Preface

Algebraic topology is one of the most important creations in mathematics which uses algebraic tools to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism (though usually classify up to homotopy equivalence). The most important of these invariants are homotopy groups, homology groups, and cohomology groups (rings). The main purpose of this book is to give an accessible presentation to the readers of the basic materials of algebraic topology through a study of homotopy, homology, and cohomology theories. Moreover, it covers a lot of topics for advanced students who are interested in some applications of the materials they have been taught. Several basic concepts of algebraic topology, and many of their successful applications in other areas of mathematics and also beyond mathematics with surprising results have been given. The essence of this method is a transformation of the geometric problem to an algebraic one which offers a better chance for solution by using standard algebraic methods.

The monumental work of Poincaré in “Analysis situs”, Paris, 1895, organized the subject for the first time. This work explained the difference between curves deformable to one another and curves bounding a larger space. The first one led to the concepts of homotopy and fundamental group; the second one led to the concept of homology. Poincaré is the first mathematician who systemically attacked the problems of assigning algebraic invariants to topological spaces. His vision of the key role of topology in all mathematical theories began to materialize from 1920. This subject is an interplay between topology and algebra and studies algebraic invariants provided by homotopy, homology, and cohomology theories. The twentieth century witnessed its greatest development.

The literature on algebraic topology is very vast. Based on the author’s teaching experience of 50 years, academic interaction with Prof. B. Eckmann and Prof. P.J. Hilton at E.T.H., Zurich, Switzerland, in 2003, and lectures at different institutions in India, USA, France, Switzerland, Greece, UK, Italy, Sweden, Japan, and many other countries, this book is designed to serve as a basic text of modern algebraic topology at the undergraduate level. A basic course in algebraic topology presents a variety of phenomena typical of the subject. This book conveys the basic language of modern algebraic topology through a study of homotopy, homology, and cohomology theories with some fruitful applications which display the great beauty of the subject. For this study, the book displays a variety of topological spaces: spheres, projective spaces, classical groups and their quotient spaces, function spaces, polyhedra, topological groups, Lie groups, CW-complexes, Eilenberg–MacLane spaces, infinite symmetric product spaces, and some other spaces. As well as, the book studies a variety of maps, which are continuous functions between topological spaces.

Characteristics which are shared by homeomorphic spaces are called *topological invariants*; on the other hand, characteristics which are shared by homotopy equivalent spaces are called *homotopy invariants*. The Euler characteristic is an integral invariant, which distinguishes non-homeomorphic spaces. The search of other invariants has established connections between topology and modern algebra in such a way that homeomorphic spaces have isomorphic algebraic structures. Historically, the concepts of fundamental groups, higher homotopy groups, and homology and cohomology groups came from such a search. The natural emphasis is: to solve a geometrical problem of global nature, one first reduces it to a homotopy-theoretic problem; this is then transformed to an algebraic problem which provides a better chance for solution. This technique has been the most fruitful one in algebraic topology. The notions initially introduced in homology and homotopy theories for applications to problems of topology have found fruitful applications in other parts of mathematics. Homological algebra and K-theory are their outstanding examples.

The materials discussed here have appeared elsewhere. Each chapter opens with a short introduction which summarizes the information that sets out its central theme and closes with a list of sources for the use of readers to expand their knowledge. This does not mean that other sources are not good. Our contribution is the selection of the materials and their presentation. Each chapter is split into several sections (and subsections) depending on the nature of the materials which constitute the organizational units of the text. Each chapter provides exercises with further applications and extension of the theory. Some exercises carry hints which should not be taken as ideal ones. Many of them can be solved in a better way. The title of the book suggests the scope and power of algebraic topology and its text is expanded over 18 chapters and two appendices displayed below.

Chapter 1 assembles together some basic concepts of set theory, algebra, analysis, set topology, Euclidean spaces, manifolds with some standard notations for smooth reading of the book.

Chapter 2 is devoted to the study of basic elementary concepts of homotopy theory with illustrative examples. A homotopy formalizes the intuitive idea of continuous deformation of a continuous map between two topological spaces. It displays a variety of phenomena and related problems such as homotopy classification of continuous maps up to homotopy equivalence introduced by Hurewicz (1904–1956) in 1935, contractible spaces, H-groups (Hopf groups) and H-cogroups through their homotopy properties. Finally, this chapter presents interesting immediate applications of homotopy in dealing with some extension problems, lifting problems, and proving "Fundamental theorem of algebra" by using homotopic concepts.

Chapter 3 continues the study of homotopy theory through the concept of fundamental groups invented by H. Poincaré in 1895 which conveys the first transition from topology to algebra by assigning an algebraic structure to the set of relative homotopy classes of loops in a functorial way. The group structure of these homotopy classes of loops is proved in Sects. 3.1 and 3.2 in two different ways. This group earlier called Poincaré group is now known as *fundamental group*. It is the first influential invariant of homotopy theory and is also the first of a series of algebraic invariants π_n, called *homotopy groups* studied in Chap. 7. This chapter computes fundamental group of the circle by using the degree of a continuous map $f : S^1 \to S^1$, and studies Brouwer fixed point theorem for dimension 2, fundamental theorem of algebra, vector field problems on D^2 and knot groups by using the tools of fundamental groups.

Chapter 4 continues the study of the fundamental groups and presents a thorough discussion of covering spaces which are deeply connected with fundamental groups. Algebraic features of the fundamental groups are expressed by the geometric language of covering spaces. This chapter is designed to utilize the power of the fundamental groups and also to establish an exact correspondence between the various connected covering spaces of a given topological space B and subgroups of its fundamental group $\pi_1(B)$, like Galois theory, with its correspondence between field extensions and subgroups of Galois groups, which is an amazing result. This chapter also studies the concepts of fibrations and cofibrations with their applications born in geometry and topology.

Chapter 5 continues the study of homotopy theory through fiber bundles, vector bundles, and K-theory. Covering spaces provide tools to study the fundamental groups. Fiber bundles provide likewise tools to study higher homotopy groups (which are generalization of fundamental groups and described in Chap. 7). The importance of fiber spaces was realized during 1935–1950 to solve several problems relating to homotopy and homology. The motivation of the study of fiber bundles and vector bundles came from the distribution of signs of the derivatives of the plane curves at each point. This chapter also discusses homotopy classification of vector bundles, Milnor's construction of a universal fiber bundle for a topological group G with homotopy classification of principal G-bundles and presents the introductory concept of K-theory born in connecting the rich structure of vector bundles over a paracompact space B with the set of homotopy classes of maps from B into the Grassmann manifold of n-dimensional subspaces in infinite-dimensional space. This theory plays a vital role in applications of algebraic topology to analysis, algebraic geometry, topology, ring theory, and number theory.

Chapter 6 builds up interesting topological spaces called *polyhedra* from simplexes followed by a study of their homotopy properties and develops some tools for computing the fundamental groups of a large class of topological spaces. The geometrical objects such as points, edges, triangles, and tetrahedra are examples of low-dimensional simplexes. Simplicial complexes provide a convenient way to study manifolds. This chapter considers how simplexes may be fitted together to construct simplicial complexes which play an important role to construct interesting topological spaces such as polyhedra for the study of algebraic topology. They form building blocks of homology theory which begins in Chap. 10. The concept of triangulation is utilized to solve extension problems and that of edge-path to show that edge-group $E(K, v)$ is isomorphic to the fundamental group $\pi_1(|K|, v)$ for any simplicial complex K. Finally, van Kampen theorem is proved by using graph-theoretic results. This chapter also proves simplicial approximation theorem given by L.E.J. Brouwer (1881–1967) and J.W. Alexander (1888–1971) around 1920 by utilizing a certain good feature of simplicial complexes introduced by Alexander. This theorem plays a key role in the study of homotopy and homology theories.

Chapter 7 continues to study homotopy theory displaying the construction of a sequence of functors π_n given by W. Hurewicz (1904–1956) in 1935 from topology to algebra by extending the concept of fundamental group invented by H. Poincaré in 1895. The basic idea of homotopy groups is to classify all continuous maps from S^n to pointed topological space X up to homotopy equivalence. To study topological spaces X of low dimension, the fundamental group $\pi_1(X)$ is very useful. But it needs refined tools for the study of higher dimensional spaces. For example, fundamental group cannot distinguish spheres S^n with $n \geq 2$. Such a limitation of low dimension can be removed by considering the natural higher dimensional analogues of $\pi_1(X)$. More precisely, this chapter defines the nth (absolute) homotopy group and generalizes it to a (relative) homotopy group of a triplet and studies algebraic, functorial and fibering properties with the exactness of homotopy sequence of fibering, Hopf maps introduced by H. Hopf (1894–1971) in 1935 for the investigation of certain homotopy groups of S^n, action of π_1 on π_n, Freudenthal suspension theorem given by H. Freudenthal (1905–1990) in 1937 for the investigation of the homotopy groups $\pi_m(S^n)$ for $0 < m < n$, weak fibration which plays a key role in the study of higher homotopy groups, and the nth cohomotopy set $\pi^n(X, A)$ on which K. Borsuk (1905–1982) endowed in 1936 with an abelian group structure under certain conditions on (X, A). This chapter also discusses some interesting applications of higher homotopy groups.

Chapter 8 continues to study homotopy theory through a suitable special class of topological spaces, called CW-complexes introduced by J.H.C. Whitehead (1904–1960) in 1949 to meet the need for further development of homotopy theory. This class of spaces is broader and has some better categorical properties than simplicial complexes, but still retains a combinatorial nature that allows for computation (often with a much smaller complex). The concept of CW-complexes is introduced as a natural generalization of the concept of polyhedra by relaxing all "linearity conditions" in simplicial complexes, instead cells are attached by arbitrary continuous maps starting with a discrete set, whose each point is regarded as a 0-cell. A CW-complex is built up by successive adjunctions of cells of dimensions $1, 2, 3, \ldots$. There is an analogy between what can be done topologically with a space, and what can be done algebraically with its chain groups. In the class of CW-complexes this analogy attains its highest strength. The category of

CW-complexes is a suitable category for a systematic study of algebraic topology. Algebraic topologists now feel that a study of CW-complex should be included in the basic course of algebraic topology, and this study should move to the theorem that every continuous map between CW-complexes is homotopic to a cellular map. This chapter studies the basic aspects of CW-complexes and relative CW-complexes with their homotopy properties and proves Whitehead theorem, Freudenthal suspension theorem (general form), and cellular approximation theorem with their applications.

Chapter 9 continues to study homotopy theory through the different products in homotopy groups such as the Whitehead product introduced by J.H.C. Whitehead in 1941, mixed product introduced by McCarty in 1964, and Samelson product. Whitehead product provides a technique at least in some cases for constructing nonzero elements of the homotopy group $\pi_{p+q-1}(X)$ of a pointed topological space X. Moreover, this chapter finds a generalization of Whitehead product, establishes certain relation between Whitehead and Samelson products, and studies mixed products corresponding to a fiber space and a topological transformation group acting on it.

Chapter 10 begins to study homology and cohomology theories. Homotopy groups are very difficult to compute. There is an alternative approach of construction of a different topological invariant, the so-called homology group, which historically came earlier than homotopy groups. Homology (cohomology) theory is a covariant (contravariant) functor from the category of topological spaces to the category of abelian groups. Homology (simplicial) invented by H. Poincaré in 1895 is one of the most fundamental influential inventions in mathematics. The basic idea of homology is that it starts with a geometric object (a space) which is given by combinatorial data (a complex). Then the linear algebra and boundary relations determined by this data are used to construct homology groups. The simplicial devices in simplicial homology theory are gradually generalized to singular homology by using the algebraic properties of the singular complex. There exist different homology theories: simplicial, singular, cellular, and Čech homology theories which are studied in this chapter. The most important homology theory in algebraic topology is the singular homology. Simplicial homology is the primitive version of singular homology. Cohomology theory is also discussed. In some sense, homology theory and cohomology theory are dual to each other. More precisely, this chapter begins with a study of the concepts of chain complex, boundary, cycle introduced by W. Mayer (1887–1947) in 1929 from a purely algebraic viewpoint. This chapter presents a construction of the homology groups of a simplicial complex in two steps: first by assigning to each simplicial complex a certain complex, called *chain complex* followed by assigning to the chain complex its homology group. This construction assigns a group structure to cycles that are not boundaries with an extension to the concept of relative simplicial homology groups and generalizes simplicial homology theory to singular homology theory. These two theories are related by the basic result that the singular homology of a polyhedron is isomorphic to the simplicial homology of any of its triangulated simplicial complexes. This chapter examines the relations between absolute homology groups of simplicial chain complexes and the relative homology groups of relative simplicial chain complexes by using the language of exact sequences and shows that the relative homology groups $H_p(K, L)$ for any pair (K, L) of simplicial complexes fit into a long exact sequence. This chapter also discusses homology groups $H_n(X; G)$ with an arbitrary coefficient group G (abelian), Mayer–Vietoris sequences in singular and simplicial homology theories, cup product, and gives the Künneth formula and Eilenberg–Zilber theorem which are used for computing homology or cohomolgy of product spaces, and Euler characteristic & Jordan curve theorem from the viewpoint of homology theory.

Chapter 11 studies a special class of CW-complexes having only one nonzero homotopy groups, called *Eilenberg–MacLane spaces* which were introduced by S. Eilenberg (1915–1998) and S. MacLane (1909–2005) in 1945. Such spaces form a very important class of CW-complexes in algebraic topology. Their importance is twofold: they develop both homotopy and homology theories. They are closely linked with the study of cohomology operations. This chapter presents Eilenberg–MacLane spaces with their construction and studies their homotopy properties. The construction process of Eilenberg–MacLane spaces $K(G, n)$ for all possible (G, n) depends on a very natural class of spaces, called *Moore spaces* of type (G, n), denoted by $M(G, n)$. This chapter also studies Postnikov towers to meet the need for construction of Eilenberg–MacLane spaces.

Chapter 12 presents an approach formulating axiomatizaton of ordinary homology and cohomology theories. These axioms, now called Eilenberg and Steenrod axioms were announced by S. Eilenberg (1915–1998) and N.E. Steenrod (1910–1971) in 1945, but first appeared in their celebrated book *Foundations of Algebraic Topology* in 1952. This approach came from the problem of comparing the various definitions of homology and cohomology given in the previous years. Eilenberg and Steenrod initiated a new approach by taking a small number of their properties (not focussing on machinery used for construction of homology and cohomology groups) as axioms to characterize a theory of homology and cohomology. This axiomatic approach simplifies the proofs of many lengthy and complicated theorems and escapes the avoidable difficulty to motivate the students who are learning homology and cohomology theories for the first time as their systematic study. This axiomatic approach classifies and unifies different homology groups on the category of compact triangulable spaces and inaugurates its dual theory called *cohomology theory*. This approach is the most important contribution to algebraic topology since the invention of the homology groups by Poincaré in 1895.

Chapter 13 continues the study of homology and cohomology theories by presenting some of their interesting properties which directly follow from the Eilenberg and Steenrod axioms for homology and cohomology theories such as homotopy equivalence in these theories, relations between cofibrations and homology theory, and finally computes the ordinary homology groups of S^n with coefficients in an arbitrary abelian group G.

Chapter 14 presents further interesting applications of the homotopy, homology, and cohomology theories. The notions initially introduced in these theories to solve problems of topology that have fruitful applications, and proves many interesting theorems such as Hopf's classification theorem, hairy ball theorem, ham sandwich theorem, Borsuk–Ulam theorem, Lusternik–Schnirelmann theorem, Lefschetz fixed point theorem, and Jordan curve theorem. It also proves some results related to graph theory, fixed point theory of continuous maps, vector fields, and applications to algebra. Moreover, this chapter indicates some applications of algebraic topology in physics, chemistry, economics, biology, and medical science with specific references.

Chapter 15 conveys the concept of a spectrum originated by F.L. Lima (1929–) in 1958 and constructs its associated spectral homology and cohomology theories, and generalized homology and cohomology theories (which have been proved to be very useful theories) to distinguish them from ordinary homology and cohomology theories. Their properties and relations to homotopy theory are also discussed. For example, the ordinary homology group of certain topological spaces X can be thought of as an approximation to $\pi_n(X)$. Moreover, this chapter constructs a new Ω-spectrum $\underline{A}$, generalizing the Eilenberg–MacLane spectrum $K(G, n)$ and also constructs its associated cohomology theory $h^*(;\underline{A})$ which generalizes the ordinary cohomology theory of Eilenberg and Steenrod. This chapter conveys K-theory as a generalized cohomology theory and also studies the Brown representability theorem, stable homotopy groups, the cohomology operations, and Poincaré duality theorem.

Chapter 16 studies a theory known as "obstruction theory" by utilizing the tools of cohomology theory to encounter two basic problems in algebraic topology such as extension and lifting problems. Obvious examples are the homotopy extension and homotopy lifting problems. The homotopy classifications of continuous maps, together with the study of extension and lifting problems, play a central role in algebraic topology. The term "obstruction theory" refers to a technique for defining a sequence of cohomology classes that are obstructions to finding solution to the extension, lifting or relative lifting problems. Obstruction theory leads to make an attempt to find a general solution. This theory originated in the classical work of Hopf, Eilenberg, Steenrod, and Postinikov in around 1940. Certain sets of cohomology elements, called obstructions, are associated with both a single map in the case of extension and with a pair of maps in the case of homotopies. These are invariants depending only on the topological spaces and their continuous mappings. In polyhedra these are the characteristics for the existence or non-existence of the desired extensions and homotopies. The underlying idea of associating cohomology elements with continuous mappings was implicitly used by H. Whitney (1907–1989) and first explicitly formulated by N.E. Steenrod (1910–1971). This chapter uses cohomology theory to yield algebraic indicators for obstacles to extension and lifting problems of continuous maps and proves Eilenberg extension theorem. It presents some applications of obstruction theory to prove a homological version of Whitehead theorem, stepwise extension of cross-section and obstruction for homotopy between relative lifts.

Chapter 17 presents some similarities and interesting relations among homotopy, homology, and cohomology. In earlier chapters, some relations between these theories have been discussed. This chapter continues to convey more relations through Hurewicz homomorphism, Eilenberg–MacLane spaces, Dold–Thom theorem, Brown's representation theorem, Hopf invariant and Adams classical theorem on Hopf invariant. Historically, L.E.J. Brouwer first connected homology and homotopy in 1912 by proving that two continuous maps of a two-dimensional sphere into itself can be continuously deformed into each other if and only if they have the same degree (i.e., if and only if they are equivalent from the view point of homology theory). Hopf's classification theorem generalizes Brouwer's result to an arbitrary dimension. The homotopy groups resemble the homology groups in many respects under suitable situations proved by Hurewicz in his celebrated "equivalence theorem". There is also a lack of similarities between these two theories essentially due to the absence in higher homotopy groups the excision property for homology and also due to the absence in higher homotopy groups a theorem analogous to van Kampen theorem for fundamental group.

Chapter 18 focuses a brief history of algebraic topology highlighting the emergences of the ideas leading to new areas of study in algebraic topology and conveys the contributions of some mathematicians who introduced new concepts or proved theorems of fundamental importance or inaugurated new theories in algebraic topology starting from the creation of fundamental group and homology group by H. Poincaré in 1895, which are the first basic and influential inventions in algebraic topology. The literature on algebraic topology is very vast. Some concepts studied now in algebraic topology had been found in the work of B. Riemann (1826–1866), C. Felix Klein (1849–1925), and H. Poincaré (1854–1912). But the foundation of algebraic (combinatorial topology) was laid in the decade beginning 1895 by H. Poincaré through the publication of his famous series of memoirs "Analysis Situs" from 1895 onwards. J.W. Alexander (1888–1971) used the word "topological" in the titles of his research papers in the 1920s. This chapter also conveys more names with their contributions in algebraic topology. The early development of homotopy theory was essentially due to H. Poincaré, L.E.F. Brouwer, H. Hopf, W. Hurewicz, H. Freudenthal, and many others. W. Hurewicz first established a connection between homology and homotopy groups for $(n-1)$-connceted spaces, when $n \geq 2$. H. Hopf pioneered a study of maps into spheres during 1926–1935 and inaugurated the homotopy theory with the discovery of the Hopf map followed by the research of W. Hurewicz, and Freudenthal. Since then homotopy theory has made a rapid progress and now plays an important role in mathematics. Homology, invented by Henry Poincaré during 1895–1901, is one of the most fundamental influential inventions in mathematics. He started with a geometric object (a space) which is given by combinatorial data (a simplicial complex), then the linear algebra and boundary relations by these data are used to construct homology groups. There are other homology theories:

(i) Homology groups for compact metric spaces introduced by L. Vietoris (1891–2002) in 1927;
(ii) Homology groups for compact Hausdorff spaces introduced by E.Čech (1893–1960) in 1932;
(iii) Singular homology groups are first defined by S. Lefschitz (1884–1972) in 1933.

All these homology theories lived in isolation. Algebraic topologists in around 1940 started comparing various definitions of homology and cohomology given in the previous years. Eilenberg and Steenrod initiated a new approach in 1945 by taking a small number of their properties (not focusing on machinery used for construction of homology and cohomology groups) as axioms to characterize a theory of homology and cohomology. This approach is the most important contribution to algebraic topology since the invention of the homology groups by Poincaré and is called the *axiomatic approach* given by a set of seven axioms announced by S. Eilenberg and N. Steenrod in 1945 and published in their book in 1952. This approach classifies and unifies different homology groups on the category of compact triangulated spaces and inaugurated its dual theory for cohomology theories. This chapter also conveys the contributions of more mathematicians, S. MacLane, J.H. Whitehead, Serre, Brown, Milnor, and Grothendieck, to name a few.

Apppendix A studies classical topological groups and Lie groups that occupy a vast territory in topology and geometry. Lie groups are special topological groups and also manifolds carrying a differential structure. For example, $GL(n,\mathbf{R}), GL(n,\mathbf{C}), GL(n,H), SL(n,\mathbf{R}), SL(n,\mathbf{C}), O(n,\mathbf{R}), U(n,\mathbf{C}), SL(n,H)$ are some important classical Lie Groups. Historically, S. Lie (1842–1899) investigated group of transformations. He developed his theory of transformation groups to solve his integration problems. Such groups are now called Lie groups after his name. The Fifth Problem of Hilbert announced at the ICM 1900, Paris, is linked to Sophus Lie theory of transformation groups which asserts that Lie groups act as groups of transformations on manifolds.

Appendix B discusses category theory through the study of categories, functors, and natural transformations with an eye to study algebraic topology which consists of the construction and use of functors from some category of topological spaces into an algebraic category. This theory plays an important role for the study of homotopy, homology, and cohomology theories, which constitute the basic text of this book in addition to *adjoint functor, representable functor, abelianization functor, Brown functor, and infinite symmetric product functor.* All constructions in algebraic topology are in general functorial. Fundamental groups, higher homotopy groups, and homology and cohomology groups are not only invariants of the underlying topological space, in the sense that two topological spaces which are homeomorphic have the isomorphic associated groups (or modules) but their associated morphisms also correspond to a continuous mapping of topological spaces an induced group (or module) homomorphism on the associated groups (modules), and these homomorphisms can be used to show non-existence (or, much more deeply, existence) of mappings. So the readers of algebraic topology cannot escape learning the concepts of categories, functors and natural transformations.

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