

Chapter 2

Ancient and Modern Statics in the Renaissance

Forza, dico essere una virtù spirituale, una potenza invisibile, la quale per accidentale esterna violenza è causata dal moto e collocata e infusa ne' corpi i quali sono dal loro naturale uso retratti e piegati, dando a quelli vita attiva di maravigliosa potenza; costringe tutte le create cose a mutazione di forma e di sito; corre con furia alla sua desiderata morte e vassi diversificando mediante le cagioni.

(da Vinci, *Ms A*, 34v)

Statics is the science of equilibrium. The term appears in the Latin version (translated by Snel) of Simon Stevin (1548–1620) most famous textbook, *Tomus quartus mathematicorum hypomnematum de statica* (Stevin 1605, p 5). This work can be considered the hinge between ancient and modern statics. Ancient statics was the science of equilibrium of weights; modern statics is the science of equilibrium of forces. In ancient Greece statics was part of mechanics, the science of transportation of bodies by means of machines. In the Middle Ages and first Renaissance, statics was known as *scientia de ponderibus* (science of weights); its main object was the study of principles of equilibrium for heavy bodies suspended from a balance. Presently, statics is part of mechanics, which is the general science studying equilibrium and motion of bodies and their assembly, of any kind.

Hereinafter we will use the term *scientia de ponderibus* to indicate ancient statics – more precisely the ancient statics of Middle Ages and Renaissance – and simply statics to indicate modern statics.

2.1 The Background

Scientia de ponderibus (science of weights) is the name given by the medieval schoolmen to the discipline that treats the equilibrium of heavy bodies with particular reference to those hanging from a balance. The *Scientia de ponderibus* was different from Greek mechanics, both for the scope – Greek mechanics placed transportation of weights, instead of their equilibrium, at the centre – and for the

methodology – *Scientia de ponderibus* charged only to the theoretical foundations of equilibrium and not applicative aspects. The *Scientia de ponderibus* was also different (i.e., see Pellicani s.d.) the mechanics of the early XVI century, the centrobaric, a discipline developed in the wake of the rediscovery of Archimedes, which was concerned mainly with the mathematical problems (Dijksterhuis 1957) of determining the geometric centres of gravity of plane figures and solids.

2.1.1 *The Scientia de ponderibus in the Middle Ages*

In the western Middle Ages, the science of weights was classified as *subalternate-science*, following the Aristotelian tradition which identified astronomy, optics, and music as the more physical of the mathematical sciences (Aristotle 1984, *Physics*, II, 2, 194a; on Aristotle's physics see also Philoponus 1993). They are *mixed sciences* (XVII century terminology), i.e. sciences with ranges both in physics and mathematics, and which are subordinate to mathematics. To these three sciences Aristotle had added a fourth, mechanics (Aristotle 1984, *Posterior analytics* I, 9, 76a; Aristotle 1984, *Metaphysics* M, 3, 1078a; Aristotle [1936] 1955b, *Problemata mechanica*, 847a). Physics – the subalternate science – can demonstrate that things are so (demonstrations *quia*) while mathematics, – the subalternating science – demonstrates why (*propter quid*) they are so. As a rule, the subject matters of the subalternating and subalternate science are not the same; if they were exactly the same, one would have a single science and not two separate sciences. Therefore, for example, the subject of geometry is geometrical lines, whereas the subject of optics is visual lines (Euclid 1945). Since a visual line is naturally associated to a geometrical line, optics falls under geometry (Bussotti and Pisano 2013). Geometry, then, can be used to study optics, but only the aspects that can modelled by it; a large portion of optics remains, which is the object of physics alone (Pisano and Casolaro 2011).

Apart from astronomy (De Pace 2009; Kesten 1945), the subalternate-sciences that attracted the greater attention by mathematicians were geometrical optics and mechanics. They were structured on the basis of the Euclidean model, based on definitions, suppositions (principles) and propositions (theorems). The main difference with respect to the Euclidean model was that some of the principles rather than being purely geometric, related to the physical world. They were the translation into mathematical terms of what belonged to physics. In the Aristotelian circles, this translation appeared unproblematic; mathematicians, instead, did not exhibit the same level of tolerance as the Aristotelian philosophers, and doubted the evidence of the principles, often assigning them the status of postulates.

Recent studies (Machamer 1978; Lennox 1985; Biener 2004–2008) have highlighted the role of the subalternate-sciences matured within Aristotelian scholarship, which provided a mathematical interpretation of the physical world quite similar to that proposed by Archimedes. In truth, these studies remain at a superficial level; for example, they do not explain why the subalternate-sciences,

once they have passed into the hands of professional mathematicians, assume a structure different from what they had in the hands of philosophers. Nevertheless, mainly they do not study in depth what professional mathematicians, and not philosophers, actually did. One of the main concerns of philosophers was to preserve the homogeneity of demonstrations, particularly in mathematics and physics. But in the treatises of science classified as subordinate (including the Archimedean ones, which will see their diffusion in the XVI century), there was no trace of this purism, and statements about the physical aspects, such as heaviness, were intermixed with statements about geometry with no concern to maintain the homogeneity of the demonstrations (Capecchi 2014a, b, c)

2.1.1.1 The Roots in the Arabic Middle Ages

The *scientia de ponderibus* saw its birth in the Arabic land; its status of a distinct *scientia* first appeared in Abū Naṣr Muḥammad ibn Muḥammad Fārābī's (ca. 870–950) *Kitab ihsa' al-'ulum* (*The Book of Enumeration of the sciences*). In particular, he definitively distinguished between science of weights and sciences of devices (or machines). In his classification of knowledge, Abū Naṣr Muḥammad ibn Muḥammad Fārābī' (hereafter Al-Farabi) took six distinct sciences: language, logic, mathematics, nature, metaphysics and politics. The mathematics were divided into seven topics: arithmetic, geometry, perspective, music, science of weights and sciences of machines or devices.¹ These last are defined as follows:

As for the *science of weights* [emphasis added], it deals with the matters of weights from two standpoints: either by examining weights as much as they are measured or are of use to measure, and this is the investigation of the matters of the doctrine of balances (*umūr al-qawl fi l-mawāzin*), or by examining weights as much as they move or are of use to move, and this is the investigation of the principles of instruments (*uṣūl al-ālāt*) by which heavy things are lifted and carried from one place to another.

As for the *science of devices* [emphasis added], it is the knowledge of the procedures by which one applies to natural bodies all that was proven to exist in the mathematical sciences. . . in statements and proofs into the natural bodies, and [the act at] locating [all that], and establishing it in actuality. The sciences of devices are therefore those that supply the knowledge of the methods and the procedures by which one can contrive to find this applicability and to demonstrate it in actuality in the natural bodies that are perceptible to the senses.²

Al-Farabi's setting was never seriously challenged, although there were different nuances in subsequent classifications (Schneider 2011). Some scholars divided the science of weights into science of balances and science of weight lifting; for

¹ For our historical epistemology aims and because the science at that time, we distinguish between the role of geometry and of other next mathematical disciplines (arithmetic, algebra, and calculus starting from the 17th century). Therefore here we historically distinguish between arithmetic, geometry and mathematics, including under this denomination all mathematical branches not belonging to classical definition of geometry.

² Al-Fārābī cited in: Abbatouy 2008, 100; see also Othman 1949.

example, Ibn Sina (980–1037). Al-Isfīzārī (1048–1116) and al-Khāzinī (1115–1130) singled out the theory of centers of gravity from the science of weight (Abattouy 2008, 103). Particularly interesting is Abu 'l-Fath al-Rahmān al-Xāzinī (fl. XII century) from Merv (Persian Greek). His *The Book of the Balance of Wisdom* (Khanikoff [1858] 1982) is one of the most important works on arabic-Islam idrostatics (Mieli 1938; Gibb and Bowen 1951; Nasr 1977; Jaouiche 1971, 1976).

The new science of weights was characterized by a strong deductive system, in which components of qualitative physics were formulated *more geometrico*. The most common historical point of view is that the science of weights originated from interplay of Aristotelian physics and the physical-geometrical approaches by Archimedes and probably Euclid, on the equilibrium of bodies. Now we did not find studies on the role played by Aristotelian conception about subalternate-science in the development of Arabic science, to contrast this point of view. Surely an important role should be assigned to Heron's writing which spread throughout the Islamic lands (Heron Alexandrinus 1893, 1899–1914; Brugmans 1785; Ferriello 2005).

From a methodological point of view, the majority of treatises in the science of weights followed what is often called dynamical or more properly kinematical approach, in which the equilibrium is seen as a balance of opposing forces and the movement, virtual or real, has an important role. In these treatments the Aristotelian dichotomy between the natural and forced, upward and downward, motions, disappears for they are considered on the balance, in which the weight is also the natural cause of lifting other weights. The pure geometrical approach, like the one carried out by Archimedes, is certainly uncommon, so that some historians do not even consider it as part of the science of weights.

The production of Arabic texts developed from the IX to the XII centuries (Giusti and Petti 2002). First, there was a phase of recovery and digestion of the works of Greek origin (Gutas 1998). Besides the translations of Aristotle's theoretical works, *Physics* and *On the Heaven*, available since the IX century, Islamic scholars surely had access to *Mechanics* by Pappus and Heron written in Greek. Also circulating were two treatises on the balance attributed to Euclid (*Euclid's book on the balance* and *De ponderoso et levi*). It seems instead that of Archimedes' mechanical works, only that on floating bodies was known, while regarding the Aristotelian *Problemata mechanica*, it can be stated with certainty that only a partial epitome was known (Abattouy 2006).

The analysis of the general significance of the Arabic medieval science of weights shows that this tradition did not represent a mere continuation of the traditional doctrine of mechanics as inherited from Greeks. Rather, it means the emergence of a new science of weights recognized very early in Arabic learning as a specific branch of mechanics, and embodied in a large scientific and technical corpus. Comprehensive attempts at collecting and systematizing (as well as updating with original contributions) the mainly fragmentary and unorganized Greco-Roman mechanical literature that had been translated into Arabic were highly successful in producing coherent and orderly mechanical systems.

The main Arabic texts on the science of weights are listed below in Table 2.1; for further information see (Abattouy 2008, 94–95).

Table 2.1 Arabic treatises on the science of weights

<i>Kitāb fī il-qarastūm</i> by Thābit ibn Qurra ^a	It is probably the first Arabic text about the steel yard. It exists into four manuscripts in Arabic: one conserved in London, one in Kraków and another in Beirut. The first manuscript was edited, translated into French and commented on by Khalil Jaouiche (Jaou 1976). The second, while in Berlin, was edited and translated into German by Eilhard Wiedmann (Wiedmann 1911), and subsequently studied by Mohammed Abattouy (2001). The third one was studied by Knorr (Knorr 1982). A fourth partial copy was recently found in the archives of the Laurentiana Library in Florence (Abattouy 2008, 94).
<i>The treatise on centres of gravity</i> , by al-Qūhī and Ibn al-Haytham, two most important mathematicians of X-XI centuries. <i>Irshād dhawī al-ʿifān ilā šināʿat al-qaffān</i> (Guiding the learned men in the art of steel-yard), by al-Isfizārī. ^b	It survived only on <i>al-Kāzinī's Kitāb mīzān al-ḥikma</i> (<i>The Book of the balance of wisdom</i>). A fundamental treatise written about 1050–1110. Here different Arabic and Greek traditions are reported, together with a unified mechanical theory.
<i>al-Kāzinī's Kitāb mīzān al-ḥikma</i> by <i>al-Kāzinī's Kitāb mīzān al-ḥikma</i> . ^c	An encyclopedia of mechanics completed in 1121–1122, well known as the <i>Book of the balance of wisdom</i> . A source of information about theoretical and practical knowledge of medieval mechanics. It is known in the West by Khanikoff's partial translation (Khanikoff [1858] 1982).

^aAl-Šābi' Thābit ibn Qurra al-arrānī (836–901) was a native of Harran and a member of the Sabian sect. He was a great scholar in mathematics and astronomy; translated and revised many of the important Greek works; particularly all the works of Archimedes that have not been preserved in the original language and Apolonius' *Conic sections* (Heath 1896; see also Panza 2008, 165–191). He was a founder of the science of weights.

^bAbū ʾātim al-Muʾaffar ibn Ismāʿīl al-Isfizārī. (ca. 1048–ca. 1116). A mathematician, astronomer and an engineer, he was born in Isfizar, a city near Herat. His study of Archimedes' book helped him in identifying the purity of gold and silver for which purpose he made a hydrostatic scale to determine the weight of alloys in the two metals His main scientific contribution was in the field of weights and mechanical designs.

^cAb ar-Raḥmān al-Khāzinī (ca. 1115–1130) was a Muslim of Greek origin who was brought to Merv as a slave by the Seljuk king after his victory over the Byzantine Emperor. Al-Khazini was a great physicist, astronomer, mathematician, philosopher and an alchemist. He is better known for his contributions to physics. His treatise; *al-Kāzinī's Kitāb mīzān al-ḥikma* written in four volumes, remained an important part of physics among the Muslim scientists

2.1.1.1.1 Thābit's *Kitāb fī il-qarastūm*

Thābit's contribution is for sure the most relevant for Arabic mechanics. Moreover, it influenced mostly Latin medieval mechanics; for this reason, it deserves a short account. The *Kitāb fī il-qarastūm* was composed of a prologue followed by eight

propositions and finally a comment. They all relate to the *karaston*, that is the steelyard or Roman balance, which is a straight-beam balance with arms of unequal length. It incorporates a counterweight, which slides along the calibrated longer arm to counterbalance the load and indicates its weight. The most important postulate Thābit assumed is the following:

PROPOSITION I. The ratio of two distances covered by two mobiles in two [equal] times is equal to the ratio of the force of the mobile [passing] the plane distance to the force of the other mobile.³

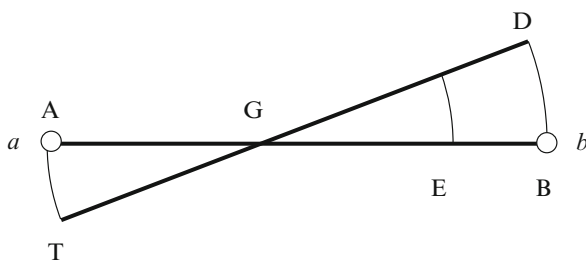
Based on the postulate, Thābit can prove the law of the lever, which is given as follows:

PROPOSITION III. Since this is manifest now, then I propose [the following with respect to] every line which is divided into two different segments and imagined to be suspended by the dividing point and where there are suspended on the respective extremities of the two segments two weights, and the proportion of the one weight to the other, so far as being drawn downward is concerned, is inversely as the proportion of the lines. [I say that in these circumstances] the line is in horizontal equilibrium.⁴

The proof of proposition III, has to rely on the following comment Thābit makes just before its enunciation:

We have already said [emphasis added] that in the case of two spaces which two moving bodies describe in the same time, the proportion of the power of the motion of one of the body to the power of the motion of the other is as the proportion of the space which the first motion cuts to the other space. And point A with the motion of the line has already cut AT and point B with the motion of the line has already cut arc BD, and this in the same time [See Fig. 2.1]. Therefore, the proportion of the power of the motion of point B to the power of the motion of point A is as the proportion, one to the other, of the two spaces which the two points describe in the same time, evidently the proportion of arc BD to arc AT. This proportion has already been shown to be the same as the proportion of line GB to line AG.⁵

Fig. 2.1 Equilibrium of the balance according to Thābit (Redrawn from Moody and Clagett [1952] 1960, 94)



³“PROPOSITION I. Le rapport de deux distances parcourues par deux mobiles en deux camps [égaux] est égal au rapport de la force du mobile [qui parcourt] la distance plane a la force de l'autre mobile” (Jaouiche 1976, 147).

⁴Moody and Clagett [1952] 1960, 92, 94.

⁵Moody and Clagett [1952] 1960, 92. English translation is ours.

Thābit clearly affirms that the ‘power of motion’ of the point B of the longest arm of the balance is greater than that of the point A, or more generally that the power of motion of a point of a balance is directly proportional to its distance from fulcrum. To note that displacements are measured according to the arcs of circles that weights describe in their motion; this is not peculiar to Thābit, but can be found also in the works by al-Isfizari (Capecchi 2012a, b, 71) and by Galilei himself (Galilei 1649, 164). Thābit justifies his affirmation by saying “We have already said” (Moody and Clagett [1952] 1960, 92) which can only refer to *Proposition I*. Nevertheless this induces, at least for modern readers, a serious interpretation of the problem (Butterfield 1957). Indeed *Proposition I* when adapted to weights seems to make sense only for downward motions, but in the previous passage, Thābit is considering both upward and downward motions. One could overcome this difficulty by assuming that if a weight suspended from one side of a balance moves upward it could move downward too the same distance in the same amount of time, when the rotation of balance is imagined to revert and then one can always make reference to a possible downward motion. The same problem occurs in Galileo’s demonstrations about equilibrium with the use of the concept of momento (hereafter also *moment*) (Galilei 1612).

2.1.1.2 Continuation in the Latin Middle Ages

The very expression *scientia de ponderibus* was derived from the Latin translation of al-Fārābī’s *Ihṣā’ al-’ulūm*. Translations of this text were due both to Gerardo da Cremona and Dominicus Gundissalinus in the XIII century. Gundissalinus in his treatise borrowed from al-Fārābī the concept of mechanics as a subalternate science, stemming from Aristotle’s division for analogous sciences. He reproduced al-Fārābī’s characterization of the sciences of weights and devices, called respectively *scientia de ponderibus* and *scientia de ingeniis*. The reason for this verbatim acquisition depends on the fact he could not rely on any scientific category in this field in Latin. Even the antique Latin tradition represented by Boece and Isidore of Sevilla (VIII AD) could not furnish any useful data.

In the Latin Middle Ages, various treatises on the *Scientia de ponderibus* circulated. They were Latin translations from Greek or Arabic between XII and XIII centuries, referred to in the following Table 2.2.

Table 2.2 Latin treatises on the science of weights

<i>Liber de canonio</i>	A short treatise on the construction of Roman scale. Translated from a Greek origin (Moody and Clagett [1952] 1960, 64–75). The law of the lever, attributed to Euclid, Archimedes and other is taken for granted (<i>sicut demonstratum est ab Euclide et Archimede et aliis</i> , Moody and Clagett [1952] 1960, 66). Basing on it the laws that regulate the balance of a ‘rod equipped with weight divided into unequal parts and loaded at the ends are determined. The problem is to find the position of the point of suspension given a certain tray so that it has equilibrium with no weights added, or vice versa given the point of suspension to find the weight of the tray.
<i>Liber Euclidis de ponderoso et levi</i>	Translated from an Arabic version attributed to Thabit, it would result from a Greek original which with many doubts can be traced back to Euclid. It consists of nine suppositions and some theorems. The version reported in (Moody and Clagett [1952] 1960) reports only five suppositions, but it is probably incomplete. Interesting the first theorem, not so much for its demonstration, but for the fact that it was assumed as a principle by Thabit in his <i>Kitāb fī il qarastūm</i> : “Of bodies which traverse unequal places in equal times, that which traverses the greater place is of greater force”. ^a
<i>Liber Archimedis de insidentibus in humidum</i> or <i>Liber Archimedis de ponderibus</i>	According to (Moody and Clagett [1952] 1960, 36–37) the text cannot be attributed to Archimedes, despite the medieval claims. It would come for the first part from Latin sources of the eighth century (Isidore of Seville), for the second part from Arab sources of the twelfth century. The text is different from the others in content since it is not centred on the equilibrium of the balance but simply arises the problem of assessing the weight of bodies immersed in a medium. Interesting is the revival of the golden crown of the famous problem solved by Archimedes (Moody and Clagett [1952] 1960, 40–53).
<i>Archimedis insidentibus in aquae and Aequiponderanti.</i>	This is the translation by William of Moerbeke of the works of Archimedes on the equilibrium of the planar and floating bodies. They had no particular success in the Middle Ages, both for the difficulties intrinsic in the mathematics, and for the inaccurate translation of the concepts by Moerbeke.
<i>Liber karastonis</i>	It is the Latin translation by Gerardo da Cremona of Thābit’s <i>Kitāb fī il-qarastūm</i> . None of the Arabic extant copies seem to be the direct model for Gerard’s translation (Moody and Clagett [1952]

(continued)

Table 2.2 (continued)

	1960, 88–117). Arabic manuscripts are quite different from the Latin one. The order of propositions, indeed not numbered, in the Arabic versions is different from the Latin one. The texts of propositions are virtually the same as those in the <i>Liber karastonis</i> , except for secondary aspects. The texts of explanations are instead very different; shorter and much less satisfactory than those of the Latin version. The Latin version was repeatedly copied and distributed in the Latin West until the XVII century, as it is documented by several extant manuscript copies. Further, the treatise was used as textbook in the quadrivium, together with works by Jordanus De Nemore and others.
<i>Excerptum de libro Thatbit de ponderibus</i>	It has the same structure as the <i>Liber karastonis</i> for the statement of principles and theorems, it is its logical abstract (Brown 1967, 24–40). According to Knorr (Knorr 1982, 42–469), it is not derived from the Latin version but from some Arabic source.
<i>Problemata mechanica</i>	There is no evidence of a Latin translation of Aristotle’s text. However, there are indications of its knowledge in the Greek version.

^a“Corporum que temporibus equalibus loca pertranseunt inequalia, quod maiorem pertransit locum maioris esse virtutis” (Moody and Clagett, [1952] 1960, 26–27)

Starting from these treatises, medieval scholars developed their own science of weight. The first texts written directly into Latin are those attributable to various ways to Jordanus de Nemore, a famous mathematician of the XIII century.⁶ We report them below with the names that have been attributed (Moody and Clagett [1952] 1960):

<i>Elementa Jordani super demonstratione de ponderibus</i>	Version E	Hereinafter version E or <i>Elementa</i>
<i>Liber Jordani de ponderibus cum commento</i>	Version P	Hereinafter version P or <i>Liber de ponderibus</i>
<i>Liber Jordani de Nemore de ratione ponderis</i>	Version R	Hereinafter version R or <i>Liber de ratione ponderis</i>

Moody and Clagett ([1952] 1960) with certainty attribute the version E to de Nemore and consider possible the attribution of version R. More uncertainty is the attribution of version P, the less refined. Brown (1976) considers the *Elementa*

⁶ Practically nothing is known about Jordanus de Nemore’s life. He appears at the beginning of the XIII century. Besides writings about mechanics, he is author of many mathematical writings. For some more information see: Klein (Kelin 1964), Høirup (1988) and Duhem (1905, I, 99–108), Ginzberg (1936).

ascribable to de Nemore but seems to opt for a different assignment for the *Liber de ratione ponderis*.⁷

De Nemore's treatises were the object of comments up to the XII century. Worthy of notice are some commentaries of XIII and XIV centuries, referred below with the name assigned to them by Moody and Clagett (Moody and Clagett [1952] 1960) and Brown (Brown 1976) (Table 2.3).

Table 2.3 Some commentaries of Jordanus de Nemore tradition

<i>Corpus Christi</i>	It contains a variant reading of the proof of the law of lever, of some interest, though controversial (Brown 1976, 570–647).
<i>Aliud commentum of Elementa</i>	Some passages of this text are of particular interest in that they testify a work of research regarding the principles of mechanics, somewhat distinct from that carried out by de Nemore (Brown 1976, 164–347).
<i>Questiones super tractatum de ponderibus.</i> By Biagio Pelacani of Parma (c. 1316–1465)	End of XIV century. A short work where three questions were raised. Contains comments on various treatises of the science of weights (Moody and Clagett [1952] 1960, 232).
<i>Tractatus Blasi de ponderibus.</i> By Biagio Pelacani of Parma	It is an independent text divided into three parts. The first two mainly refer to <i>De ponderibus</i> and <i>De canonio</i> , without new arguments. The third part refers to the <i>Liber Archimedis de insidentibus in humidum</i> (Moody and Clagett [1952] 1960, 238–279).

2.1.1.2.1 Jordanus de Nemore's *Liber de ratione ponderis*

Of the three versions (E, P, R) attributed to Jordanus de Nemore that denoted by R or *Liber de ratione ponderis*, is the most complete. It is quite a complex treatise, ideally divided into four parts with 7 suppositions (principles) and 43 (or 45 according to the manuscripts) propositions (theorems) of the science of weights. The first part has a theoretical aim and collects the suppositions and the most interesting propositions, among which the proof of the laws of the lever and inclined plane; the second and third parts are more technical and concern the solutions of some of the problems of the balance, with arms endowed or not with

⁷ There are various hypotheses about the roots of Jordanus' mechanical works. Quite convincing is the hypothesis of the Arabic roots: Abattouy (2006, p 17), Folkerts and Lorch (2007, 4, 12); Brown (1967), Clagett (1959).

natural weight. The fourth part is about various issues, among which the fall and breaking of bodies. The version P assumes the same suppositions (7) and 13 propositions, the first seven coinciding with the propositions of the first book of version R, the other with other propositions of the second book. The version E is the shorter; it has the same suppositions but only 9 propositions corresponding to the first nine propositions of version P. All versions use two, not independent, fundamental laws:

1. The first law assumes the concept of gravity position for which the efficiency of a weight to descend or its resistance to be raised depends on the kinematic constraints to which it is subject. The law states that the effectiveness and strength are the greater the closer the path (made possible by constraints) to the vertical.
2. The second law has a precise mathematical expression and says that “what can raise a weight p at height h , can lift a weight p/n at a height nh , or vice versa a weight np to the height h/n ”. In other words, the discriminant magnitude is the product ph , as requested by the modern principle of virtual displacement.

The first law is presented by de Nemore as a principle, it could have been derived by Aristotle’s considerations in his *Problemata mechanica* on the amazing properties of the circle, but could also have origins in everyday experience of practical mechanicians; de Nemore says nothing about it. Weights are considered both as active elements, which push down and as passive elements, which offer resistance to be raised. The second law has a logical status that does not appear clear from the reading of texts. According to our interpretation, as argued later on, it is a theorem proved from simple principles. The weight in this case is considered only as a passive element.

Jordanus de Nemore only uses the first law to demonstrate propositions of a qualitative nature, such as the demonstration that the lever ab of Fig. 2.2 with unequal arms and equal weights tilts on the side of a . The rationale is that the path ag of a is closer to the vertical than that bf of b .

Fig. 2.2 Balance with unequal arms and equal weights (Redrawn from de Nemore 1565, 5r)

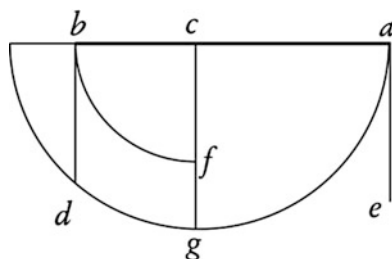
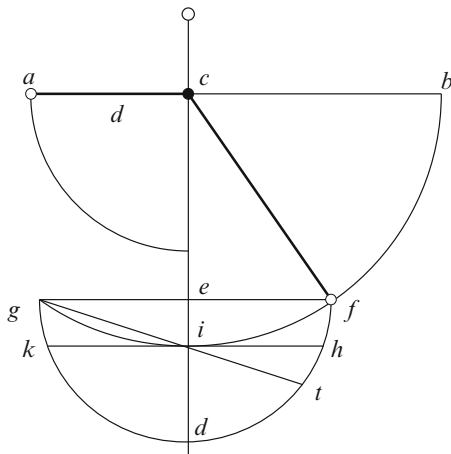


Fig. 2.3 Equilibrium of the angular balance (Redrawn from de Nemore 13th, 4r. See Fig. 2.2bis)



Consider the angular balance-or-lever *acf* (Fig. 2.3) at whose ends two equal weights *a* and *b* are suspended symmetrically with respect to the vertical *cd*. Fixed a vertical segment *em*, weight *b* passes after covering the arc *fm*; weight *a* instead passes the same vertical *cd* by covering a shorter arc *fh*. It is clear from figure that the path *fh* is closer to the vertical than the path *fm*, and then the gravity position of *a* is greater than that of *b*. As a result, there should be no equilibrium and the angular lever should rotate anticlockwise.⁸

Actually, things do not go this way and the angular balance remains in equilibrium. De Nemore will correctly prove this fact (R version) in which the angular lever is studied with the use of the second law without making any reference to the concept of gravity position (de Nemore 1565, 6rv).

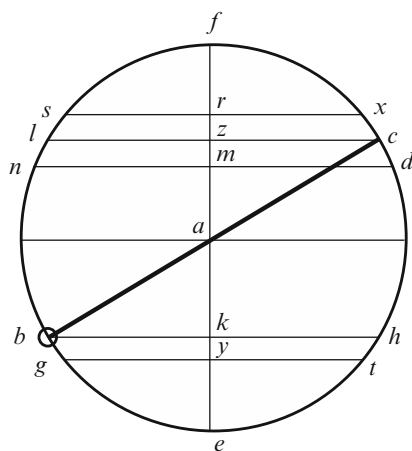
One more case where the concept of gravity of position is used, this time successfully, is in the study of the balance with equal arms and weighs, which is the object of proposition II:

[PROPOSITION II] When a horizontal position is gained [for a balance of equal arms], then, if equal weights are suspended [from its extremities], the balance will not leave the horizontal position; and if it is moved from the horizontal position, it will revert to the horizontal position. If instead unequal [weights] are suspended, [the balance] will fall on the side of the heavier [weight] until it reaches the vertical position.⁹

⁸ Cfr.: Moody and Clagett [1952] 1960, 136.

⁹ “Quum aequilibris [aequilibriis] fuit positio aequalis aequis ponderibus appensis ab aequalitate non discedet: et si a rectitudine separatur, ad aequalitatis situm revertetur. Si vero inaequalia appendantur, ex parte gravioris usque ad directionem declinare cogetur.” (de Nemore 1565, 3v). English translation is ours.

Fig. 2.4 Equilibrium of the balance with equal weights and arms (Redrawn from de Nemore 1565, 4r)



The first part of the proposition, equal weights hanging from a balance with equal arms are equilibrated in the horizontal position, rather than being taken as a postulate, is demonstrated in the same manner as Thabit did, arguing that the two weights are moving with the same obliquity, so they have the same gravity of position and equilibrate themselves. The second part is proved by showing that when the balance assumes a position different from the horizon, the gravity of position of the weight that is lower (b in Fig. 2.4) is less than the weight that is higher (c in Fig. 2.4) because in a virtual rotation of each arm of the balance, the higher c is lowered more than the lower b , when passing equal arcs. So its gravity of position is greater and the balance returns horizontally:

Let it now be supposed that the balance is tilted down on the side of b , and up on the side of c [Fig. 2.4]. I say that it will revert to the horizontal position. The descent from c toward the horizontal position is indeed less oblique than the descent from b toward e . Assume indeed equal arcs, *as small as you please* [emphasis added], cd and bg ; and draw the lines parallel to the horizontal czl and dmn , and also bkh and gyt , and draw, vertically, the diameter *frmakye*. Then zm will be greater than ky , because if an arc, equal to cd , is taken in the direction of f , and if the line xrs is drawn transversally, then rz will be smaller than zm , what is easy to show. And since rz equals ky , zm will be greater than ky . Since because any arc you please, which is beneath c , takes more of the vertical than an arc equal to it, taken beneath b , the descent from c is more direct than the descent from b ; and then c will be heavier in the most elevated position, than b . Therefore, [the balance] will revert to the horizontal position.¹⁰

¹⁰“Ponatur item quod submittatur ex parte b , et ascendat ex parte c , dico quoniam redibit ad aequalitatem. est enim minus obliquus descensus a , ad aequalitatem, quam a , b , versus e . Sumanter enim sursum arcus aequales, quantumlibet parvi qui sint c , d , et h , b , et ductis lineis ad aequidistantiam aequalitatis, quae sint, c , h , l , et d , m , n . Item b , k , h , g , y , t , dimittatur orthogonaliter descendens diametrum quae sit f , m , a , k , y , e , erit quod z , m , maior k , y , quia sumpto versus f , arcu ex eo quod sit aequalis c , d , et ducta ex transverso linea x , r , s , erit r , z , minor z , m , quod facile demonstrabis. Et quia r , z , est aequalis k , y , erit z , m , maior k , y . Quia igitur quilibet arcus sub c , plus capiat de directo quam ei aequalis sub b , directo est descensus a , c , quam a , b , et ideo in altiori situ gravior erit c , quam b , redibit ergo ad aequalitatem” (de Nemore 1565, 3v). English translation is ours.

Note that a part of the secondary literature considered usually de Nemore's assumption of arcs "as small as you like" (*Ivi*) the adoption of reasoning about infinitesimals.¹¹ According to them de Nemore did not make the passage to the limit and then he "failed" (*Ivi*) to notice that in the limit, for infinitesimal arcs, vertical displacements of c and b are equal, then the gravity of their positions are equal, then equilibrium is indifferent. Actually things are not so, as will be explained in Sect. 3.1.2.4, *Proposition VI*, of the present book.

However, de Nemore's failure can hardly be blamed since his way of reasoning was still maintained long after infinitesimals were introduced. In his criticism to Lagrange, Joseph Louis François Bertrand (1822–1900) and Carl Gustav Jacob Jacobi (1804–1851), two important mathematicians of the XIX century, would have subscribed to de Nemore's position to assume finite arcs.¹² The error of de Nemore in this case would have been to consider the gravity of position of the two heavy bodies, c and b , as both moving downward. If he had assumed a congruent motion according to which when one weight raises the other falls, he would have found equality of gravity of positions for c and b .

However, the reduction to infinitesimal motion, according to the modern view,¹³ would lead to an evaluation of the gravity of position different from that proposed by de Nemore. If the motion on a given circle with infinitesimal displacements is assumed; gravity of position is maximum at the horizontal position of the balance and is zero in the vertical position; in an intermediate position, the gravities of the weights are equal and the balance is in equilibrium. Nevertheless, if circles of different radius are considered, the infinitesimal displacements do not attribute the greater gravity to the weights that are on the larger circle. Considering finite displacements instead enables this attribution. The concept of gravity of position, although interesting and suggestive, seems to take more than a simple infinitesimal reinterpretation in order to be adopted by modern statics.

2.1.2 Revival During the Age of Humanism

In XV century Italy there was a sparkling situation for economic, social and political conditions, on the one hand and cultural achievements on the other hand.

¹¹ Cfr.: Clagett (1959, Chapter 2).

¹² Bertrand's criticism is reported in the third issue of Lagrange's *Mecanique analytique* (Lagrange 1811, 1870–1873, 1889), first volume edited by Bertrand himself (Lagrange 1853, 22). Jacobi gave profound criticisms of Lagrange's mechanics (Pisano and Capecchi 2013; on Lagrangian as a methodological approach in other scientific fields see Pisano 2013e) in his *Vorlesungen über analytische Mechanik*, Berlin, 1847–1848, particularly concerning the role of mathematics in the empirical sciences. For details and references, see Pulte 1998. Note that Bertrand and Jacobi, as well as Jordanus, considered infinitesimals as small as you like but always finite quantities.

¹³ In the modern view, infinitesimals are considered in the limit, and the infinitesimal motion is closer to a velocity than a displacement.

A situation, which then would be established in the rest of Europe (Garin 1993, 2008; Tenenti 1990). Regarding cultural aspects, besides emergence of the culture of the middle class, which played an important part in accounting calculations, geography, economics and financial technique, the emergence of the humanist movement should be highlighted.¹⁴ This was made possible by the new social and economic conditions, offering new perspectives on the world, which on one hand allowed the members of the middle class to be able to devote time to study and on the other hand allowed the courts to play a more or less disinterested activity of patronage.

The XV century records a check on growth in the development of science and the publication of scientific papers. The check existed of course for the science of weights too. In this case it also depended on the fact that the discipline, formulated axiomatically, had reached its complete internal maturity and only the proposition of new problems could have lead to an evolution. Although until the early years of the XVI century no new major scientific treatise was written,¹⁵ except the *Summa de arithmetica, geometria, proportioni et proportionalità* (Pacioli 1992) and *De divina proportione* (Pacioli 1509a)¹⁶ by Luca Pacioli (c. 1445–1517), it must be said that in this period the foundations of a major renovation were laid down, with the breaking of the spirit of the scholasticism system and the repudiation of the principle of authority, particularly that of Aristotle, the rediscovery of Plato and Pythagoras and the valorization of mathematics which was the premise for the new philosophy of nature (van Ophuijsen 2005; Vanderputten 2005), of the second half of the XVI century.

2.1.2.1 A Variety of Approaches to Mathematics

At the end of the Middle Ages, mathematics was taught essentially at universities and at abacus schools. In the history of the universities (De Ridder-Symoens 1992), mathematics was taught in the *quadrivium* (arithmetic, geometry, astronomy and music) of the faculties of arts that, while maintaining their autonomy, were instrumental to the training of future physicians and theologians.¹⁷ The medical faculties of the early Renaissance were usually those in which mathematics had

¹⁴ Since different intellectual schools of thought are identified with the term *Humanism*, here are just a few words to remark that by this term we mean in particular the Italian humanist group (*human nature*) busy with lecturing, transcription, and studies of the mathematical sciences from Greek and Latin manuscripts.

¹⁵ The last book of some importance toward the end of the XIV century was *Questiones super tractatum de ponderibus* by Biagio (or Blasius) Pelacani da Parma.

¹⁶ The *De divina proportione* is well known also for the famous Leonardo da Vinci's engravings it contains. (Pisano 2013a; see also Pacioli 1496–1508).

¹⁷ For the role of European universities in the XV century, refer to: Duhem 1988, X; Grant 2001; de Ridder-Symoens 2003; Rüegg 2004. For the Italian universities see the Annals of the history of Italian universities (CLUEB, Bologna) and Grendler's work (Grendler 2002).

more space.¹⁸ Medicine was, in fact, connected to the study of astrology, which required the students to have rudiments of Ptolemaic astronomy and early cosmology (Duhem 1913–1959) and then knowledge of elements of geometry and arithmetic. Professors of these subjects were the masters of liberal arts of the *quadrivium*, whose teaching and research many of the mathematical works of the XV century are connected. However, the place occupied by mathematics was still marginal¹⁹ and the level of mathematical knowledge was, except for some teachers, limited to what was indispensable for the exercise of astrology. In fact, it did not cover the study of many Greek classics that at the time were already available in Latin translations from Arabic of the XII century. However not to be forgotten is that, for instance, Galileo was nurtured at a university and by a shared knowledge as clearly exposed in his correspondence (Galluzzi and Torricini 1975–1984). The University of Padova in particular was an important centre for training in science (Favaro 1883). Among its students in the XV and XVI centuries the following people should be noted: Paolo da Pozzo Toscanelli (1397–1482), Leon Battista Alberti (1404–1472), Francesco della Rovere alias Pope Sixtus IV (1414–1484), Giovanni Pico della Mirandola (1463–1494), Pietro Bembo (1470–1547), Nicolaus Copernicus (1473–1543), Francesco Guicciardini (1483–1540), Girolamo Cardano (1501–1576?), Bernardino Telesio (1509–1588), Torquato Tasso (1544–1595), Roberto Bellarmino (1542–1621), Paolo Sarpi (1552–1623), Giovanni Domenico Campanella called Tommaso Campanella (1568–1693), William Harvey (1578–1657).

Different considerations hold for the schools of abacus. They were born in the XIII century with the spread of *Liber abaci* (Fibonacci 2004; Giusti 2002) by Leonardo Pisano's also called Fibonacci (1170–1250; see Pisano and Bussotti 2013a, Pisano and Bussotti 2015a; Ulivi 2002). Some of these schools were subsidized by the municipalities, some others by private organizations or individuals. The practical mathematics that emerged from the abacus treatises of XIV and XV centuries had so many characteristics that quite clearly distinguished it from the traditional Euclidean axiomatic-deductive mathematics. The main features of the abacus treatises were the use of the vernacular, mercantile writing, a great amount of examples and the presence of important drawings for illustrative purposes. The treatises on the abacus had different quality levels, which reflected the skills of teachers who had drawn them up: some were very simple and neglected those parts of mathematics (algebra, practical geometry, speculative arithmetic) that were not immediately applicable in the art of the merchant. Others, however, showed a certain organic quality, aesthetically cured, mainly in the miniatures illustrating the drawings, and treatment of some algebraic problems, which involved the solution of quadratic and higher degree equations (Ciocchi 2011, 266–271). Even

¹⁸ This is the case for example of Padova, where the introduction of mathematics into the undergraduate curriculum preceded that of astronomy-astrology related to medicine (Kusukawa 2012).

¹⁹ Considering the small number of chairs of mathematics in the University of Padova and Bologna compared to those of medicine until the time of Galileo, it can be seen that the academic discipline was marginal (Ciocchi 2011, 261).

mathematical textbooks used by the artists had characteristics similar to those of the schools of abacus, where, however, drawings and operational rules prevailed over theoretical aspects.

Piero della Francesca, Michelangelo Buonarroti (1475–1564), Niccolò Machiavelli (1469–1527), Leonardo da Vinci (1452–1519) and Alberti were influenced by the mathematics of this environment. Most studies of the history of science, including mechanics, focus on the influence of Euclidean and Archimedean mathematics and neglect that of abacus mathematics, which should not have been small, especially in view of its non-axiomatic approach (Pisano 2013a, b, d; Pisano 2009a, b, c).

With the Renaissance in the XV century (Laird 1986, 1987; Laird and Roux 2008), medieval mathematics is joined by the new mathematics, or rather the rediscovered ancient Greek mathematics to which the humanist movement gave a great contribution. The essential role of Italian humanism in the Renaissance of mathematics during the XV and XVI centuries was well documented in (Rose 1975). Many humanists returned from their travels to Byzantium with codes of Apollonius, Ptolemy, Pappus and Heron written in Greek. In the early XVI century, within a few decades, many revisions and translations of classics were delivered. Some of the most important were: the *De expetendis et fugiendis rebus* (1501) by Giorgio Valla (1447–1500), a rich encyclopaedic anthology of Greek scientific texts,²⁰ a new translation of Euclid (Venezia, 1505) led by Bartolomeo Zamberti (fl. second half XV c.), the first Archimedean texts published (Venezia 1503) by Luca Gaurico (1476–1558), the *editio princeps* of Euclid's *Elements* (Basel, 1533), the translation of Apollonius' *Conic sections* (Venezia, 1537) by Giovanni Battista Memmo (1503/1504–1579), the Italian translation of Euclid and the publication of several works of Archimedes (Venezia 1543) presented by Niccolò Tartaglia (1499/1500–1557), and the *editio princeps* of Archimedes with Greek and Latin text (Basel, 1544). It was however a non-Italian humanist, Johannes Müller von Königsberg, whose Latin toponym was Johannes Regiomontanus (1436–1476), the first to embark on a complete restoration of mathematics and astronomy (Pisano and Bussotti 2012) based on his acquaintance with Italian classicists and humanists related to Basilio Bessarione (1403–1472). In effect, the scientific knowledge spread by humanists during the Renaissance depended on the scientific aptitudes of translators and many other factors related to circulation of information:

As we have seen, the starting point for this renaissance of mathematics was the correction of Greek mathematical texts, to be undertaken by those who were expert in both the Greek language and astronomy. To make the refurbished traditions of Greek mathematics available to mathematicians generally, Regiomontanus from at least 1461 was engaged on a series of Latin translations. But by 1471, this means of communication was revolutionised by Regiomontanus' discovery of the new invention of printing. Through printing, an

²⁰ Printed for Aldo Manuzio's types, *De rebus expetendis et fugiendis* consisted of 49 books, 30 of which were devoted to sciences. The first book presents a classification of philosophy, within which the mathematical sciences plays a dominant role as given on the basis of the commentary to Euclid's *Elements* made by Proclus. Valla's book contains references to Archimedes' works.

astonishingly rapid and accurate dissemination of texts and translations become possible that had been inconceivable in an age where manuscripts represented the sole means of circulating the written word. In its fusion of mathematics, Greek and printing Regiomontanus' publishing *Programme* of 1474 marks the formal beginning of the renaissance of mathematics.²¹

Thus, the reacquisition of mathematical techniques was rather slow. What the humanist movement had since carried on was of a meta-mathematical character and concerned the new role that mathematics acquired within the philosophy of the Platonic and Pythagorean schools of thought. Important to this purpose was the role played by Luca Pacioli, who was at the same time a teacher of abacus and magister theologiae, which allowed him to mediate the culture of technicians and learned men. The biblical-metaphysical idea inspired²² Luca Pacioli in his dedicatory letter (Fig. 2.5) to Guidobaldo da Montefeltro (1472–1508). It regarded a book of nature that – later resumed by Galileo Galilei (1564–1642) as well – was written in geometrical characters.

²¹ Rose (1975, 110).

²² “[...] Fratr̃is Luca de Burgo Sancti Sepulcri, ordinis minorum, sacre theologiae Magistri [...]” “Ad Illustrissimum principem sui Ubaldum Duces Montis Feretri, Mathematicae discipline cultorem serventissimum [...]” (Pacioli 1494, *Summa*, 3r).

tria. *proportio* & *proportionalita* possi intendere. Certo nullo sia che tal l'uade se attribuesca. Lascio
 boemai ogn'altra cosa che longolonia di dire: ma solo tutte le cose create sù nostro specchio che nian
 si trouera che sotto numero. peso e misura non sia costituita commo e dicto da salamone: nel secondo
 de la sapientia. hanc deniq; peccatis summus opifex in celestium terrestriumq; rerum dispositione
 semper habuit. Dum orbium motus: cursumq; syderum & planetarum omnium ordinatissime dispone
 rit. Nec quando etera firmabat sursum. Et appendebat fundamenta terre: & liberabat fontes aqua
 rum. Et mari terminum suum circumdabat legemq; ponens aquis ne transirent fines suos: cum co
 erat cuncta componens &c. Non sia chi temerariam e giudicatio dica quel che fin qua de le *Libarbo*
 manici discorso habiamo i persuasoi a. U. D. S. sia fatto. Alla qual (stando di loro ede ogn'altra cred
 lence) non accadeua per connumeratione de lutilita siegue in ogni doctrina e pratica per esse persuader
 lie infiammarla a seguirle e abbracciarle. Uda solo a finatione aperimento de la nobilita e utilita
 grandissima (commo sopra dicemmo) de li Reuerenti di. U. D. S. quali in simili exercitandose lorri
 ta sustentano. Commo per tutte degne terre a. U. D. S. subiecte si fa chi al traffico. E altri laudabili
 exercitij sonno dati di quali la digna. U. Cita de Urbino principalmente e piena. Lascio de la cita de
 Ugobio essentia membo de. U. D. S. La quale de ogni traffico reduce. Lascio *Solambone*. Cagli e
Obacrata alere. U. D. S. degne cita. Castel durant *Aragnilo* e *Abercatello*. E molti altri luogghi al. U.
 D. S. *Isotopoli* ne li qual non me curo stenderne per che da se sia manifesto. Chi con poco e chi con
 assai sua vita exercitando semper in la famose fiere per aqua e per terra. Ora auinegia. Ora a Roma.
 O a fiorenza se ritrouano. per le qual cose non dubio la presente opera summamente esseri grata: co
 cio sia che in lei a tutte occurrentie (commo habian deducto) li sia suffragatoria e seruiente. Non altro
 e per lo presente a. U. D. S. da exponere se non che in tutti versi vie e modi lo infimo de quella figliu
 lo e seruo frate Luca dal Borgo san sepulcro de loadine de li minoi bunilie de sacra Theologia pro
 fessore deuotamente alci se ricomanda. La qual lo onnipotente dio secondo ogni suo bon desiderio li
 piaccia aacrescere e conseruare con tutti de la casa sua eccelsa: e di quella deuotoli e aderenti. Tale.

Ad illustrissimum Principem Sui. Ubaldu Urbini Ducem. Montis feretri: ac durantis
 Comitum. Graecis latinisq; literis. Ornatissimum: & Mathematicis discipline cultore seruientissimus.
 Fratri Luce de Borgo sancti Sepulchri: Ordinis minorum: & sacre Theologie Magistri: in arte
 Arithmetice: & Geometrie. Epistola.

Non animaduertem Illustrissime Princeps inmensa dulcedine: ac
 maximas utilitates quas ex his scientiis assequimur: que graeci mathe
 mata nostri disciplinas possunt appellare: si recte praece & Theorice
 animo comandentur. Constitui nouum hoc uolumen pro ingenij nostri
 tenuitate componere maxime in eorum usum ac voluptatem edere qui
 virtutum solo affecti essent. In quo (ut ex subscripto indice facile perspi
 ci potest) varias diuersasq; Arithmetice Geometrie proportionis: et
 proportionalitatis partes plurimum necessariae: tum in praxi: tum in
 Theorica collegimus: firmissimisque rationibus & canonibus perfectissi
 mis subieciimus: et antiquis & recentibus philosophis cuiuscumq; pra
 ctis indubitata fundamenta. Quamobrem non immerito libri titulus.

Summa Arithmetice Geometrie proportionum & proportionalitatum dicatur. Ubi ante omnia
 studium eracram in huiusmodi facultatibus praxim tradere quemadmodum ex ordinatissima eius
 serie haud difficulter inueniri licet. Ceterum quia temporibus nostris uerba propria mathematicos ob rari
 tatem bonorum preceptorum apud Latinos ferme interire: capiens ego uisi esse his qui vestre cunctis
 parent (non ignarus stilo elegantiori. Eloquio Cicroniano te salientem eloquentie vndas adari opor
 tere) quid quod unusquisque non bene caperet: si latine per scripta essent: potius vernaculo sermone descri
 ptum. Literature itaq; perita pariter. Et imperitis hoc commodum et iocunditatem afferens: si in
 eis se exercuerint vacent quibuslibet facultatibus et artibus: ob per tractata que communia vnicuique vi
 dentur & optime applicari posse. Et primo quis non dico doctus: sed multo minus quod mediocriter
 eruditus sit: qui non perspicue videat quantum beneant quantumque necessaria sint. Astrologie cuius
 principes hac tempestate vigent auiunculus tuus princeps. Et uianus: una cum Reuerendissimo foel
 simpsoni Episcopo paulo mindesburgensi quos in omnibus semper admiro: & reuere: quorumque
 tractis iudicio hoc ipsum opus non immerito caritate subieciimus: ut que bene scripta sunt approbaret

Fig. 2.5 Plate from the initial part of the dedicatory letter by Pacioli (Pacioli 1494, *Summa*, 3r; see also 4r. Source: Max Planck Institute for the History of science-Echo/Archimedes Project)

Let all created beings be our mirror, as no one will be found to be constituted but as number, weight and measure, as said by Salomon in the second book of the Sapiaentia.²³

2.1.2.2 The Emergence of a New Type of Intellectual Technician: The Engineer

Regarding economic aspects of the times, the emergence of a middle class of which the merchant was a key element should be emphasized. The middle class had long since conquered a great economic and social weight and had acquired the consciousness of its social role and the possession of a culture, independent of universities and various humanist circles. The evolution of the economy and society was strongly influenced by three fundamental technological discoveries: circumnavigability of the earth, gunpowder, and printing. The possibility to circumnavigate the globe was perhaps the most important discovery leading to a boost in the economy of many nations. It also entailed the development of navigation techniques with invention of the compass, the representations of geographic maps, the improvement of astronomy for navigation using the stars, and the crafting of ships, which no doubt provided a stimulus to the improvement of many applied sciences (Singer 1954, II–III).

The spread of modern artillery based on the propellant effect of gunpowder was important, especially for the development of new mechanics (Costabel 1973; Crombie 1957; Dugas 1950). Knowing what causes the beginning of motion, and its sequel, was considered important by commanders of the armies and therefore also by states. This was true especially since the XVI century, when artillery had become extremely effective. The development of artillery had as a natural consequence the development of defensive techniques. This gave birth to the bastioned fortresses, first appearing in Italy and then becoming a real battleground for numerous national and foreign armies. Perhaps even more than artillery, fortress design mobilized engineers and architects, leading to the development of methods of construction and a better understanding of the strength of materials (Pisano 2009a, b, c, d; Pisano and Capecchi 2008, 2009, 2010a, b, 2012, 2013).

The emergence of the engineer as an intellectual technician, seen as a new kind of technician in some way educated in sciences, is a characteristic feature of the XV century and the first half of the XVI. Indeed this is perhaps the main feature of science, where the reduced creativity (real or apparent) of ‘pure’ scientists, was counterbalanced by the great creativity of “applied” scientists. A short list is sufficient to give an idea of the dimension of the phenomenon: Mariano di Jacopo, called Taccola, (1381–1458), Leon Battista Alberti, Francesco di Giorgio Martini (1439–1501), Leonardo da Vinci, Vannoccio Vincenzio Austino Luca Biringuccio also known as Vannuccio, Biringuccio (1480–1539), Francesco de’ Marchi (1504–1576), Giovanni Battista Bellucci (1506–1554), and Daniele Barbaro (1513–1570).

²³ Pacioli (1494, *Summa*, 4r).

Although there was no public funding to encourage scientists to devote their efforts to the study of technical applications and to improvement of their knowledge, a common ground arose, particularly in Central and Northern Italy. A link between engineers and scientists emerged, at least in part, through the creation of some technical centres in the courts of the principalities which had been set up. This was the case of the Medici's court in Florence, but also, and perhaps more importantly, the court of Milan under Francesco Sforza with its very rich library. Another important centre was Urbino. Here among others the presences of Francesco di Giorgio Martini (1480–1490), who translated Marcus Vitruvius Pollio's (ca. 80–70 BC – after 15 BC; see Mussini 2003) *De Architectura* into Italian,²⁴ which although questionable from a philological point of view, made this author known to all technicians²⁵ and Piero della Francesca (1415–1492), one of the greatest mathematicians and painters at that time (Grendler 1955), are to be reported.

2.1.2.3 Leonardo da Vinci's Science of Weights

It is not easy to understand how the science of weights may have influenced the training of technicians. Certainly, some basic aspects on the working of the lever and the block and tackle needed for the construction of building and industry machinery was available independently of mechanics treatises. There was a long tradition of transmission of technological knowledge from antiquity that found concrete expression in the regular use of construction machinery designed during the Hellenistic era. There is however no doubt that when a certain culture of mathematics and drawing began to spread, a precise knowledge of the basic laws of mechanics, which could be acquired with limited scientific knowledge (Capecchi and Pisano 2008), gave the opportunity for the design of machines at the work table (Lefèvre 2004).

In the hands of technicians, the theoretical medieval science of weights could evolve toward a more mature discipline, in the attempt of its application to situations required by the technology of the time (Pisano and Bussotti 2014d, 2015e). This possibility of evolution was widely exploited by a man who is today universally regarded as the engineer of the XV century par excellence: Leonardo da Vinci (Marcolongo 1932; Galluzzi 1988; Pedretti 1978, 1998). In the following, we will expose how the science of weights will be transformed in his hands. The choice of studying Leonardo is partly motivated by the fact that the studies conducted so far on him, not always exhaustive, have shown the great theoretical significance of his writings, but it is also motivated by the fact that now we have access to a complete set of Leonardo's works (Pisano 2013). His many interests were considered in the early 1400s by Taccola (Knobloch 1981) who was interested in the

²⁴ Probably one of the first partial translations from Latin to Italian, which was not published. On our side, no historical documents we know of has claimed that it was really the first.

²⁵ Francesco di Giorgio Martini added elements of theory of machines and construction in book X already devoted to use and construction of machines. For an English edition, see: Rowland and Howe (Rowland and Howe 1999).

writings of mechanics and military technics. In more recent times Giambattista Venturi published, in 1797, a famous essay on the scientific work of Leonardo da Vinci (Venturi 1797). In the years 1880–1940 da Vinci's notebooks were published in facsimile and nearly all the manuscripts were printed with a diplomatic transcription²⁶ and translation in different languages, resulting in approximately a thousand drawings and propositions. However, an organic edition is still lacking, with the happy exception presented by Arturo Uccelli who edited with a critical transcription²⁷ nearly all the mechanical writings, ordering them according to a criterion inspired by Leonardo himself (da Vinci 1940).

Between 1482 and 1499 Leonardo da Vinci²⁸ was in the service of the Duke of Milan. During his service he also advised on architecture, fortifications and military matters and worked as a hydraulic and mechanical engineer and became interested in geometry. He read Leon Battista Alberti's *De re ædificatoria* on architecture (ca. 1450) and Piero della Francesca's *De prospectiva pingendi* on perspectives studies. He illustrated Pacioli's *Divina proportione*²⁹ (1498) and worked with him. Leonardo studied Euclid and Pacioli's *Summa* and began his own research on geometry, sometimes giving mechanical solutions. In 1499 Leonardo left Milan together with Pacioli; in 1506 he returned there for a second period. Again his scientific work took precedence over his painting and he was involved in hydrodynamics, anatomy, mechanics, mathematics and optics. In 1513 Leonardo accepted an invitation from King Francis I to enter his service in France (Gillispie 1971–1980, VIII, 199–244).

Leonardo da Vinci is a difficult subject to be confined within a fixed frame and it is difficult to give a full account of the opinions of historians on Leonardo's role in science in general and mechanics in particular. One goes from an enthusiastic vision of the early XIX century, especially on the side of historians of science educated in literature, to a more mature appreciation of Duhem and finally to a

²⁶ A transcription that respects the original spelling and punctuation marks, spaces included.

²⁷ A transcription that is faithful to the original but avoids typos, resolves “u” in “v” according to the modern practice, uses a standard character for “s”, unifies the writing of words with the same meaning to the most used form, and so on.

²⁸ Leonardo da Vinci was born in 1452 in *Vinci*, a small village near Empoli and province of Firenze, in the Toscana department, Italy. He died in 1519 at the *Château du Clos Lucé*, in the *Indre-et-Loire* department, France. He was educated in his father's house, receiving thereby the usual elementary notions of reading, writing and arithmetic.

²⁹ Leonardo da Vinci's (written down at an earlier meeting with Pacioli) transcripts of his handful of whole passages of the *Summa* (Pisano 2013a, b, c, d). On 10th November 1494 (Venice) finally released in print in Latin, Luca Pacioli's *Summa arithmetica, geometria, proportionibus et proportionality*. Luca Pacioli inspired Leonardo da Vinci (Pisano 2013a) and was his counselor, teacher and translator. Da Vinci purchased the *Summa* (119 soldi) as he himself claimed (da Vinci, *Codex Atlanticus*, 288r f. 104r, 331r) and noted: “Learn multiplication of the roots by master Luca” (da Vinci, *Codex Atlanticus*, 331r [120r]). From 1496 to 1504 Leonardo studied Luca Pacioli's works and summarized his theory of proportions (da Vinci, *Codex Madrid*, 8936). Particularly, geometrical figures were presented for the first time in the *Codex Forster* and finally included in the *De divina proportione* (Pisano 2013a). For Leonardo's sources see the *Pinacoteca Ambrosiana* in Milan and *Museo Galileo-Istituto e Museo di Storia della Scienza* in Florence.

fierce criticism by Clifford Ambrose Truesdell (1919–2000) who minimised (Truesdell 1968, 1–29) both the originality and the contribution to the subsequent science development of Leonardo's work and George Sarton (1884–1956) who affirmed (Sarton 1953, 11–22) that the development of mechanics would have been the same without Leonardo. Eduard Jan Dijksterhuis (1892–1965) eventually considered studying Leonardo as being of interest not for his contributions to science, but for the opportunity offered by his copious notes that were written to follow the maturation of various scientific concepts (Dijksterhuis 1961).

A better understanding of the history of mechanics and a different conception of history of science with a trend to greater contextualization of the work of scientists has certainly contributed to this change of opinions. Today there is a phase of stagnation on the studies of Leonardo as a scientist, probably due to the concerns aroused by the latest criticisms and the concern to approach a job seemingly titanic at first glance. It is with reverential awe and humility that we have set about the study. One of the difficulties in reading Leonardo's texts is that they consist largely of scattered notes, often repeated with slight variations, sometimes with inconsistencies. Although attempts were made to reach a chronologically consistent order, different scholars have not yet obtained results sufficiently shared, also because Leonardo had the habit of putting his own hands to the manuscripts and editing them with continuous additions and deletions. The only valid criterion is the search for logical consistency and the persistence of certain statements over others. Arturo Uccelli (da Vinci 1940), Roberto Marcolongo (1862–1943; Marcolongo 1937), Pierre Maurice Marie Duhem (1861–1916; Duhem 1906) and others, among which we want to name at least Edmondo Solmi (1874–1912; Solmi 1908), attempted to find the source of the thought of Leonardo da Vinci. The enterprise is difficult because in the XV century they were not particularly generous in quotations; Leonardo specifically names only: Aristotle (384 BC – 322 BC), Archimedes, Euclid (ca. 323 BC–286 BC), Abū l Hasan Thābit ibn Qurra' ibn Marwān al-Sābi' al-Harrānī (826–901), Jordanus de Nemore (fl. XII or XIIIth), Biagio of Parma (c. 1365–1416), Albertus Magnus (1193/1206–1280) also known as Albert the Great and Albert of Cologne, Albert of Saxony (ca. 1316–1390), Alberti and perhaps Richard Swineshead³⁰ (fl. 1340–1354). Moreover, it is also difficult to understand the influence of Leonardo on posterity because it seems that he had not made his works known, except to a very restricted circle. We have set ourselves an easier task in trying to decipher Leonardo's thought by framing it within his time on the basis of medieval texts of mechanics known to us but maybe not to him. Stating that Leonardo's claims are original with him is perhaps misleading and at best uninteresting, since we are convinced that he was not an isolated genius, but probably a representative engineer with beliefs common to others (Favaro 1916).

³⁰ Cfr.: Arturo Uccelli (da Vinci 1940).

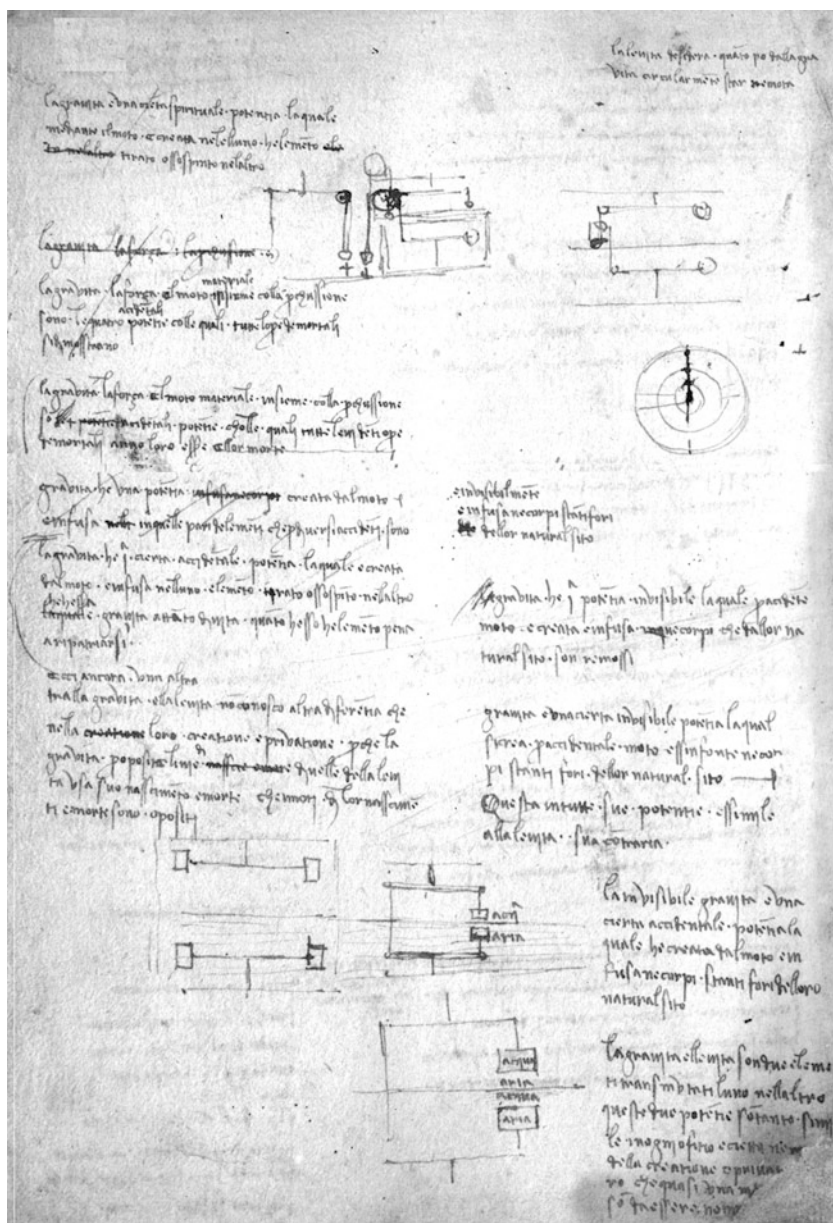


Fig. 2.5bis Plate from the studies on gravity and force (da Vinci, *Codex Arundel*, 37r)

The science of weights in the hands of Leonardo became a discipline similar to modern statics, closer to that of Simon Stevin, a century after, than to that of Guidobaldo del Monte (and even Galileo). In addition, del Monte proposed restoring Greek mechanics (Stevin 1955) limiting the study to simple machines, the lever, an axle with a wheel, the wedge, the screw and the inclined plane.³¹

2.1.2.3.1 Powers: Gravity and Force

Before moving on to analyse the more technical contributions of Leonardo to mechanics we should make a clarification of the meaning of certain terms, including: power, gravity and weight. The following quotes give a first idea:

Gravity is an accidental power, which is created by motion and infused into bodies out of their natural site.³²

[...] Gravity, force and accidental motion (material motion), together with percussion are the four accidental powers, by which all the evident work of mortal beings have their origin and their death.³³

In this passage, Leonardo da Vinci refers to the four powers (with a modern language, forces). Regarding the gravity, it can be said that Leonardo married the traditional Aristotelian school thesis considering it as the tendency of bodies to reach their natural place (Duhem, I, 16–17). For Leonardo gravity is caused by motion (Fig. 2.6):

³¹ Screw was also applied to an inclined plane but in a rotating motion. In addition it is the only simple machine which offers the possibility to turn and drive inward. The idea of a *simple machine* originated with Archimedes who, as well known, studied three machines: *lever, pulley and screw*. Later on, Heron of Alexandria (see *Mechanica*, in Heron 1899–1914, vol. II) studied five machines: *winch, lever, pulley, wedge, screw*. Guidobaldo del Monte in *Mecanicorum Liber* (1577) supplied an advanced – for that period – theory of simple machines, also taking into account *gravitas*. He pointed out the limits of the approach held by the ancients to this subject, in particular as far as Aristotle’s approach was concerned (Aristotle 1955b, pp. 329–411). Galilei in *Le Meccaniche* added the inclined plane, so that the simple machines became six. With regard to the definition of machine, for our historical epistemology aims, we refer to the intuitive conception according to which a machine is a device or a system of devices consisting of fixed and moving parts, which modifies mechanical energy and transforms (machineries) it in a more useful form. Very interesting is its development during 19th century between mechanics and thermodynamics (Gillispie and Pisano 2014). Machines studies also concern the history of science in social context (technoscience) of *machines drawings* traits (e.g. see Popplow 2002, 2003). Recently on how science works and how technique works see Pisano and Bussotti 2014d; 2015a, e, f.

³² “Gravità è una potentia invisibile la quale per accidente moto è creata, e infusa ne’ corpi che dal lor natural sito sono remossi.” (da Vinci, *Codex Arundel*, 37r. See also: da Vinci 1940, 31). English translation is ours.

³³ “La gravità, la forza, e’l moto accidentale, insieme colla percussione, son le quattro accidentali potenza, colle quali tutte le evidenti opere de’ mortali hanno loro essere e loro morte.” (da Vinci, *Codex Forster II*, 116v. See also: da Vinci 1940, 32). English translation is ours.

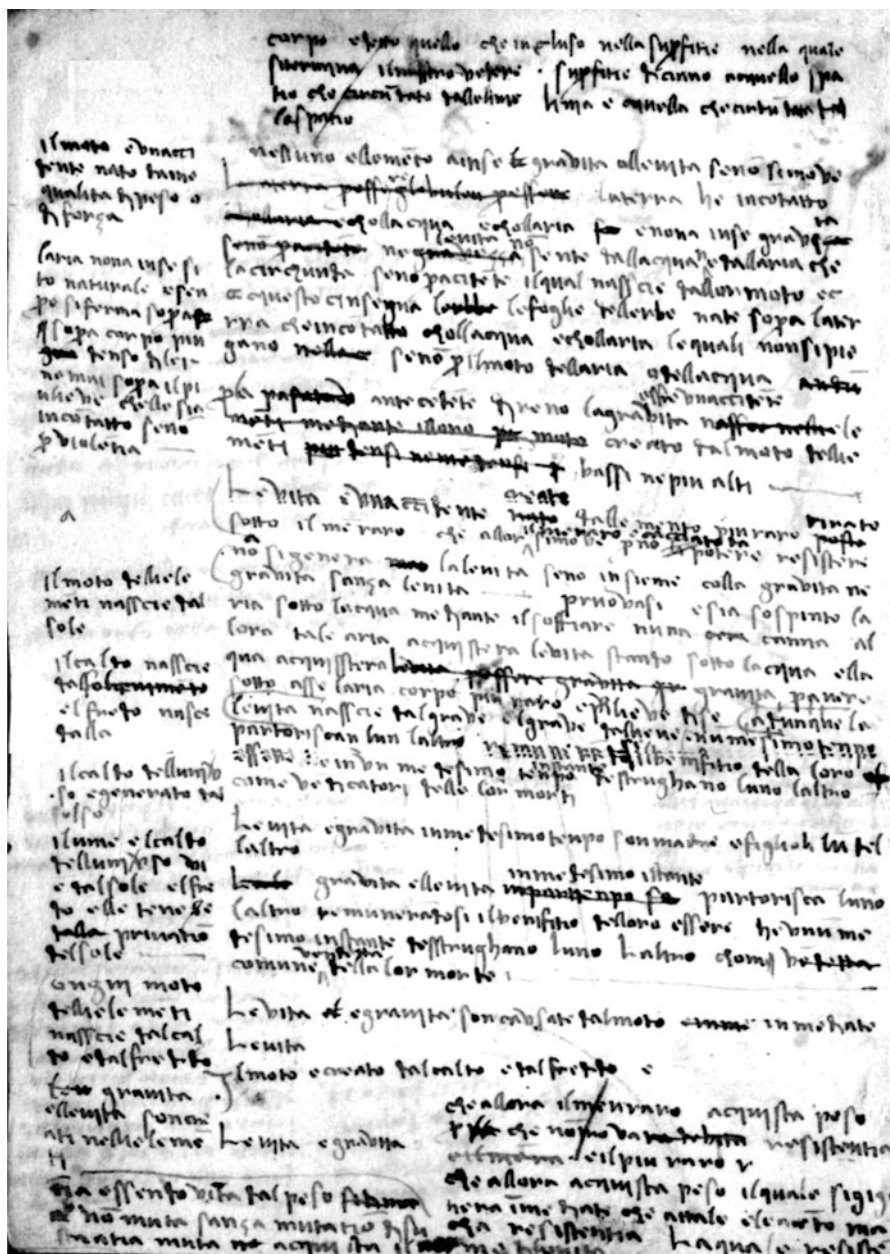


Fig. 2.6 Plate from the studies on gravity (da Vinci, *Codex Arundel*, 205r)

No element has in itself gravity or levity if it does not move. The earth is in contact with the air and water and has in itself neither gravity nor levity; it has not stimulus neither from the water nor from the surrounding air, unless by accident, which originates by motion. And this teaches us the leaves of herbs, born above the earth, which is in contact with the water and the air, which do not bend if not for the motion of air or water.³⁴

To this statement, a bit cryptic for a contemporary, Leonardo adds an explanation:

Gravity is an accident created by the motion of the lower elements into the upper.³⁵

That is, a body shows its gravity if, following an upheaval of the underlying parts, an imbalance of the upper parts is determined. More problematic is the interpretation of the term force (Stinner 1994). On the purpose, quite clarifying was the following famous quotation, which is interesting from a literary point of view also, as a very effective example of scientific prose, in which someone wanted to see the influence of the neo-Platonic philosophy of universal animation (Fig. 2.7).

³⁴ “Nessun elemento ha in sè gravità o levità se non si move. La terra è in contatto coll’aria e coll’acqua e non ha in sè gravità nè levità; non sente dall’acqua nè dall’aria che la circunda se non per accidente, il qual nasce dal lor moto. E questo c’insegna le foglie dell’erbe nate sopra la terra ch’è in contatto coll’acqua e coll’aria, le quali non si piegano se non per il moto dell’aria o dell’acqua.” (da Vinci, *Codex Arundel*, 205r. See also: da Vinci 1940, 30). English translation is ours.

³⁵ “La gravità essere un accidente creato dal moto delli elementi bassi ne’ più alti.” (da Vinci, *Codex Arundel*, 205r. See also: da Vinci 1940, 30). English translation is ours.



Fig. 2.7 Plate from the studies of the equilibrium of weights and of impact (“percossa”) (da Vinci, *Ms. A f. 1v*)

Force I say is a strong spiritual virtue, an invisible power, caused by accidental external violence of motion and located and instilled into bodies, which are moved from their natural habit [the rest] and determined by giving them active life of wonderful power: constrains all created things to change form and site, runs with fury to her desired death and comes diversifying through the causes. Slowness makes it great and swiftness weak, it comes into being from violence and dies for freedom and the greater the sooner is it consumed. Drives away in a rage what is opposed to her decay; she wants winning, to kill by its causes any constraints and winning, it kills herself. It becomes stronger where it finds a stronger contrast. Nothing will move without its. The body from which it originates does not change form or weight.³⁶

It seems to define the impetus of scholastic conception, which is generated in the bodies by the motion transmitted to it by another body, for example by the hand that launches a stone.

Leonardo distinguishes between natural gravity and accidental gravity. The former is the ordinary one and is invariant; the latter is not clearly defined or at least is not defined in a unique way. According to Duhem (Duhem 1906, I, 114–115), the schoolmen used this term as a synonym of impetus and Leonardo, following the ideas of Albert of Saxony who assumed the natural gravity concentrated in the centre of gravity, would consider also the accidental concentrated in a point, named the centre of accidental gravity:

Each body has three centres of figure, one of which is a natural centre of gravity, the other of the accidental gravity and the third one of the magnitude.³⁷

In other cases, Leonardo seems to give a different meaning to the accidental gravity. For instance (cfr. Marcolongo 1937, 64) the centre of accidental gravity coincides with the centroid of a system, composed by accident of many components. This description could be compatible with the other, because in the forced motion, by accident, actions are focused on the accidental centre, so in the case of weights joined by *accident* all motion behaves as if the centre of gravity were a point that is the centre of gravity of no body. As regards the term “weight”, Leonardo uses it as in the modern Italian, to indicate either a heavy body, or the weight of a heavy body. When a body is constrained, the weight is often understood as power, a measure of the effectiveness of gravity according to site. For example, a weight of three pounds that slides on an inclined plane with a ratio between height and length of 2:3,

³⁶ “Forza, dico essere una virtù spirituale, una potenza invisibile, la quale per accidentale esterna violenza è causata dal moto e collocata e infusa ne’ corpi i quali sono dal loro naturale uso retratti e piegati, dando a quelli vita attiva di maravigliosa potenza; costringe tutte le create cose a mutazione di forma e di sito; corre con furia alla sua desiderata morte e vassi diversificando mediante le cagioni. Tardità la fa grande e prestezza la fa debole; nasce per violenza more per libertà. E quanto è maggiore, più presto si consuma. Scaccia con furia ciò che si oppone a sua disfazione, desidera vincere, uccidere la sua cagione, il suo contrasto e, vincendo, sè stessa occide. Fassi più potente, dove trova maggior contrasto. Ogni cosa volentieri fugge sua morte. Essendo costretta, ogni cosa costringe. Nessuna cosa senza lei si move. Il corpo dove nasce non cresce in peso nè in forma.” (da Vinci, *Ms. A*, 34v. See also: da Vinci 1940, 253–254). English translation is ours.

³⁷ “Ogni corpo di disforme figura ha 3 centri, de’ quali l’uno è centro della gravità naturale, l’altro dell’accidentale e l’3° della magnitudine.” (da Vinci, *Codex Atlanticus*, 188v(b); See also: da Vinci 1940, 45). English translation is ours.

weighs two lbs. Leonardo speaks of weight also to indicate the tension of ropes, designed as a portion of the weight carried by them, considered as the portion of the weight supported.

2.1.2.3.2 The Balance and Lever

Leonardo da Vinci, instead of the term *lever* (*lieva*), prefers *balance* – sometimes *scale* – which for him does not necessarily have equal arms. The *lieva* is thus to indicate the balance arm placed where resistance is located, while the *contro-lieva* is the other arm to which power is applied. Note that Leonardo avoids separate treatments of the lever, balance and wheel and axle, as done by del Monte (Renn and Damerow 2010a), considering all of one type, as defined by the balance. Of course, da Vinci knows the law of the lever. He does not report, however, demonstrations of it but merely terms. The applications of Leonardo are of such richness that they have a theoretical value in themselves because they both offer new issues, which could only be imagined by an engineer and not a mathematician or a humanist, and because the proposed solutions, although not supported by experiments, are very stimulating. One of the innovations in the texts of Leonardo da Vinci compared to the traditional science of weights is the use of forces (modern term) applied to the arms of the balance or lever by means of ropes connected to weights with the use of pulleys which modify the direction of application.

In order to understand da Vinci's use of quantitative expressions, the mathematics of time based on proportions must be taken into account. Here the determination of an unknown term was not immediate and instead of writing a simple algebraic equation, as we would do today, it required algorithms now obsolete, including that of the three simple steps derived by the treatise of the abacus. According to the use of this treatise, Leonardo da Vinci often exposes his results, not with propositions having general character, but with numerical examples. They have the function to exemplify the general laws for it is not difficult to imagine that the chosen numbers could be replaced by other numbers. It would therefore represent the need for Leonardo da Vinci to move from his geometrical language based on arithmetical proportions to an early algebraic language which is not formalized enough because of the difficulties in deposing of efficient algebraic rules.

Even with the rule of three one can say: in arms *ab* and *bf* that are 2 and 3, who exchanges suspended weights according to the proportions, they will resist to the descent one of; thus the 5, weight placed in the arm of two spaces resists to weight of 2 placed in the 3 spaces. So you will say for rule of 3: if the 2 of *ab* located in *f* would change in 6 and *f*, which would as to change 5 of *bf* placed, it would be 9 and so inversely, knowing the weight *a* and looking for weight *f*.³⁸

³⁸ “Ancora colla regola del 3 potrà dire: ne’ bracci *ab* e *bf*, che son 2 e 5, chi scambia e’ pesi attaccati secondo le proporzioni, essi resisteranno al dissenso luno dell’altro, onde il 5, peso posto nel braccio di due spazi resiste al peso di 2 posto ne li 5 spazi; onde dirai per la regola del 3: se ‘l 2 di *ab* posto in *f* trasmutassi in 6 che in *f*, il che sarebbe a trasmutare il 3 di *bf* posto in [?] sarebbe 9 e così de converso, sapendo il peso *a* e cercando del peso *f*.” (da Vinci, *Codex Windsor*, 12602v. See also da Vinci 1940, 76). English translation is ours.

In the following passage, Leonardo da Vinci proposes a rule much more complex to calculate the counter-weight:

RULE TO FIND A COUNTERWEIGHT TO A GIVEN WEIGHT IN ONE OF THE ARMS OF THE BALANCE. Multiply the number of times the arm [b] of the counter-weight contains the other arm [a] by the number of the given weight [p], then divide the weight [p] with this result [q], and multiply the result by the number of weight [p]. This result will give the searched counterweight [r] to the given weight.³⁹

Basically if p is the weight, a the length of the lever, b that of the counter-lever, r the counterweight, Leonardo performs the following calculations: multiply the weight p by the ratio of the lengths of the arms getting the result $q = p \times b / a$; divide then p by the result q and multiply again by p : $p : q \times p$ and obtain $a / b \times p$, which is not difficult to verify to be the correct value of the counterweight (r). Marcolongo (Marcolongo 1937, 31–32) argues that the previous quotation was written before 1500, subsequently Leonardo would have given up this complicated rule for the simpler rule of the three. In addition to the relationship between forces in the lever, Leonardo also knows that between displacements:

That proportions that the length of the lever will have with its counter-lever, this same proportion you will find in their weights and similarly in the slowness of motion and in the path made by each of their ends when they arrive to the permanent height of their pole.⁴⁰

Leonardo also knows how to handle balances with more weight hanging from them (cases also considered by Thābit and de Nemore) and thus addresses the case of balances whose arms are endowed with weight, by concentrating it in their centre of gravity.

Of interest is Leonardo's comment on the triangular balance, the *Equilibra*, proposed by Leon Battista Alberti (Alberti 15th, Alberti 1973; Di Pasquale 1992).

³⁹ “REGOLA DA TROVARE IL CONTRAPPEO A UN DATO PESO NELL' UN DE' BRACCI DELLA BILANCIA. Moltiplica il braccio del contrappeso per tante volte il numero del dato peso, quante sono le volte che esso riceve in sè il suo opposite braccio, e colla somma parti il numero del peso, e quel che ne viene rimoltiplica con esso numero del peso, e co' la resultata somma arà fatto il debito contrappeso al già dato peso.” (da Vinci, *Codex Atlanticus*, 309r(d). See also da Vinci 1940, 86; Author's capital letters). English translation is ours.

⁴⁰ “Quella proporzione, che arà in sè la lieva colla sua contralievà, tale proporzione troverai in nelle qualità de' pesi, in nella tardità del moto e in nella qualità del cammino fatta da ciascuna loro stremità, quando sieno pervenute alla permanente altezza del loro polo.” (da Vinci, *Codex Atlanticus*, 173r(a). See also da Vinci 1940, 165). English translation is ours.

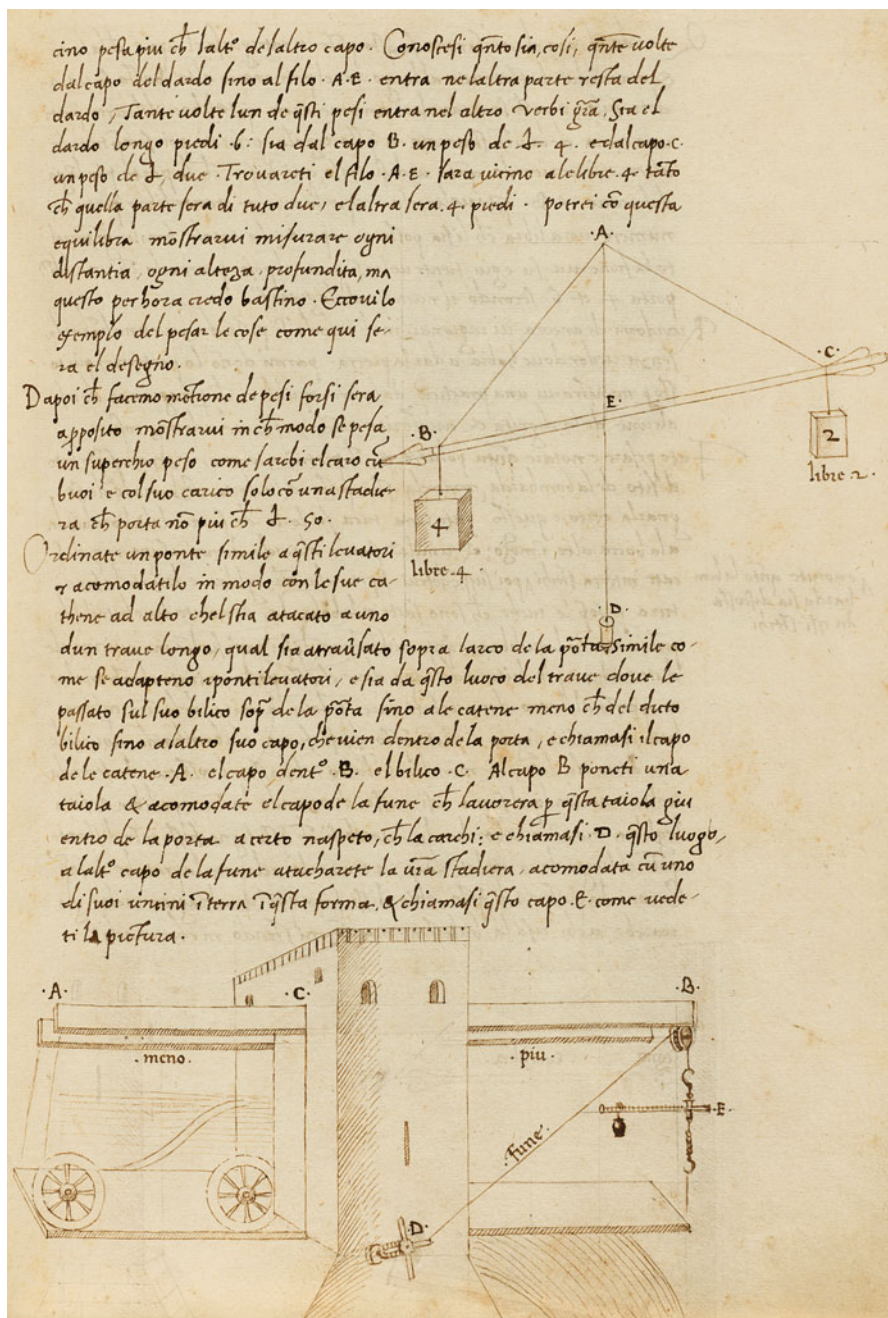


Fig. 2.8a Leon Battista Alberti *Equilibria* (Alberti 15th, Ms 422.2, 10r. With Permission of the President and Fellows of the Harvard College Copyright. The Houghton Library. The Harvard University Cambridge, MA, U.S.A.)

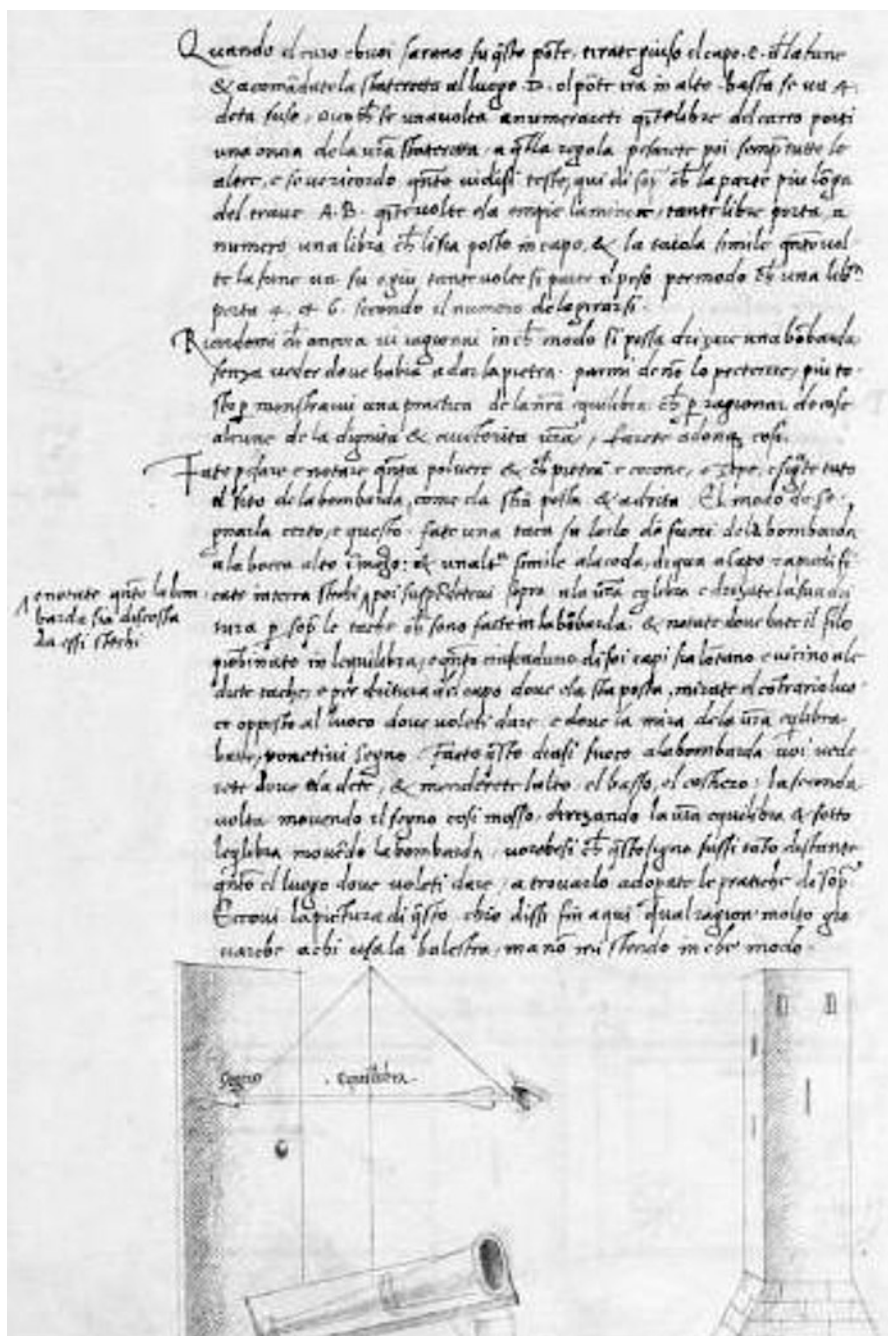
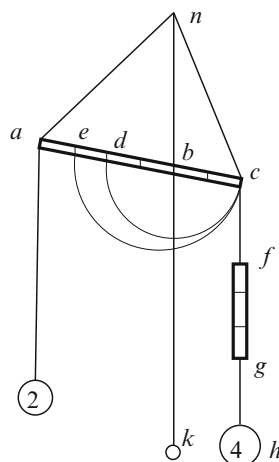


Fig. 2.8b Leon Battista Alberti, *Equilibria* (Alberti 15th, Ms 422.2, 10v. With Permission of the President and Fellows of the Harvard College Copyright. The Houghton Library. The Harvard University Cambridge, MA, U.S.A.)

Fig. 2.8c The triangular balance according to Leonardo da Vinci (Redrawn from da Vinci, *Codex Arundel*, 66r)



Although Alberti suggests building the *Equilibra* with a rod connecting the ends of a wire longer than the rod and suspended in the middle point (see Fig. 2.8a), Leonardo considers from a theoretical point of view the *Equilibra* as a balance with equal arms with the fulcrum located at the top. With this balance one can determine a weight P of any one value with a fixed known counterweight p . With reference to Fig. 2.8c the following relation of proportionality holds true: $ab : bc = P : p$.

Leonardo da Vinci argues that in reality things do not go that way because of the weight of the rod:

Battista Alberti says in a work titled *Ex ludi rerum mathematicarum*: that when the balance abc will have the arms ba and bc in double proportion, with weights suspended from its ends, that dispose it such way, they are in the same proportion of arms, but converse, that is, the more the weight the smaller the arm [See Fig. 2.8c].⁴¹ [...] Which the experience and reason show to be a false proposition, because he puts the opposite weights 2 vs 4 in a balance, which in itself weighs 6 pounds, it is 7 vs 2, and so the balance will remain at rest with equal resistance of arms. And here he wandered, for not to mention the weight of the beam of the balance which is unequal in weight.⁴²

It must be said that Leonardo is not consistent and when he uses Alberti's *Equilibra* he does this without taking into account its own weight. Leonardo da Vinci is not

⁴¹ The title should be in Italian: *Ludi matematici*, as the book was in Italian vulgare. Its original dedication was however: "Leonis Baptistae Albertis ad Illustrissimum principem dominum Meliadusium Marchionem Estensem ex Ludis Rerum mathematicarum". From that, it can be deduced that the original title was probably *Ludi rerum mathematicarum*. Indeed a Latin title for a work in vernacular was a quite common use of the time".

⁴² "Alla qual cosa la sperienza e la ragion li mostra essere falsa proposizione; perché dove lui mette li pesi oppositi 2 contro 4 nella bilancia che in sé pesa 6 libbre, vole essere 7 contro 2; e così resterà la bilancia ferma con equali resistenza di braccia. E qui errò esso altore per non far menzione del peso dell'aste della bilancia, che è ineguale di peso." (da Vinci, *Codex Arundel* 66r. See also da Vinci 1940, 101–102). Here Leonardo's calculations do not sound right.

exempted from the examination of the equal arm balance and weights, which had been and would be a key paradigm of the science of weights. His conclusion is the same as de Nemore; when the arms are horizontal, the balance is in a stable equilibrium configuration and resumes its configuration if moved so

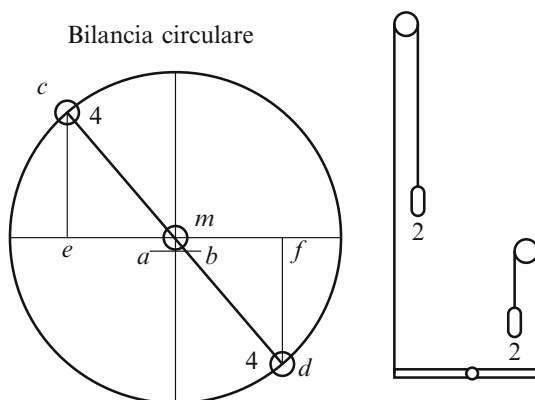
[...] balance with equal arms and weights removed from the site of equality will make unequal arms and weights, so necessity constraints it to acquire again the lost equality of arms and weights.⁴³

Here it is not entirely clear why Leonardo speaks of unequal arms, unless he wants to consider, as shown in some of his drawings, and differently from the medieval science of weights, the descents of weights converging toward the centre of the earth.

The circular balance instead is for Leonardo da Vinci in a state of neutral equilibrium, because of polar symmetry. The indifference changes into stability, however if two consistent weights are added:

CIRCULAR BALANCE. This circular scale [See Fig. 2.9] for it be of uniform gravity, to any lines around its pole, does not completely make the office which would do the common scale, i.e., that which, when moved from the site of equality, it returns there by itself. But this, having heavy weights equally distant from its centre, being removed from the site of equality, it itself does return there. But I think it would return, if the weights attached to it largely overcomes the weight of that wheel.⁴⁴

Fig. 2.9 Equilibrium for the circular balance
(Redrawn from da Vinci, *Codex Atlanticus* 1018 [new numeration])



⁴³ “La bilancia di braccia e pesi uguali, remossa del sito dell’equilibrà farà braccia e pesi inequali, onde necessità la costringe a riacquistare la perduta equalità di braccia e di pesi.” (da Vinci, *Ms. E*, 59r. See also da Vinci 1940, 74–75). English translation is ours.

⁴⁴ “BILANCIA CIRCULARE. Questa bilancia circolare, per essere lei d’uniforme gravità, per qualunque linia intorno al suo polo, essa non fa totalmente tutto l’uffizio che farebbe la bilancia comune, cioè, che quella, essendo mossa del sito della equalità, essa per sè medesima vi ritorna; e questa, avendo e’ pesi equahmente pesanti e distanti dal suo centro, essendo remossa del sito della equaùtà, essa per sè non vi ritorna” (da Vinci, *Codex Atlanticus*, 365r(a). See also da Vinci 1940, 103). Author’s capital letter. English translation is ours.

2.1.2.3.3 The Inclined Plane Law

The equilibrium of weights posed on inclined planes was studied by Heron, Pappus and de Nemore. Only the latter had obtained a *correct* solution.

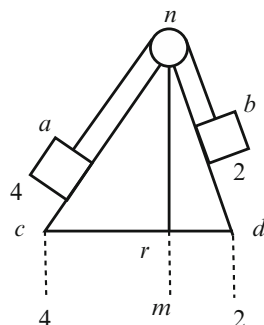
Leonardo da Vinci, as happens in the school of Jordanus de Nemore, does not typically consider a single inclined plane but two opposing planes, on each of which two weights are arranged connected by a rope that passes over a pulley disposed at the intersection of the planes. He does not always refer to the law of the inclined plane in the same way. Generally speaking he correctly states that the effectiveness of the weight decreases with the obliquity, using a term and a concept typical of the school of de Nemore: the term obliquity to mean the inclination of a plane from the vertical and the concept of gravity of position according to which the effectiveness of a weight varies with the obliquity. The problem is that Leonardo does not always measure obliquity in the same way. Sometimes he measures it as the ratio between base and height, sometimes as the ratio between length and height of the plan; this way, as well known today, is the correct one. Leonardo provides an explanation of the different efficacy of weights disposed on an inclined plane, stating that the weight that moves on the more oblique plane undergoes a greater resistance (da Vinci 1940, p 109). Therefore, Leonardo seems to consider the effectiveness of the weight determined by the effectiveness of the constraints and not by the variation of gravity, which often he claims to be invariable.

In the following passage, the obliquity is clearly measured by the ratio between the base and the height, in this way the effectiveness of the weights depends on the cotangent of the angle formed by the inclined plane with the horizontal. Leonardo da Vinci did not realize that in this case, when the plane becomes vertical, one faces a relationship between a finite value and zero.

If the weights a , b [See Fig. 2.10] do not push toward the centre of the world, for they are separated, their combined centre tends to the centre of the world, as the central line nm teaches us passing through the proportions of weights 2 and 4 and for the proportions of the basis of triangles 2 and 4; but the site of them has no proportionate spaces, because in the same obliquity a weight may be high and the other low and [the obliquity] will not vary in this situation; the double ratio of the weights will vary in height.⁴⁵

⁴⁵ “Se a , b , pesi, non spingono inverso il centro del mondo, essendo come son separati, il lor congiunto attende a esso centro del mondo, come ci insegna la linia centrale nm che passa per le proporzioni de’ pesi 2 e 4 e per le proportioni delle base che hanno li triangoli 2 e 4; ma il sito d’essi pesi non ha spazi proporzionati, perchè nelle medesime obbliquita un peso pò stare alto e l’altro basso e non varierà in tal situazione; varia in altezza, la proporzion de’ pesi dupla” (da Vinci, *Ms. G*, 77v. See also da Vinci 1940, 109). English translation is ours.

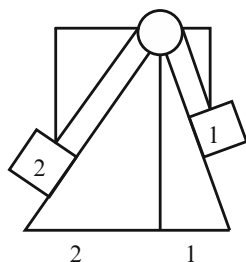
Fig. 2.10 Equilibrium of two weights on an inclined plane by Leonardo da Vinci (Redrawn from da Vinci, *Ms G*, 77v)



The reading of the following passage seems to show that this time obliquity is measured by the ratio between the length and height of the inclined plane, if for *obliqua* it means the inclined plane.

The equality of declinations in accord with the equality of weights. If the proportion of weights and the *obliqua* [emphasis added] where will they stay will be the same but inverse, the said weight will remain the same in gravity and in motion.⁴⁶

Fig. 2.11 Second case-study concerning the equilibrium of two weights on an inclined plane (Redrawn from da Vinci, *Codex Atlanticus*, 981b [new numeration])



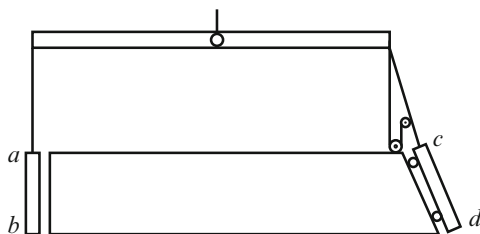
Unfortunately, examination of Fig. 2.11 next to the quotation does not allow this interpretation and in this case also the obliquity should be understood as the ratio between the base and height. In the third case-study, Leonardo asserts quite clearly that the obliquity can be measured ‘correctly’ by the ratio between length and height of the plane, with the following Fig. 2.12 commented with a few words. The balance will be to weight ab as weight cd .⁴⁷

From Fig. 2.12 (see below), is indeed clear how the weights, given by the two prisms of the same thickness, are proportional to the length of the inclined planes. Marcolongo (1937, 54) saw in this figure, an analogy with Stevin’s modelling of weights on the inclined plane by means of a necklace. On the basis of the above and other passages not reported, it can thus be stated with certainty that Leonardo did not possess the law of the inclined plane, except for the observation derived from daily experience

⁴⁶ “La equalità della declinazione osserva la equalità de’ pesi. Se le proporzioni de’ pesi e dell’obliqua dove si posano saranno equali ma converse, essi pesi resteranno uguali in gravità e in moto” (da Vinci, *Codex Atlanticus*, 981b [new numeration]). See also da Vinci 1940, 110; English translation and is ours.

⁴⁷ Redrawn from da Vinci, *Ms H*, 81v.

Fig. 2.12 Third case-study: concerning the equilibrium of two heavy prisms on an inclined plane (Redrawn from da Vinci, *Ms H*, 81v)



that the effectiveness of the weight decreases with the obliquity, and it is also possible that the results he shows are simply uncritical replication of current views of the time.

Finally, one more case-study should be reported that relates to motion rather than the equilibrium of the inclined plane, but which still gives information even for the equilibrium case (See Fig. 2.13):

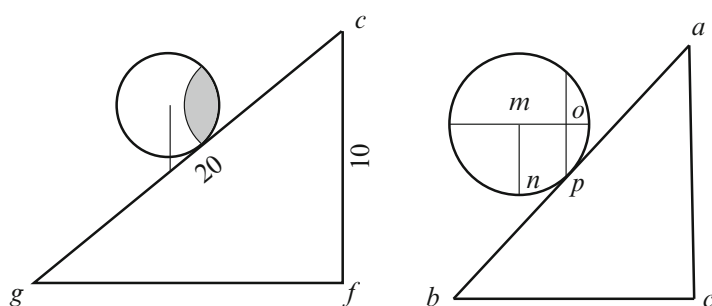


Fig. 2.13 Fourth case-study concerning the motion of a sphere on an inclined plane (Redrawn from da Vinci, *Ms A*, 52r)

ON MOTION. The spherical body will take by itself a motion the faster the more the contact with the site is farther from the vertical passing through its centre. As much as ab is longer than bc , so the ball will fall slower for its line ab , and as much slower, as the part o is less than m , because being p the pole of the ball, the part m , being over p , would fall with faster motion, if it there were not but the small resistance which the counterweight o makes [Fig. 2.13b]. And without this counterweight the ball would descend on the line bc the sooner the more o is close to m , i.e. if the part o enters m 100 times, [the ball] would descend faster than one hundredth of his time than when the part o is missing; mn is the line from the centre and p is the pole where the ball touches its plane, and the more the space np , the faster its way.⁴⁸

⁴⁸ “DEL MOTO. Il corpo sferico e ponderoso piglierà per sè tanto più veloce moto, quanto il contatto suo col loco dove corre fia più lontano dal perpendicolare della sua linia centrica. Tanto quanto ab è più lungo che ac , tanto caderà più tardi la palla per la sua linia che per la linia ab , e tanto più tardi, quanto la parte o è minore che la parte m ; perchè essendo p il polo della palla, essendo sopra p la parte m , caderebbe con più veloce moto, se non fussi quel poco della resistenza che gli fa di contrappeso la parte o ; e se non fussi detto contrappeso, la palla discenderebbe per la linia af tanto più presto, quanto o entra in m ; cioè se la parte o entra nella parte m 100 volte, mancando sempre nel voltare della palla la parte o , discenderebbe più presto il centesimo del suo tempo; mn è la linia centrica e p sia il polo dove la palla tocca il suo piano, e quanto Ha maggiore spazio da np , tanto fia più veloce il suo corso.” (da Vinci, *Ms. A*, 52r. See also da Vinci 1940, 343). English translation is ours.

In the previous passage, Leonardo asserts that the ball moves with the greater velocity the greater the ratio between the segment o and the segment m (the sum of which is the diameter of the sphere), and that the part o opposes the descent. This analysis seems intermediate between those by Pappus of Alexandria⁴⁹ (ca. 290 AC – ca. 350) and by Heron of Alexandria. The idea that we should consider p as a pole is Pappus's, of whom, however, the idea that a force different from zero is necessary to make the ball roll on a horizontal plane is not taken up. The similarity with the analysis of Heron is evident from Fig. 2.13⁵⁰ (on the left) where it is shown how much the left side exceeds the right one. This is not the only point where Leonardo seems to refer to Heron's *Mechanica* (Heron Alexandrinus 1893, see also: *Id.*, 1900, 1999), normally considered to be unknown in the West at least until the XVIII century. One can then make a reasonable guess that the text of Heron was not completely unknown and that Leonardo has become aware of it either directly or indirectly.

2.1.2.3.4 The Pulley, Block and Tackle

Leonardo considers in depth a subject that was completely ignored by the Middle Ages science of weights; i.e. pulleys and the assembly of pulleys or block and tackles. They were commonly used in machines for lifting weights for military and civil constructions (Knobloch 2004), so it is no wonder that Leonardo considered them. He however knows also the rule that connects power to resistance; this information could have been obtained from his reading regarding traites concerning mechanics, or other available sources.⁵¹

The pulley is seen by Leonardo da Vinci sometimes as a mere device to divert the action of a tight rope, other times as a circular lever. The following comments are interesting:

⁴⁹ Cuomo (2004), Hultsch (1878).

⁵⁰ Note that this figure will be used again by Nicola Antonio Stigliola (1546–1623), also known as Colantonio Stelliola (Cfr.: Gatto 1996).

⁵¹ A reasonable conjecture would be that he could have obtained information by some epitome of Heron's text of mechanics (a book intended for architects, containing means by which to lift heavy objects). Nevertheless, even if Heron's *Mechanica* (3 Books) was quite close to the Archimedian ideas circulating in the Renaissance, i.e. shapes, proportion statics problems and balance (Taisbak 1981–1982; Drachmann 1963), it is remarkable that it was preserved only in an Arabic language (Tybjerg 2000). Instead, the idea that theoretical information may be derived also by *Book X* of Vitruvius' *De architectura*, could be less conjectural. In fact, da Vinci could have reasonably known it from the Italian translation due to Francesco di Giorgio Martini.

I call circular scale the pulley or the wheel, with which water from wells is drawn, with which it will never be raised more weight than the weight of the drawn water. No heavy body will lift by means of the circular scale with the strength of its sheer weight more weight than its own.⁵²

The circular scale, said pulley, being of such relevance in mechanical instruments (maximum in transmutations of forces), is not to be neglected; for with it the power of the motor of said machine is increased, as seen in the block and tackles, where the power grows as much as the number of pulleys. Thus we will define its nature and power, and before will show as the strings without motion support the weight due to the supported heavy bodies, and this we will call natural weight, then we will say of motion, varying the weight supported by the strings and we will name this weight accidental weight, i.e., forces, which grows the more the more the [motion] is faster, but the natural weight never varies. The power of the engine varies with the resistance of moved thing and the air which condenses and resists, as the air in fat of watches.⁵³

For assemblies of pulleys, the block and tackles (See Fig. 2.14), Leonardo da Vinci refers laws both for forces and displacements:

THE ROPE, which passes among the pulleys, is named in two ways, the part that gives cause to motion which is fixed to the winch, is named *arganica*, and that which is fixed to the superior pulley and which makes the pulleys neither falling nor slipping is called *ritenente*.

ON MOTION. The longer the motion of the arganica rope, that moves the weight, which is not the motion of the weight which by means of block and tackles, by this rope is moved, the larger the number of wheels that are in the block and tackle.

ON TIME. The larger the number of wheels, which forms the block and tackle, the faster the motion of the arganica rope than that of the ritenente rope.

ON Weight. The larger the number of wheels of block and tackle, the greater the supported weight than that which supports.⁵⁴

⁵² “Bilancia circolare chiamo la rotella ovver carrucola, colla quale si trae l’acqua de’ pozzi, colla quale non si leverà mai più peso che si pesi quello che attigne l’acqua. Nessuno corpo ponderoso leverà in bilancia circolare con forza del suo semplice peso più peso di sè medesimo.” (da Vinci, *Ms A*, 62r. See also da Vinci 1940, 104). English translation is ours.

⁵³ “La bilancia circolare, detta carrucola, essendo di tanta importanza nelli strumenti machinali (e massime nelle trasmutazioni delle forze), non è da preterire; con ciò sia che mediante quella si multiplla la potenza al motore delle dette machine, come si vede nelle taglie, dove tanto cresce la potenza, quanto cresce il numero di tal carrucole; adunque definiren la sua natura e potenza, e prima mostreremo come le corde senza moto sentano equal peso della gravita da lor sostenuto, e questo domanderen peso naturale; poi diren del moto, e che varia il peso che nelle corde si comparte e questo nomineren peso accidentale, cioè forza, la quale tanto si cresce, quanto più si fa veloce; ma il peso naturale mai si varia, variasi la potenza nel motore insieme colla resistenza della cosa mossa e della resistenza dell’aria, che si condensa e resiste, come fa l’aria alla ventola delli orologi.” (da Vinci, *Codex Atlanticus*, 566 [new numeration]. See also da Vinci 1940, 104). English translation is ours.

⁵⁴ “LA CORDA, che passa infra le taglie ai sua stremi, in due modi nominati, quella parte che dà causa al moto che si ferma all’argano, si nomina, arganica; e quella ch’è ferma alla superiore taglia, che non lascia scorrere nè cadere le taglie, è detta ritenente. DEL MOTO. Tante volte fia più lungo il moto della corda arganica che ‘l peso move, che non è il moto del peso, che, mediante le taglie, per essa corda è mosso, quanto è il numero delle rote che in esse taglie stanno. DEL TEMPO. Tanto quanto fia il numero delle rote, che nelle taglie stanno, tanto fia più veloce il moto fatto dalla corda atganica, che quello fatto dalla corda ritenente. DEL PESO. Quanto fia il numero delle rote delle taglie, tanto fia maggiore il peso sostenuto, che quello che sostiene.” (da Vinci, *Codex Atlanticus*, 882 [new numeration]. See also Vinci 1940, 496. Author’s italic). English translation is ours.

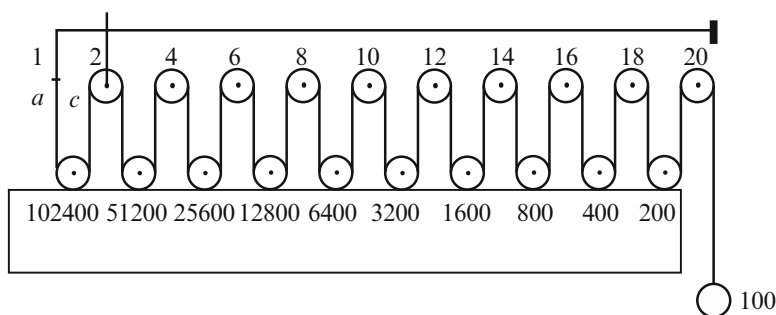


Fig. 2.14 An example of a large block and tackle (Redrawn from da Vinci, *Ms A*, 52r)

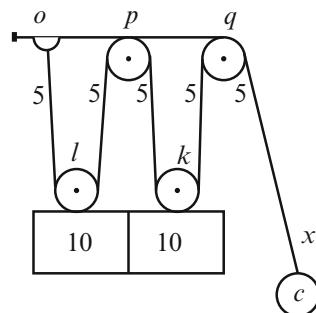
If you want to know the weight [the force] of the rope that supports the latest pulley, always multiply the applied weight at the bottom by the number of pulleys, and what this multiplication gives, be the number of pounds that the last rope receives of said weight attached at the bottom. Let thus, may the attached weight be 4, so you say: 4 pounds times 4 pulleys is 16 numbers, and then say: 4 times 16 is 64, and the rope it supports 64 pounds for the 4 applied by at the bottom, and if they were 6 pulleys, you would say: 4 times 6, 24, and 4 times 24, 98,⁵⁵ and this the weight that the last rope of 4 pounds attached at the bottom sustains.⁵⁶

No explicit rule is proposed but examples sufficiently clear are made, as typical in the mathematics of abacus. The explanation of the operation of the block and tackle sometimes seems that proposed in *Problemata mechanica* which calls for the law of lever (Aristotle [1936] 1955b, 852b, 367–370), sometimes that of Heron who assumed a constant stress in the ropes which encircles the pulleys and thus the whole weight lifted is given by the resultant of all the rope forces of the block and tackle. This type of reasoning is reported in the following quotation:

⁵⁵ It should be 96; 24 times 4.

⁵⁶ “Se voi sapere che peso ha la corda che sostiene l’ultima carrucola, moltiplica sempre cubicamente il peso appiccato da piè col numero delle carrucole, e quel che di tal moltiplicazione resulta, fia il numero delle libbre che tale ultima corda riceve di detto peso attaccato da piè. Diciamo adunque ch’esso peso attaccato da piè sia 4, onde tu dirai: 4 libbre vie 4 carrucole fa 16 numeri; e poi di: 4 vie 16 f. 64; ed è moltiplicato cubicamente, e essa corda di sopra sostiene 64 libbre per le 4 appiccate da piè; e se esse carrucole fussino 6, diresti: 4 via 6, 24, e 4 vie 24, 98; e tanto peso sostiene l’ultima corda delle 4 libbre attaccate da piè.” (da Vinci, *Codex Foster II*, 82v. See also da Vinci 1940, 501). English translation is ours.

Fig. 2.15 Model concerning the evaluation of the power necessary to lift a given weight by means of a block and tackle (Redrawn from da Vinci, *Ms A*, 62r)



If you want to supply the block and tackle of 4 ropes, which block and tackle has to lift 20 pounds [Fig. 2.15]. I say that the wheel *l* will support 10 pounds, and the wheel *k* will support 10, which are transferred to they higher supports, that is, *o* takes 5 pound from *l* and *p* also takes 5 from *l*, and 5 from *k*, and this same *k* will take 5 from *q*. And whoever wanted to win the 5 of *q*, put 6 into the counterweight *x*, and putting the last place 6 against 5 of each of the 4 ropes that support 20 pounds, not supporting itself more than 5 pounds, the one pound more that I put in the rope *qx*, find no resistance in the opposed ropes equal to it, all wins and all moves.⁵⁷

Note that Leonardo distinguishes motion from equilibrium and to obtain motion the power should be a little greater than the resistance; in the previous quotation 6 vs 5. Quite interesting is the Fig. 2.16. This is a situation that actually occurs in practice when the pull of the rope is relatively low compared to the friction.

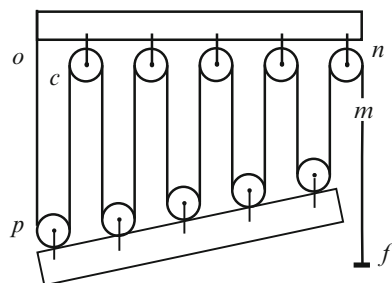


Fig. 2.16 Behaviour of a block and tackle for the effect of friction (da Vinci, *Codex Arundel*, 96r. On the friction in Leonardo da Vinci's studies see also the *Banco for studies on friction* (da Vinci, *Codice Arundel*, 40v–41r; da Vinci, *Ms L* 11v; see also Pisano 2009a, b, c, d, 2013a))

⁵⁷ “Se tu voi incordare le taglie in 4 doppi, le quali taglie abbino a le- vare 20 libbre di peso, dico che la girella *l* sosterrà 10 libbre, e 10 ne sosterrà la rotella *k*, le quali si trasferiscano a’ sua superiori sostentaculi, cioè *o* piglia da 1 5 libbre, e 5 ne piglia ancora *p* da *l*, e 5 da *k*, e questo medesimo *k* ne da 5 a *q*; e chi volessi vincere le 5 di *q* ne metta 6 nel contrappeso *x*, e mettendo in l’ultimo loco 6 contra 5 in ciascuna delle 4 corde che sostengono le 20 libbre, non sentendo per sè se non quelle 5 libbre, quella libbra più ch’io metto nella corda *gx*, non trovando in nessuna delle opposte corde pari peso a sè, tutte le vince e tutte le move.” (da Vinci, *Ms A*, 62r. See also da Vinci 1940, 499). English translation is ours.

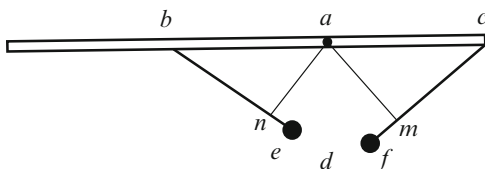
Leonardo also poses other problems in block and tackles, such as the way the stress in the rope varies with motion, the location where the rope is more stressed and thus where it breaks more easily, the effect of the diameter of the ropes on the effectiveness of the pulleys, the load carried by the supports of the pulleys. His comments are not always flawless, but are notwithstanding interesting to any readers, and are perhaps the most interesting of Leonardo's contributions to block and tackle theory.

2.1.2.3.5 The Concept of *Momento* of a Force

In presenting some of Leonardo's quotations, because of the uncertainty of dating, we attempted a *rational reconstruction*. According to this reconstruction Leonardo would have developed the idea of *potential arm* in his study on the equilibrium of levers, introducing the concept, if not the term of *moment of a force*. The potential arm of a lever for Leonardo da Vinci is both the distance between the line of action of a *power* from the fulcrum and the imaginary-material arm, orthogonal to the power, which could replace the real arm. Then he would have extended this concept to the study of the composition of forces. It is however possible, that there were not two distinct phases and the idea of potential arm was driven by the need to solve the problem of the composition of powers. Notice that Leonardo da Vinci to indicate what we commonly call force uses terms like power and weight, so we will do the same in the following.

The first time the idea of potential arm appears, according to our reconstruction, is in the study of the balance in which weights are suspended through pendants. In this situation, Leonardo assumes that the weights tend toward the centre of the world and then the pendants are not vertical but convergent (Fig. 2.17):

Fig. 2.17 Balance with converging pendants (Redrawn from da Vinci, *Ms A*, 62r)



Leonardo da Vinci is not explicit but everything suggests that the potential arms are those marked with *an* and *am* (See Fig. 2.17). The closer the balance is to the centre of the world *d*, the shorter they are.

Each of the arms of the balance is double; one of which is real, the other potential and they are located in different places with ends distant from each other.⁵⁸

⁵⁸ “Ciascuno de’ bracci della bilancia è duplo; de’ quali l’uno è reale e l’altro potenziale e son posti in diversi siti distanti con l’estremi l’un dall’altro, e son di varie lunghezze.” (da Vinci, *Codex Atlanticus*, 338 [new numeration]. See also da Vinci 1940, 70). English translation is ours.

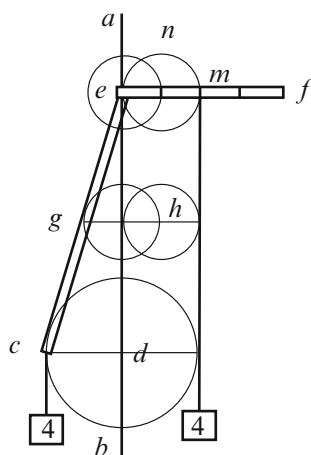
The real arms are always longer than the potential arms and the longer the closer to the centre of the world.

The real arms are not in the same proportion between them as the potential arms, but the more different the closer to the centre of the world.⁵⁹

Subsequently the idea of potential arm, although not explicitly named, is used in the study of equilibrium of an angular balance. In the following quotation the rule of angular balance is worded clearly enough and makes clear that equilibrium is determined by weights and their distances from the fulcrum measured horizontally.

RULE OF THE ANGULAR BALANCE. The angular balance is a balance for which the conjunction of its arms is angular; the pole being located in the angle. Arm means where the centre of suspended weight falls. The distances of the opposite ends of the angular balance from the central line of the pole have always the same proportion of the lengths of the arms of the balance, but with inverse order. Let us consider the angular balance *cef* [See Fig. 2.18] the pole of which is in the corner *e*; the opposite extremes *f* and *c*, have their distances from the central line *ab* in the same proportion of the length of the arms *c* and *f*, but converse: i.e. the smallest arm has its end farther from the centre as much as it is smaller than the greatest. And so the distance of the greatest arm from the central line, is as lower as its arm is greater than the lowest. Here the portions of circles are not equal to the motion of the arms, but in the distances from the central line.⁶⁰

Fig. 2.18 Leonardo da Vinci's angular balance (Redrawn from da Vinci, *Codex Arundel*, 32v)



⁵⁹ “Sempre le braccia reali della bilancia sono più lunghe di quelle potenziali e tanto più quanto esse sono più vicine al centro del mondo.” (da Vinci, *Ms E*, 64r. See also da Vinci 1940, 72). English translation is ours.

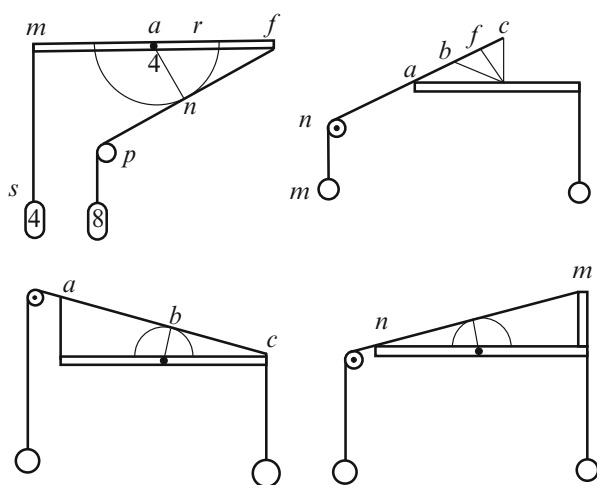
⁶⁰ “REGOLA DELLA BILANCIA ANGULARE. La equilibra angulare è quella della quale la congiunzione delle sue diritte braccia è angulare; nel quale angulo il suo polo è collocato. Braccio si intende dove cade il centro del peso appiccatovi. Sempre le distanzie che hanno li oppositi stremi della bilancia angulare dalla linia centrale del polo suo han nella medesima proporzione qual’è quella che hanno le lunghezze delle braccia d’essa bilancia infra loro; ma sia proporzione conversa. Come dire della bilancia angulare *cef*, de la quale il polo è nell’angolo *e*, che lo stremo *f* e *c* oppositi hanno nelle loro distanzie dalla linia centrale *ab* tal proporzione qual’è quella della

Notice the particular type of angular balance studied by da Vinci, made up of two arms with uniform section endowed with weight. The weight of the arms is concentrated in their centre of gravity; equilibrium requires inverse proportionality between the weights and the distances of their centres of gravity from the vertical line passing through the fulcrum. The idea for the study of this particular kind of balance is likely to have derived from Leon Battista Alberti's *Equilibra*.

Leonardo da Vinci continues his analysis of the angular balance extending the concept of potential arm to the case of straight levers in which weights are applied obliquely by means of ropes. Probably the most explicit statement regarding the introduction of the term potential arm occurs in some pages from which the drawings (See Fig. 2.19) are obtained. In particular, for the first drawing Leonardo da Vinci writes:

This is told the true end of the arm of the balance, the connection of which with the line of the rope, loaded by the weight, will be made according to the right angle as you can see in s with ma and similarly in pn with na (spiritual arm).⁶¹

Fig. 2.19 Instances of potential arms in various kinds of lever (From top to bottom and left to right: da Vinci, redrawn from *Ms H*, 40r, 50v, 39v, 39v)



lunghezza delle sue braccia ec e ef ; ma è conversa: cioè che 'l braccio minore ha il suo estremo tanto più discosto dalla centrale quant'egli è minor del suo maggiore. E così lo spazio, che ha il braccio maggiore da tale linea centrale, è tanto minore quanto il suo braccio è maggiore che 'l suo minore. Qui le porzion de' cerchi non sono equali nel moto de' bracci, ma sì nelle distanzie dalla linea centrale." (da Vinci, *Codex Arundel*, 32v. See also da Vinci 1940, 99). English translation is ours.

⁶¹ "Quello è detto vero termine di braccio di bilancia, il quale giungendo la sua retta colla rettitudine della corda, tirata dal peso, questa congiunzione sarà fatta componendo l'angolo retto come si vede in s con ma e similmente pn con na (braccio spirituale)." (da Vinci, *Ms M*, 40r. See also da Vinci 1940, 170). English translation is ours.

These figures (see Fig. 2.19) leave no doubt that at least from a certain period Leonardo considered the effectiveness of a power in order to equilibrate a balance, or more generally of a rigid body constrained to a point, a ‘circonvolubile’, determined only by the value of the powers and the distance of its line of action from the fulcrum. This is also the opinion of Duhem (1905–1906, I, pp 24–25) who however balances his positive opinion with the statement that Leonardo’s mechanical writings there are not essential ideas, which were not present in the writings of the mathematicians of the Middle Age.⁶² Duhem certainly refers to the fact that the idea of potential arm was contained *de novo* in the writings of Jordanus de Nemore. The latter in *Liber de ratione ponderis*⁶³ (de Nemore 1565, 6r) studied the case of an angled lever with equal weights, arguing and demonstrating that equilibrium is achieved when the two weights are at the same distance measured horizontally from the fulcrum. In another point, de Nemore also stated that the parameter determining the balance of a body is given by the horizontal distance measured from the fulcrum (de Nemore 1565, 10v). One could go further back and climb up to Heron and Archimedes who knew the law of angular balances (Capecchi 2012a, 53). However, Leonardo in our opinion has gone much farther. The argumentation of de Nemore on the angular lever, only referred to weights hanging from the balance, was based on the analysis of their descent and ascent. He could hardly have carried out his argumentation in the case of weights suspended from inclined ropes.

In common expositions of the history of mechanics, this discovery of Leonardo is often attributed to Giovanni Battista Benedetti in his *Diversarum speculationum mathematicarum physicarum et liber* (Benedetti 1585; see also Favaro 1900). This attribution can find a partial justification in the fact that Benedetti proved his result, albeit not entirely convincingly, and that the text of Benedetti had a wide circulation while Leonardo da Vinci’s has remained hidden to most.

2.1.2.3.6 The Law of Composition of Forces

Probably the most important of Leonardo da Vinci’s contributions to statics concerns the rule of composition-decomposition of a force along two given directions. The problem to be solved was to find the tensions of two inclined ropes supporting a weight. To remove any ambiguity, the forces of the ropes also were associated with weights.

⁶² Cfr.: Duhem (1905–1906, I, 192).

⁶³ As discussed in Chapter 1, this works belongs to Jordanus de Nemore and generally assigned to be edited by Tartaglia and posthumously published by Curtio Troiano in 1565.

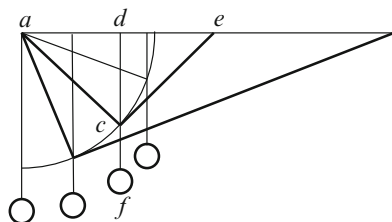
Da Vinci, besides formulating the rule, also correctly proved it. This fact is normally not recognized by historians and even Duhem suggested only as a possibility that Leonardo understood the rule; only Marcolongo asserted his priority with no doubt.⁶⁴ The analysis of texts has however led us to believe that in this case Marcolongo's analysis is correct and actually Leonardo recognized the rule of weight distribution in two ropes supporting a weight. There are of course, as typical in Leonardo, situations in which the rule is loosely worded, and sometimes wrongly. But, although there are no certain dating criteria, the analysis of the manuscripts shows a long series of examples with a lot of correct arguments that can leave no doubt that Leonardo reached a conscious knowledge of the rule of composition of forces (Capecchi 2012b).

The following quotations start from the intuitive finding that the weight distribution depends on the obliquity of the ropes.

ON WEIGHT. If two ropes converge to support a heavy body, one of which is vertical the other oblique, the oblique one does not sustain any part of the weight. But if two oblique ropes would support a weight, the proportion of weight to weight would be as the obliquity to obliquity. For ropes that descend with different obliquity from the same height, to support a weight, the proportion of the accidental weight of the ropes is the same as that of the length of these ropes.⁶⁵

From these passages it could be deduced that by the term *obliquity* Leonardo refers to the slope rather than to the length of the ropes – see the final part of the previous quotation – while the accidental weight could be understood as the tension of the ropes. The statement is patently incorrect, but one could think that Leonardo had become confused and meant to speak of the inverse ratio of obliquity, which is still wrong but at least the tendency is correct. The analysis of the following passage (See Fig. 2.20) shows however, that Leonardo's statement is not a typo, because he clearly states that the weight is divided into proportion of the angles formed by the ropes with the vertical, which is clearly false:

Fig. 2.20 A wrong instance of decomposition of forces (Redrawn from da Vinci, *Ms E*, 71r)



⁶⁴ Note that Duhem did not study the fundamental *Codex Arundel*.

⁶⁵ “DEL PESO. Se due corde concorrono alla sospensione d’un grave e che l’una sia diritta e l’altra obliqua, essa obliqua non sostiene parte alcun d’esso peso. Ma se due corde oblique concorreranno al sostenere d’un peso, tal proporzione fia da peso a peso, qual fia da obbliquità a obbliquità. Delle corde che da una medesima altezza che con diverse obbliquità discendano alla sospensione d’un peso, tal proposizione fia quella che a tal corda del peso accidentale si congiugne, qual’è quella delle lunghezze d’esse corde.” (da Vinci, *Ms E*, 70r. See also da Vinci 1940, 142). English translation is ours.

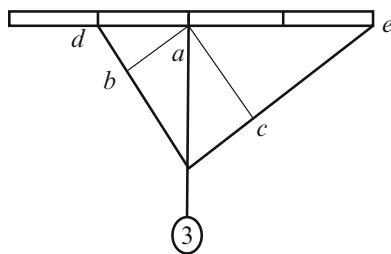
Let us consider two lines concurring in the angle which sustains the weight, if you draw the perpendicular which divides this angle, then the weights [tensions] of the two ropes have the same ratio as that of the two angles generated by the above division. If between the two lines ac and ec , which form the angle c , from which the weight f is suspended, the perpendicular dc is drawn that divides this angle into two angles acd and dfe , we say that these ropes will receive the weight in proportion equal to that of the two angles they form and equal to the proportion of the two triangles. And the perpendicular that divides the angle of this triangle will split the gravity suspended in two equal parts, because passing through the centre of such gravity.⁶⁶

It is difficult to understand how Leonardo could present so clearly wrong examples. Perhaps he is thinking of a weight hanging from the middle of a rope in which the greater the obliquity – i.e. the angle they form with the vertical – the larger the tensions in the rope.

Marcolongo (1937) argues, however, that these wrong results date back to the years before 1508, when Leonardo had not yet reached his final idea, which is well expressed in the passage:

For the 6th and 9th [propositions], the weight 3 [See Fig. 2.21] does not split into the two real arms of the balance in the same proportion of these arms, but in the proportion of the potential arms.⁶⁷

Fig. 2.21 A correct instance of decomposition of forces by Leonardo da Vinci (Redrawn from da Vinci, *Codex Arundel*, 1v)



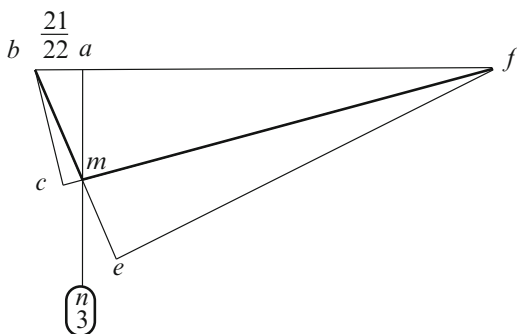
⁶⁶ “Quando dalla linea equigiacente discenderan due linee concorrenti all’angolo sospensore del grave, caderà la perpendicolare dividitrice di tale angolo, allora sarà diviso il peso alle due corde d’esso sospensore infra li quali pesi fia la medesima proporzione ch’è quella de’ due angoli, generata dalla predetta division del primo angolo; come se dalla equigiacente a e discendessi le due linee ac e ec, concorrenti alla composizione dell’angolo c, al quale angolo si sospenda il peso f, cadessi la perpendicolare dc dividitrice d’esso angolo in due altri angoli acd e dfe; dico che tale corde riceveranno il predetto peso in tal proporzione qual’è quella che hanno infra loro li due angoli predetti e qual fia la proporzione delle quantità de’ due triangoli infra loro. E sempre la perpendicolare dividitrice dell’angolo di tal triangolo sarà dividitrice della gravità sospesa in due parti equali, perché passa per il centro di tal gravità.” (da Vinci, *Ms E*, 71r; See also da Vinci 1940, 143). English translation is ours.

⁶⁷ “Per la 6° del 9°, il grave 3 non si distribuisce alle braccia reali della bilancia nella medesima proporzione che è quella d’esse braccia, ma in quella proporzione che hanno infra loro le braccia potenziali.” (da Vinci, *Codex Arundel*, 1v. See also da Vinci 1940, 171). English translation is ours.

Here Leonardo asserts, without proving it, that the suspended weight is supported by tensions b (left) and c (right) having inverse ratio to the potential arms ab and ac , i.e.: $b : c = ac : ab$. The relation, correctly, allows us to find the ratio of tensions in the two ropes.

In other passages, Leonardo proves the asserted relation and also indicates the way to evaluate the absolute value of the tension in each rope. He introduces the terms: *potential lever* and *potential counter lever* (See Fig. 2.22). The potential lever corresponds somewhat to the potential arm; the potential counter lever is the horizontal segment connecting one support of a rope to the vertical from the suspended weight. The reading of the following quotation is useful to illustrate the use of these terms. The potential lever associated to the arm fm is fe , the potential counter lever is fa .

Fig. 2.22 Example of a potential lever and potential counter lever (Redrawn from da Vinci, *Codex Arundel*, 7v)



Here the weight is sustained by two powers, i.e. mf and mb . Now we have to find the potential lever and counter lever of the two powers. The lever fe and the counter lever fa will correspond to the power mb . The appendix eb is added to the lever fe , which is connected with the engine b ; and the appendix ab is added to the counter lever fa , which sustains the weight n . By having endowed the balance with the power and the resistance of engine and weight, the proportion between the lever fe and the counter lever ab should be known. Let fe be $21/22$ of the counter lever fa . Then b supports 22 when the weight n is 21 .⁶⁸

⁶⁸ “Qui è il peso n sostenuto da due potenzie varie, cioè mf e mb . Ora mi bisogna trovare la lieva e contralievà potenziale d’esse due potenzie bm e fm . Delle quali alla potenza b sarà data la lieva fe e la contralievà fa . Alla lieva fe si dà l’appendiculo eb , al quale sta appiccato il motore b ; e alla contralievà fa si dà l’appendiculo an , che sostiene il peso n . Avendo ordinata la bilancia della potenza e resistenza del motore e peso, è necessario vedere che proporzione ha la lieva fe colla contralievà, fa . La quale fe è li $21/22$ della contralievà fa . Adunque b sente 22 , quando il peso n fusi 21 .” (da Vinci, *Codex Arundel*, 7v. See also da Vinci 1940, 179). The translation is ours.

Attention is centred on the rope bm with the aim to find its tension. A similar argument can be repeated for the rope fm . Basically Leonardo imagines the rope fm as ‘solidified’, i.e. as a rigid beam hinged at f . According to his embryonic concept of moment of a force, Leonardo asserts the validity of the following relation: $b : n = fa : fe$, where b is the tension of the rope bm and n is the suspended weight. He gives as an example $fa : fe = 21 : 22$; for $n = 21$ it results $b = 22$.

The previous quotation deserves some comments. First: the idea to solidify the rope anticipates what is commonly called the solidification principle, according to which if a body is in equilibrium its state is not perturbed by adding additional constraints. This principle has been used to study deformable bodies by many scientists, including Stevin, Lagrange, Cauchy, Louis Poincaré (1777–1859) and Duhem.

2.1.3 *Tartaglia’s Legacy. A Transition between Science of Weights and Modern Statics*

At the beginning of the XVI century there was in Italy a broad debate on the role of mathematics in the natural sciences as a result of the increasing use of mathematics in applications and the fact that mathematicians were beginning to give a distinct form of knowledge to their discipline; debate which became even more pressing in the second half of the century. While almost no one denied the fundamental role of mathematics in itself, not everyone agreed on the status of knowledge in regard to the physical world. The importance of the role of mathematics was certainly carried out by supporters of Platonist instances, which in addition to their diffusion through the humanist circles, found their support from a professional mathematician, Luca Pacioli, whose *Summa* (Pisano 2009a, 2013a) was read and appreciated by all the major mathematicians of the early XVI century, Tartaglia, Cardano, Giovanni Battista Benedetti (1530–1590), Federico Commandino (1509–1575). There were, however, even within Aristotelism advocates of the use of mathematics in physics, some who made reference to the Aristotelian theory of subalternate-sciences.

The second half of the XVI century saw the dissemination of Archimedean mathematical (and mechanical) work, which deeply modified the approach to mechanics. Though Archimedes work was influential everywhere, its stimulus was different in different regions. In the Northern school, formed by Benedetti, Tartaglia, Cardano, Archimedes texts received less attention than Jordanus de Nemore or *Problemata mechanica*. The contrary holds for the centre school, formed by Commandino, del Monte, Bernardino Baldi (1553–1617) and the southern formed by Francesco Maurolico (1494–1575), Nicola Antonio Stigliola (1546–1623) and Luca Valerio (1553–1618) (Gatto 1988, 1996, 2006; Galileo 2002; Nastasi 1985) (Table 2.4).

Table 2.4 Heron, de Nemore, Archimedes' texts published in Italy during the XVI century

	Title	Author
Heronian		
1501	<i>De expetendis et fugientis rebus</i>	Valla
1521	<i>Di Lucio Vitruvio Pollione de architectura libri dece traducti de latino in vulgare affigurati</i>	Cesariano
1550	<i>De subtilitate</i>	Cardano
1575	<i>Spiritualium liber</i>	Commandino
1588	<i>Mathematica collectiones</i>	Commandino
1589	<i>Gli artificiosi et curiosi moti spirituali</i>	Aleotti
1589	<i>Automata.</i>	Baldi
1581	<i>Pneumatica</i>	Baldi
1592	<i>Spirituali di Herone Alexandrino, ridotte in lingua volgare</i>	Giorgi
Nemorean		
1533	<i>Liber de ponderibus.</i>	Apianus
1546	<i>Quesiti et inventioni diverse.</i>	Tartaglia
1565	<i>Jordani opusculum de ponderositate</i>	de Nemore
Archimedean		
1543	<i>Opera Archimedis.</i>	Tartaglia
1544	<i>Archimedis Syracusani philosophi ac geometrae excellentissimi Opera</i>	Cremonensis
1551	<i>Archimedis de insidentibus aquae</i> (into Italian)	Tartaglia
1558	<i>Archimedis opera non nulla</i>	Commandino ^a
1570?	<i>Momenta omnia mathematica</i> (published 1685)	Maurolico
1565	<i>Archimedis De iis quae vehuntur in aqua libri duo</i>	Commandino ^b
1588	<i>In duos Archimedis aequponderantium libros paraphrasis</i>	del Monte

^aArchimedes 1558^bArchimedes 1565

2.1.3.1 Statics in Italy During the XVI Century

The medieval science of weights was not sufficient for the needs of the XVI century, because it was confined to a small number of cases, and because it was founded on principles not always shared. In the text of Jordanus de Nemore, *Liber de ratione ponderis*, the most advanced, except for various types of scales, only the inclined plane case is reported. Nothing is said about pulleys or aspects regarding situations of practical interest, such as for example, the horizontal transport of weights, which had also been addressed in the Aristotelian *Problemata mechanica*. Regarding the laws of equilibrium formulated in the *Liber de ratione ponderis*, only those of the lever were unanimously accepted while that of the inclined plane was not known or was not shared. Leonardo da Vinci, Girolamo Cardano, Guidobaldo del Monte, Stigliola, offered alternative solutions, unfortunately not correct.

The reworking of the science of weights carried out by the engineers of the XV century, including that of Leonardo da Vinci, was not sufficient to meet the new requirements of mathematical rigor and development of general laws that would have allowed going beyond the rigid schematism of the medieval statics. This

demand was picked up by a new generation of engineer-scientists, with a greater mathematical and philosophical training.

The first representative of this new generation was Niccolò Tartaglia. He gave a clear place to mechanics and introduced many ideas. In ballistics he asserted that the trajectory of a projectile is curved everywhere and nowhere, i.e., there are both straight and circular paths. He also stated that the maximum range of a projectile is obtained by firing with an inclination of 45° and that any intermediate distance may be covered by firing with two different angles. Moreover, he made clear, against the Aristotelian thesis, that the air is an impediment and does not aid motion. He was the ultimate champion of the science of weights adding mathematical rigour to traditional presentations. Starting from a manuscript of de Nemore's *Liber de ratione ponderis* in his possession (de Nemore 1565), he wrote an important section, the book VIII, of his treatise *Quesiti et inventioni diverse* where he revisited in a more organic way Jordanus de Nemore's theory. Nevertheless mainly Tartaglia was the first to use mathematics as the fundamental theoretical tool in the study of mechanical and physical problems, as it will be manifest in the subsequent chapter. Tartaglia, although according to the Aristotelian epistemology conceived mathematical objects as abstracted from matter, assumed that conclusions derived from mathematics are 'true' and should necessarily be verified from an empirical point of view. If that were not the case it would not depend on mathematics but on experience, which was not well exploited.

After Tartaglia, and somehow their heirs, Giovanni Battista Benedetti, Guidobaldo del Monte and Galilei follow. Benedetti made (Maccagni 1967) important contributions to the analysis of natural motion of bodies (see also Borelli 1686a, b; Drabkin 1964). In statics, he made clear and universally known that the effect of a force depends on the distance of its line of actions from the fulcrum, results that now historians call the law of static moment.

Guidobaldo del Monte attempted the restoration of Greek mechanics in the spirit of Pappus Alexandrinus, whose work was published by Federico Commandino, basing it on an Archimedean mechanical approach (Palmieri 2008). He attempted however, a synthesis with the Aristotelian approach of subalternate-science in which physical aspects were clearly present. For example, when studying the balance, he treats of a physical body and not simply a geometrical figure, giving substance also to the fulcrum, which for Archimedes was a simple geometrical point. Del Monte's mechanics was not only a science of the principles of equilibrium of weights on a balance. It was rather a science of machines, Greek meaning; and, even if the equilibrium was crucial as well, the role of the displacement of bodies was examined.

Galileo, well known as the founder of modern dynamics (Drake 1990; Grant 1965, 1996; Grant and Murdoch 1987), also made fundamental contributions to statics, somehow managing to reconcile the medieval science of weights, with references to kinematics, with Archimedes' mechanics, purely geometric. However, it was not a true synthesis because he flanked medieval methodologies alongside Archimedean ones without making a decisive choice of field, while expressing a preference for the Archimedean approach.

Below we list a few details about the contribution to statics of the authors mentioned above, highlighting the legacy of Niccolò Tartaglia, whose contribution has been and will be studied in depth in other chapters.

2.1.3.1.1 Giovanni Battista Benedetti

Giovanni Battista Benedetti received his first and only systematic education in philosophy, music and mathematics from his father. Though never mentioned by Tartaglia, Benedetti was nevertheless one of his pupils for a short time. In mechanics, his chief work was the *Diversarum speculationum mathematicarum et physicarum liber* of 1585 (Benedetti 1585). The book deals largely with questions of dynamics; there were however fundamental contributions to statics. Here a concept of static moment of a force, more precisely defined than Leonardo's, is referred to. Though the *Diversarum speculationum mathematicarum et physicarum liber* may be considered a commentary on the *Problemata mechanica*, Benedetti's approach was essentially Archimedean. He criticised both Tartaglia and de Nemo for their kinematic analysis.

The Concept of Static Momento

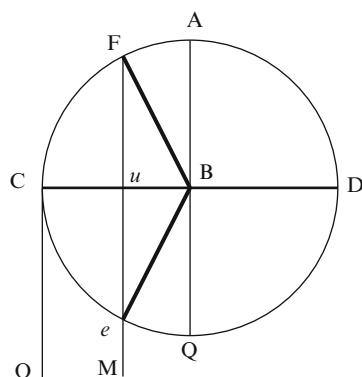
In *Diversarum speculationum mathematicarum et physicarum liber* (Benedetti 1585, Chapter 3, section *De mechanicis*, 141–167) Benedetti made considerations of quantitative character about the effect of a force – associated to a weight attached to a rope or a muscle – on an arm of a balance, however inclined, obtaining the result that it is proportional to the distance of the line of action of the force from the fulcrum and to the force itself. This was at the time an already known result, but Benedetti for the first time formulated this fact as a general law, that now historians call the law of static moment. The main difference between Leonardo and Benedetti's static moment laws does not so much concern the generality of the law but the reference, for Benedetti, to a proof, or at least an intuitive justification.

First Benedetti's argument is developed for the not problematic case of vertical forces:

From what we have already shown it may easily be understood that the length of Bu [Fig. 2.23], which is virtually perpendicular from centre B to the line of inclination Fu , is the quantity that enables us to measure the force of F itself in a position of this kind, i.e., a position in which line Fu constitutes with arm FB the acute angle BFu .⁶⁹

⁶⁹ “Ex iis quae nobis hucusque sunt dicta, facile intelligi potest, quantitatis $B.u.$ quae fere perpendicularis es a centro $.B.$ ad lineam $.F.u.$ inclinationis, ea est, quae non ductis in cognitionem quantitatis virtutis ipsius F in huiusmodi situ constituens videlicet linea $.F.u.$ cum brachio $.F.B.$ angulum acutum.” (Benedetti 1585, 142–143. See also Drake and Drabkin 1969, 169). Drake and Drabkin's translation.

Fig. 2.23 Evaluation of the static moment of a weight
(Redrawn from Benedetti 1585, 142–143)



Then the argument is referred to forces or weights, which act along inclined directions, an argument that— even if unequivocal — in substance does not appear entirely convincing.

To understand this better, let us imagine [Fig. 2.24] a balance *boa* fixed at its centre *o*, and suppose that at its extremities two weights are attached, or two moving forces, *e* and *c*, in such a way that the line of inclination of *e*, that is *be*, makes a right angle with *ob* at point *b*, but the line of inclination of *c*, that is *ac*, makes an acute angle [Fig. 2.24, on the left] or an obtuse angle [Fig. 2.24, on the right] with *oa* at point *a*. Let us imagine, then, a line *ot* perpendicular to the line of inclination *ca*. [. . .]. Imagine, then, that *oa* is cut at point *i*, so that *oi* is equal to *ot*, and that a weight is suspended at *i*, equal to *c* and with a line of inclination parallel to that of weight *e*. But we assume that the weight or force *c* is greater than *e* in proportion as *bo* is greater than *ot*. Obviously, then, according to Archimedes, *De ponderibus*, *boi* will not move from its position. Again, if in place of *oi* we imagine *ot* rigidly connected [in the same line] with *ob* and subjected to force *c* acting along line *tc*, the result will obviously be the same, *bot* will not move from its position.⁷⁰

⁷⁰“Ut hoc tamen melius intelligamus, imaginemur libram .b.o.a. fixam in centro .o. ad cuius extrema sint appensa duo pondera, aut duae virtutes moventes .e. et .c. ita tamen, linea inclinationis .e. idest .be. faciat angulum rectum cum .o.b. in puncto .b. linea vero inclinationis .c. idest .a.c. faciat angulum acutum, aut obtusum cum .o.a. in puncto .a. Imaginemur ergo lineam .o.t. perpendicularem lineae .c.a. inclinationis [. . .] secetur deinde imaginatione .o.a. in puncto .i. ita ut .o.i. aequalis sit .o.t. & puncto .i. appensum sit a pondus aequale ipsi .c. cuius inclinationis linea parallela sit linea inclinationis ponderis .e. supponendo tamen pondus aut virtutem .c. ea ratione maiorem esse ea, quae est .e. qua .b.o. maior est .o.t. absque dubio ex 6 lib. primi Archi. de ponderibus .b.o.i. non movebitur situ, sed si loco .o.i. imaginabimur .o.t. consolidatam cum .o.b. & per lineam .t.c. attractam virtute .c. similiter quoque contingent ut .b.o.t.; communi quadam scientiam, non moveatur situ.” (Benedetti 1585, Chapter 3, p 143. See also Drake and Drabkin 1969, 169–170). Drake and Drabkin’s translation.

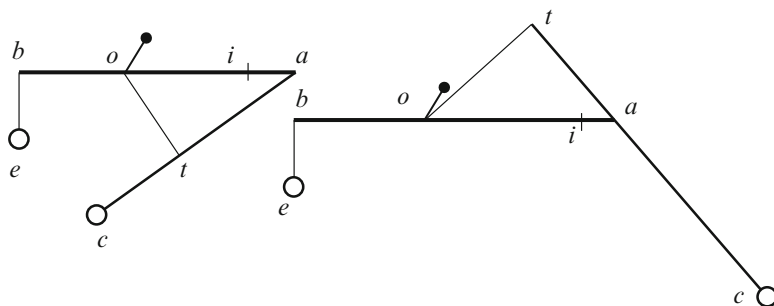


Fig. 2.24 Evaluation of the static moment for inclined forces by Benedetti (Redrawn from Benedetti 1585, 143)

Benedetti's Criticisms of Tartaglia

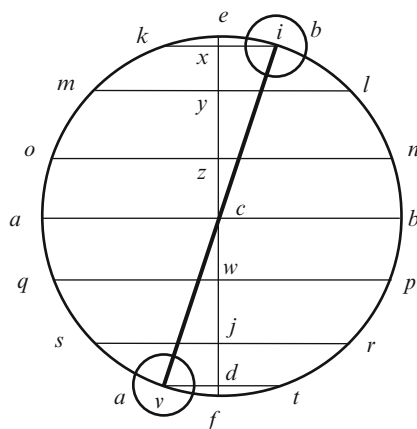
Benedetti knew Tartaglia's work very well, considering that for some time he was his pupil, and was surely influenced by him. Though only one generation younger, Benedetti's approach to mechanics is very different from Tartaglia's. As Tartaglia, he assumes mathematics at the foundation of mechanics, but he has a different cultural background; different because he received an education in philosophy, different because he became acquainted with Archimedes' mathematics and physics (Drachmann 1967–1968), also thanks to Tartaglia's editorial work. Benedetti is completely outside Aristotelianism. He fights against Aristotle in all his physical assumptions: on the existence of voids; on the law of fall of heavy bodies, on the nature of forced motion and so on. He is also outside the medieval science of weights, which interests him only for marginal aspects. From Archimedes he derived a greater attention to rigour in mathematical proofs but he also renounced the important resource that Tartaglia had: the algebraic calculus to solve geometrical and mechanical problems.

In Chapter VII of the section *De mechanicis*, Benedetti refers some criticisms toward Tartaglia's consideration on the science of weights (that, he specified, were partially "[...] taken from a certain ancient writer Jordanus [...]]" (Benedetti 1585, VII, 148). Benedetti's criticism refers both to general assumptions and to defects in the exposition of the matter. He criticizes⁷¹ in particular the proof of Tartaglia's Propositions III–VIII of the *Quesiti et inventioni diverse* (Tartaglia 1554, *Book VIII*, Qs XXX–XXXI Propositions III–VIII, 87rv–88rv), commenting that Archimedes had proved it more properly (Archimedes 2002, *Book I*, Propositio VI, 192–194).

⁷¹ Benedetti really started his criticisms by with comments on Tartaglia's proposition II concerning his errors in external resistance on motion (Tartaglia 1554, *Book VIII*, Quesito XXIX [in the book "XIX" is wrongly reported], Propositione II, 86rv–87r).

More important, for us, is the criticism about Proposition V (Tartaglia 1554, *Book VIII*, Q XXXII, Proposition V, 88v–89rv) on the equilibrium of the balance with equal arms and weights in the *Quesiti et invention diverse*. Differently from Tartaglia (and de Nemo) who considers the tendency of both the two weights hanging from the opposite sides of the balance to go down, he assumes the congruent situation for which while one weight descends the other ascends. In such a case, he notices, the path along the vertical is the same for the two weights, so the balance is in equilibrium whichever its inclination is:

Fig. 2.25 Equilibrium of balance with equal weights and arms according to Tartaglia (Redrawn from Tartaglia 1554, *Book VIII*, Q XXXII, Proposition V, 90v)



And in the second part of the fifth proposition he [Tartaglia] fails to see that no difference in weight is produced by virtue of position in the way in which he argues. For if body *b* must descend on arc *il*, body *a* must ascend on arc *vs*, equal and similar to arc *il* and placed in the same way. Therefore, just as it is easy for body *a* to ascend on arc *vs* it is easy for body *b* to descend on arc *vs*. And this fifth proposition is the second proposed by Jordanus [de Nemo 1565, Quaestio secunda, 3v–4r].⁷²

One more criticism, that will be made again by Guidobaldo del Monte, concerns the cause for which the tendency to descend of a body suspended from a hinged rod decreases with its inclination. According to Benedetti the cause of this fact is the greater resistance the weight receives from the rod and the fulcrum – mechanical cause.⁷³ According to Tartaglia (and de Nemo) it depends on a lower facility of

⁷² “Sed in secunda parte quinte propositionis non videt vigore situs eo modo, quo ipse disputat, nulla elicitur ponderis differentia quia si corpus .B. descendere debet per arcum .IL. corpus .A. ascendere debet per arcum .VS. Haec autem quinta propositio Tartalea est secunda quaestio a Iordano proposita.” (Benedetti 1585, VII, 148. Drake Drabkin’s translation 1969, 174–175). The figure in the text belongs to Tartaglia (Tartaglia 1554 *Book VIII*, Q XXXII, Propositione V, 89v; see also de Nemo 1565, 3r–5r) since Benedetti did not report it in his Chapter VII of the section *De mechanicis* (Benedetti 1585, VII, 148–149).

⁷³ Benedetti (1585, VII, 147–148).

descent as a kinematic constraint. Other criticisms seem to us simply a way to quibble to show his superiority. As when Benedetti criticizes Tartaglia for having considered as parallel the lines of the descents of heavy bodies, while he himself in some situation does the same, or when he blames Tartaglia for not having considered the resistance due to medium on motion (virtual) of weights hanging from a balance.

2.1.3.1.2 Guidobaldo del Monte

Guidobaldo del Monte attended the university of Padova in 1564 as along with a companion Torquato Tasso (1544–1595). He studied mathematics with Federico Commandino and was a teacher of Bernardino Baldi. He was one of the greatest mathematicians and mechanicians of the late XVI century. In 1577 he published the *Mechanicorum liber* (del Monte 1577, 2010, 2013), translated into Italian vulgare by Filippo Pigafetta in 1581 as *Le mecaniche* (del Monte 1581). The book had an enormous editorial success and was read throughout the whole XVII century.

Del Monte's Criticisms of Tartaglia and Jordanus de Nemore

Del Monte was one of the major critics of the approach of Jordanus de Nemore and Niccolò Tartaglia. According to him those of de Nemore and Tartaglia, are not valid demonstrations and goes so far as to say that de Nemore should not even be counted among the true mathematicians. Bernardino Baldi went still further and considered as paralogisms the demonstrations of de Nemore (Baldi 1621, 32). Del Monte, like Benedetti, knew Tartaglia's work very well and does not share his position. Like Benedetti, and differently from Tartaglia, he individuates the lower tendency to go down of heavy bodies suspended from a more inclined arm in the greater resistance the weight receives from the rod and the fulcrum (mechanical cause). Other criticisms, very often repeated, concern the approximation adopted by Tartaglia for the lines of descent of heavy bodies.

Criticisms of del Monte must be placed in his time to be understood. As noted in the introduction of this section, scholars of mathematics of the period, particularly those of Centre and South Italy, could not fail to be charmed by the elegance and rigor of geometry as it was revealed by the recently published Greek translations of Euclid and Archimedes. Archimedes, flanking his mathematical theory, developed a consistent mechanical theory with the same standards of rigor.

It was therefore natural to accept the argument of Archimedes in mechanics and reject those by de Nemore. Although to a modern observer, the full refusal of de Nemore seems unjustified because the *Liber de ratione ponderis* has an Euclidean approach based on definitions, axioms and theorems: it is certainly the ancient text in which the Euclidean approach is extended further outside geometry, in the wake of subalternate sciences. It is overall a very modern text. Del Monte, however, could hardly accept to reason with concepts such as gravity of position, which remained a bit undefined.

Given that de Nemore's and Tartaglia theses were then quite common in Italy, del Monte somehow felt the need to re-establish the *truth*, by writing the *Mechanicorum liber Archimedis aequeponderantium* and the *Mechanicorum liber* (del Monte 1577) that can be seen as the natural completion of the work of spreading Archimedes' mechanical thought. The hostility towards the approach of de Nemore also led del Monte to refuse the correct proof of the inclined plane for the incorrect one by Pappus of Alexandria. However one can show that del Monte was not as strict an Archimedes' follower as normally accepted. His mechanics is less abstract than Archimedes', and if he refused the concept of gravity of position because of its physical pregnancy, he contaminates the Archimedean approach. For instance he gives a material consistence to the fulcrum of the lever (which for Archimedes was a simple geometrical point), which is also capable of delivering forces; he gives a physical definition to the centre of gravity; he used the concept of muscle force. Although the *Mechanicorum liber* on the one hand had given up the fertility of de Nemore's approach, based on the concept of gravity of position and a law of virtual work, playing in some way a conservative role, it expanded the scope of mechanics. The medieval science of weights, in which attention was focused on demonstrating the law of the lever, is led back to the Greek tradition of mechanics as a science of machines, influenced in this by the *Problemata mechanica*, but especially by Heron's approach, then known only through the work of Pappus of Alexandria just translated by Commandino (1588; see also *Id.*, 1970).

The Balance with Equal Weights and Arms

In order to show del Monte's way of reasoning, below we report a summary of the way he studies the equilibrium of the balance with equal arms and weights. Proving this balance is in a position of indifferent equilibrium is a crucial point for him. In fact if that could not be the case the whole Archimedean building of centrobaric would collapse, because the fourth proposition of Archimedes' *Aequiponderanti* (Archimedes 2002, 191), according to which two equal weights have their centre of gravity in the middle of the segment joining them, would be false. Indeed if and only if the balance with two equal weights is in an indifferent situation of equilibrium its fulcrum – the middle point – coincides with the centre of gravity of the two weights and Archimedes' proposition is verified. Del Monte's strategy to defend the Archimedean centrobaric is twofold. In the first step he 'proves' that Tartaglia and de Nemore's result, for which the balance will revert to its horizontal position when disturbed (stable equilibrium) is false. The proof is carried out by provisionally accepting the concept of gravity of position but assuming the convergence of the lines of descent of heavy bodies (that for de Nemore were parallel to each other). From Fig. 2.26 it is clear that the weight in E has a greater gravity of position than the weight in D (its descent is less oblique, for the angle between the line of descent ES and the path of E – the tangent in E to the circle FBGA – is less than the angle of LS and the descent of D); thus the balance rotates until it will reach the vertical configuration FS (unstable equilibrium). After having falsified de Nemore's results on his own ground, in the second step del Monte leaves the concept of gravity of

places [...]. Thus the descent of the weight at D will be equal to the rise of the weight at E, and the weight at D will not raise the weight at E. From which it follows that the weights at D and E, considered in conjunction, are equally heavy.⁷⁴

It is one thing, he says, to consider the weights in D and E separately, in which case they would move toward the centre of the world S along DS or ES respectively, the other is to consider them together, so their centre of gravity would move to S along CS, while the weights in D and E along DH and EK, as shown in Fig. 2.26. But since C cannot sink, the weights remain at their place, D and E.

Del Monte claims to have verified empirically the indifference of equilibrium. And if the result of some scholar does not correspond to his theory it is because the experiment was not well executed and there were differences between the ideal and real situations (Thorndike 1923–1958). The following excerpt from the Italian version of the *Mechanicorum liber*, clearly expresses del Monte's ideas:

[...] that being the balance supported in its center by gravity it still remains wherever it is, which effect in particular has no longer been expressed by anyone, save only by the author. Indeed so far it was considered false, and impossible to put by all our predecessors; who with many reasons have endeavored to prove not only the opposite, but also have said for sure, that experience shows the scale never stops except when it is equally distant from the horizon. This thing is contrary to all reason, first, to be the demonstration of such fourth proposition as clear, simple, and true, and I do not know, how it can be contradicted, and then the experience which the author did with very finely balances, right on purpose to clarify this truth, one of which I have seen in the hands of the Illustrious Mr. Vincenzo Pinello,⁷⁵ sent to him by the author himself, which supported from the center of its gravity,

⁷⁴ “DELLA BILANCIA. [...]. Se dunque il peso posto in E è più grave del peso posto in D, la bilancia DE non starà giamai in questo sito, la qual cosa noi habbiamo proposto di mantenere, ma si moverà in FG. Alle quali cose rispondiamo che importa assai, se noi consideriamo i pesi ovvero in quanto sono separati l'uno dall'altro, ovvero in quanto sono tra loro congiunti: perche altra è la ragione del peso posto in E senza il congiungimento del peso posto in D, et altra di lui con l'altro peso congiunto, si fattamente che l'uno senza l'altro non si possa muovere. Imperoche la diritta, et naturale discesa dal peso posto in E, in quanto egli è senza altro congiungimento di peso, si fa per la linea ES, ma in quanto egli è congiunto col peso D, la sua naturale discesa non sarà più per la linea ES, ma per una linea egualmente distante da CS percioche la magnitudine comporta de i pesi ED, et della bilancia DE il cui centro della gravezza è C, se in nessun luogo non sarà sostenuta, si muoverà naturalmente in giù nel modo che si trova, secondo la grandezza del centro per la linea diritta tirata dal centro della gravezza C al centro del mondo S, finche il centro C pervenga nel centro S [...]. Ma se i pesi posti in ED sono l'un l'altro fra se congiunti, et gli considereremo in quanto sono congiunti, sarà la naturale inclinazione del peso posto in E per la linea MEK, percioche la gravezza dell'altro peso posto in D fa sì, che il peso posto in E non gravi sopra la linea ES, ma nella EK. Il che fa parimente la gravezza del peso posto in E, cioè, che'l peso posto in D non gravi per la linea reta DS, ma secondo DH impedirsi ambedue l'uno l'altro, che non vadino a propri luoghi [...]. Adunque il peso posto in D non moverà in su il peso posto in E. Dalle quali cose segue che i pesi posti in DE, in quanto tra loro sono congiunti, sono egualmente gravi.” (del Monte [1581] 1615, 34–36. See also Drake and Drabkin 1969, 281–282). Drake and Drabkin's translation.

⁷⁵ Gian Vincenzo Pinelli (1535–1601), erudite Neapolitan and bibliophile man, was a friend of Galileo.

moved in any position and then left, stops at every point it comes left. It is true, in making this experience, that we must not striving so to rage, for it is something very difficult, as the author says above, to make a scale, which is supported precisely in the middle of its arms at the center of its own gravity.⁷⁶

2.1.3.1.3 Galileo Galilei

Galileo Galilei was born in Pisa in 1564 and died in Arcetri (Florence) in 1642. In Pisa he undertook the study of mathematics under the guidance of Ostilio Ricci (1540–1603), a pupil of Niccolò Tartaglia. In 1638 he published *Discorsi e dimostrazioni matematiche sopra due nuove scienze* (Galileo 1638; Koyré 1996; Drake 1999, 2000; Galilei 1914; Banfi 1966).

The contribution Galileo provided to statics is far less decisive than that to dynamics, nonetheless it is important. Though there may be doubts on the originality of some of his writings, it is certain that no one before him had formulated and solved his own problems with extraordinary clarity. Differently from Benedetti and del Monte he does not disdain the science of weights and maintains for some important respects its kinematic approach. In his first important writing in statics, *Le mecaniche*,⁷⁷ which is related in part to del Monte's *Mechanicorum liber*, Galileo prevalently adopts an Archimedean approach (Galilei 1585; 2009a, b) and presents an elegant proof of the law of lever, based on purely geometrical arguments. He then reduces all the simple machines to the lever, including the inclined plane (Palmieri 2011) which escaped Guidobaldo del Monte. Nevertheless, the Archimedean approach is flanked by the kinematic approach both for the lever and inclined plane laws. The kinematic approach will become dominant for the problem of equilibrium in the subsequent works: *Discorso intorno alle cose che*

⁷⁶ “[...] che essendo la bilancia sostenuta nel suo centro dalla gravezza sta ferma dovunque el la si trova, il quale effetto in particolare non è piu stato tocco, ne veduto, ne man co da niuno manifestato, fuor che dall'autore: anzi fin hora tenuto falso, & impossibile da tutti gli predecessori nostri; i quali con molte ragioni si sono sforzati di provare non solamente il contrario, ma hanno etiandio affermato per certo, che la sperienza mostra la bilancia non dimorare gia mai ferma se non quando ella è egualmente distante dall'orizzonte. La qual cosa in tutto è contraria alla ragione prima, per essere la dimostrazione della sudetta quarta propositione tanto chiara, facile, & vera, che non sò, come se le possa in modo alcuno contradire: & poi all'esperienzia concio sia che l'autore habbia fatto sottilissimamente lavorare bilancie giuste a posta per chiarire questa verità, una delle quali hò io veduto in mano dell'Illustre Signor Gio. Vincenzo Pinello, mandatagli dall'istesso autore, la quale per essere sostenuta nel centro della sua gravezza, mossa dovunque si vuole, & poi lasciata, sta ferma in ogni sito dove ella vien lasciata. Ben è egli vero, che non bisogna, nel fare cotesta esperienza, correr così a furia, per essere cosa oltra modo difficile, come dice l'autore di sopra, il fare una bilancia, la quale sia nel mezzo del le sue braccia sostenuta à punto, & nel centro proprio della sua gravezza.” (del Monte [1581] 1615, 56). Drake and Drabkin's translation.

⁷⁷ In the 1593–1594 the early manuscripts was and first printed in a French version by Mersenne (Galilei 1634; See also Festa and Roux, *forthcoming*). It was published into Italian (1649) after the death of Galileo (Galilei 1649; for completeness see also: Galilei 1610, 1632, 1656. About Galilei's Opere (Works) see: Galilei 1846–1856, 1890–1909c, 1888–1905, 2005; recently on Galileo and Hobbes see Jessephe 2004).

stanno in su l'acqua e scritte varie, printed in 1612 (Galilei 1612 in Galilei 1888–1905, IV) and the already cited *Discorsi e dimostrazioni matematiche sopra due nuove scienze*.

The Galilean Concept of Momento

In *Le mecaniche*⁷⁸ Galileo introduced a concept and a term, that of *moment* (*momento*), that will be of great fortune and adopted, at least in Italy, until the early nineteenth century. The concept, formulated in *Le mecaniche*, was taken up and elaborated in the *Discorso intorno alle cose che stanno in su l'acqua* (Galilei 1612):

Moment for mechanics, means that virtue, that force, that effectiveness with which the motor moves and the *mobile resists* [emphasis added], virtue which depends not only on the simple gravity, but on the speed of motion, from the different angles of the spaces over which the motion is made, because a heavy body makes more impetus in a very inclined space than in one less inclined. The second principle [the first was that equal weights with equal speed have equal forces and moments] is, that the moment and the force of gravity is increased by the speed of motion so that absolutely equal weights, but combined with unequal velocities, are of force, moment and virtue unequal, and the fastest is more powerful, according to the proportion of its speed to the speed of the other. Of this we have very suitable example in the balance with unequal arms, where absolutely equal weights do not press and are not equally strong, but that which is at the greatest distance from the centre, around which the balance moves, sinks and rises the other, and it is the motion of the ascending fast, the other slow: and such is the force and virtue that the speed of motion gives to the mobile that receives, and it can compensate as much weight is added to the other mobile; so that if one arm of a balance were ten times longer than the other, in order to move the balance around his middle, the end of that passed ten times more space that the end of this, a weight placed at the greater distance can sustain and equilibrate another ten times heavier than it is, and this because, moving the balance, the lower weight will move ten times faster than the other.⁷⁹

⁷⁸ In 2014 Galileo's anniversary is celebrated. 1564–2014. *Homage to Galileo Galilei. History and Historical Epistemology of Sciences within Iuvenilia-Early Galilean Works*. It is a Special issue of *Philosophia Scientiae* (21/1: February 2017). Raffaele Pisano and Paolo Bussotti Guest editors.

⁷⁹ “Momento, appresso i meccanici, significa quella virtù, quella forza, quella efficacia, con la quale il motor muove e ‘l mobile resiste; la qual virtù dipende non solo dalla semplice gravità, ma dalla velocità del moto, dalle diverse inclinazioni degli spazii sopra i quali si fa il moto, perché più fa impeto un grave descendente in uno spazio molto declive che in un meno. Il secondo principio è, che il momento e la forza della gravità venga accresciuto dalla velocità del moto: sì che pesi assolutamente eguali, ma congiunti con velocità diseguali, sieno di forza, momento e virtù diseguale, e più potente il più veloce, secondo la proporzione della velocità sua alla velocità dell'altro. Di questo abbiamo accomodatissimo esempio nella libra o stadera di braccia disuguali, nelle quali posti pesi assolutamente eguali, non premono e fanno forza egualmente, ma quello che è nella maggior distanza dal centro, circa il quale la libra si muove, s'abbassa sollevando l'altro, ed è il moto di questo che ascende, lento e l'altro veloce: e tale è la forza e virtù che dalla velocità del moto vien conferita al mobile che la riceve, che ella può compensare altrettanto peso che all'altro mobile più tardo fosse accresciuto; sì che, se delle braccia della libra uno fosse dieci volte più lungo dell'altro, onde nel muoversi la libra circa il suo centro, l'estremità di quello passasse dieci

From the reading of passages quoted above it is clear that Galileo espoused the view that the downward velocity of a heavy body increases its efficacy or *force* to go down while the upward velocity increases its resistance to be lifted. His conception is rather uncommon in statics and differed from del Monte and Benedetti's who instead believed that there was no increase of 'force' due to velocity, but only a greater velocity due to lower resistance of constraints. It also differs from Tartaglia's who equally saw an increase of gravity but justified because of a virtual – determined by the kinematics – not real velocity. Galileo specified that moment is also the resistance to gain speed.⁸⁰ Therefore, the equilibrium is not from the equality of two trends to go down, but from the balance of the impetus to go down and the resistance to go up, both increased by the speed.

In *Le mecaniche*, after having proved the law of the lever according to Archimedes and similarly to what he will do in the first day of the *Discorsi*, Galileo examined the equilibrium of the lever using the concept of moment. The principle he invoked for the equilibrium is the equality of moments. He stated that this principle could be deduced from the *Problemata mechanica* (Galileo [1612] 1888–1905, IV, 275). Nevertheless, that is probably a rhetorical artifice only and he more simply took his inspiration from the science of weights tradition and perhaps from Tartaglia.

About the origin of Galileo's concept of moment many pages have been written; for a historical reconstruction philologically based, reference can be made to Paolo Galuzzi (Galluzzi 1979). It seems, however, that a reconstruction based on similarity of concepts is of more interest to us. This obviously can lead only to demonstrate the possibility and not the need – but even an accurate historical reconstruction is not necessarily conclusive. There is no doubt that the concept of moment in Galileo has some similarities with that of gravity position in de Nemo, and that some of its connotations are also present in the Aristotelian *Problemata mechanica*. It seems, however, that apart from these rather general similarities Galileo could have found some more specific ideas in the writings of Tartaglia and Benedetti. The idea that led Galileo to express the moment as proportional to the (virtual) velocity could have come from the proposition IV of *Book VIII* of *Quesiti et inventioni diverse* (Tartaglia 1554, *Book VIII*, Q XXXI Propositione IIII, 88rv) which says that the gravity of the position of a heavy body, suspended from a lever, grows linearly with its distance from a fulcrum. As the speed increases linearly with distance, it is natural for a reader of Tartaglia to imagine the gravity of position (and then the moment) as proportional to speed. From Benedetti, Galileo

volte maggiore spazio che l'estremità di questo, un peso posto nella maggiore distanza potrà sostenerne ed equilibrarne un altro dieci volte assolutamente più grave che non egli è; e ciò perché, muovendosi la stadera, il minor peso si moveria dieci volte più velocemente che l'altro." (Galilei's *Discorsi intorno alle cose che stanno in su l'acqua e o che in quella si muovono* (1612) in Galilei 1888–1905, IV, pp 68–69). The translation is ours.

⁸⁰ On proportion theory and force-resistance-and-velocity see Bradwardine 1955 and recently Rommevaux 2013).

may have drawn the idea that in the study of the equilibrium of heavy bodies one must consider motions congruent with each other. In the case of the balance, the congruence of the motions implies that when a weight drops the other raises. From this Galileo's idea to consider moment proportional to speeds even in the case of upward motion would have come up (see Benedetti above).

Inclined Plane Law

Today the inclined plane is seen as a conceptual model different from that represented by the lever and essentially not reducible to it. The inclined plane is representative of virtual displacement laws, it is somehow its geometric representation; the lever is representative of the virtual velocity laws (Capecchi 2004; Pisano 2015b; Pisano and Drago 2013). In the past however, things were not seen this way. That the inclined plane had its peculiarities was understood by Aristotle who did not treat it and by Heron who treated it apart from the other machines. However, after Pappus of Alexandria had reduced it to the lever, the difficulties in the study of the inclined plane seemed to vanish. In the Renaissance the problem reappeared because some scholars did not accept Pappus' solution, considering it both logically unconvincing and empirically inadequate. For example, it featured an infinite value of the force required to lift a weight on a vertical plane, and this is patently absurd. Other scholars did not accept it because in contrast with de Nemore's solution, whose demonstration seemed more consistent, though the principles adopted could appear not very obvious.

With Galileo the reductionist project, started with Pappus and strongly supported by Guidobaldo del Monte, to reduce all simple machines including the inclined plane to the lever, was perfected. Note that Galileo's attempt to reduce the inclined plane to the lever was accepted not because verified empirically – with the conceptions of experiment (Rogers 2005) of the times also the results of Heron or Cardano were verified – but because he finally presented a rigorous reasoning and employed reasonable assumptions. Moreover, Galileo's result along with that of de Nemore coincided with that of Stevin more or less of the same period, very elegant and based on different assumptions. Note also that if the reasoning of Galileo was corroborated by the result of de Nemore and Stevin, the reasoning of de Nemore and Stevin was corroborated by that of Galileo and from now on the problem of the inclined plane was considered definitively solved by all mathematicians.

In the section devoted to the mechanics of the screw, Galilei (1649) showed how the inclined plane can be reduced to the lever and furnished a simple mathematical law. The proof reproduces what he had reported in *De motu* (Galilei [1590] 1888–1905, I, 297–298), differing mainly for the use of the word *moment* instead of *gravitas*.

The present Speculation hath been attempted by Pappus Alexandrinus in Lib. 8. de Collection. Mathemat. but, if I be in the right, he hath not hit the mark, and was overseen in the Assumption that he maketh.

Figure, such as this here present, the Moment of Descending which the Moveable hath upon the inclined Plane CA hath to its total Moment wherewith it gravitates in the Perpendicular to the Horizon CP the same proportion that the said Line PC hath to CA. And if thus it be, it is manifest, that like as the Force that sustaineth the Weight in the Perpendicular PC ought to be equal to the same, so for sustaining it in the inclined Plane CA, it will suffice that it be so much lesser, by how much the said Perpendicular CP wanteth of the Line CA: and because, as sometimes we see, it sufficeth, that the Force for moving of the Weight do insensibly superate that which sustaineth it, therefore we will infer this universal Proposition, that upon an elevated plane the force hath to the weight the same proportion.⁸²

The key assumptions to demonstrate the law of the inclined plane are:

- (a) For static purposes, moving on the inclined planes like to NO or GH is the same as moving on the circumference described by the lever arms BL or BF (see Fig. 2.27)
- (b) The effectiveness of a heavy body on an angled lever is determined by the horizontal distance from the fulcrum.

The second assumption is an accepted theorem of statics, but the first has a logic status not completely clear. It indeed appears quite intuitive, at least after its formulation, because to study the equilibrium it seems sufficient to verify that also very small displacements cannot occur. In this way the displacements at the extremity of the lever and on the inclined plane are the same, the two kinds of constraints are locally equivalent and can be replaced the one with the other. But this intuitive character stems more from empirical than logical considerations; it would be then a postulate which could even not be accepted. Moreover the first assumption has a kinematic connotation, which makes it closer to the science of weights approach than the Archimedean's.

⁸² “Vedesi dunque come, nell’inclinare a basso per la circonferenza CFLI il peso posto nell’estremità della linea BC, viene a scemarsi il suo momento ed impeto d’andare a basso di mano in mano più, per esser sostenuto più e più dalle linee BF, BL. [. . .]. Se dunque sopra il piano HG il momento del mobile si diminuisce dal suo totale impeto, quale ha nella perpendicolare DCE, secondo la proporzione della linea KB alla linea BC o BF; essendo, per la similitudine de i triangoli KBF, KFH, la proporzione medesima tra le linee KF, FH che tra le dette KB, BF, concluderemo, il momento integro ed assoluto che ha il mobile nella perpendicolare all’orizzonte, a quello che ha sopra il piano inclinato HF, avere la medesima proporzione che la linea HF alla linea FK, cioè che la lunghezza del piano inclinato alla perpendicolare che da esso cascherà sopra l’orizzonte. Sì che, passando a più distinta figura, quale è la presente, il momento di venire al basso che ha il mobile sopra il piano inclinato FH, al suo totale momento, con lo qual gravita nella perpendicolare all’orizzonte FK, ha la medesima proporzione che essa linea KF alla FH. E se così è, resta manifesto che, sì come la forza che sostiene il peso nella perpendicolare FK deve essere ad esso eguale, così per sostenerlo nel piano inclinato FH basterà che siano tanto minore, quanto essa perpendicolare FK manca dalla linea FH. E perché, come altre volte s’è avvertito, la forza per muover il peso basta che insensibilmente superi quella che lo sostiene, però concluderemo questa universale proposizione: sopra il piano elevato la forza al peso avere la medesima proporzione, che la perpendicolare dal termine del piano tirata all’orizzonte, alla lunghezza d’esso piano.” (Galilei [1649] 1888–1905, II, 182–183). Salusbury’s translation (Salusbury 1661–1665a, II, 294–296).

2.1.3.2 Stevin's Legacy. The Circulation of Statics in Europe

Mechanics in the XVI century developed mainly in Italy. In the XVII century, things began to change and the dominance, for that which concerns the science of balance too, went to France, the Netherlands and England. In this process, is appropriate to mention a very important transitional figure: Simon Stevin. A contemporary of Galileo, and therefore a man of the XVI century, he is not Italian. In an ideal temporal representation of the evolution from the science of weights to the modern statics, fairly regular in truth, the presence of Stevin marks a net-discontinuity (Pisano and Gaudiello 2009). He transformed the science of equilibrium of weights into a science of equilibrium of forces for which he proposed a composition rule. The very Latin word for statics⁸³ (a neologism from *status*), while giving a unique name to a discipline, also demarcated areas, emphasizing its main reference to equilibrium. Statics is distinct from mechanics, which also deals with motion, but is also separated from the science of weights, as statics is centred because it also deals with forces and not weight only.

Simon Stevin (1548–1620) was for some years book-keeper in a business house at Antwerp; later he secured employment in the administration of the Franc of Bruges. In 1583, he entered the University of Leiden. From 1604 Stevin was an outstanding engineer who advised on building windmills, locks and ports. Author of many books, he made significant contributions to trigonometry, mechanics, architecture, musical theory, geography, fortification, and navigation. He introduced the use of decimals in mathematics in Europe (Struik 1981).

Stevin wrote important works on mechanics. Mainly dealing with equilibrium they are his books *De Beghinselen der Weegconst* (*The elements of the art of weighing*) (Stevin 1586a) and *De Beghinselen des Waterwichts* (*The elements on the weight of water*) (Stevin 1586b) both published in 1586 into Flemish language. Although he undertook his mathematical work earlier in his life, Stevin collected together some of his mathematical writings and edited and published them during

⁸³ The word *statica* appears in the title of the fourth parts of the translation of Stevin's major work in mechanics *Wisconstige Gedachtenissen* (Stevin 1605–1608c). In addition, Willebrord Snel van Royen (1580–1626), in his Latin translation as *Hypomnemata mathematica*, published into two volumes (Stevin 1605–1608b) immediately after the original Flemish publication, uses also the term “*Statica*”. Particularly, Snel uses the word “*Statica*” in the volume 2 (Stevin 1605–1608b, II [*Tomus quartus mathematicorum hypomnematum de Statica*] *Liber Primus Staticae, de Staticae Elementis*, 5). Jean Tuning (see next footnote) in his French translation of Stevin's work as *Memoires mathematiques* (Stevin 1605–1608a) uses the word “*art pondéraire*”. Then, in 1634 Albert Girard (1595?–1632) reused – in his French work, as *Les Œuvres Mathematiques de Simon Stevin de Bruges* (Stevin 1634), the term “*L'art pondérarire ou de la statique* [...]”. This word was not so much of succesful, at least until to *Nouvelle mécanique* (1725) by Pierre Varignon (1654–1722). Therefore, in agreement with Patricia Radelet de-Grave, it seems that the introduction of the notation *Statica* should be attributed to Snel rather than to Stevin. Nevertheless, as suggested by Radelet de-Grave, since Snel translated in collaboration with Stevin, it is hard to establish the history of genesis of the scientific term “*Statica*” in Stevin context.

the years 1605 to 1608 in *Wiskonstighe Ghedachtenissen*⁸⁴ – mathematical memoirs, in Latin *Hypomnemata mathematica* – (Stevin 1605–1608b; see also 1955). As a custom of the times, he did not quote his predecessors with the exception of Archimedes, Commandino and Cardano but in the last case only to criticize his (wrong) result for the inclined plane. Assessing Stevin's contribution to the history of mechanics is not simple because his ideas were originally written in Flemish and thus read by few. When they were translated into Latin and later, again into French language (Stevin 1634) by Girard the state of mechanics had already changed. He is indeed, in any case, the founder of statics in the modern sense.

Stevin's major work, *Tomus quartus mathematicorum hypomnematum de statica* (Stevin 1605) is divided into five books, plus an *Appendix* and some *Additions* to the Flemish edition of 1586. The approach is of Euclidean type, in the sense that for every book there is a different topic, first there are definitions, then postulates and finally theorems that are linked together. In the first part of the first book (Stevin 1605, *Book I*) Stevin demonstrates the law of the lever, with an argument similar to that used by Galileo in *Le mecaniche*. Starting from a continuous prismatic body with geometric considerations in the wake of Archimedes, he finds the law of inverse proportionality between weight and arm length. In the second part of the same book, Stevin gives his famous demonstration of the law of the inclined plane, determining the value of the force parallel to the slope enough to maintain a heavy body in balance. Stevin extends his result to the case where the uplifting force is not parallel to the inclined plane. Gilles Personne de Roberval (1602–1675) found Stevin's proof not satisfactory and gave a much more convincing proof (de Roberval 1636). Basing on the law of the inclined plane generalized to a force of any direction, with a rather complex argument that is developed with many theorems and corollaries, Stevin puts the groundwork for the proof of the rule of the parallelogram of forces which is satisfactory if the generalized law for the inclined plane is accepted. In the *Additions* Stevin considers and devises demonstrations for pulleys and treats with some generality the case of forces applied by means of ropes in a section called *Spartostatica*. In this section statics has already become the science of equilibrium of force – modern meaning – and no longer of weights. It contains the wording of the rule of the parallelogram, which is a rule of composition of forces, even though it is presented as a way to determine the tension of two ropes which sustain a weight (Stevin 1605). This change of attitude is a fundamental Stevin's contribution to modern statics, and it does not matter if the rule of composition of forces is given an imperfect proof; it is however a rule which works. In the final part of the *Spartostatica* Stevin considers for the first time fundamental arguments that can be conceived only in the new frame of reference based on forces, i.e. the funicular polygon, the weight sustained by more than two ropes in the plane and out of plane. It is worth noticing however that Stevin never uses the terms *force* or

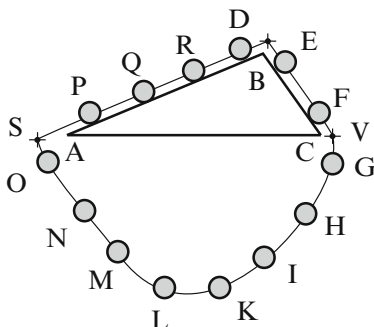
⁸⁴ This Stevin's book was immediately – but partially – translated both into French language as *Memoires mathematiques: contenant ce en quoy s'est exercé le très-illustre, très-excellent Prince & Seigneur Maurice, Prince d'Orange, Conte de Nassau, Catzenellenbogen, Vianden, Moers* [...] (Stevin 1605–1608a) by Jean Tuning and into Latin language as *Hypomnemata mathematica, hoc est eruditus ille pulvis, in quo se exercuit* [...] *Mauritius, princeps Auræicus* [...] a Simone in two volumes (Stevin 1605–1608b) by Willebrord Snel van Royen.

power. This holds good also when it is clear that he is concerning with a muscle force; as well as when in his drawings he shows the images of human hands sustaining or lifting a weight. The reading of Stevin's mechanical work offers a much more modern view than that of Guidobaldo del Monte (del Monte 1577) and Galileo (Galilei [1649] 1888–1905, II). The approach of Archimedean kind is equally rigorous, but less verbose. Unlike Galilei, Stevin does not bother to set up statics on a single principle, that of lever. He uses the Archimedean geometric proof for the lever, but when he relies on the law of the inclined plane he uses an empirical principle, in part still controversial, that of the impossibility of perpetual motion.

2.1.3.2.1 The Law of the Inclined Plane

Although he declares his opposition to the kinematic approach for which the equilibrium of a body depends on its possible motion, in the proof of the law of the inclined plane Stevin seems to contradict himself by deducing the equilibrium from the impossibility of motion. He considers a chain that wraps around the inclined plane, as shown in Fig. 2.28, which is accurately described:

Fig. 2.28 Equilibrium of the necklace wrapped around an inclined plane by Stevin (Redrawn from Stevin 1605, 34)



PREPARATION. Let consider around the triangle ABC a necklace of fourteen equal globes, like E, F, G, H, I, K, L, M, N, O, P, Q, R, D, so that they can rotate around their centres and that there are two globes on the side BC and four on the side BA, so that as the line is to the line, the number of globes is to the number of globes. Let S, Γ, V be three fixed points, on which the line, or the lace, could slide. And the two parts above the triangle be parallel to its sides AB, BC; so that the whole should rotate freely and without friction on the said sides AB, BC.⁸⁵

⁸⁵ “PRAEPARATIO. Triangulum A B C quatuordecim globorum pondere et magnitudine æqualium, quasi corona ut E, F, G, H, I, K, L, M, N, O, P, Q, R, D, cunctum fingamus, qui omnes lineâ per centro ipsorum, ut in illis moveri possint, transeunte, colligati æquali inter se spacio distent, ut illorum bini lateri B C, quaterni vero B A accommodentur, hoc est, quemadmodum linea ad lineam; ita globi sint ad globos. Insuper in S, Γ, V tria sint puncta immota ac fixa, quae a linea sive globorum funiculo, cum movetur, raduntur, ac stringuntur: duaeque funiculi partes, quae supra trianguli basin, lateribus A B, B C sint parallelæ, ut, quando connexio illa seriesque; globorum adscendit, descenditve, globi pes crura A B, B C volui possint.” (Stevin 1605, 34). The translation is ours. See also important works by Radelet-de grave 1996.

The proof is conducted by reduction to the absurd. Suppose, says Stevin, the necklace is not in equilibrium and moves to reach equilibrium. Since the relative configuration of the necklace cannot change, if it is not equilibrated in one configuration it is not equilibrated in any other configuration. Thus perpetual motion would occur, which is absurd. The necklace is so in equilibrium:

It is not possible that a given motion has no end.⁸⁶

Thus a comparison of weights of the necklace that rely on the two opposing inclined planes (see Fig. 2.28) immediately gives the law of the inclined plane according to which two heavy bodies on two inclined planes are equilibrated when their weights are proportional to the length of the planes. Notice that Stevin considers the negation of the perpetual motion as unproblematic, but does not assume it explicitly as a principle of statics, though it is as fundamental for his mechanics at least as the law of the lever. The simple justification for this is that probably Stevin did not want his book to appear too new by introducing since from the beginning a non-standard statement. Stevin pretends to extend the law of the inclined plane to cases where the force to uplift the load is not parallel to the inclined plane. To this purpose, he concentrates his attention on a prism sliding on the plane.

In corollary V to the law of the inclined plane reported in the second half of the first book (Stevin 1605, p 36) it is easy for Stevin to show that the ratio between the weight M of the prism (Fig. 2.29), i.e. the force to lift it, called the *direct uplifting*, and the force E needed to move it on the inclined plane, called the *oblique uplifting*, is equal to the ratio of the segments LD and DI identified by the intersection of the ropes with the prism (because $M : E = AB : BC = LD : DI$).

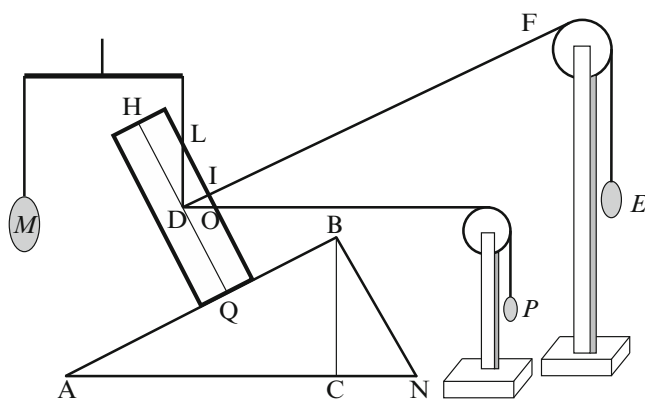


Fig. 2.29 Uplifting forces for various directions (Redrawn from Stevin 1605, 36)

⁸⁶ “[...] ipsique globi ex sese continuum et aeternum motum efficient, quod est falsum” (Stevin 1605, 35). The translation is ours.

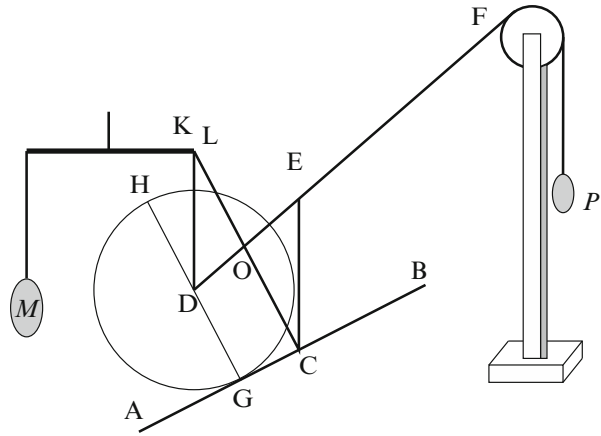
In corollary VI (Stevin 1605, 36–37) Stevin considers a horizontal uplifting measured by weight P (see Fig. 2.29). Imagining a rotation of ninety degrees, the horizontal uplifting becomes vertical and the plane ABC turns into a tilted plane whose slope is as NL of the triangle NCB. Following this rotation the ratio between direct and oblique upliftings is equal to that between the segments DO and DI. Stevin believes that this relationship is maintained even when the rope carrying the load P is effectively horizontal. At this point, he can say that in the vertical, in the inclined and in the horizontal directions the values of the ‘forces’ necessary to keep the prism in balance are proportional to the length of the segments DL, DI, DO, intercepted by the ropes on the prism, to conclude (improperly) that this fact applies to all directions. Stevin’s argument is interesting only for its strong rhetorical value, at least for the generalization to the case of any direction. The belief of the reader is made possible by the choice of a prism as the body to be lifted. It should be stressed however, that even if the reasoning cannot convince us the result is correct.

Below Stevin’s proof of *Consectarium* (corollary) VI follows, to allow the reader to judge the lawfulness of the reasoning:

Let BN be conducted cutting AC and extended to N, and the same DO cutting in O the extension of LI, so that the angle IDO be equal to the angle CBN, and then the uplifting P be applied along DO, taking the column in its position (with weights M and E balanced); then as LD is homologous to BA in the triangle BAC and DI with BC, it follows that BA is to BC as the weight on BA is to the weight on BC [...]. And also DL is to DI as the weight belonging to DL is to that to DI, i.e. M to E . Similarly the three lines LD, DI, DO being homologous to the three segments AB, BC, BN, then BA is to BN as the weights that belong to them, and also LD to DO will be like the weights that belong to them, i.e. M to P . Because this proportion is not valid only at that elevation as DI perpendicular to the axis, but for all sorts of angles.⁸⁷

⁸⁷ “BN ducatur, secans AC continuatam in N, consimiliter D O secans continuatam LI, hoc est, latus columnæ in O, ut angulus IDO aequalis sit angulo CBN. Appendatur quoque ad DO pondus P oblique attollens, quod (amotis M, E ponderibus) columnam in suo situ conservet. Quia vero DL et BA, item DI et BC latera triangulorum DLI et BAC homologa sunt, hujusmodi conclusio inde deducitur. Quemadmodum BA ad BC: ita sacoma lateris B A ad anti sacoma lateris BC (per 2 consecarium) item quemadmodum DL ad DI: ita sacoma lateris DL ad antisacoma lateris DI, hoc est ita M ad E. sed homologa latera triangulorum similium ABN, LDO sunt AB et DL, item BN, et DO. Itaque ut supra, quemadmodum BA ad BN: ita sacoma B A ad anti sacoma B N (per 1 consecarium) Et quemadmodum DL ad DO: ita illius sacoma ad hujus anti sacoma, id est, M ad P. si linea BN à puncto B alioversum; A scilicet versus, ultra BC fuisset ducta, etiam recta DO à D ultra DI cecidisset, hoc est, ut nunc citra: ita tunc ultra cecidisset, et praecedens demonstratio etiam isti situi accommoda fuisset, hoc est, quemadmodum BA ad BN ita sacoma lateris BA, ad anti sacoma lateris BN esset: et quemadmodum DL ad DO: ita sacoma lateris DL, ad anti sacoma lateris DO. hoc est M ad P. Ut ista proportio non tantum in exemplis valeat, in quibus linea attollens, ut DI, perpendicularis est axi, sed etiam in aliis cuiusmodi cunque sint anguli.” (Stevin 1605, 36–37). The translation is ours.

Fig. 2.30 Uplifting forces for a generic direction (Redrawn from Stevin 1605, 37)



Stevin extends his result to non-cylindrical bodies, for example the sphere of Fig. 2.30. Stevin claims that the ratio between the direct uplifting M (the weight of the sphere) and the oblique uplifting P is as DL to DO , with LC orthogonal to the inclined plane, i.e.:

$$M : P = DL : DO$$

For similar triangles, it also holds good the relation:

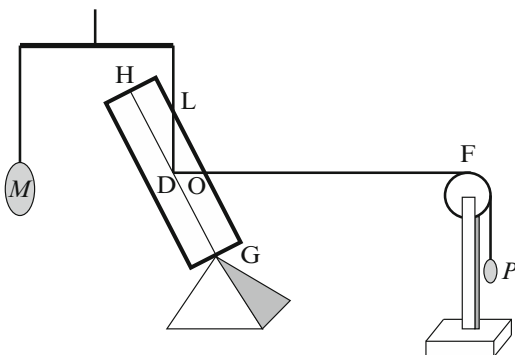
$$M : P = EC : OE$$

EC , OE are the sides of the triangle OEC . A modern reader can easily see that Stevin's result is in fact correct, as the weight of the sphere is balanced by the tension of the rope and the constraint reaction orthogonal to the plane AB .

2.1.3.2.2 Forces' Composition: The Rule of Parallelogram

The demonstration of the rule of the parallelogram for composition of *forces* is carried out by Stevin with a long series of theorems and corollaries (about twenty) that leave the modern reader a little upset (Capecchi and Drago 2005). This happens also because as each theorem and corollary is proved with rather slender mathematical reasoning, very close to the modern sensibility, it is difficult to understand the reason for Stevin's prolixity. A part of this difficulty might be overcome by assuming that Stevin's objective originally was not to formulate the rule of composition, of which perhaps he did not understand the full extent, but only to make a series of comments on the way weights can be lifted. In fact, the explicit formulation of the rule of the parallelogram is in the section of the *Additions* named *spartostatica*.

Fig. 2.31 Uplifting forces for a punctiform support by Stevin (Redrawn from Stevin 1605, 39)



Consider the prism (See Fig. 2.31) with direct and horizontal upliftings applied to its centroid. Stevin assumes that the ratio between the direct and horizontal upliftings is the same as that of the segments DL and DO . Stevin does not pause to justify the lawfulness of the replacement of the inclined plane with the fixed point G .

Reading between the lines it can be understood that since for every inclination of the rope the intercept with the side of the prism provides the ‘force’ necessary to maintain the equilibrium whichever is the inclination of the inclined plane, the inclined plane can be replaced with a constraint that performs its essential function, i.e. to offer a support to the prism.

By means of the following Theorem 12, Propositio 20, Stevin extends his result to the case where the fixed point and the upliftings are applied anywhere in the axis of the prism.

12 THEOREM. 20 PROPOSITION. If in the axis of the prism there are a fixed point and a movable point, which could be maintained in any disposition by means of a direct uplifting, the line of direct uplifting is to the line of inclined uplift as the direct uplifting is to the oblique uplifting.⁸⁸

The result of Stevin, namely the determination of the *force* necessary to support the prism constrained to a fixed point could have been extended quite easily to the case of a body of any shape to get a rule of equilibrium as efficient as the vanishing of the static moments. But Stevin does not do it.

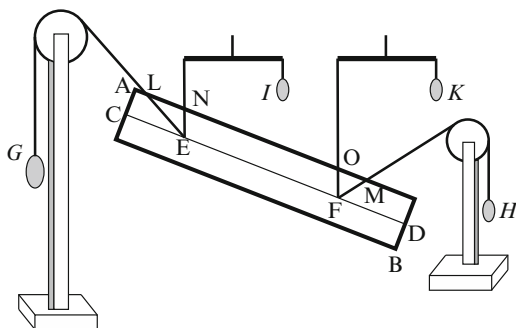
The next step – basically the definitive one – consists in the consideration of the situation of following Fig. 2.32 for which Stevin states the following theorem:

18 THEOREM. 27 PROPOSITION. If a column is maintained in equilibrium by two oblique uplifting as the line of the oblique uplifting is to the line of the direct uplifting, so each oblique uplifting is to its direct uplifting.⁸⁹

⁸⁸ “12 THEOREM. 20 PROPOSITIO. Si axis columnae puncta habeat, firmum, et mobile, et ex isto dependentia pondera, unum rectè, alterum obliquè extollens, in dato situ columnam conservant: erit quemadmodum linea recte extollens ad lineam oblique extollentem; ita illius pondus, ad pondus hujus”. (Stevin 1605, 41).

⁸⁹ “18 THEOREM. 27 PROPOSITIO. Si columna, et duo pondera oblique extollentia situ aequilibria sunt, erit quemadmodum linea oblique extollans, ad lineam recte extolletem: ita ponderum quodque obliquum ad suum pondus rectum”. (Stevin 1605, 48).

Fig. 2.32 Equilibrium of a column supported into two points (Redrawn from Stevin 1605, 49)



Notice that if points E and F have the same distance from the centre of gravity of the prism the vertical upliftings I and K will be the same, so LE and FM have the ratio of G and H . Indeed from theorem XVII the relations can be written:

$$LE : NE = G : I$$

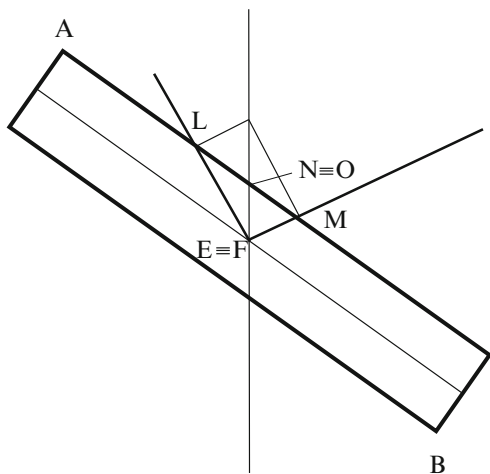
$$OF : MF = K : H,$$

which if $I = K$ can be combined to give:

$$LE : FM = G : H.$$

From this result, it is very easy to arrive at the parallelogram *Rule of the additions*. To get the rule of the parallelogram from Theorem 18 (Stevin 1605, 48) it suffices to consider the case where the two points E and F (see Fig. 2.32) coincide with each other and with the centroid of the prism as shown in the following Fig. 2.33:

Fig. 2.33 Reconstruction of the application of parallelogram of forces rule according to Stevin



In this case it can be affirmed that the proportion between segments FQ, EL, EM is the same as the weight, and inclined upliftings (Stevin 1605, Corollary III, 35) of the *Additions*; but this is the rule of the parallelogram. In order to prove this it is enough to consider that the whole uplifting, i.e., the weight of the prism, in Fig. 2.33 is given by $I + K = 2I = 2K$, which is proportional to $2NE = 2OF$.

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