

Chapter 2

Incompressible and Compressible Analytical Viscoelastic Models

Abstract In this chapter we will derive the analytical solutions for two most representative cases of Earth's models: the stratified viscoelastic incompressible model and the self-compressed viscoelastic compressible sphere. Within the scheme of the Correspondence Principle, these analytical solutions make possible to gain a deep insight into the relaxation spectra of incompressible and compressible Earth's models, via normal modes or complex contour integration. Love numbers, for surface, unitary, constant loads, are finally explored during their time evolution.

2.1 Analytical Solution

By considering the general solution for the spheroidal and toroidal vector solution $\mathbf{y}_{\ell m}$ given by Eq. (1.169) in Chap. 1, its analytical expression can be obtained once specified the analytical expressions for the core-mantle boundary matrix $\mathbf{I}_C$ and the propagator matrix $\mathbf{\Pi}_\ell$. This is possible by imposing some restrictions on the material parameters of the Earth's model that will be introduced later. Here, we recall that the vector $\mathbf{b}$, Eq. (1.128), describes the Earth's surface boundary conditions and it differs from zero only for surface loading and tidal forcing, while the vector $\mathbf{w}$, Eq. (1.165), describes internal loading and seismic forcing.

2.2 Green Functions for Incompressible and Compressible Stratified Viscoelastic Earth's Models

Four major layers build our Earth's models, the lithosphere, the upper and lower mantle and the core. Each one of these layers can in principle be subdivided into other layers, but this four-layer scheme continues to be valid, since any further stratified model would continue to carry an elastic lithosphere, a viscoelastic upper mantle, a viscoelastic lower mantle and an inviscid core. Thus from the point of view of rheology, the Maxwell Model of Fig. 1.2 will be appropriate for the upper and lower

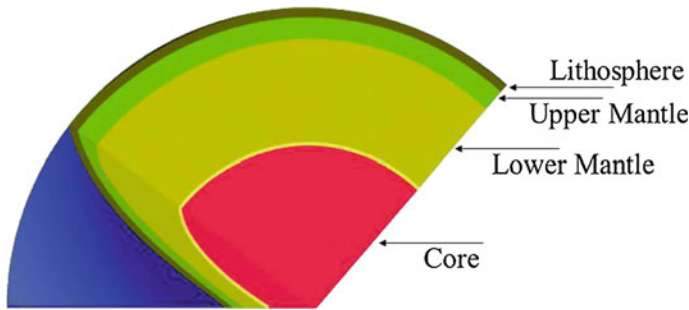


Fig. 2.1 Schematic Earth's model, including the lithosphere, *gray*, the upper mantle, *dark green*, the lower mantle, *light green*, and the core, *red*

mantle, the green layers of Fig. 2.1, while the lithosphere would be characterized solely by the spring of Fig. 1.2.

The core will always be inviscid, and we will not consider the differentiation between the inner and the outer core. When dealing with other planetary bodies, as Europa in Chap. 9, we will modify the outermost layers, in such a way to account for an icy “lithosphere” and a fluid global ocean underneath, and the mantle will also be considered different from the Earth’s one in terms of its dimension. Earth’s models can be differentiated in many more layers than those of Fig. 2.1, but the major discontinuities in density, at the CMB and upper-lower mantle, remain fixed at 3485 and 5701 km from the centre of the planet. It is worthwhile to remind that the mantle layers are green in our cartoon of the Earth since olivine and pyroxene, the major constituents of the mantle, carry beautiful green colors in the visible frequency range, as observed in samples lifted at the Earth’s surface from the mantle by tectonic processes. Peridot (i.e., the gem variety of olivine—The Egyptians fashioned peridot beads as long as 3500 years ago!) is used in jewelry, which makes our planet a precious source of gemstones in the Universe!

2.2.1 Core-Mantle Boundary (CMB) Matrix

In order to specify the core-mantle boundary (CMB) matrix, Eq. (1.150), we must obtain the solution ψ_ℓ of the Poisson equation for the inviscid core, Eq. (1.142), that satisfies the regularity condition at the Earth’s centre, Eq. (1.143). Within the assumption of an homogeneous core, i.e., with a constant density profile, the Eq. (1.142) becomes the Laplace equation for ψ_ℓ because its right-hand side yields zero

$$\nabla_r^2 \psi_\ell = 0 \quad (2.1)$$

This equation admits two independent solutions

$$\psi_\ell(r) = c r^\ell + c^* r^{-(\ell+1)} \quad (2.2)$$

where c and c^* are two integration constants. However, by considering the regularity condition at the Earth's centre, Eq. (1.143), we are left only with

$$\psi_\ell(r) = r^\ell \quad (2.3)$$

and the quantity q_ℓ defined in Eq. (1.148) yields

$$q_\ell(r) = 2(\ell - 1) r^{\ell-1} \quad (2.4)$$

where we used the following expression for the gravity within an homogeneous sphere of density ρ_C

$$g(r) = \frac{4\pi}{3} G \rho_C r \quad (2.5)$$

Then, in the case of an homogeneous core, we can further specify the general expression of the CMB matrix, Eq. (1.150), as follows

$$\mathbf{I}_C = \begin{pmatrix} -r^\ell/g(r_C) & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & g(r_C) \rho_C \\ 0 & 0 & 0 \\ r^\ell & 0 & 0 \\ 2(\ell - 1) r^{\ell-1} & 0 & 4\pi G \rho_C \end{pmatrix} \quad (2.6)$$

This is Eq. (63) of Sabadini et al. (1982).

2.2.2 Propagators and Fundamental Matrices

As already argued in Sect. 1.7, the propagator matrix solves the homogeneous system given by Eq. (1.159), with the Cauchy datum and the continuity condition across internal interfaces of radius r_j , as in Eqs. (1.160) and (1.162). In this chapter we derive the fundamental solutions for an incompressible model and for the self-compressed compressible sphere, which allow us to build the propagator.

Considering that the solution of a system of N homogeneous first order differential equations can be expressed as a linear combination of N independent solutions, both spheroidal and toroidal vector solutions $\mathbf{y}_\ell$ of the homogeneous differential system within the same layer, say the j th viscoelastic layer,

$$\frac{d\mathbf{y}_\ell(r)}{dr} = \mathbf{A}_\ell(r) \mathbf{y}_\ell(r) \quad r_{j-1} < r < r_j \quad (2.7)$$

can be expressed as follows

$$\mathbf{y}_\ell(r) = \mathbf{Y}_\ell(r) \mathbf{C} \quad r_{j-1} < r < r_j \quad (2.8)$$

where $\mathbf{Y}_\ell$ is the fundamental 6×6 - or 2×2 -matrix, whose columns are independent solutions of the spheroidal or toroidal homogeneous differential system (2.7) in the j th layer, and $\mathbf{C}$ is a vector of integration constants. Note that only the harmonic degree ℓ enters, being the order m dependence due to the non-homogeneous forcing. This allows us to cast the propagator matrix $\mathbf{\Pi}_\ell(r, r')$, when r and r' belong to the j th layer of the Earth's models, in the following form

$$\mathbf{\Pi}_\ell(r, r') = \mathbf{Y}_\ell(r) \mathbf{Y}_\ell^{-1}(r') \quad r_{j-1} < r < r_j \quad (2.9)$$

in such a way that it solves the homogeneous differential system (2.7) and satisfies the Cauchy datum $\mathbf{\Pi}_\ell(r', r') = \mathbf{1}$. When r and r' belong to different layers, say j th and i th layers, with $j > i$, the propagator matrix can be obtained using the continuity condition Eq. (1.134) at the chemical interfaces between these two layers, i.e. $\mathbf{y}_\ell(r_k^+) = \mathbf{y}_\ell(r_k^-)$ at r_k for $k = i, \dots, j - 1$. We thus obtain, from Eqs. (1.134), (2.8) and (2.9)

$$\mathbf{\Pi}_\ell(r, r') = \mathbf{\Pi}_\ell(r, r_j) \left(\prod_{k=i+1}^j \mathbf{\Pi}_\ell(r_k, r_{k-1}) \right) \mathbf{\Pi}_\ell(r_i, r') \quad (2.10)$$

In order to obtain an analytical expression for the propagator matrix $\mathbf{\Pi}_\ell$ we thus need to obtain the linear independent solutions of the homogeneous differential system (2.7) in each layer of the Earth's model. They can be obtained analytically both for incompressible and compressible Earth's models with some restrictions on the material parameters, leading to incompressible models, or on the density profile, leading to a new analytical compressional model. Particularly, for layered models we will require that the two Lamé parameters are constant in each layer of the Earth, i.e., $\partial_r \lambda = 0$ and $\partial_r \mu = 0$. Furthermore, rather than solving Eq. (2.7), we will consider the original momentum and Poisson equations, after spherical harmonic expansion, Eqs. (1.84)–(1.87), where the forcing terms will be omitted because we are now interested in the solution of the homogeneous problem.

In order to simplify the following treatment, it is useful to introduce the quantity H_ℓ defined by

$$H_\ell = \partial_r V_\ell + \frac{V_\ell - U_\ell}{r} \quad (2.11)$$

and to consider the radial (1.84), tangential (1.85) and toroidal (1.86) components of the momentum equation (without the forcing terms) multiplied by $1/\rho_0$, r/ρ_0 and $1/\mu$, respectively, and cast them as follows within the assumption of constant elastic parameters

$$\frac{\beta}{\rho_0} \partial_r \chi_\ell - \partial_r (g U) + g \chi_\ell - \partial_r \Phi_\ell + \frac{\mu}{\rho_0} \frac{\ell(\ell+1)}{r} H_\ell = 0 \quad (2.12)$$

$$\frac{\beta}{\rho_0} \chi_\ell - g U_\ell - \Phi_\ell + \frac{\mu}{\rho_0} \partial_r (r H_\ell) = 0 \quad (2.13)$$

$$\nabla_r^2 W_\ell = 0 \quad (2.14)$$

It is interesting to note that the terms β/ρ_0 and μ/ρ_0 entering Eqs. (2.12) and (2.13) correspond to the square of the compressional and shear wave velocities.

The Poisson equation without the forcing terms becomes

$$\nabla_r^2 \Phi_\ell = -4 \pi G (\rho_0 \chi_\ell + U_\ell \partial_r \rho_0) \quad (2.15)$$

Exercise 9 Verify that the spheroidal radial (1.84) and tangential (1.85), and toroidal tangential (1.86) components of the momentum equation can be cast as in Eqs. (2.12)–(2.14)

2.3 Layered Incompressible Models

In the case of incompressibility, volume variations Δ are constrained to zero

$$\chi_\ell = 0 \quad (2.16)$$

From Eq. (1.74), we then express the tangential displacement V_ℓ as function of the radial displacement U_ℓ

$$V_\ell = \frac{r \partial_r U_\ell + 2 U_\ell}{\ell(\ell+1)} \quad (2.17)$$

and the quantity H_ℓ defined in Eq. (2.11) becomes

$$H_\ell = \frac{\nabla_r^2 (U_\ell r)}{\ell(\ell+1)} \quad (2.18)$$

Furthermore, as pointed out in Sect. 1.2.2, incompressible materials are characterized by an infinitely large bulk modulus κ in order that they are able to react to isotropic stresses. We thus require that the product $\kappa \chi_\ell$ remains finite in the limit of χ_ℓ going to zero and of κ going to infinity (Love 1911, Sect. 154). We define this limit with the quantity p_ℓ according to Eqs. (1.31) and (1.37)

$$p_\ell = \lim_{\chi_\ell \rightarrow 0, \kappa \rightarrow \infty} -\kappa \chi_\ell \quad (2.19)$$

Note that the same limit for $\lambda \chi_\ell$ and $\beta \chi_\ell$ converges to p_ℓ since $\mu \chi_\ell$ goes to zero. In view of this, the radial stress component R_ℓ , Eq. (1.79), becomes

$$R_\ell = -p_\ell + 2\mu \partial_r U_\ell \quad (2.20)$$

We also assume that the initial density is homogeneous within each layer of the Earth's model

$$\partial_r \rho_0 = 0 \quad (2.21)$$

even though it may differ from a layer to another, due to the specific chemical composition and phase of each layer of the Earth. From the generalized Williamson-Adams equation (1.42), this corresponds to the case in which the compositional coefficient γ defined in Eq. (1.43) is zero, i.e., every incompressible layer is in a neutral state of equilibrium.

In view of Eqs. (1.75), (2.16) and (2.21), there are no local density perturbations ρ^Δ within the layers, and the Poisson equation (2.15) becomes the Laplace equation

$$\nabla_r^2 \Phi_\ell = 0 \quad (2.22)$$

and it admits two independent solutions which takes the following form

$$\Phi_\ell = c_3 r^\ell + c_3^* r^{-(\ell+1)} \quad (2.23)$$

where r^ℓ and $r^{-(\ell+1)}$ are the regular and irregular solutions at the Earth's centre, respectively. Here c_3 and c_3^* are two constants of integration and the subscript 3 is used for convenience.

The absence of local density perturbations ρ^Δ within the layers does not mean that there are no gravitational perturbations Φ_ℓ^Δ . Indeed, density perturbations occur at the Earth's surface, as well as at any internal chemical interfaces separating two neighboring layers of the Earth, where the density contrast $\Delta\rho_j$ differs from zero

$$\Delta\rho_j = \rho_0(r_j^+) - \rho_0(r_j^-) \quad (2.24)$$

Here r_j is the radius of the interface, with the superscripts $+$ and $-$ indicating that the density is evaluated just above and below of the interface. Indeed, within the assumption of infinitesimal perturbations, the radial displacement $u_r(r_j, \theta)$ of the interface (i.e., the perturbation of the topography of the interface) yields a surface density anomaly given by the negative of the product between the density contrast $\Delta\rho_j$ and the radial displacement $u_r(r_j, \theta)$

$$\rho^\Delta = -\Delta\rho_j u_r(r_j, \theta) \delta(r - r_j) \quad (2.25)$$

with δ being the Dirac delta function. Equation (2.25) can be also obtained from the right-hand side of Eq. (1.75) by considering that the radial derivative of a function

with a step-like discontinuity at $r = r_j$ yields the Dirac delta $\delta(r - r_j)$ multiplied by the value of the step, the density contrast $\Delta\rho_j$ in our case

$$\partial_r \rho_0 = \Delta\rho_j \delta(r - r_j) \quad (2.26)$$

For each interface there is a surface density perturbation that contributes to the gravitational perturbation via the Poisson equation and it affects the momentum equation in terms both of gravity perturbations and buoyancy forces, i.e., the fourth and third terms in Eq. (1.19). Note that the effects of these surface density perturbations are already accounted for by the way in which we have defined the potential flux $Q_{\ell m}$, Eq. (1.123), and by the requirement that it is continuous across the internal interfaces.

In view of Eqs. (2.16) and (2.19), the radial (2.12) and tangential (2.13) components of the momentum equation simplify into

$$\partial_r \Gamma_\ell + \frac{\mu}{\rho_0} \frac{\ell(\ell+1)}{r} H_\ell = 0 \quad (2.27)$$

$$\Gamma_\ell + \frac{\mu}{\rho_0} \partial_r (r H_\ell) = 0 \quad (2.28)$$

where Γ_ℓ is defined by

$$\Gamma_\ell = -\frac{p_\ell}{\rho_0} - g U_\ell - \Phi_\ell \quad (2.29)$$

By applying the operator $\partial_r + 2/r$ to Eq. (2.27) and subtracting Eq. (2.28) multiplied by $\ell(\ell+1)/r^2$ we obtain

$$\nabla_r^2 \Gamma_\ell = 0 \quad (2.30)$$

the terms in H_ℓ canceling each other

$$\left(\partial_r + \frac{2}{r} \right) \frac{\ell(\ell+1)}{r} H_\ell - \frac{\ell(\ell+1)}{r^2} \partial_r (r H_\ell) = 0 \quad (2.31)$$

Since Γ_ℓ satisfies the Laplace equation (2.30), it thus takes the following form, with the same dependence of Φ_ℓ obtained in Eq. (2.23) on the radial distance r from the Earth's centre

$$\Gamma_\ell = -\frac{\mu}{\rho_0} c_1 r^\ell - \frac{\mu}{\rho_0} c_1^* r^{-(\ell+1)} \quad (2.32)$$

where c_1 and c_1^* are two constants of integration that have been multiplied by $-\mu/\rho_0$ for convenience, as it will be apparent in the following. Then, by substituting this solution into Eq. (2.27) and by making use of Eq. (2.18), we obtain a dishomogeneous differential equation of the second order in U_ℓ

$$\nabla^2 (U_\ell r) = c_1 \ell r^\ell - c_1^* (\ell+1) r^{-(\ell+1)} \quad (2.33)$$

It is solved by the two particular solutions

$$U_\ell = c_1 \frac{\ell r^{\ell+1}}{2(2\ell+3)} + c_1^* \frac{(\ell+1)r^{-\ell}}{2(2\ell-1)} \quad (2.34)$$

and by the two linearly independent solutions of the homogeneous differential equation, i.e., when c_1 and c_1^* are set to zero in Eq. (2.33),

$$U_\ell = c_2 r^{\ell-1} + c_2^* r^{-(\ell+2)} \quad (2.35)$$

Summing up all the contributions, we obtain

$$U_\ell = c_1 \frac{\ell r^{\ell+1}}{2(2\ell+3)} + c_2 r^{\ell-1} + c_1^* \frac{(\ell+1)r^{-\ell}}{2(2\ell-1)} + c_2^* r^{-(\ell+2)} \quad (2.36)$$

and, using this into Eq. (2.17), the tangential displacement V_ℓ yields

$$V_\ell = c_1 \frac{(\ell+3)r^{\ell+1}}{2(2\ell+3)(\ell+1)} + c_2 \frac{r^{\ell-1}}{\ell} + c_1^* \frac{(2-\ell)r^{-\ell}}{2\ell(2\ell-1)} - c_2^* \frac{r^{-(\ell+2)}}{\ell+1} \quad (2.37)$$

Exercise 10 Verify that the radial, R_ℓ , tangential, S_ℓ , and potential, Q_ℓ , stress components of the spheroidal solution $\mathbf{y}$, as defined in Eqs. (2.20), (1.80) and (1.93), respectively, take the form

$$\begin{aligned} R_\ell = & c_1 \frac{\ell \rho_0 g r + 2(\ell^2 - \ell - 3)\mu}{2(2\ell+3)} r^\ell + c_2 [\rho_0 g r + 2(\ell-1)\mu] r^{\ell-2} \\ & + c_3 \rho_0 r^\ell + c_1^* \frac{(\ell+1)\rho_0 g r - 2(\ell^2 + 3\ell - 1)\mu}{2(2\ell-1)} r^{-(\ell+1)} \\ & + c_2^* [\rho_0 g r - 2(\ell+2)\mu] r^{-(\ell+3)} + c_3^* \rho_0 r^{-(\ell+1)} \end{aligned} \quad (2.38)$$

$$\begin{aligned} S_\ell = & c_1 \frac{\ell(\ell+2)}{(2\ell+3)(\ell+1)} \mu r^\ell + c_2 \frac{2(\ell-1)}{\ell} \mu r^{(\ell-2)} \\ & + c_1^* \frac{(\ell^2-1)}{\ell(2\ell-1)} \mu r^{-(\ell+1)} + c_2^* \frac{2(\ell+2)}{\ell+1} \mu r^{-(\ell+3)} \end{aligned} \quad (2.39)$$

$$\begin{aligned} Q_\ell = & c_1 \frac{2\pi G \rho_0 \ell}{2\ell+3} r^{\ell+1} + c_2 4\pi G \rho_0 r^{\ell-1} + c_3 (2\ell+1) r^{\ell-1} \\ & + c_1^* \frac{2\pi G \rho_0 (\ell+1)}{2\ell-1} r^{-\ell} + c_2^* 4\pi G \rho_0 r^{-(\ell+2)} \end{aligned} \quad (2.40)$$

For each layer of the Earth's model (assuming that each layer has material parameters ρ_0 and μ which are constant inside it), on the basis of Eqs. (2.23), (2.36)–(2.37) and (2.38)–(2.40), the spheroidal vector solution $\mathbf{y}$ can be written as

$$\mathbf{y}_\ell(r) = \mathbf{Y}_\ell(r) \mathbf{C}_\ell \quad (2.41)$$

where $\mathbf{Y}_\ell$ is the fundamental matrix

$$\mathbf{Y}_\ell(r) = \begin{pmatrix} \frac{lr^{l+1}}{2(2l+3)} & r^{l-1} & 0 \\ \frac{(l+3)r^{l+1}}{2(2l+3)(l+1)} & \frac{r^{l-1}}{l} & 0 \\ \frac{(l\rho_0gr+2(l^2-l-3)\mu)r^l}{2(2l+3)} & (\rho_0gr+2(l-1)\mu)r^{l-2} & \rho_0r^l \\ \frac{l(l+2)\mu r^l}{(2l+3)(l+1)} & \frac{2(l-1)\mu r^{l-2}}{l} & 0 \\ 0 & 0 & r^l \\ \frac{2\pi G\rho_0l^{l+1}}{2l+3} & 4\pi G\rho_0r^{l-1} & (2l+1)r^{l-1} \end{pmatrix} \dots \begin{pmatrix} \frac{(l+1)r^{-l}}{2(2l-1)} & r^{-l-2} & 0 \\ \frac{(2-l)r^{-l}}{2l(2l-1)} & -\frac{r^{-l-2}}{l+1} & 0 \\ \frac{(l+1)\rho_0gr-2(l^2+3l-1)\mu}{2(2l-1)r^{l+1}} & \frac{\rho_0gr-2(l+2)\mu}{r^{l+3}} & \frac{\rho_0}{r^{l+1}} \\ \dots & \dots & \dots \\ \frac{(l^2-1)\mu}{l(2l-1)r^{l+1}} & \frac{2(l+2)\mu}{(l+1)r^{l+3}} & 0 \\ 0 & 0 & \frac{1}{r^{l+1}} \\ \frac{2\pi G\rho_0(l+1)}{(2l-1)r^l} & \frac{4\pi G\rho_0}{r^{l+2}} & 0 \end{pmatrix} \quad (2.42)$$

and $\mathbf{C}_\ell$ is the vector of integration constants

$$\mathbf{C}_\ell = (c_1, c_2, c_3, c_1^*, c_2^*, c_3^*)^T \quad (2.43)$$

The regular and singular part of the fundamental matrix are the first three columns and last three columns respectively, the first going to zero at the Earth's centre and they are given by Eqs. (46) and (48) in Sabadini et al. (1982), where the singular part of the incompressible solution was given for the first time for a stratified viscoelastic Earth. The asterisk for denoting the part of the vector of integration constant multiplying the singular part of the fundamental matrix is also taken from Sabadini et al. (1982).

Each column of the fundamental matrix $\mathbf{Y}_\ell$ represents an independent solution of the linear differential system (1.161) in the incompressible case, i.e., in the limit of the matrix $\mathbf{A}$ defined in Eq. (1.95) for the bulk modulus κ going to infinity

$$\partial_r \mathbf{Y}_\ell = \left(\lim_{\kappa \rightarrow \infty} \mathbf{A}_\ell \right) \mathbf{Y}_\ell \quad (2.44)$$

Note that, from Eqs. (1.77) and (1.96), also λ and β go to infinity

The analytical expression of the fundamental solution (2.42), which includes the regular and singular part at the Earth's centre was first obtained in Sabadini et al. (1982), after the regular part was given first in Wu and Peltier (1982), their Eqs. (30a)–(30d). The reader should note that the terms of the third and sixth columns of the fundamental matrix $\mathbf{Y}_\ell(r)$ have the opposite sign with respect to the corresponding terms in Eqs. (46) and (48) of Sabadini et al. (1982) due to the sign definition of the ϕ gravitational potential in this book with respect to Sabadini et al. (1982), to match herein the definition of the perturbation of the gravitational potential in agreement

with Eqs. (1.60) and (1.2). The results are of course the same, for both Sabadini et al. (1982) and what we show in this book: we now simply avoid a somehow confusing artificial change of sign in the definition of the gravitational potential. Several works and results from other groups streamed from this fundamental solution describing a self-gravitating, viscoelastic, stratified Earth. It should be noted on the other hand, that some applications of $\mathbf{Y}_\ell(r)$, as for the case of co-seismic deformation, is not appropriate because in that case the effects of compressibility play a key role, but this will be analyzed in Chap. 5.

The inverse of the fundamental matrix $\mathbf{Y}_\ell$ has the form

$$\mathbf{Y}_\ell^{-1}(r) = \mathbf{D}_\ell(r) \bar{\mathbf{Y}}_\ell(r) \quad (2.45)$$

with $\mathbf{D}_\ell$ being a diagonal matrix with elements

$$\text{diag} [\mathbf{D}_\ell(r)] = \frac{1}{2\ell+1} \left(\frac{\ell+1}{r^{\ell+1}}, \frac{\ell(\ell+1)}{2(2\ell-1)r^{\ell-1}}, -\frac{1}{r^{\ell-1}}, \ell r^\ell, \frac{\ell(\ell+1)}{2(2\ell+3)} r^{\ell+2}, r^{\ell+1} \right) \quad (2.46)$$

and

$$\bar{\mathbf{Y}}_\ell(r) = \begin{pmatrix} \frac{\rho g r}{\mu} - 2(\ell+2) & 2\ell(\ell+2) & -\frac{r}{\mu} & \frac{\ell r}{\mu} & \frac{\rho r}{\mu} & 0 \\ -\frac{\rho g r}{\mu} + \frac{2(\ell^2+3\ell-1)}{\ell+1} & -2(\ell^2-1) & \frac{r}{\mu} & \frac{(2-\ell)r}{\mu} & -\frac{\rho r}{\mu} & 0 \\ 4\pi G \rho & 0 & 0 & 0 & 0 & -1 \\ \frac{\rho g r}{\mu} + 2(\ell-1) & 2(\ell^2-1) & -\frac{r}{\mu} & -\frac{(\ell+1)r}{\mu} & \frac{\rho r}{\mu} & 0 \\ -\frac{\rho g r}{\mu} - \frac{2(\ell^2-\ell-3)}{\ell} & -2\ell(\ell+2) & \frac{r}{\mu} & \frac{(\ell+3)r}{\mu} & -\frac{\rho r}{\mu} & 0 \\ 4\pi G \rho r & 0 & 0 & 0 & 2\ell+1 & -r \end{pmatrix} \quad (2.47)$$

Although it would be quite laborious to derive such an analytical compact form of a 6×6 inverse matrix by hand, this can be done nowadays by means of an algebraic software package like *Mathematica*. It was first done by (Spada et al. 1990, 1992). Of course, it is not difficult to show analytically that $\mathbf{Y}_\ell(r) \mathbf{Y}_\ell^{-1}(r) = \mathbf{1}$, with $\mathbf{1}$ being the identity matrix, by hand! With respect to (Spada et al. 1990, 1992), the elements of $\mathbf{D}_\ell(r)$ and $\bar{\mathbf{Y}}_\ell(r)$ are changed accordingly to account for our own definition of the sign of the perturbation of the gravitational potential.

2.4 Relaxation Times for Incompressible Earth's Models

In order to gain insights into the physics of the relaxation processes, it is important to take a close look at the relaxation times corresponding to the modes excited by discontinuities in the physical parameters of simple Earth's models. We will consider the spheroidal case. Equations (2.42) and (2.45) allow us to build the propagator, Eq. (2.9), and to derive the normalized secular determinant $D(s)$, Eq. (1.199).

Fig. 2.2 With s normalized by 1 kyr, plot of the function defined as $f(D_\ell(s)) = \text{sgn}(D_\ell(s)) \times \log(|D_\ell(s)|)$ if $|D(s)| > 10.0$ and $f(D_\ell(s)) = D_\ell(s)/10.0$ if $|D_\ell(s)| \leq 10.0$ as function of $\log(-1/(s \times \text{kyr}))$ and its zero crossings, providing the relaxation times $T_i = -1/s_i$, for the 5-layer incompressible model of Table 2.1 and $\ell = 2, 10, 100$ from top to bottom. This calculation and corresponding figure have been kindly provided by Shuang Yi, from the Key Laboratory of Computational Geodynamics from the University of the Chinese Academy of Sciences (UCAS), January 2015

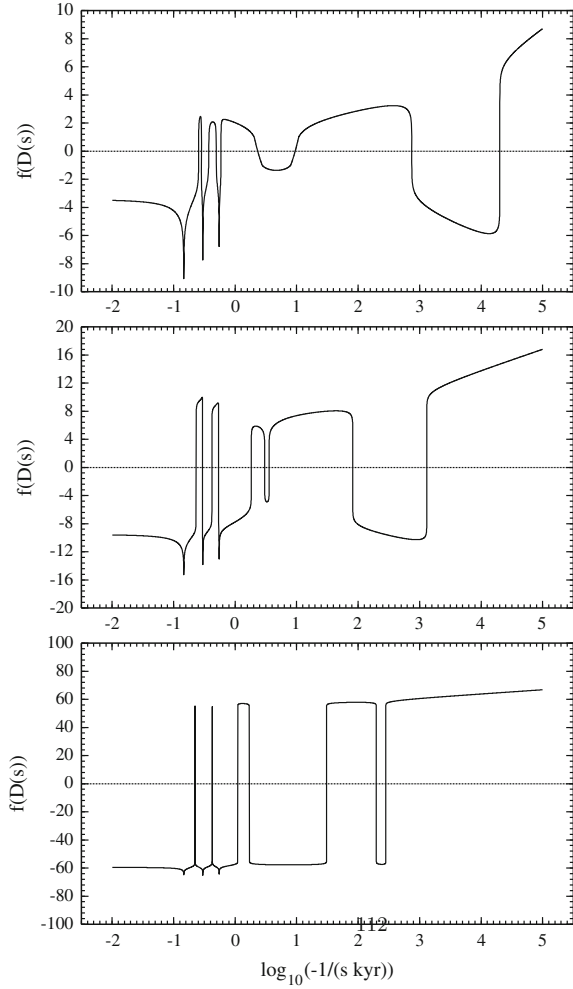


Figure 2.2, kindly provided by S. Yi (personal communication), portrays $f(D_\ell(s)) = \text{sgn}(D_\ell(s)) \times \log(|D_\ell(s)|)$ if $|D(s)| > 10.0$ and $f(D_\ell(s)) = D_\ell(s)/10.0$ if $|D_\ell(s)| \leq 10.0$ as function of $\log(-1/(s \times \text{kyr}))$, which means that s , negative, expressed in $1/s$ is normalized by 1 kyr = 3.153×10^{10} s, for the harmonic degrees $\ell = 2, 10$ and 100 , from top to bottom and for the 5-layer, incompressible Earth's model displayed in Table 2.1. The zero crossings are the relaxation times corresponding to the normal modes $T_i = -1/s_i$ of the 5-layer Earth's model. According to the rules established in Sect. 1.8 for counting the normal modes for incompressible models, Fig. 2.2 portrays 9 zero crossings, or normal modes of the secular determinant $D(s)$ as expected for the five-layer model of Table 2.1: four transient modes $T1$ – $T4$ are triggered at the two interfaces between the viscoelastic layers, at 5951 and

Table 2.1 Parameters for the 5-layer fixed-boundary contrast Earth's model

Layer	r (km)	ρ (kg/m ³)	ν (Pa s)	μ (N/m ²)	
1	6371 – 6251	3070	Elastic	5.76×10^{10}	Lithosphere
2	6251 – 5951	3070	10^{21}	5.76×10^{10}	Shallow upper mantle
3	5951 – 5701	3850	10^{21}	1.06×10^{11}	Transition zone
4	5701 – 3480	4970	10^{21}	2.16×10^{11}	Lower mantle
5	3480 – 0	10,750	Inviscid	0	Inviscid fluid core

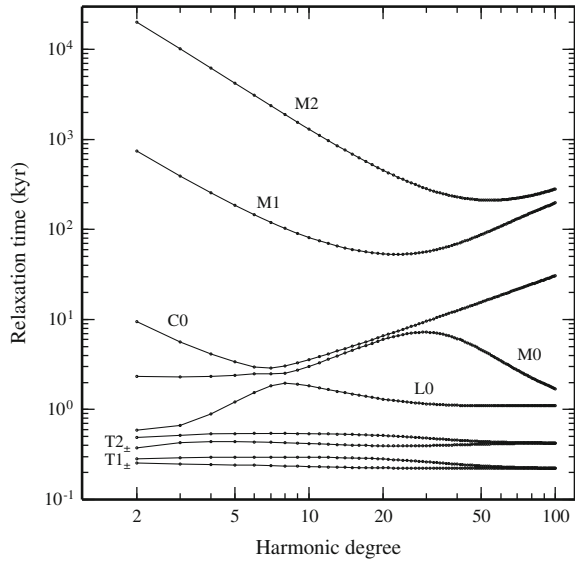
r is the distance with respect to the centre of the Earth, ρ the density of the layer, and μ the rigidity

5701 km, because their Maxwell times are different (from Table 2.1); two buoyancy modes, $M1$ and $M2$ are triggered at these two interfaces between viscoelastic layers, $M1$ at 5701 and $M2$ at 5951 km; the density contrast between the viscoelastic mantle and the outer atmosphere triggers the mantle mode $M0$, the rheological contrast between the viscoelastic mantle and the elastic lithosphere triggers $L0$ and finally the density contrast between the mantle and the core triggers the core mode $C0$, giving a final total count of nine modes. The slowest modes have been named $M1$ and $M2$ by Wu and Peltier (1982) and are associated with the two internal chemical boundaries at 670 and 420 km. These $M1$ and $M2$ modes will be quoted several times in the book when dealing with the geophysical processes affected by the slow readjustment of the 670 and 420 km density discontinuities.

The longest T_i correspond to the buoyancy $M2$ and $M1$ modes in the extreme right of the abscissa, and for $\ell = 2$ they are $T_{M2} = 2.01 \times 10^7$ yr, $T_{M1} = 7.44 \times 10^5$ yr, due to density contrasts between viscoelastic internal layers. To the left, towards shorter times, we catch the core mode $C0 = 9.48 \times 10^3$ yr, the mantle mode $M0 = 2.33 \times 10^3$ yr, the lithospheric mode $L0 = 5.89 \times 10^2$ yr and then, at the shortest times, the couple of transient relaxation times $T1$ – $T4$, 4.92×10^2 , 3.73×10^2 , 2.83×10^2 and 2.55×10^2 yr. Increasing the harmonic degree at $\ell = 10$ the transient relaxation times remain constant, while the three modes $L0$, $M0$ and $C0$ move to the right, towards longer times, while $M1$ and $M2$ do the opposite. When $\ell = 100$ the transient modes merge together and the two adjacent modes become indistinguishable at this scale, apparently reducing the nine modes to seven modes. The lithospheric mode $L0$ moves towards the slowest $M1$, $M2$ modes, while the latter get close to one another. Figure 2.3, kindly provided by S. Yi (personal communication), shows for the same Earth's model of Table 2.1 the relaxation times as a function of the harmonic degree from $\ell = 2$ to $\ell = 100$, in order to provide a more precise determination of the zero crossings and relaxation time values with respect to Fig. 2.2.

Figure 2.3 portrays in detail the relaxation times for $\ell = 2$ –100, which in Fig. 2.2 are shown solely for $\ell = 2, 10$ and 100. This representation allows us to perceive that each viscoelastic mode follows its own peculiar branch for varying harmonic degree ℓ , so the $M1$, $M2$ modes become faster with increasing ℓ compared to their slowest $\ell = 2$ counterparts. The $\ell = 100$ $M2$ for example is faster by almost two orders of magnitude compared to $\ell = 2$, from about twenty million years to three hundred thousand years. On the opposite, the $C0$ mode becomes slower, while the

Fig. 2.3 Relaxation times T_i in kyr for the 5-layer incompressible model of Table 2.1 and $l = 2-100$. As for Fig. 2.2, this calculation and corresponding figure have been kindly provided by Shuang Yi, from UCAS, January 2015



$M0$ mode is the slowest for $\ell = 30$, about seven thousand years compared to two 2.5 thousand years for $\ell = 2$, becoming fast again for $\ell = 100$. The $L0$ mode becomes slower for increasing ℓ in a non monotonic way, and the transient modes become indistinguishable in pairs. This result shows the richness in time scales caused by the Earth's stratification and wavelength decomposition.

We now have a closer look at the relaxation times as a function of the harmonic degree ℓ as in Fig. 2.3 but for 5-layer Earth's models with varying viscosity ratio between the lower and upper mantle and between the transition zone and the upper mantle. The relaxation times for these 5-layer models are shown in Figs. 2.4 and 2.5 as a function of the harmonic order ℓ , for varying ratio $B = \nu_2/\nu_1$ between the lower and upper mantle viscosity from 1 to 200, and for varying ratio $C = \nu_3/\nu_1$ between the viscosity of the transition zone and that of the upper mantle by the same amount. All the ℓ patterns resemble that of Fig. 2.3.

The relaxation times $T_i = -1/s_i$ are expressed in years, ranging from $\ell = 2$ to $\ell = 100$. Figure 2.4 deals with a viscosity increase in the lower mantle, with the ratio B between the lower and upper mantle viscosity ranging from 1 to 200. OM stands for an old viscosity model in which the upper mantle viscosity is fixed at 10^{21} Pa s, while NM stands for a new viscosity model in which ν_1 is fixed at 0.5×10^{21} Pa s, in agreement with the recent analyses by Lambeck et al. (1990), Vermeersen and Sabadini (1999), Devoti et al. (2001) based on postglacial rebound modeling from different perspectives, sea-level changes in the far field and long-wavelength geopotential variations. These models are chemically stratified at 420 and 670 km depth and the viscosity is uniform in the whole upper mantle; this stratification supports nine relaxation modes.

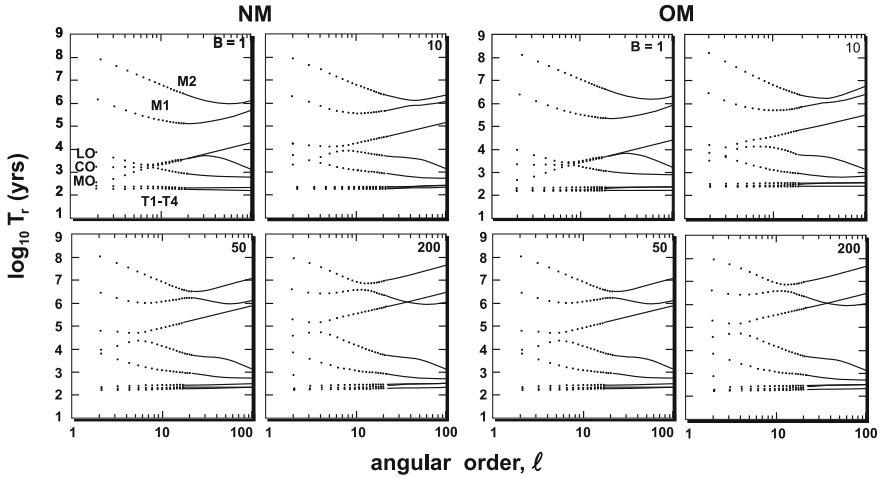


Fig. 2.4 Relaxation times in years as a function of the harmonic degree ℓ and varying lower mantle viscosity. The parameter $B = \nu_2/\nu_1$ is varied from 1 to 200. *OM* corresponds to $\nu_1 = 10^{21}$ Pa s, while *NM* corresponds to $\nu_1 = 0.5 \times 10^{21}$ Pa s (Fig. 2 in Spada et al. 1992)

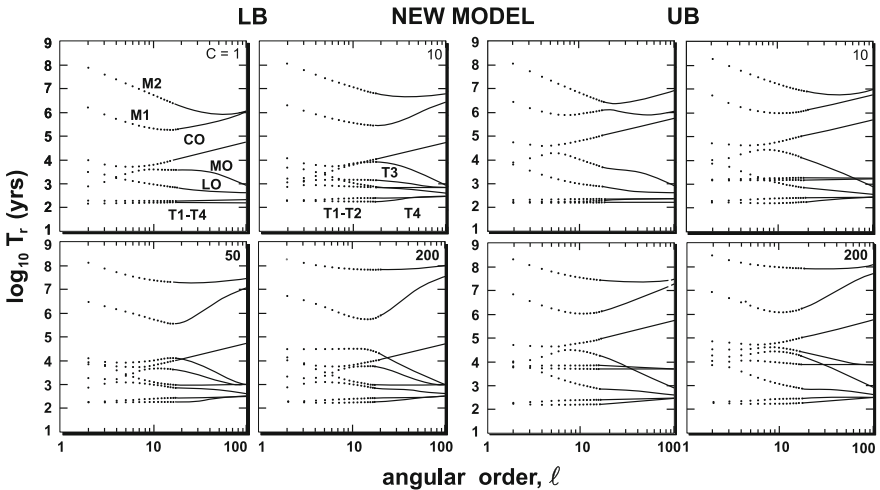


Fig. 2.5 Relaxation times in years as a function of the harmonic degree ℓ and varying lower mantle viscosity. The parameter $C = \nu_3/\nu_1$ is varied from 1 to 200. *LB* corresponds to $\nu_1 = 0.5 \times 10^{21}$ Pa s and $\nu_2 = 2 \times 10^{21}$ Pa s, while *UB* corresponds to $\nu_1 = 0.5 \times 10^{21}$ Pa s and $\nu_2 = 2 \times 10^{22}$ Pa s (Fig. 3 in Spada et al. 1992)

At low degrees, Fig. 2.4, the transient times $T_1 - T_4$ are followed by the lithospheric ($L0$) mode and by the core ($C0$) and mantle ($M0$) modes, as portrayed in the panel NM by $B = 1$, with $B = \nu_2/\nu_1$ denoting the ratio between the lower to upper mantle viscosity. When B is increased from 1 to 200, all the curves are moved upward toward

slower relaxation times. This upward migration occurs first for longer wavelengths, say lower than $\ell = 10$, followed by the shorter ones, which are less affected by lower mantle viscosity. For shorter wavelengths only the $M1$, $M2$ and core modes have slower relaxation times, while the lithospheric and mantle modes are rather unaffected, the deformation at such high harmonic degrees being concentrated in the upper mantle and thus unaffected by lower mantle viscosity variations. The NM curves, in the left panel, can be obtained from their counterparts in the right panel by a uniform downward shift towards faster relaxation times, in agreement with the lowering of the global mantle viscosity of this model.

Figure 2.5 shows the effects of a viscosity increase in the transition zone for the new model NM, with $C = \nu_3/\nu_1$ denoting the ratio between the viscosity in the transition zone ν_3 with respect to the viscosity in the upper mantle. These models, with a stiff transition zone at the upper lower mantle boundary, are based on the laboratory studies by Karato (1989) and Meade and Jeanloz (1990), which suggest that the transition zone may form a layer of relatively high viscosity between the upper and lower mantle. Panel LB, with LB standing for lower branch, corresponds to an upper mantle viscosity of 0.5×10^{21} Pa s and to $\nu_2 = 2 \times 10^{21}$ Pa s in the lower mantle, while UB, with UB standing for upper branch, corresponds to the same upper mantle viscosity and to a higher lower mantle viscosity of $\nu_2 = 2 \times 10^{22}$ Pa s. LB and UB for the lower mantle viscosity stand for the two possible viscosity solutions when true polar wander data and variations in the long-wavelength gravity field are used to constrain the viscosity of the lower mantle (see Chaps. 4 and 5). Viscosity increase in the hard layer influences all the modes for all the models, in particular the $M1$ and $M2$ modes, which is not surprising as these modes are excited by the discontinuities that bound the region where the viscosity is varied. With respect to the previous figure, all the modes are now affected by the viscosity increase in the transition layer which, lying close to the surface, is also able to affect the short wavelength, high-degree modes.

Exercise 11 Making use of the analytical fundamental solution for the incompressible, homogeneous Earth's model, show that the loading Love number k_2^L can be cast in the following form in the Laplace transform domain

$$k_2^L(s) = \frac{-1}{(1 + \hat{\mu}(s))} \quad (2.48)$$

while the tidal Love number k_2^T becomes

$$k_2^T(s) = \frac{3/2}{(1 + \hat{\mu}(s))} \quad (2.49)$$

where $\hat{\mu}(s)$ is given by Eq. (1.54), with the elastic μ multiplied by the normalization factor $19/(2\rho g(a)a)$, valid for $\ell = 2$.

2.5 The Self-compressed, Compressible Sphere

Cambiotti and Sabadini (2010) found the analytical solution of viscoelastic perturbations in the Laplace domain for a specific self-gravitating compressible Maxwell Earth's model, called “self-compressed compressible sphere”. This model is composed of an incompressible inviscid core and a compressible Maxwell mantle with constant shear modulus, μ , bulk modulus, κ , and viscosity, ν . In order to account for the self-compression of the mantle at the initial state of hydrostatic equilibrium, the initial density profile within the mantle varies with the radial distance from the Earth's centre r according to

$$\rho_0(r) = \begin{cases} \frac{3\alpha}{2r_C} & 0 \leq r \leq r_C \\ \frac{\alpha}{r} & r_C < r \leq a \end{cases} \quad (2.50)$$

where r_C , a and α are the radius of the core, the Earth's radius and a constant related to the total Earth mass M_E by

$$M_E = 2\pi\alpha a^2 \quad (2.51)$$

This choice of the initial density profile fixes the initial gravity acceleration g within the mantle to

$$g = 2\pi G\alpha \quad (2.52)$$

with G being the universal gravitational constant.

Depending on the bulk modulus κ , the self-compressed compressible sphere describes compressional or compositional stratifications of the mantle. Indeed, from the generalized Williamson-Adams equation (1.42) with the compositional coefficient set to zero, $\gamma = 0$, we obtain that compressional stratifications are characterized by a constant bulk modulus to which we will refer as compressional bulk modulus κ_0

$$\kappa_0 = g\alpha = 2\pi G\alpha^2 \quad (2.53)$$

Departures from this value result in compositional stratifications. Particularly, the compositional coefficient γ yields

$$\gamma = -\frac{\epsilon\alpha}{r^2} \quad (2.54)$$

with

$$\epsilon = \frac{\kappa - \kappa_0}{\kappa} \quad (2.55)$$

As it results from the analysis of the relaxation spectrum of the self-compressed compressible sphere, that we will discuss later in Sect. 2.5.2, the compositional stratification is stable if $\kappa > \kappa_0$ and unstable if $\kappa < \kappa_0$. This also results from the

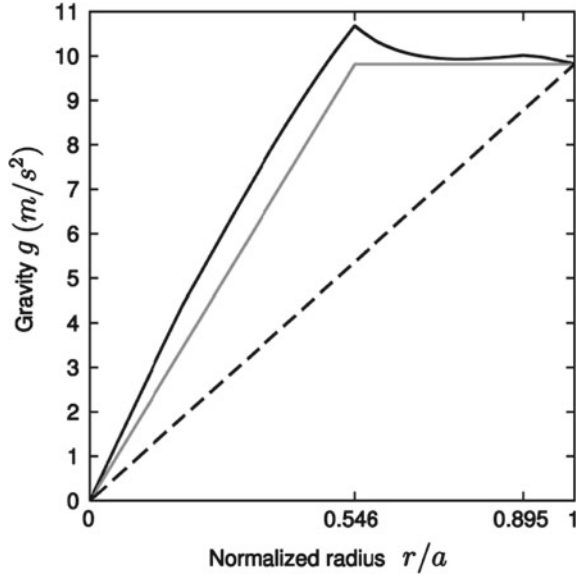
comparison between the generalized Williamson-Adams equation (1.42) and the expression for the square of the Brunt-Väisälä frequency ω

$$\omega^2 = -\frac{g}{\rho_0} \left(\partial_r \rho_0 + \frac{g \rho_0^2}{\kappa} \right) = -\frac{g \gamma}{\rho_0} \quad (2.56)$$

The Brunt-Väisälä frequency ω characterizes the motion of a particle in the ideal fluid that adiabatically moves away from its equilibrium position. The particle will oscillate around its equilibrium position with frequency ω if $\omega^2 > 0$, while it will continue to move away from its equilibrium position if $\omega^2 < 0$. Differently, the particle remains in the new position due to the perturbation if $\omega = 0$. Although the present theoretical framework is based on the assumption of quasi-static deformations (we neglect the inertial forces in the momentum equation), the analysis of the sign of ω^2 allows to establish if the viscoelastic model is stable or unstable (Plag and Jüttner 1995; Vermeersen and Mitrovica 2000). In view of Eq. (2.56), the stability only depends on the sign of the compositional coefficient γ . The model is stable if $\gamma \leq 0$ and unstable if $\gamma > 0$. For instance layered compressible models present the unstable Rayleigh-Taylor modes (Plag and Jüttner 1995) and, indeed, their compositional coefficient becomes positive because the radial derivative of the density is zero in this case. An alternative way to describe layered compressible models consists in the assumption that they are incompressible at the initial state of hydrostatic equilibrium, i.e., they have an infinitely large bulk modulus at the initial state, $\kappa \rightarrow \infty$, and a finite bulk modulus during the perturbations. This would imply that $\gamma = 0$ from Eq. (2.56), but it is not a self-consistent with compressibility during deformation and so we reject this interpretation, which also contrasts with the presence of Rayleigh-Taylor instabilities.

In view of the way in which we have defined the self-compressed compressible sphere, we have the possibility of studying the effects of the compressional and compositional stratifications on the relaxation process of Maxwell Earth's models. Previous analytical solutions were obtained assuming material or local incompressibility and for the case of the “homogeneous compressible sphere” (Gilbert and Backus 1968). Only the latter model actually accounts for compressibility during perturbations, but all the material parameters, included the initial density, are constant from the centre to the surface of the Earth. Its analytical solution has been widely used, first, in seismology and, after, in viscoelastic modelling (Vermeersen et al. 1996b). Nevertheless, it neglects the self-compression at the initial state of hydrostatic equilibrium since it has a constant density profile: in this respect, the homogeneous compressible sphere is always unstable. Instead, our self-compressed compressible sphere (Cambiotti and Sabadini 2010) takes into account compressibility both during the perturbations and at the initial state, having a depth dependent density profile, Eq. (2.50). In addition to this qualitative improvement with respect to the homogeneous compressible sphere, our model also reproduces the density contrast at the core-mantle boundary, although it neglects other density contrasts within the mantle due to the simple Darwin-law used to describe the compressibility at the initial state. This also results into a better reproduction of the actual initial

Fig. 2.6 Initial gravity acceleration g of PREM (black solid line), the self-compressed compressible sphere (our new model, grey line) and the homogeneous compressible sphere (Gilbert and Backus 1968 black dashed line). The core-mantle boundary corresponds to the normalized radius 0.547



gravity acceleration predicted by PREM, which is indeed almost constant within the mantle as in Eq. (2.52). We show this in Fig. 2.6 where we compare the initial gravity acceleration predicted by PREM and the gravity from the self-compressed and homogeneous compressible spheres.

2.5.1 The Analytical Solution

In order to solve the spheroidal radial and tangential components of the momentum equation and the Poisson equation, Eqs. (2.12), (2.13) and (2.15), for compressible Earth's models with constant elastic parameters κ and μ , it is convenient obtaining first two differential equations that involve only the radial and tangential displacements. This is possible owing to the specific initial density and gravity of the self-compressed compressible sphere, Eqs. (2.50) and (2.52). The first differential equation is obtained by subtracting to the radial component (2.12) of the momentum equation the derivative of the tangential component (2.13) with respect to the radial variable r

$$\left(\frac{\beta}{\alpha} - g\right) \chi_\ell + \frac{\mu}{\alpha} \left[r^2 \partial_r^2 H_\ell + 3r \partial_r H_\ell + (1 - \ell(\ell + 1)) H_\ell \right] = 0 \quad (2.57)$$

The second differential equation is obtained by applying the operator $\partial_r + 2/r$ to the radial component Eq. (2.12) of the momentum equation and subtracting to it the tangential component Eq. (2.13) multiplied by $\ell(\ell + 1)/r^2$

$$\begin{aligned}
& -\nabla_r^2 \Phi_\ell + \nabla^2 \left(\frac{\beta}{\alpha} \chi_\ell - g U_\ell \right) \\
& + \left(g - \frac{\beta}{\alpha} \right) \frac{1}{r} \partial_r (r^2 \chi_\ell) + \frac{\mu}{\alpha} \frac{\ell(\ell+1)}{r} H_\ell = 0
\end{aligned} \tag{2.58}$$

Here, we also substitute the Laplacian of the potential by means of the Poisson equation (2.15) together with Eqs. (2.50) and (2.52)

$$\nabla_r^2 \Phi_\ell = -\frac{2g\alpha}{r} \left(\chi_\ell - \frac{1}{r} U_\ell \right) \tag{2.59}$$

This yields

$$\begin{aligned}
& \frac{2g\alpha}{r} \left(\chi_\ell - \frac{1}{r} U_\ell \right) + \nabla^2 \left(\frac{\beta}{\alpha} \chi_\ell - g U_\ell \right) \\
& + \left(g - \frac{\beta}{\alpha} \right) \frac{1}{r} \partial_r (r^2 \chi_\ell) + \frac{\mu}{\alpha} \frac{\ell(\ell+1)}{r} H_\ell = 0
\end{aligned} \tag{2.60}$$

Let us now suppose that the six linearly independent solutions of Eqs. (2.57) and (2.60) may have the following form

$$U_\ell = u r^z \tag{2.61}$$

$$V_\ell = v r^z \tag{2.62}$$

with u , v and z as constants, and substitute these trial solutions. From Eqs. (2.57) and (2.60), we thus obtain

$$\frac{\mu}{\alpha} r^{z-1} \{u [Z - z(\zeta + 1) - 2\zeta] - v [Zz - \ell(\ell+1)(1 + \zeta)]\} = 0 \tag{2.63}$$

$$\begin{aligned}
& \frac{\mu}{\alpha} r^{z-2} \left\{ u \left[\frac{g\alpha}{\mu} (Z+2)(z+1) + (Z-2)(z+1)\zeta - \ell(\ell+1)(\zeta+1) \right] \right. \\
& \left. - v \ell(\ell+1) \left[\frac{g\alpha}{\mu} (Z+2) + Z\zeta - (\zeta+1)(z+1) \right] \right\} = 0
\end{aligned} \tag{2.64}$$

Here, Z is the second order polynomial in z

$$Z = z^2 + z - \ell(\ell+1) \tag{2.65}$$

and ζ is given by

$$\zeta = \frac{\beta - g\alpha}{\mu} \tag{2.66}$$

Since it has been possible to collect the dependence on the radial variable r in Eqs. (2.63) and (2.64), the latter can be seen as equations for the constants u , v and z . Solving Eq. (2.63) for v , we obtain

$$v = u \frac{Z - z(\zeta + 1) - 2\zeta}{Zz - \ell(\ell + 1)(1 + \zeta)} \quad (2.67)$$

and, using this in Eq. (2.64), after some straightforward algebra, it yields the following third-order polynomial in Z

$$a_0 + a_1 Z + a_2 Z^2 + Z^3 = 0 \quad (2.68)$$

with a_0 , a_1 and a_2 being constant coefficients, which depend solely on the material parameters and the harmonic degree ℓ

$$a_2 = 4 \frac{g\alpha}{\beta} - 2 \quad (2.69)$$

$$a_1 = \ell(\ell + 1) \left(\frac{g\alpha}{\beta} (\zeta + 3) - 4 \right) \quad (2.70)$$

$$a_0 = 2\ell(\ell + 1) \frac{g\alpha}{\beta} (\zeta - 1) \quad (2.71)$$

In order to satisfy Eq. (2.68), Z has to be one of the three roots Z_j , with $j = 1, 2, 3$, of the third-order polynomial of the LHS. We do not report here the lengthy expressions for Z_j . However, we note that they only depend on the harmonic degree ℓ and on the material parameters of the self-compressed compressible sphere via $g\alpha/\beta$ and ζ . Then, by considering that Z is a second order polynomial in the constant z , the latter can assume only two values z_j and z_{j+3} for each root Z_j

$$z_j = -\frac{1}{2} \left(1 + \sqrt{1 + 4(\ell(\ell + 1) + Z_j)} \right) \quad (2.72)$$

$$z_{j+3} = -\frac{1}{2} \left(1 - \sqrt{1 + 4(\ell(\ell + 1) + Z_j)} \right) \quad (2.73)$$

We thus have obtained six constants z_j that, once substituted into Eqs. (2.61), (2.62) and (2.67), yield six linearly independent solutions of the two differential Eqs. (2.63)–(2.64)

$$U_\ell = \sum_{j=1}^6 u_j r^{z_j} \quad (2.74)$$

$$V_\ell = \sum_{j=1}^6 u_j v_j r^{z_j} \quad (2.75)$$

where u_j are the constants of integration and, according to Eq. (2.67), v_j are given by

$$v_j = \frac{Z_j - z_j (\zeta + 1) - 2\zeta}{Z_j z_j - \ell(\ell + 1)(1 + \zeta)} \quad (2.76)$$

Since Z_j has been defined only for $j = 1, 2, 3$, we impose that Z_4 , Z_5 and Z_6 coincide with Z_1 , Z_2 and Z_3 .

The solution for the gravitational potential $\Phi_{\ell m}$ is obtained by substituting Eqs. (2.74)–(2.75) into the Poisson equation (2.59). This yields the following non-homogeneous differential equation of the second order in $\Phi_{\ell m}$

$$\nabla^2 \Phi_\ell = -2 \sum_{j=1}^6 u_j r^{z_j-2} g \left[(\zeta_j + 1) - \ell(\ell + 1) v_j \right] \quad (2.77)$$

It is solved by the particular solution

$$\Phi_\ell = \sum_{j=1}^6 u_j p_j r^{z_j} \quad (2.78)$$

with

$$p_j = 2g \frac{\ell(\ell + 1)(1 - \zeta) - Z_j^2}{Z_j (Z_j z_j - \ell(\ell + 1)(\zeta + 1))} \quad (2.79)$$

and by the two solutions of the homogenous differential equation (i.e., the Laplace equation)

$$\Phi_\ell = c r^\ell + c^* r^{-(\ell+1)} \quad (2.80)$$

with c and c^* being constants of integration. The latter, however, must not be considered. Indeed they solve neither the radial nor the tangential components of the momentum Eqs. (2.12)–(2.13), once set $U_{\ell m}$ and $V_{\ell m}$ to zero. This is due to the fact that we have already used the Poisson equation (2.59) to obtain Eq. (2.60) from Eq. (2.58).

Within the solid mantle of the self-compressed compressible model, on the basis of Eqs. (2.74), (2.75) and (2.78), the spheroidal vector solution $\mathbf{y}$ defined in Eq. (1.92) yields

$$\mathbf{y}_\ell(r) = \mathbf{Y}_\ell(r) \mathbf{C} \quad (2.81)$$

where $\mathbf{Y}_\ell$ and $\mathbf{C}$ are the so called fundamental matrix for the self-compressed compressible sphere and the vector of constants of integration

$$\mathbf{Y}_\ell = \left(\mathbf{y}_\ell^{(1)}, \mathbf{y}_\ell^{(2)}, \mathbf{y}_\ell^{(3)}, \mathbf{y}_\ell^{(4)}, \mathbf{y}_\ell^{(5)}, \mathbf{y}_\ell^{(6)} \right) \quad (2.82)$$

$$\mathbf{C} = (u_1, u_2, u_3, u_4, u_5, u_6)^T \quad (2.83)$$

with $\mathbf{y}^{(j)}$ being the six linearly independent solutions

$$\mathbf{y}_\ell^{(j)}(r) = \begin{pmatrix} r^{z_j} \\ v_j r^{z_j} \\ [\beta z_j + 2\lambda - \ell(\ell + 1)v_j \lambda] r^{z_j-1} \\ \mu [1 + (z_j - 1)v_j] r^{z_j-1} \\ p_j r^{z_j} \\ [2g + (z_j + \ell + 1)p_j] r^{z_j-1} \end{pmatrix} \quad (2.84)$$

Note that the fundamental matrix $\mathbf{Y}_\ell$ describes the dependence on the radial distance from the Earth's centre r of the propagator matrix $\mathbf{\Pi}_\ell$, which solves the homogeneous differential system (1.159). We have

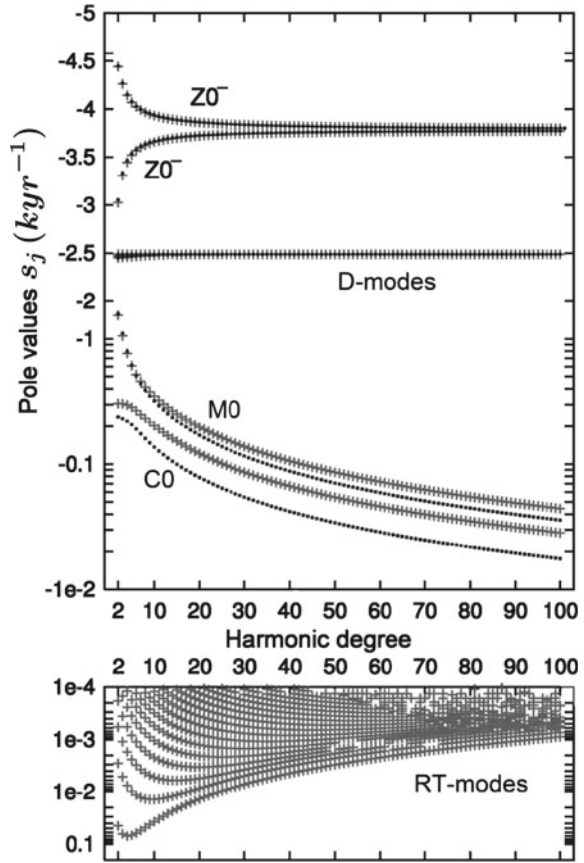
$$\mathbf{\Pi}_\ell(r, r') = \mathbf{Y}_\ell(r) \mathbf{Y}_\ell^{-1}(r') \quad (2.85)$$

This result can be used to solve more sophisticated models composed of different layers within the mantle, each with different (but constant) shear and bulk moduli, and viscosities. Nevertheless, the density profile must be the same given by Eq. (2.50). It is sufficient to use the fundamental matrix $\mathbf{Y}_\ell$ to obtain the propagator matrix $\mathbf{\Pi}_\ell$ in each layer and, then, impose chemical boundary conditions at the internal interfaces for propagating the solution from the inner to the outer layers. In this way, the self-compressed compressible sphere also takes into account the contrasts of the rheological parameters at the main Earth's interfaces, but not the density contrasts.

2.5.2 The Relaxation Spectrum of the Self-compressed Compressible Sphere

Let us now consider the self-compressed compressible sphere with compressional stratification, that we denote with CC₀, where the compositional coefficient is zero, $\gamma = 0$. The viscoelastic mantle is characterized by shear modulus $\mu = 1.45 \times 10^{11}$ Pa and viscosity $\nu = 10^{21}$ Pa s. The radius of the core is 3480 km and the Earth's radius is 6371 km. In order to respect the total Earth's mass $M_E = 5.97 \times 10^{24}$ kg, the density profile given by Eq. (2.50) is characterized by $\alpha = 2.34 \times 10^{10}$ kg/m² and, from Eq. (2.53), the compressional bulk modulus is $\kappa_0 = 2.23 \times 10^{11}$ Pa, which is comparable to the range of PREM bulk modulus in the transition zone, from 1.53×10^{11} Pa to 2.56×10^{11} Pa. The resulting density of the core is 10096.3 kg/m³, which differs by 8 % from the volume-averaged PREM density of the core and its density profile within the mantle differs from that of PREM by 9, 6 and 21 % at the Moho discontinuity, the 670 km discontinuity and the core-mantle boundary, respectively. Nevertheless, the model density differs from PREM by 41 % at the Earth's surface, due to the compositional decrease of the Earth's density within the crust.

Fig. 2.7 Pole values s_j of the relaxation modes of the models CC_0 (*dot points*) and MC (*cruciform points*). The inverse Maxwell and compressional transient times of both models are $\tau^{-1} = 4.58 \text{ kyr}^{-1}$ and $\zeta^{-1} = 2.49 \text{ kyr}^{-1}$



In Fig. 2.7 we compare the relaxation spectra (up to the harmonic degree $\ell = 100$) of the self-compressed compressible sphere with compressional stratification, CC_0 , and of a two layered compressible model, that we denote with MC , consisting of homogeneous core and mantle, where the material parameters are constant and obtained from the model CC_0 by means of volume averages, with mantle density of 4623 kg/m^3 . Note that these two models share the $M0$ and $C0$ buoyancy modes, the pair of compressional transient modes, $Z0^+$ and $Z0^-$, and the dilatational modes (also abbreviated as D -modes). The pair of compressional transient modes, $Z0^+$ and $Z0^-$, have been identified and discussed for the first time in Cambiotti et al. (2009) in the case of layered compressible models. Nevertheless, only the layered model MC has the Rayleigh–Taylor modes (also abbreviated as RT -modes) that are, indeed, absent in the relaxation spectrum of the self-compressed model CC_0 .

The transient relaxation spectra of the models CC_0 and MC , characterized by the D -modes and the pair of the modes $Z0^+$ and $Z0^-$, differ mainly at low harmonic degree, while the differences decrease at high harmonic degrees. Instead, the $C0$ buoyancy mode presents important differences at all harmonic degrees and the $M0$

buoyancy mode agrees only at the first 10 harmonic degrees. Such a circumstance is due to the different density profiles of the two models, since the elastic parameters and the viscosity are the same. We thus safely argue that the differences in the pole values are caused by the different density contrasts of the two models at the core-mantle interface and Earth's surface, respectively, which affect mainly the buoyancy modes $C0$ and $M0$.

Let us now consider two representative self-compressed compressible spheres characterized by stable and unstable compositional stratifications, where the bulk modulus κ differs from the compressional bulk modulus κ_0 . Contributions from compositional and non-adiabatic stratification do not amount to more than 10–20 % of that of the compressional stratification (Birch 1952, 1964; Wolf and Kaufmann 2000); we assume bulk modulus of 2.62×10^{11} and 1.94×10^{11} Pa to describe stable and unstable compositional stratifications, respectively. They correspond to values of -0.15 and 0.15 for the parameter ϵ , Eq. (2.55), and, in this respect, we denote these two models with $CC_{-0.15}$ and $CC_{0.15}$.

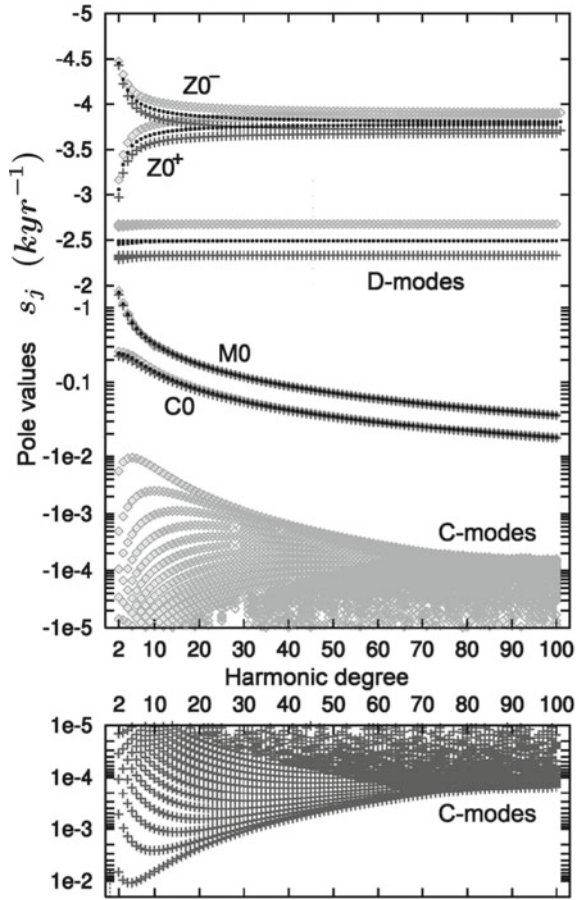
Figure 2.8 compares the relaxation spectra of the self-compressed compressible spheres with compressional and compositional stratifications. Note that the compositional models $CC_{-0.15}$ and $CC_{0.15}$ share the same relaxation modes of the compressional model CC_0 , but they have further relaxation modes that are infinite denumerable, with the origin of the Laplace domain as cluster point. The latter relaxation modes are stable for $\epsilon = -0.15$ and unstable for $\epsilon = 0.15$, in agreement with the analysis of the gravitational stability based on the sign of the Brunt–Väisälä frequency ω^2 , Eq. (2.56). They thus describe relaxation processes involved by the compositional stratification and, for this reason, we called them compositional modes (also abbreviated as C-modes). With respect to the common relaxation modes of the three models, we note that they differ mainly in the transient region where the characteristic relaxation times of the D-modes and of the pair of transient compressional modes $Z0^+$ and $Z0^-$ of the models $CC_{-0.15}$ and $CC_{0.15}$ are greater and lower than those of the model CC_0 , reflecting the different compressional transient times ς , Eq. (1.193), that decrease for increasing bulk modulus, 0.43, 0.40 and 0.37 kyr for $\epsilon = -0.15$, 0 and 0.15.

2.5.3 The Compositional Modes

The denumerable set of C-modes originates from the oscillating behavior of the secular determinant $D_\ell(s)$ near the origin. It occurs on the positive or negative real axis of the Laplace domain, depending on the sign of the compositional coefficient γ . Such a behavior is like that of the secular determinant of layered compressible models to which the Rayleigh–Taylor modes are associated, with the exception that the RT-modes are always unstable because layered compressible Earth's models have always an unstable compositional stratification.

Let us now derive an approximated analytical expression for the pole values of the C-modes. After substitution of the analytical expression for the propagator matrix

Fig. 2.8 Pole values s_j of the relaxation modes of the models CC_0 (*dot points*), $CC_{-0.15}$ (*diamond points*) and $CC_{0.15}$ (*cruciform points*). The inverse Maxwell relaxation time of all three models is $\tau^{-1} = 4.58 \text{ kyr}^{-1}$ while the inverse compressional relaxation times are $\varsigma^{-1} = 2.49, 2.67$ and 2.33 kyr^{-1} , respectively



Π_ℓ of the self-compressed compressible sphere, Eq. (2.85) into the expression for the secular determinant D , Eq. (1.199), and after expansion in Taylor series of the function $\hat{\mu}(s)$, Eq. (1.54), we obtain that the dominant terms of the secular determinant D is proportional to

$$D_\ell(s) \propto \left(\frac{a}{r_C}\right)^i \left(\frac{\kappa_0 \ell(\ell+1)\epsilon}{\hat{\mu}(s)}\right)^{\frac{1}{4}} - 1 \quad (2.86)$$

By equating to zero the RHS of Eq. (2.86), we thus obtain the following approximated analytical expression for the roots of the secular equation (1.200), which are the poles s_{C_m} of the compositional modes

$$s_{C_m} = -\ell(\ell+1) \frac{\kappa_0 \epsilon}{\nu} \left(\frac{\log\left(\frac{r_C}{a}\right)}{\pi m}\right)^4 + O(m^{-6}) \quad (2.87)$$

for $m = 1, \dots, \infty$. It confirms that the compositional modes are an infinite denumerable set of relaxation modes and that the origin of the Laplace domain, $s = 0$, is the cluster point of the poles s_{C_m} for $m \rightarrow \infty$ since they converge to zero as m^{-4} . Besides this, the dependence of Eq. (2.87) on the parameter ϵ gives us the possibility to show analytically that a little deviation from the completely compressional stratification is sufficient to activate the compositional modes. Particularly, they are stable if $\epsilon > 0$ and unstable if $\epsilon < 0$. This suggests that the Rayleigh–Taylor modes are actually a particular case of the compositional modes and that the compositional modes describe buoyancy relaxation processes arising from deviations of the stratification from the neutral state of equilibrium. This interpretation is furthermore supported by the characteristic relaxation times of the compositional and Rayleigh–Taylor modes. As shown in Figs. 2.7 and 2.8, their upper limits are of similar order of magnitude, that is greater than $10\text{--}10^2$ kyr. This short time scales, however, are due to the use of simplified models. Indeed, more realistic Earth’s models based on PREM predict much larger characteristic relaxation times of order $1\text{--}100$ Myr (Plag and Jüttner 1995; Vermeersen and Mitrovica 2000).

These findings contrast with the interpretation of Han and Wahr (1995) that a continuous density profile yields a continuous spectrum of buoyancy modes. This interpretation was based on the investigation of the relaxation spectrum of layered compressible models, where each density contrast contributes with a buoyancy mode M_i . For very fine layered models, where small density contrasts are introduced in order to simulate the continuous variations of the PREM density, the number of buoyancy modes M_i is large. This large number of modes was interpreted by Han and Wahr (1995) as evidence that continuous variations of the initial density imply a continuous set of buoyancy modes in the region of small Laplace variable s . In view of the relaxation spectrum of the self-compressed compressible sphere, however, we can say that this is not the case. Indeed, despite the continuous variations of the initial density described by the Darwin-law profile, Eq. (2.50), it is remarkable that no additional buoyancy modes other than the M_0 and C_0 modes are present in the compressional model CC_0 and that only a discrete, although infinite denumerable, set of compositional modes are triggered by compositional stratifications.

Even if Han and Wahr (1995) supported the normal mode approach, we note that their conclusion about the presence of a continuous spectrum in the buoyancy region has weakened the normal mode approach, discouraging further investigations of the relaxation spectrum for not layered Earth’s models, where the continuous variations of the material parameters within the layers of the Earth are taken into account. On the contrary, our results indicate that such an analysis can be done and interesting physical knowledge of the viscoelastic relaxation processes at large timescales can be obtained as we will show in Sect. 2.6.

2.6 Viscoelastic Perturbations Due to Surface Loading

In order to investigate the role of the initial state of the viscoelastic Earth's model, characterized by compressional or compositional stratifications, we focus on Love numbers $\mathbf{k}$ for loads seated at the Earth's surface (we will omit the superscript L and the harmonic degree ℓ to not overwhelm the text): they are relevant in Glacial Isostatic Adjustment (GIA) studies. We consider herein perturbations within the mantle for which the Love numbers for surface loading in the Laplace s -domain read, according to Eqs. (1.172), (1.177) and (1.179)

$$\tilde{\mathbf{k}}(r, s) = \mathbf{N}_L^{-1} \mathbf{B}_\ell(r) \mathbf{b}^L \Big|_{\mu=\hat{\mu}(s)} \quad (2.88)$$

with the tilde standing for the Laplace transform and the dimensional matrix $\mathbf{N}_L$ given by Eq. (1.174). According to Eq. (1.179), $\tilde{\mathbf{k}}(r, s)$ provides $\tilde{h}$, $\tilde{l}$ and $\tilde{k}$, the radial, tangential and gravitational viscoelastic Love numbers in the Laplace domain and their dependence on the radial distance from the Earth's centre r refers to where we calculate the perturbations. In this respect, note that the matrix $\mathbf{B}_\ell$ also depends on r , according to Eq. (1.172).

In view of the study of relaxation spectrum of the self-compressed compressible sphere, the viscoelastic Love numbers can be recast by a spectrum of relaxation modes

$$\tilde{\mathbf{k}}(r, s) = \mathbf{k}_E(r) + \sum_{j \in \mathcal{S}} \frac{\mathbf{k}_j(r)}{s - s_j} \quad (2.89)$$

where $\mathbf{k}_E$ consists of the elastic Love numbers, $\mathbf{k}_j$ contains the residues of the j th relaxation mode and s_j is the corresponding pole. Here, $\mathcal{S}$ denotes the whole set of relaxation modes, which is denumerable but infinite (Cambiotti and Sabadini 2010). The set $\mathcal{S}$ of relaxation modes is split into two types:

$$\mathcal{S} = \mathcal{F} \cup \mathcal{C} \quad (2.90)$$

The set $\mathcal{F}$ of fundamental modes appears both for compressional and compositional stratifications: the M0 and C0 buoyancy modes (associated with the Earth's surface and CMB), the pair of transient compressible modes, Z_- and Z_+ , and the infinite and denumerable set of dilatational modes, D_m , with $m = 1, \dots, \infty$. The set $\mathcal{C}$ of compositional modes, C_m , with $m = 1, \dots, \infty$ is again denumerable and infinite but is triggered only in the case of compositional stratifications.

The fundamental modes describe the transition from the elastic to the Newtonian-fluid behavior, while the compositional modes control the long time-scale perturbations towards the isostatic equilibrium described by the inviscid fluid. Accordingly, we split the perturbations $\mathbf{K}$, due to a point-like surface load of unit mass with Heaviside time history, into contributions describing the elastic response, the transition to the Newtonian fluid and the final transition towards the isostatic equilibrium

$$\mathbf{K}(r, t) = \begin{pmatrix} U(r, t) \\ V(r, t) \\ \Phi(r, t) \end{pmatrix} = \mathbf{k}_E(r) - \sum_{j \in \mathcal{F}} \frac{\mathbf{k}_j(r)}{s_j} (1 - e^{s_j t}) - \sum_{m \in \mathcal{C}} \frac{\mathbf{k}_{C_m}(r)}{s_{C_m}} (1 - e^{s_{C_m} t}) \quad (2.91)$$

where U , V and Φ are the degree- ℓ non-dimensional radial and tangential displacements (normalized by a/M_E), and gravitational potential perturbation (normalized by $a g/M_E$). In view of the fact that the poles s_j of the fundamental modes are negative and that their characteristic relaxation times, $|1/s_j|$, are shorter than those of the compositional modes, $|1/s_{C_m}|$, we can write the final transition towards the isostatic equilibrium as

$$\mathbf{K}(r, t) = \mathbf{k}_S(r) + \mathbf{K}_C(r, t) \quad (2.92)$$

where $\mathbf{k}_S$ is the secular perturbation due to the elastic response and the relaxation of the fundamental modes

$$\mathbf{k}_S(r) = \mathbf{k}_E(r) - \sum_{j \in \mathcal{F}} \frac{\mathbf{k}_j(r)}{s_j} \quad (2.93)$$

and where $\mathbf{K}_C$ is the perturbation due to the only compositional modes

$$\mathbf{K}_C(r, t) = \begin{pmatrix} U_C(r, t) \\ V_C(r, t) \\ \Phi_C(r, t) \end{pmatrix} = - \sum_{m=1}^{\infty} \frac{\mathbf{k}_{C_m}(r)}{s_{C_m}} (1 - e^{s_{C_m} t}) \quad (2.94)$$

As compressional stratifications have no compositional modes, the viscoelastic Love number $\tilde{\mathbf{k}}(s)$ is an analytic function in a neighbourhood of the origin of the Laplace domain, $s = 0$. Thus, $\tilde{\mathbf{k}}(s = 0)$ exists and is finite. From Eqs. (2.89) and (2.93), we obtain the following identity

$$\mathbf{k}_S(r) = \tilde{\mathbf{k}}(r, s = 0) \quad (2.95)$$

This implies that the summation over the strengths $\mathbf{k}_j/s_j$ of the fundamental modes entering Eq. (2.93) converges to a finite value. Furthermore, the secular perturbations describe the isostatic equilibrium to surface loading (Wu and Peltier 1982). In this respect, the secular radial displacement and gravitational-potential perturbation satisfy the isostatic conditions at the Earth's surface a

$$\lim_{t \rightarrow \infty} U(a, t) = h_S(a) = -\frac{2\ell + 1}{2} \quad (2.96)$$

$$\lim_{t \rightarrow \infty} \Phi(a, t) = k_S(a) = -1 \quad (2.97)$$

Perturbations below the Earth's surface, as well as the tangential displacement at the Earth's surface, are instead unconstrained due to the indeterminateness of static perturbations of the inviscid body discussed by (Longman 1962, 1963). They must be obtained solving the whole viscoelastic problem and using Eq. (2.93).

2.7 Toroidal Solution

The toroidal vector solution Eq. (1.101) does not depend whether the material is compressible or incompressible. In fact, the only material parameter entering the toroidal component of the momentum Eq. (1.86) is the rigidity μ . By assuming the latter constant within each layer of the Earth's model, $\partial_r \mu = 0$, Eq. (1.86) simplifies into

$$\mu \nabla_r^2 W_\ell = 0 \quad (2.98)$$

This is the Laplacian equation for $W_{\ell m}$ and the two independent solutions are

$$W_\ell = c r^\ell + c^* r^{-(\ell+1)} \quad (2.99)$$

with c and c^* as integration constants. By substituting Eq. (2.99) into Eq. (1.81), we obtain the solutions for the toroidal component of the stress

$$T_\ell = c\mu (\ell - 1) r^{\ell-1} - c^*\mu (\ell + 2) r^{-\ell-2} \quad (2.100)$$

and, thus, we can write the toroidal solution vector as follows

$$\mathbf{y}_\ell(r) = \mathbf{Y}_\ell(r) \mathbf{C} \quad (2.101)$$

where $\mathbf{Y}_\ell$ is the fundamental matrix for toroidal deformations

$$\mathbf{Y}_\ell = \begin{pmatrix} r^\ell & r^{-\ell-1} \\ \mu (\ell - 1) r^{\ell-1} & -\mu (\ell + 2) r^{-\ell-2} \end{pmatrix} \quad (2.102)$$

and $\mathbf{C}$ is the vector of integration constants

$$\mathbf{C} = (c, c^*)^T \quad (2.103)$$

$$\mathbf{y}_\ell(r_C) = \mathbf{I}_C \mathbf{C}_c \quad (2.104)$$

2.8 Time Dependent Loading Love Numbers

When our Earth's models are forced by loading or tidal forcing, the Green functions are named Love numbers, for loads and tidal potentials. Love numbers can in principle be generalized to earthquake forcing. In this Section we consider only the loading case, since the tidal case will be considered in Chap. 3 when dealing with the readjustment of the rotational bulge in Sect. 3.6. Figure 2.9 portrays the time evolution in $\log(t)$ of the convolution between the Love numbers and the Heaviside function, given by Eq. (2.94). The time interval spans a wide range of time scales, from one tenth of an year to 10^3 Gyr. The Love numbers are all amplified in time,

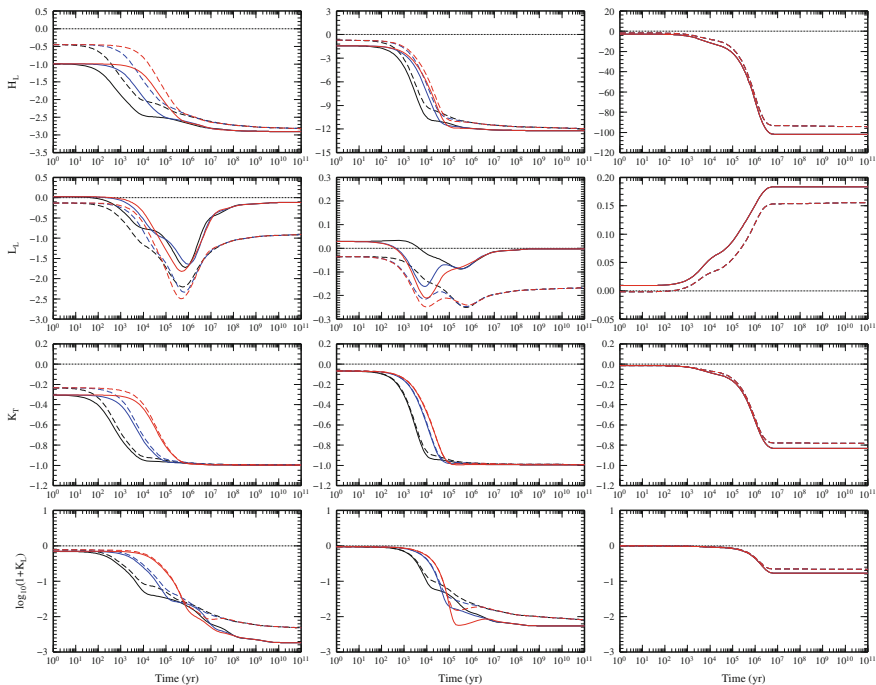


Fig. 2.9 Loading Love numbers convolved with the Heaviside function, from the elastic limit, *left*, to the fluid (long time) limit, *right*, in the abscissa. Columns from *left to right*, refer to the harmonic degree $\ell = 2, 10, 100$. The *first row* stands for the radial Love number H_ℓ , the *second row* for the tangential Love number L_ℓ , the *third row* for the gravitational potential Love number K_ℓ and the *bottom fourth row* stands for the gravitational potential $\log(1 + K_\ell)$, including the direct effect of the unitary load. Capital letters are used since the Love numbers are convolved with the Heaviside function. *Continuous* and *dashed curves* stand for compressible and incompressible models, all based on PREM stratification. The viscosity in each layer below the 80 km thick elastic lithosphere is fixed at 10^{21} Pa s in the *upper mantle*. In the *lower mantle*, the viscosity is 10^{21} Pa s, *black*, 10^{22} Pa s, *blue* and 10^{23} Pa s, *red*

from the elastic limit in the left of the abscissa to the fluid (long time) limit, right in the abscissa, due to viscoelastic relaxation: in terms of normal modes, such an amplification is caused by the decay of the exponentials $(1 - e^{s_i t})$ in Eq. (2.94) which sum up to the elastic contribution.

The largest differences among the various models, in terms of incompressible versus compressible or low versus high lower mantle viscosities, occur for the lowest harmonic degree $\ell = 2$ both for the elastic limit and transients, and such differences tend to diminish for $\ell = 100$, with the three viscosity models becoming indistinguishable. The $\ell = 2$ is in fact sensitive to the global properties of the Earth, in terms of elastic, density and viscosity stratification, while the sensitivity of the highest harmonics is limited to the outermost properties of the Earth.

The $\ell = 2$ Love numbers show clearly the effects of lower mantle viscosity increase from 10^{21} to 10^{23} Pa s, causing the corresponding increase from 10^3 to 10^5 yr for global mantle relaxation, during which Love numbers are amplified. It is also notable that H_ℓ and K_ℓ portray a monotonic increase carrying the same sign, while the horizontal Love number L_ℓ portrays a non-monotonic behavior, changing also the sign when increasing the harmonic degree. It is also notable that incompressibility, dashed lines, impacts in particular the horizontal Love number, and so the component of the displacement field for surface loads, in the whole range of time windows, from the elastic limit, through the intermediate time scales when the Love numbers are amplified, to the fluid (long time) limit. It is interesting to look at the effects of viscosity changes in the outermost part of the planet, in the asthenosphere, as in Fig. 2.10. A striking difference from Fig. 2.9 is that the highest harmonic $\ell = 100$ is sensitive to the viscosity of the outermost part of the Earth as visible in the third column, to confirm that the longest wavelengths sample the deepest mantle, and the shortest ones the shallowest portion of the Earth. For $\ell = 100$, in particular the compressible Earth portrays for the three Love numbers H_ℓ , L_ℓ , K_ℓ an important sensitivity to the asthenospheric viscosity, with time scales in the amplification of the Love numbers ranging from 10^1 to 10^4 yr, corresponding to a viscosity increase from 10^{19} to 10^{21} Pa s.

The horizontal Love number L_ℓ and $\ell = 10$ portrays a variability in time at shorter time scales once compared to Fig. 2.9, with the amplification starting at $t = 3$ yr instead of $t = 10^2$ yr. The Love number K_ℓ for $\ell = 2, 10$ is not sensitive to the asthenospheric viscosity, since the gravitational field is most affected by the properties of the mantle as in Fig. 2.9, while the quantity $1 + K_\ell$ is sensitive to the outermost asthenospheric viscosity, but solely when K_ℓ , for $\ell = 2, 10$, is close to its long term (fluid) value -1 .

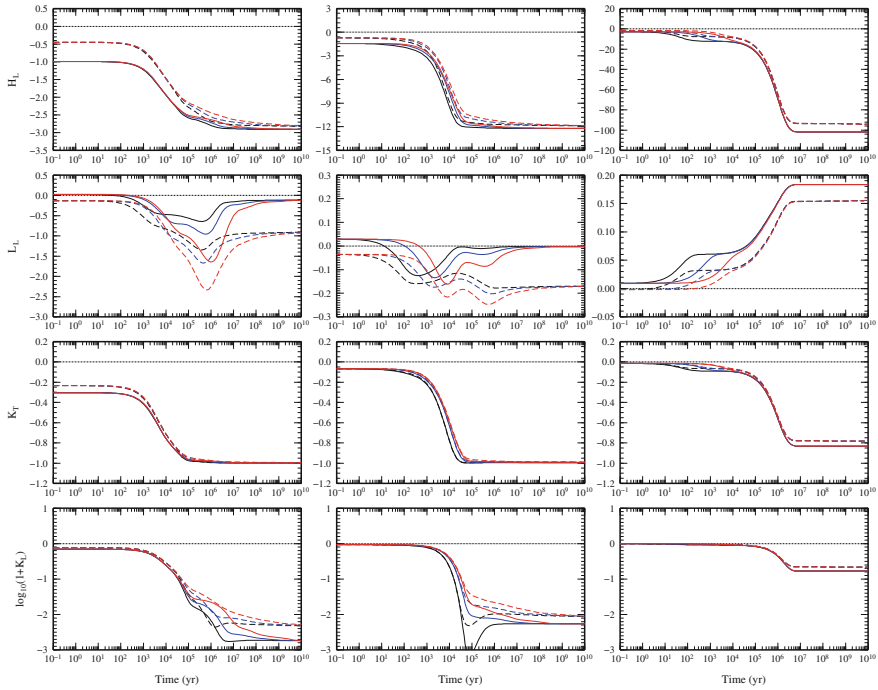


Fig. 2.10 As Fig. 2.9, except for viscosity changes in the 200 km asthenosphere below the 80 km thick elastic lithosphere. Note that the abscissa with respect to Fig. 2.9 has been lowered by one order of magnitude. *Black, blue and red curves stand for 10^{19} , 10^{20} and 10^{21} Pa s in the asthenosphere. The viscosity is 10^{21} Pa s in the *upper mantle* below the asthenosphere and 10^{22} Pa s in the *lower mantle**

It is interesting to note that the viscosity of the asthenosphere affects to some extent also the slowest time scales, due to the interplay between the viscosity and the density stratification in the outermost part of the Earth.

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Sabadini, R.; Vermeersen, B.; Cambiotti, G.

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