi_i(S^n)$ for $1 \leq i, n \leq 8$ . . . . .	292
7.10.1	Freudenthal Suspension Theorem . . . . .	293
7.10.2	Table of $\pi_i(S^n)$ for $1 \leq i, n \leq 8$ . . . . .	294
7.11	Action of $\pi_1$ on $\pi_n$ . . . . .	294
7.12	The $n$ th Cohomotopy Sets and Groups . . . . .	295
7.13	Applications . . . . .	297
7.14	Exercises . . . . .	300
7.15	Additional Reading . . . . .	302
	References . . . . .	303
<b>8</b>	<b>CW-Complexes and Homotopy . . . . .</b>	<b>305</b>
8.1	Cell-Complexes and CW-Complexes: Introductory Concepts . . . . .	306
8.1.1	Cell-Complexes . . . . .	307
8.1.2	CW-Complexes . . . . .	308
8.1.3	Examples of Spaces Which Are Neither CW-Complexes Nor Homotopy Equivalent to a CW-Complex . . . . .	312
8.2	Cellular Spaces . . . . .	312

8.3	Subcomplexes of $CW$ -Complexes . . . . .	313
8.4	Relative $CW$ -Complexes . . . . .	314
8.5	Homotopy Properties of $CW$ -Complexes, Whitehead Theorem and Cellular Approximation Theorem . . . . .	315
8.5.1	Homotopy Properties of $CW$ -Complexes . . . . .	315
8.5.2	Whitehead Theorem . . . . .	317
8.5.3	Cellular Approximation Theorem . . . . .	318
8.6	More on Homotopy Properties of $CW$ -Complexes . . . . .	319
8.7	Blakers–Massey Theorem and a Generalization of Freudenthal Suspension Theorem . . . . .	320
8.8	Applications . . . . .	321
8.9	Exercises . . . . .	323
8.10	Additional Reading . . . . .	325
	References . . . . .	326
<b>9</b>	<b>Products in Homotopy Theory . . . . .</b>	<b>329</b>
9.1	Whitehead Product Between Homotopy Groups of $CW$ -Complexes . . . . .	329
9.2	Whitehead Products Between Homotopy Groups of $H$ -Spaces . . . . .	333
9.3	A Generalization of Whitehead Product . . . . .	335
9.4	Mixed Products in Homotopy Groups . . . . .	336
9.4.1	Mixed Product in the Homotopy Category of Pointed Topological Spaces . . . . .	336
9.4.2	Mixed Product Associated with Fibrations . . . . .	337
9.5	Samelson Products . . . . .	338
9.5.1	The Samelson Product . . . . .	338
9.5.2	The Iterated Samelson Product . . . . .	339
9.6	Some Relations Between Whitehead and Samelson Products . . . . .	340
9.7	Applications . . . . .	341
9.7.1	Adams Theorem Using Whitehead Product . . . . .	341
9.7.2	Homotopical Nilpotence of the Seven Sphere $S^7$ . . . . .	342
9.8	Exercises . . . . .	342
9.9	Additional Reading . . . . .	344
	References . . . . .	345
<b>10</b>	<b>Homology and Cohomology Theories . . . . .</b>	<b>347</b>
10.1	Chain Complexes . . . . .	349
10.2	Simplicial Homology Theory . . . . .	352
10.2.1	Construction of Homology Groups of a Simplicial Complex . . . . .	353
10.2.2	Induced Homomorphism and Functorial Properties of Simplicial Homology . . . . .	360
10.2.3	Computing Homology Groups of Polyhedra . . . . .	361

10.3	Relative Simplicial Homology Groups . . . . .	362
10.4	Exactness of Simplicial Homology Sequences . . . . .	364
10.5	Simplicial Cohomology Theory: Introductory Concepts. . . . .	365
10.6	Simplicial Cohomology Ring . . . . .	367
10.7	Singular Homology . . . . .	368
10.7.1	Singular Homology Groups . . . . .	369
10.7.2	Reduced Singular Homology Groups. . . . .	372
10.7.3	Relative Singular Homology Groups . . . . .	373
10.8	Eilenberg–Zilber Theorem and Künneth Formula . . . . .	374
10.8.1	Eilenberg–Zilber Theorem . . . . .	374
10.8.2	Künneth Formula . . . . .	374
10.9	Singular Cohomology . . . . .	375
10.10	Relative Cohomology Groups . . . . .	376
10.11	Hurewicz Homomorphism . . . . .	377
10.12	Mayer–Vietoris Sequences. . . . .	379
10.12.1	Mayer–Vietoris Sequences in Singular Homology Theory . . . . .	379
10.12.2	Mayer–Vietoris Sequences in Simplicial Homology Theory . . . . .	380
10.13	Computing Homology Groups . . . . .	381
10.13.1	Homology Groups of a One-Point Space . . . . .	381
10.13.2	Homology Groups of CW-complexes . . . . .	382
10.14	Cellular Homology . . . . .	383
10.15	Čech Homology and Cohomology Groups. . . . .	384
10.16	Universal Coefficient Theorem for Homology and Cohomology . . . . .	385
10.16.1	Homology with Arbitrary Coefficient Group. . . . .	385
10.16.2	Universal Cohomology Theorem for Cohomology . . . . .	387
10.17	Betti Number and Euler Characteristic . . . . .	388
10.17.1	Euler Characteristics of Polyhedra. . . . .	388
10.17.2	Euler Characteristic of Finite Graphs. . . . .	390
10.17.3	Euler Characteristic of Graded Vector Spaces. . . . .	390
10.17.4	Euler–Poincaré Theorem for Finite CW-complexes . . . . .	391
10.18	Cup and Cap Products in Cohomology Theory . . . . .	392
10.18.1	Cup Product. . . . .	393
10.18.2	Cap Product. . . . .	395
10.19	Applications . . . . .	396
10.19.1	Jordan Curve Theorem . . . . .	396
10.19.2	Homology Groups of $\bigvee_{\alpha \in a} S_{\alpha}^n$ . . . . .	397

10.20	Invariance of Dimension . . . . .	398
10.21	Exercises. . . . .	399
10.22	Additional Reading. . . . .	404
	References . . . . .	406
<b>11</b>	<b>Eilenberg–MacLane Spaces . . . . .</b>	<b>407</b>
11.1	Eilenberg–MacLane Spaces: Introductory Concept . . . . .	407
11.2	Construction of Eilenberg–MacLane Spaces $K(G, n)$ . . . . .	409
11.2.1	A Construction of $K(G, 1)$ . . . . .	409
11.2.2	A Construction of $K(G, n)$ for $n > 1$ . . . . .	409
11.2.3	Moore Spaces . . . . .	410
11.2.4	Killing Homotopy Groups . . . . .	410
11.2.5	Postnikov Tower: Its Existence and Construction . . . . .	411
11.2.6	Existence Theorem . . . . .	413
11.3	Applications . . . . .	413
11.4	Exercises. . . . .	415
11.5	Additional Reading. . . . .	416
	References . . . . .	417
<b>12</b>	<b>Eilenberg–Steenrod Axioms for Homology and Cohomology Theories . . . . .</b>	<b>419</b>
12.1	Eilenberg–Steenrod Axioms for Homology Theory. . . . .	420
12.2	The Uniqueness Theorem for Homology Theory . . . . .	422
12.3	Eilenberg–Steenrod Axioms for Cohomology Theory . . . . .	424
12.4	The Reduced 0-dimensional Homology and Cohomology Groups. . . . .	427
12.5	Applications . . . . .	427
12.5.1	Invariance of Homology Groups. . . . .	428
12.5.2	Invariance of Cohomology Groups . . . . .	428
12.5.3	Mayer–Vietoris Theorem . . . . .	428
12.6	Exercises. . . . .	429
12.7	Additional Reading. . . . .	430
	References . . . . .	431
<b>13</b>	<b>Consequences of the Eilenberg–Steenrod Axioms. . . . .</b>	<b>433</b>
13.1	Immediate Consequences. . . . .	433
13.2	Applications . . . . .	440
13.2.1	Cofibration and Homology. . . . .	440
13.2.2	Computing Ordinary Homology Groups of $S^n$ . . . . .	441
13.3	Exercises. . . . .	442
13.4	Additional Reading. . . . .	443
	References . . . . .	443



<b>14 Applications.</b>	445
14.1 Degrees of Spherical Maps and Their Applications.	445
14.1.1 Degree of a Spherical Map	446
14.1.2 Hopf Classification Theorem	449
14.1.3 The Brouwer Fixed Point Theorem.	450
14.2 Continuous Vector Fields	451
14.3 Borsuk–Ulam Theorem with Applications	453
14.3.1 Borsuk–Ulam Theorem	453
14.3.2 Ham Sandwich Theorem	454
14.3.3 Lusternik–Schnirelmann Theorem.	455
14.4 The Lefschetz Number and Fixed Point Theorems	455
14.5 Application of Euler Characteristic	458
14.6 Application of Mayer–Vietoris Sequence.	460
14.7 Application of van Kampen Theorem	461
14.8 Applications to Algebra	462
14.9 Application of Brown Functor	463
14.10 Applications Beyond Mathematics	464
14.10.1 Application to Physics.	464
14.10.2 Application to Sensor Network.	465
14.10.3 Application to Chemistry.	465
14.10.4 Application to Biology, Medical Science and Biomedical Engineering.	466
14.10.5 Application to Economics	466
14.10.6 Application to Computer Science	467
14.11 Exercises.	467
14.12 Additional Reading.	471
References	472
<b>15 Spectral Homology and Cohomology Theories.</b>	475
15.1 Spectrum of Spaces	476
15.2 Spectral Reduced Homology Theory.	477
15.3 Spectral Reduced Cohomology Theory	480
15.4 Generalized Homology and Cohomology Theories	481
15.5 The Brown Representability Theorem	481
15.6 A Generalization of Eilenberg–MacLane Spectrum and Construction of Its Associated Generalized Cohomology Theory.	484
15.6.1 Construction of a New $\Omega$ -Spectrum $\underline{A}$ .	484
15.6.2 Construction of the Cohomology Theory Associated with $\underline{A}$ .	485
15.7 $K$ -Theory as a Generalized Cohomology Theory	487
15.8 Spectral Unreduced Homology and Cohomology Theories.	488
15.9 Cohomology Operations	489
15.9.1 Cohomology Operations of Type $(G, n; T, m)$ and Eilenberg–MacLane Spaces	490

15.9.2	Cohomology Operation Associated with a Spectrum . . . . .	491
15.9.3	Stable Cohomology Operations. . . . .	492
15.9.4	A Characterization of the Group $\{\overline{O}_m^k\}$ . . . . .	493
15.10	Stable Homotopy Theory and Homotopy Groups Associated with a Spectrum. . . . .	496
15.10.1	Stable Homotopy Groups. . . . .	496
15.10.2	Homology Groups Associated with a Spectrum . . . .	498
15.10.3	Homotopy Groups Associated with a Spectrum . . . .	499
15.11	Applications . . . . .	499
15.11.1	Poincaré Duality Theorem . . . . .	500
15.11.2	Homotopy Type of the Eilenberg–MacLane Space $K(G, n)$ . . . . .	502
15.11.3	Application of Representability Theorem of Brown. . . . .	503
15.11.4	More Applications of Spectra. . . . .	504
15.11.5	Homotopical Description of Singular Cohomology Theory . . . . .	505
15.12	Exercises. . . . .	506
15.13	Additional Reading. . . . .	508
	References . . . . .	509
<b>16</b>	<b>Obstruction Theory . . . . .</b>	<b>511</b>
16.1	Basic Aim of Obstruction Theory. . . . .	512
16.1.1	The Extension Problem . . . . .	513
16.1.2	The Lifting Problem . . . . .	514
16.1.3	Relative Lifting Problem . . . . .	514
16.1.4	Cross Section Problem . . . . .	515
16.2	Notations and Abbreviations . . . . .	515
16.3	The Obstruction Theory: Basic Concepts. . . . .	516
16.3.1	The Obstruction Cochains and Cocycles . . . . .	516
16.3.2	The Deformation and Difference Cochains. . . . .	519
16.3.3	The Eilenberg Extension Theorem . . . . .	520
16.3.4	The Obstruction Set for Extension . . . . .	521
16.3.5	The Homotopy Index . . . . .	522
16.4	Applications . . . . .	523
16.4.1	A Link between Cohomolgy and Homotopy with Hopf Theorem . . . . .	523
16.4.2	Stepwise Extension of A Cross Section . . . . .	524
16.4.3	Homological Version of Whitehead Theorem . . . . .	526
16.4.4	Obstruction for Homotopy Between Relative Lifts . . . . .	526
16.5	Exercises. . . . .	527
16.6	Additional Reading. . . . .	530
	References . . . . .	530

<b>17</b>	<b>More Relations Between Homology and Homotopy</b>	533
17.1	Some Similarities and Key Links	534
17.1.1	Some Similarities	534
17.1.2	Hurewicz Homomorphism Theorem: A Key Link	534
17.2	Relative Version of Hurewicz Homomorphism Theorem	536
17.3	Alternative Proof of Homological Version of Whitehead Theorem	537
17.4	Dold–Thom Theorem	538
17.5	The Hopf Invariant and Adams Theorem	539
17.5.1	Hopf Invariant	539
17.5.2	Vector Field Problem and Adams Theorem	541
17.6	Exercises	542
17.7	Additional Reading	545
	References	545
<b>18</b>	<b>A Brief History of Algebraic Topology</b>	547
18.1	Poincaré and his Conjecture	548
18.2	Early Development of Homotopy Theory	550
18.3	Category Theory and CW-Complexes	554
18.4	Early Development of Homology Theory	555
18.5	Hopf Invariant	557
18.6	Eilenberg and Steenrod Axioms	558
18.7	Fiber Bundle, Vector Bundle, and $K$ -Theory	559
18.8	Eilenberg–MacLane Spaces and Cohomology Operations	561
18.9	Generalized Homology and Cohomology Theories	562
18.10	$\Omega$ -Spectrum and Associated Cohomology Theories	563
18.11	Brown Representability Theorem	564
18.12	Obstruction Theory	565
18.13	Additional Reading	566
	References	567
	<b>Appendix A: Topological Groups and Lie Groups</b>	569
	<b>Appendix B: Categories, Functors and Natural Transformations</b>	581
	<b>List of Symbols</b>	599
	<b>Author Index</b>	607
	<b>Subject Index</b>	609

<http://www.springer.com/978-81-322-2841-7>

Basic Algebraic Topology and its Applications

Adhikari, M.R.

2016, XXIX, 615 p. 176 illus., Hardcover

ISBN: 978-81-322-2841-7