Lecture 2: Pseudo-differential Operators. Berezin, Kohn–Nirenberg, Born–Jordan Quantizations

Weyl quantization is strictly linked to Wigner transform.

If $l(q, p)$ is a linear function of the $q's$ and of the $p's$ (coordinates of the cotangent space at any point $q \in R^d$) the *Weyl quantization* is defined, in the Schrödinger representation, by

$$Op^w(e^{it(q,p)}) = e^{it(x,-i\hbar\nabla_x)} \tag{1}$$

Let $\mathcal{S}$ be the Schwartz class of functions on R^{2d}. It follows from the definition of Wigner function $W_\psi(q, p)$ that in the Weyl quantization one can associate to a function $a \in \mathcal{S}(R^{2d})$ an operator $Op^w(a)$ through the relation

$$(\psi, Op^w(a)\psi) = \int W_\psi(q, p)(\mathcal{F}a)(q, p)dpdq, \quad q, p \in R^d \quad \psi \in L^2(R^d) \tag{2}$$

where the symbol $\mathcal{F}$ stands for Fourier transform *in the second variable* (a map $-\frac{i\nabla}{\hbar} \to p$).

To motivate this relation recall that the Weyl algebra is formally defined a twisted product (twisted by a phase).

Introducing the parameter $\hbar$ to define a microscopic scale, we define the operator $Op^w_\hbar(a)$, as operator on $L^2(R^d)$, by

$$[Op^w_\hbar(a)\ \phi](x) \equiv (2\pi\hbar)^{-d} \int\int a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{\hbar}(x-y,\xi)}\phi(y)dy\, d\xi \tag{3}$$

or equivalently

$$(Op^w_\hbar(a)\phi)(x) = \int \tilde{a}\left(\frac{x+y}{2}, x-y\right)\phi(y)dy,$$
$$\tilde{a}(\eta, \xi) \equiv \left(\frac{1}{2\hbar}\right)^N \int a(\eta, z)e^{\frac{i}{\hbar}(z,\xi)}dz \tag{4}$$

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Notice that (3) can also be written (from now on, for brevity, we omit the symbol $\hbar$ in the operator)

$$(Op^w(a)\ \phi)(x) = \left(\frac{1}{2\pi\hbar}\right)^d \int\int e^{\frac{i}{\hbar}(y-\xi),\xi)} a\left(\frac{1}{2}(x-y), \xi\right)\phi(y)d\xi\ dy \tag{5}$$

In this form it can be used *to extend the definition* (at least as quadratic form) to functions $a(x, y))$ that are not in $\mathcal{S}$.

1 Weyl Symbols

We will call the function *a Weyl symbol* of the operator $Op^w_\hbar(a)$.

Some Authors refer to the function a in (3) as *contravariant symbol* and define as *covariant symbol* the following expression

$$a^\#(z) = (2\pi\hbar)^{-d} a(Jz) \tag{6}$$

With this definition one has

$$Op^w(a) = (2\pi\hbar)^{-1} \int a^\#(z)\hat{T}(z)dz \qquad \hat{T}(z) = e^{\frac{i}{\hbar}(y\cdot\hat{q} - x\cdot\hat{p})} \quad z = x + iy \tag{7}$$

Notice that $T(z)$ is translation by z in the Weyl system. The use of covariant symbols is therefore most convenient if one works in the Heisenberg representation, regarding q, p as translation parameters (hence the name *covariant*).

It is easy to prove

$$(\phi, Op^w(a)\psi) = \left(\frac{1}{2\pi\hbar}\right)^{-d} \int a^\#(z)(\phi, \hat{T}(z)\psi)dz \qquad \forall\phi, \psi \in \mathcal{S}(R^d) \tag{8}$$

In particular in the case of coherent states centered in z

$$(\psi_z, Op^w(a)\psi_z) = \left(\frac{1}{2\pi\hbar}\right)^{-d} \int a^\#(z')e^{-\frac{|z-z'|^2}{4\hbar} - \frac{i}{2\hbar}\sigma(z',z)}dz' \tag{9}$$

Recall that

$$\psi_z = \hat{T}(z)\psi_0, \qquad \psi_0 = \left(\frac{1}{\pi\hbar}\right)^{\frac{d}{2}} e^{-\frac{x^2}{2\hbar}} \tag{10}$$

where $\hat{T}(z)$ is the operator of translation by z in the Weyl representation.

The covariant symbol $a^\#$ is therefore suited for the analysis of the semiclassical limit in the coherent states representation and in real Bergmann–Segal representation.

2 Pseudo-differential Operators

Definition 1 (*Pseudo-differential operators* [1–4]) The operators obtained by Weyl's quantization are called *pseudo-differential operators*.

They are a subclass of the Fourier Integral Operators [5] which are defined as in (4) by substituting the factor $e^{\frac{i}{\hbar}(x-y,\xi)}$ with $e^{\frac{i}{\hbar}(f(x,y),\xi)}$ where f is a regular function.

Notice that when $\hbar$ is very small, this function is fast oscillating in space.

◇

As a remark we mention that the notation *pseudo-differential* originates from that fact that if $a(q,p) = P(p)$ where P is a polynomial, the operator $Op^w(a)$ is the differential operator $P(-i\nabla)$ and if $a(q,p) = f(q)$, the operator $Op^w(a)$ acts as multiplication by the function $f(x)$.

In the case of a generic function a the operator $Op^w(a)$ is far being a simple differential operator (whence the name *pseudo-differential*).

Later we discuss other definitions of *quantization*; Weyl quantization has the advantage of being invariant under symplectic transformations (since it is defined through a symplectic form) and therefore is most suited to consider a semiclassical limit.

In the analysis of the regularity of the solutions of a P.D.E. with space dependent coefficients other quantization procedures may be more useful, e.g. the one of Kohn–Nirenberg [7] that we shall define later.

For the generalization to system with an infinite number of degrees of freedom other quantizations (e.g.the Berezin one) [6] are more suited because they stress the role of a particular element in the Hilbert space of the representation, the *vacuum*.

In a finite dimensional setting this vector is represented by function $\iota(z)$ which takes everywhere the value one and therefore satisfies $\frac{\partial \iota(z)}{\partial z_k} = 0 \ \forall k$ (is *annihilated* by all destruction operators) in the Berezin-Fock representation.

In this representation a natural role is taken by the operator $N = \sum_k z_k \frac{\partial}{\partial z_k}$, which has as eigenvalues the integer numbers and as eigenvectors the homogeneous polynomials in the z_k's.

In the Theoretical Physics literature this representation is often called the *Wick representation* and the operator N is called *number operator*.

For a detailed analysis of pseudo-differential operators, also in connection with the semiclassical limit, one can consult e.g. [1–3, 8].

Let us notice that one has

$$Op^w_\hbar(a) = \int\int e^{i[(p,x)+\hbar(q,D_x)]} \mathcal{F}a(p,q)dpdq \tag{11}$$

where $\mathcal{F}a$ is the Fourier transform of a in the second variable. In particular

$$\|Op^w_\hbar(a)\|^2_{L^2} = \int\int |(\mathcal{F}a)(p,q)|^2 dqdp \tag{12}$$

Remark that integrability of the absolute value of $\mathcal{F}a$ is a *sufficient* (but not a necessary) condition for $Op^w_\hbar \in \mathcal{B}(\mathcal{H})$.

The relation between Weyl symbols and Wigner functions associated to vectors in the Hilbert space (or to density matrices) is obtained by considering the pairing between bounded operators and bilinear forms in $\mathcal{S}$.

More explicitely on has

$$(Op^w(a)f, g) = \int a(\xi, x) W_{f,g}(\xi, x) d\xi \; dx \quad \forall f, g \in \mathcal{S} \tag{13}$$

where

$$W_{f,g}(\xi, x) \equiv \int e^{-i(\xi,p)} f\left(x + \hbar\frac{p}{2}\right) g\left(x - \hbar\frac{p}{2}\right) dp \tag{14}$$

From this one concludes that the Wigner function associated to a density matrix ρ *is the symbol of* ρ as a pseudo-differential operator.

Define in general, for $f \in \mathcal{S}'$

$$W_f(\xi, x) \equiv \int e^{-i(p,\xi)} f\left(x + \hbar\frac{p}{2}, x - \hbar\frac{p}{2}\right) dp \tag{15}$$

Notice that it is the composition of Fourier transform with a change of variable that preserves Lebesgue measure:

It follows that (11) preserves the classes $\mathcal{S}$ and $\mathcal{S}'$ and is unitary in $L^2(R^{2n})$.

One has moreover

$$Op^w_\hbar(\bar{a}) = [Op^w_\hbar(a)]^* \tag{16}$$

and therefore if the function a is real the operator $Op^w_\hbar(\bar{a})$ is symmetric.

One can prove that if the symbol a is sufficiently regular this operator is essentially self-adjoint on $\mathcal{S}(R^d)$.

One can give sufficient conditions in order that a pseudo-differential operator belong to a specific class (bounded, compact, Hilbert–Schmidt, trace class...).

We shall make use of the following theorem

Theorem 1 ([2, 3]) *Let $l_1, \dots l_k$ be independent linear function on R^{2d} and $\{l_h, l_i\} = 0$. Let $\tau : R^k \to R$ be a polynomial.*

Define

$$a(\xi, x) \equiv \tau(l_1(\xi, x), \dots l_k(\xi, x)) \tag{17}$$

Then
(i) $a(\xi, x)$ maps $\mathcal{S}$ in $\mathcal{B}(L^2(R^d))$ and is a self-adjoint operator
(ii) For every continuous function g one has

$$(g.a)(\xi, x) = g(a(\xi, x)) \tag{18}$$

$\diamond$

We leave to the reader the easy proof.

From the relation between Wigner functions and pseudo-differential operators one derives the following properties ($\mathcal{L}$ denotes a linear map).
(1) $Op^w(a)$ is a continuous map from $\mathcal{S}(R^d)$ to $\mathcal{L}(\mathcal{S}(R^d), \mathcal{S}'(R^d))$
(2) $Op^w(a)$ extends to a continuous map $\mathcal{S}'(R^d) \to \mathcal{L}(\mathcal{S}(R^d)\mathcal{S}'(R^d))$
(3) If $a(z) \in L^2(C^d)$ one has

$$|Op^w(a)|_{H.S.} = (2\pi\,\hbar)^{-\frac{n}{2}} \left[\int |a(z)|^2\,dz\right]^{1/2} \tag{19}$$

(4) If $a, b \in L^2(R^d)$, then the product $Op^w(a) \cdot Op^w(b)$ is a trace class operator and

$$Tr\,(Op^w(a)\;Op^w(b)) = (2\pi\,\hbar)^{-d} \int \bar{a}(z)\;b(z)\;dz \tag{20}$$

In order to find conditions on the symbol a under which $Op^w(a)$ is a bounded operator on $\mathcal{H}$ one can use the duality between states and operators and

$$(\psi, Op^w(a)\psi) = \int \mathcal{F}a(p,q)W_\psi(p,q)dqdp \tag{21}$$

One can verify in this way that $\|Op(a)\| \leq |\hat{a}|_1$, but $\hat{a} \in L^1$ is *not necessary* in order $Op(a)$ be a bounded operator.

Remark that using this duality one can verify that Weyl quantization is a *strict quantization* (see Volume I).

One can indeed verify that, if $\mathcal{A}_0$ is the class of functions continuous together with all derivatives, introducing explicitly the dependence on $\hbar$.
(i) *Rieffel condition*. If $a \in \mathcal{A}_0$ then $\hbar \to Op^w_\hbar(a)$ is continuous in $\hbar$.
(ii) *von Neumann condition*. If $a \in \mathcal{A}_0$

$$lim_{\hbar\to 0}\|Op^w_\hbar(a)Op^w_\hbar(b) - Op^w_\hbar(a \otimes b)\| = 0 \tag{22}$$

where $\otimes$ is convolution.
(iii) *Dirac condition*. If $a \in \mathcal{A}_0$

$$lim_{\hbar\to\infty}\|\frac{1}{2\hbar}\left[Op^w_\hbar(a)Op^w_\hbar(b) - Op^w_\hbar(b)Op^w_\hbar(a) - Op^w_\hbar(\{a,b\})\right]\| = 0 \tag{23}$$

where $\{a, b\}$ are the Poisson brackets.

If one wants to make use of the duality with Wigner function to find bounds on $Op^w_\hbar(a)$ in term of its symbol $a(x, \hbar\nabla)$ one should consider that Wigner's functions can have *strong local oscillations* at scale $\hbar$.

3 Calderon–Vaillantcourt Theorem

The corresponding quadratic forms are well defined in $\mathcal{S}$ but to obtain regular operators on $L^2(R^d)$ these oscillations (which become stronger as $\hbar \to 0$) *must be smoothed out* by using regularity properties of the symbol.

This is the content of the theorem of Calderon and Vaillantcourt.

From the proof we shall give one sees that the conditions we will put on the symbol a in order to estimate the norm of the pseudo-differential operator $Op^w(a)$ *are far from being necessary*.

We give an outline of the proof of this theorem because it is a prototype of similar proofs and points out the semiclassical aspects of Weyl's quantization.

Theorem 2 (Calderon–Vaillantcourt [2, 3, 8]) *If*

$$A_0(a) \equiv \sum_{|\alpha|+|\beta|\leq 2d+1} |D_\xi^\alpha D_x^\beta a(x,\xi)|_\infty < \infty, \qquad x,\xi \in R^d \tag{24}$$

then $Op^w(a)$ is a bonded operator on $L^2(R^d)$ and its norm satisfies

$$||Op^w(a)|| < c(d)A_0(a) \tag{25}$$

where the constant $c(d)$ depends on the dimensions of configuration space.

$\diamond$

The proof relies on the decomposition of the symbol a as

$$a(x,\xi) = \sum_{j,k} a(x,\xi)\zeta_{j,k}(x,\xi) \qquad \sum_{j,k} \zeta_{j,k} = 1 \tag{26}$$

where $\zeta_{j,k}$ are smooth function providing a covering of R^{2d} each having support in a hypercube of side $1+\delta$ centered in $\{j,k\}$ and taking value one in a cube of side $1-\delta$ with the same center.

One gives then estimates of the norm of $Op^w(\sum_\Gamma a(x,\xi)\zeta_{j,k})$, where Γ is a bounded domain in terms of the derivatives of $a(x,\xi)$ up to an order which depends on the dimension of configuration space.

These bounds rely on embeddings of Sobolev spaces $H^p(R^{2d})$ in the space of continuous functions for a suitable choice p (that depends on d).

The convergence $\Gamma \to R^d$ is controlled by the decay at infinity of the symbol $a(x,\xi)$.

A standard procedure is to require at first more decay, and prove by density the theorem in the general case.

The estimates on $Op^w(a\zeta_{j,k})$ are obtained noticing that the symbols of these operators are the product of a function that is *almost* the product the characteristic function of a set on configuration space and of a function that is *almost* the characteristic function of a set on momentum space.

The word *almost* refers to the fact that the partition is smooth, and the functions one uses tend to characteristic functions as $\hbar \to 0$.

If one chooses the side of the hypercubes to be of order $\sqrt{\hbar}$ these qualitative remarks explain why the estimates that are provided in the analysis of pseudo-differential operators have relevance for the study of the semiclassical limit.

And explains why pseudo-differential calculus is relevant if one considers a macroscopic crystal, the partition is at the scale of the elementary cell and one must analyze the properties of projection operator in a Bloch band (these pseudo-differential operators are far from being simple polynomials).

The proof of the theorem of Calderon–Vaillantcourt is based on two results of independent interest.

The first is the theorem of Cotlar–Knapp–Stein; we give the version by L. Hormander [5]; this paper is a very good reference for a detailed analysis of pseudo-differential operators.

In what follows we shall use units in which $\hbar = 1$.

Theorem 3 (Cotlar–Knapp–Stein) *If a sequence $A_1, A_2, \ldots, A_N$ of bounded operators in a Hilbert space $\mathcal{H}$ satisfies*

$$\sum_{k,j=1}^{N} \|A_j^* A_k\| \leq M \qquad \sum_{k,j=1}^{N} \|A_j A_k^*\| \leq M \tag{27}$$

then

$$\sum_{k=1}^{N} \|A_k\| \leq M \tag{28}$$

$\diamondsuit$

Proof The proof follows the lines of the corresponding proof for finite matrices. For each integer m

$$\|A\|^{2m} = \|(A^*A)^m\| \tag{29}$$

Also

$$(A^*A)^m = \sum_{1 \leq j_1 \leq j_2 \ldots \leq j_m} A_{j_1}^* A_{j_2} \ldots A_{j_{2m-1}}^* A_{j_{2m}} \tag{30}$$

and

$$\|A_{j_1}^* A_{j_2} \ldots A_{j_{2m-1}}^*, A_{j_{2m}}\| \leq \min\{\|A_{j_1}^* A_{j_2}\| \ldots \|A_{j_{2m-1}}^* A_{j_{2m}}\|, \\ \|A_{j_1}^*\| \|A_{j_1} A_{j_2}^*\| \ldots \|A_{j_{2m}}\|\} \tag{31}$$

Making use of the inequality for positive numbers $min\{a,\ b\} \le \sqrt{ab}$ and taking into account the assumption $\|A_j\| \le M$ e $\|A_j^*\| \le M$ one has

$$\|A_{j_1}^* A_{j_2} \dots A_{j_{2m-1},j_{2m}}^*\| \le M \|A_{j_1}^* A_{j_2}\|^{\frac{1}{2}} \dots \|A_{j_{2m-1}}^* A_{j_{2m}}\|^{\frac{1}{2}} \tag{32}$$

Performing the summation j_2, j_3, j_{2m} one obtains

$$\|A\|^{2m} \le NM^{2m} \tag{33}$$

so that, taking logarithms, for $m \to \infty$

$$\|A\| \le M \frac{logN}{m} \tag{34}$$

$\diamond$

It is possible [2, 5] to generalize the theorem replacing the sum by the integration over a finite measure space Y. In this case the theorem takes the form

Theorem 4 (Cotlar–Knapp–Stein, continuous version) *Let $\{Y, \mu\}$ be a finite measure space and $A(y)$ be a measurable family of operators on a Hilbert space $\mathcal{H}$ such that*

$$\int \|A(x)A(y)^*\| d\mu \le C \qquad \int \|A(x)^* A(y)\| d\mu \le C \tag{35}$$

Then the integral $A = \int A(x) d\mu$ is well defined under weak convergence and one has $\|A\| \le C$.

$\diamond$

Outline of the Proof of the Theorem of Calderon–Vaillantcourt

We build a smooth partition of the identity by means of functions $\zeta_{j,k}(x, \xi)$ of class C^∞ such that

$$\zeta_{j,k}(x, \xi) = \zeta_{0,0}(x - j, \xi - k), \qquad \sum_{j,k \in Z} \zeta(x - j, \xi - k) = 1 \quad x,\ \xi \in R^d \tag{36}$$

We choose $\zeta_{0,0}(x, \xi)$ to have value one if $|x|^2 + |\xi|^2 \le 1$ and zero if $|x|^2 + |\xi|^2 \ge 2$.

Define

$$a_{j,k} = \zeta_{j,k} a \qquad A_{j,k} = Op^w(a_{j,k}) \tag{37}$$

We must verify that the corresponding operators are bounded and that their sum converges in the weak (or strong) topology. We shall see that these requirements can be satisfied provided the symbol a is sufficiently regular as a function of x and ξ.

The regularity conditions do not depend on the value of the indices j, k since the functions $\zeta_{j,k}$ differ from each other by translations.

It follows from the definitions that $\sum A_{j,k}$ converges to $Op^w(a)$ in the weak topology of the functions from $\mathcal{L}(\mathcal{S}(R^d))$ to $\mathcal{L}(\mathcal{S}'(R^d))$. We are interested in conditions under which convergence is in $\mathcal{B}(L^2(R^d))$.

For this it is sufficient to prove that there exists an integer $K(d)$ such that, for any finite part Γ of the lattice with integer coordinates

$$\| \sum_{j,k\in\Gamma} A\| \leq Csup_{|\alpha|\leq K(d),|\beta|\leq K(d),(x,\xi)\in R^d} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \tag{38}$$

This provides conditions on the symbol and at the same time provides bounds for the operator norm.

From Theorem 3 it follows that it is sufficient to obtain bounds on the norm of

$$A_\gamma^* \cdot A'_\gamma \tag{39}$$

for any choice of the index $\gamma \equiv \{j, k\}$.

Define $a_{\gamma,\gamma'}$ by

$$Op^w(a_{\gamma,\gamma'}) = A_\gamma^* \cdot A'_\gamma \tag{40}$$

One derives

$$\begin{gathered} a_{\gamma,\gamma'} = e^{\frac{i}{2}\sigma(D_x,D_\xi;D_y,D_\eta)} (\bar{a}_\gamma(x,\xi).a_{\gamma'}(y,\eta))_{y=x,\eta=\xi} \\ \forall\gamma \in Z^{2d} \quad a_\gamma \in L^2(R^d) \end{gathered} \tag{41}$$

where σ is the standard symplectic form.

Notice that we have used estimates on Sobolev embeddings to obtain an estimate of the norm of $Op^w(a)$ in term of Sobolev norms of the symbol a; recall that the operator norm of $Op(a)$ is the L^2 norm of its Fourier transform. Remark that a_γ has support in R^{2d} of radius $\sqrt{2}$.

The partition of phase space serves the purpose of localizing the estimates; the number of elements in the Cottlar–Kneipp–Stein procedure depends on the dimension $2d$ of phase space.

Remark that $\sum_{k\in Z^{2d}} A(\gamma)$ converges to $A = Op^w a$ in the topology of linear bound operators from $\mathcal{S}(R^d)$ to $\mathcal{S}'(R^d)$.

Therefore it is sufficient to prove, for any bounded subset $\Gamma \subset Z^{2d}$,

$$\| \sum_{\gamma\in\Gamma} A(\gamma)\|_{L^2(R^d)} \leq C(d)sup_{|\alpha|\leq 2d+1,|\beta|\leq 2d+1,(x,\xi)\in R^{2d}} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \tag{42}$$

We must have a control over $\|A_\gamma^* \cdot A_{\gamma'}\|$ and therefore of the norm of the operator with symbol $a_{\gamma,\gamma'}$.

We use Sobolev-type estimates. If B is a real quadratic form on R^{2n}, for every $R > 0$ and integer $M \geq 1$ there exists a constant $C(R, M)$ such that

$$|(e^{iB(x,D)}u(x)| \leq C(R, M)(1 + |x - x_0|^2)^{-d} sup_{|\alpha| \leq 2M+d+1, x \in B(x_0,R)} |\partial_x^\alpha u(x)| \quad (43)$$

for every function $u \in C_0^\infty(B(x_0, R))$ and every $x_0 \in R^{2d}$.

This is a classical Sobolev inequality for $x_0 = 0, \quad M = 0$; it holds $x_0 \neq 0$ since the operator commutes with translation and it is satisfied for every M since

$$\mathcal{F}_{x \to \zeta}[(1 + q|x|^2)^M e^{iB(D)} u](\zeta) = e^{iB(\zeta)} \sum_{|\alpha|+|\beta| \leq 2M} C_{\alpha,\beta} \mathcal{F}_{x \to \zeta}[x^\beta \, \partial^\alpha u](\zeta) \quad (44)$$

where $\mathcal{F}_{x \to \zeta}$ denotes total Fourier transform and the constants $c_{\alpha,\beta}$ depend only on the dimension n and on the quadratic form B.

This ends our sketch of the proof of the theorem of Calderon–Valliantcourt.

♡

4 Classes of Pseudo-differential Operators. Regularity Properties

To characterize other classes of pseudo-differential operators we introduce two further definitions.

Definition 2 We shall denote by *tempered weight* on R^d a continuous positive function $m(x)$ for which there exist positive constants $C_0, \ N_0$ such that

$$\forall x, \ y \in R^d \qquad m(x) \leq C_0 m(y)(1 + |y - x|)^{N_0} \quad (45)$$

◇

Definition 3 If Ω is open in R^d, $\rho \in [0, 1]$ and m is a tempered weight, we denote *symbol of weight* (m, ρ) in Ω a function $a \in C^\infty(\Omega)$ such that

$$\forall x \in \Omega \qquad |\partial^\alpha a(x)| \leq C_\alpha \cdot m(x)(1 + |x|)^{-\rho|\alpha|} \quad (46)$$

We shall denote by $\Sigma_{m,\rho}$ the space of symbols of weight (m, ρ); in particular $\Sigma_\rho \equiv \Sigma_{\iota,\rho}$ where ι is the function identically equal to one. ◇

With these notations one can prove (following the lines of the proof of the Theorem of Calderon–Vaillantcourt).

Theorem 5 *(1) If $a \in \Sigma_{\iota,0}$, there exists $T(d) \in R$ such that*

$$\|Op^w(a)\|_{Tr} \le T(d) \sum_{|\alpha|+|\beta|\le d+2} \int\int |\partial_x^\alpha \partial_\eta^\beta a(x,\eta)| dx d\eta \tag{47}$$

(α and β are multi-indices).
(2) If $a \in \Sigma_{m,0}$ and $lim_{|x|+|\eta|\to\infty} a(x,\eta) = 0$ then the closure of $Op^w(a)$ is a compact operator on $L^2(R^d)$. ◇

The proof is obtained exploiting the duality with Wigner's functions taking into account that both trace class operators and Hilbert–Schmidt operators are sum of one-dimensional projection operators and the eigenvalues converge respectively in l^1 and l^2 norm, and that a compact operator is norm-limit of Hilbert–Schmidt operators.

A *more stringent* condition which is easier to prove (making use of the duality with Wigner's functions) and provides an estimate of the trace norm is given by the following theorem

Theorem 6 *Let $a \in \Sigma_{\iota,0}$ be such that for all multi-indices α, β*

$$\partial_x^\alpha \partial_\eta^\beta a \in L^1(R^{2d}) \tag{48}$$

Then $Op^w(a)$ is trace class and one has

$$trOp^w(a) = \int\int a(x,\eta) dx d\eta \tag{49}$$

◇

Since Hilbert–Schmidt operators form a Hilbert space, it is easier to verify convergence and then to find conditions on the symbol such that the resulting operator be of Hilbert–Schmidt class. A first result is the following

Theorem 7 *Let $a \in \Sigma_{m,0}$, $b \in \mathcal{S}(R^{2n})$. Then*

$$tr[Op^w(a) \cdot Op^w(b)] = \int\int a(x,\xi) b(x,\xi) dx d\xi \tag{50}$$

◇

Proof If $B \equiv Op^w(b)$ is a rank one operator $B = \psi \otimes \phi \qquad \psi, \quad \phi \in \mathcal{S}$ one has

$$tr(A \cdot B) = (\phi, A\psi) \qquad A = Op^w(a) \tag{51}$$

From the definition of $Op^w(a)$ it follows

$$(\phi, A\psi) = \int\int a(x,p)\left[\int e^{ip\zeta}\bar{\phi}(x+\frac{\zeta}{2})\psi(x-\frac{\zeta}{2})d\zeta\right]dxdp \tag{52}$$

and (50) is proven in this particular case.

The proof is the same if B has finite rank, and using the regularity of $a(x,p)$, $b(x \cdot p)$ one achieves the proof of (50).

♡

The bilinear form

$$A,\ B \to Tr(A^*B) \equiv < A, B > \tag{53}$$

can be extended to

$$\mathcal{L}(\mathcal{S}(R^{2d}), \mathcal{S}'(R^{2d})) \times \mathcal{L}(\mathcal{S}'(R^{2d}), \mathcal{S}(R^{2d})) \tag{54}$$

with the property $< A, B >=< B, A >^*$.

This duality can be used to extend the definition of the symbol $\sigma^w(a)$ to an operator-valued tempered distribution A by

$$Tr(A \cdot Op^w(b)) = 2\pi^{-n}(b, \sigma^w(a)) \tag{55}$$

and the duality can extended to symbols belonging to Sobolev classes dual with respect to $L^2(R^d)$. Remark that one has

$$a \in L^2(R^{2d}) \leftrightarrow Op^w(a) \in H.S. \tag{56}$$

but $|a|_\infty < \infty$ does not imply that $a(D,x)$ be bounded.

For example if $a(\xi, x) = e^{i(\xi,x)}$ one has $(a(D,x))f(x) = \int f(y)dy \cdot \delta(x)$.

It is convenient to introduce a further definition.

Definition 4 (*Class O(M)*) A function a on $C^d \equiv R^{2d}$ belongs to $O(M)$ if and only if $f \in C^\infty(R^d)$ and for every multi-index $m\ :\ |m| = M$ one has

$$|\frac{\partial^m}{\partial z^m}a(z)| \le C|z^M|, \qquad \forall z \in C^d \tag{57}$$

We shall denote by Σ_M the collection of functions in $O(M)$.

◇

Following the lines of the proof of Theorem 7 one proves

Theorem 8 *(i) If $a \in O(0)$, then $Op^w(a)$is a bounded operator*
(ii) If $a \in O(M)$, $M \leq -2d$ then $Op^w(a)$ is trace-class and

$$TrOp^w(a) = (2\pi\,\hbar)^{-d} \int |A(z)|\, dz \tag{58}$$

(iii) If $a \in O(M)$ is real, then $Op^w(a)$ is essentially self-adjoint on $C_0^\infty(R^d)$.

◇

5 Product of Operator Versus Products of Symbols

The next step is to establish the correspondence between the product of symbols and the product of the corresponding operators. We can inquire e.g. whether, given two symbols a e b, there exists a symbol c such that $Op^w(c) = Op^w(a) \cdot Op^w(b)$.

The answer is in general no. To obtain a (partially) positive answer it will be necessary to *enlarge the class of symbols considered* and add symbols that depend *explicitly* on the small parameter $\hbar$.

In their dependence on $\hbar$ they must admit an expansion to an order M such that the remainder has the regularity properties that imply that the corresponding operator is a bounded operator with suitable estimates for its norm.

This possibility to control the residual term is *an important advantage* of the (strict) quantization with pseudo-differential operators as compared to formal power series quantization. We limit ourselves to consider pseudo-differential operators with symbols in $O(M)$.

Definition 5 ($\hbar$ *admissible symbol*) A $\hbar-$ admissible symbol of weight M is a C^∞ map from $\hbar \in]0, \hbar_0]$ to Σ_M such that there exists a collection of functions $a_j(z) \in O(M)$ with the property that, for every integer N and for every multi-index γ with $|\gamma| = N$ there exists a constant C_N such that

$$sup_z \left[\left(\frac{1}{1+|z|^2} \right)^{d/2} \left| \frac{\partial^\gamma}{\partial z^\gamma} a(z,\ \hbar) - \sum_1^N \hbar^j a_j(z) \right| < c_N \hbar^{N+1} \tag{59}$$

◇

Definition 6 ($\hbar$*-admissible operator*) An $\hbar$-admissible operator of weight M is a C^∞ map

$$A_\hbar \quad : \hbar \in]0, h_0 \Rightarrow \mathcal{L}(\mathcal{S}(R^d), L^2(R^d)) \tag{60}$$

for which there exists a sequence of symbols $a_j \in \Sigma_M$ and a sequence $R_N \in \mathcal{L}(L^2(R^d))$ such that for all $\phi \in \mathcal{S}$

$$A_\hbar = \sum_1^N h^j Op_h^w a_j + R_N(\hbar), \quad sup_{0<\hbar\leq\hbar_0} |R_N(h)\phi|_2 < \infty \quad \forall \phi \in L^2(R^d) \tag{61}$$

The function $a_0(z)$ is called *principal symbol* of the $\hbar$-admissible operator $A_\hbar$; it will be denoted $\sigma_P(A_\hbar)$.

The function $a_1(z)$ is called *sub-principal symbol* of the $\hbar$-admissible operator $A_\hbar$; it will be denoted $\sigma_{SP}(A_\hbar)$.

$\diamond$

Definition 7 (*class* $\hat{O}^{s.c.}$ *operators*) We shall denote $\hat{O}^{sc}_M$ the set in $\mathcal{L}(\mathcal{S}(X))$ (the collection of all bounded operators in $\mathcal{S}(X)$) that is obtained associating to each function in Σ_M the operator obtained by Weyl quantization.

This class of operators is sometimes called $\hbar$-admissible.

$\diamond$

The following theorem states that the $\hbar$-admissible operators form an algebra:

Theorem 9 *For any pair $a \in O(M)$ and $b \in O(P)$ there exists unique a semiclassical observable $\hat{C} \in O^{sc}_{M+P}$ such that*

$$Op^w(a) \cdot Op^w(b) = \hat{C} \tag{62}$$

The semiclassical observable has the representation

$$\hat{C} = \sum \hbar^j Op^w(c_j) \tag{63}$$

$$c_j = 2^{-j} \sum_{|\alpha|+|\beta|=j} \frac{(1)^\beta}{\alpha!\ \beta!} (D_x^\beta\ D_\xi^\alpha a)\ (D_x^\beta\ D_\xi^\alpha b)(x, \xi) \tag{64}$$

Moreover

$$\frac{i}{\hbar}[Op^w(a), Op^w(b)] \in \hat{O}_{sc}(M + P) \tag{65}$$

with principal symbol the Poisson bracket $\{a, b\}$.

$\diamond$

Sketch of the Proof

The proof follow the same lines as the proof of the Theorem of Calderon–Vaillantcourt and makes use of the definition of pseudo-differential operator, the duality with Wigner's functions and the explicit form of the phase factor in Weyl product.

Notice that the structure of Weyl algebra implies that if L is a linear form on R^{2d} and $a \in O(M)$ one has for any linear operator L

$$L(x, \hbar\nabla) Op^w_\hbar(a) = Op^w_\hbar(b), \qquad b = L \cdot a + \frac{\hbar}{2i}\{L, a\} \tag{66}$$

where $\{., .\}$ denotes Poisson brackets.

This remark is useful to write in a more convenient form the product of the phase factors that enter in the definition of the product $Op^w(a) \cdot Op^w(a_1)$.

Recall that by definition

$$(Op^w_\hbar a\phi)(x) = \hbar^{-n} \int\int a(y,z) e^{\frac{2\pi i}{\hbar}(\frac{x+z}{2},\xi)} \phi(y) dy d\xi \tag{67}$$

and that $Op^w_\hbar(b)$ is given by a similar expression.

The integral kernel $K_{Op^w_\hbar(a).Op^w_\hbar(b)}$ of the operator $Op^w_\hbar(a),\ Op^w_\hbar(b)$ is then given by

$$K_{Op^w_\hbar(a)\cdot Op^w_\hbar(b}(x,y)$$
$$= \hbar^{2n} \int\int e^{\frac{i}{\hbar}[(x-z,\xi)+(z-y,\eta)]} a\left(\frac{1}{2}(x+z),\xi\right)\ b\left(\frac{1}{2}(y+z),\eta)\right) dz d\xi d\eta \tag{68}$$

In general there is no symbol c such that $C_\hbar \equiv Op^w_\hbar(a) Op^w_\hbar(b) = Op^w(c)$

One can verify, making use of (67), that if $a_\hbar,\ b_\hbar \in O(M)$ the operator $C_\hbar$ is $\hbar$-admissible, i.e. for some $N \in \mathcal{Z}$ it can be written as

$$C_\hbar = \sum_{n=0,\ldots N} c_n(\hbar) + \hbar^{N+1} R_{N+1}(h) \tag{69}$$

where $a_n \in O(M)$ and $sup_{h\in[0,\hbar_0]} \|R_{N+1}(h)\|_{\mathcal{L}(L^2(R^d))}$.

♡

With the new definition Theorem 9 *can be extended to all semiclassical observables*.

For each $A \in O^{sc}(M),\ B \in O^{sc}(N)$ there exists *a unique semiclassical observable* $C \in \hat{O}^{sc}(N{+}M)$ such $\hat{A}\cdot\hat{B} = \hat{C}$.

Moreover the usual composition and inversion rules apply.

6 Correspondence Between Commutators and Poisson Brackets; Time Evolution

From the analysis given above one derives the following relations.

Let $A_\hbar$ and $B_\hbar$ be two $\hbar$-admissible operators and denote by $\sigma_P(A)$ the principal symbol of A and by $\sigma_{SP}(A)$ its sub-principal symbol. Then

(1)

$$\sigma_P(A_\hbar \cdot B_\hbar) = \sigma_P(A_\hbar)\cdot\sigma_P(B_\hbar) \tag{70}$$

(2)

$$\sigma_{SP}(A_\hbar \cdot B_\hbar) = \sigma_P(A_\hbar)\cdot\sigma_P(B_\hbar) + \sigma_{SP}(A_\hbar)\cdot\sigma_P(B_\hbar) + \frac{\hbar}{2i}\{\sigma_P(A_\hbar)\cdot\sigma_P(B_\hbar)\} \tag{71}$$

These relations give the correspondence between the commutator of two quantum variables and the Poisson brackets of the corresponding classical variables.

The introduction of semiclassical observables is also useful in the study of time evolution. One has [2–4].

Theorem 10 *Let $H \in O^{sc}(2)$ be a classical hamiltonian satisfying*

$$|\partial_z^\gamma H_j(z)| < c_\gamma, \qquad \gamma + j \geq 2 \tag{72}$$

$$\hbar^{-2}(H - H_0 - \hbar H_1) \in \hat{O}_{sc}(0) \tag{73}$$

Let $a \in O(m)$, $m \in \mathcal{Z}$. Then
(i) For any sufficiently small value of $\hbar$, $\hat{H}$ is essentially selfadjoint with natural domain $S(X)$. Therefore $exp\{-i\hbar^{-1}\hat{H}t\}$ is well defined and unitary for each value of t and continuous in t in the strong topology.
(ii)

$$\forall t \in R, \quad Op^w(a(t)) \equiv e^{i\frac{t}{\hbar}\hat{H}} Op^w(a) e^{-i\frac{t}{\hbar}\hat{H}} \in \hat{O}_{sc}(m) \tag{74}$$

Moreover

$$a(t) = \sum_{k \geq 0} \hbar^k a_k(t) \quad a_k(t) \in \quad O_{sc}(m) \tag{75}$$

uniformly over compacts. ◇

Proof (outline) Under the conditions stated the classical flow $z \to z(t)$ exists globally. From the properties of the tangent flow it easy to deduce that $a(z(t)) \in O(m)$ uniformly over compacts in t.

With $U_H(t) = exp\{-i\frac{t}{\hbar}\hat{H}\}$ Heisenberg equations give

$$\begin{aligned} &\frac{d}{ds}(U_H(-s)Op^w(a(z(t-s)))U_H(s) \\ &\quad = U_H(-s)(\frac{i}{\hbar}[H, Op^w(a(z(t-s)))] - Op^w(\{H, a_0\})])U_H(s) \end{aligned} \tag{76}$$

From the product rule one derives then that the principal symbol of

$$\frac{i}{\hbar}[H, Op^w a_0(t-s)] - Op^w(\{H_0, a_0(t-s)\}) \tag{77}$$

vanishes. Therefore the right hand side of (76) is of order one in $\hbar$ and the thesis of the follows by a formal iteration as an expansion in $\hbar$ through Duhamel series.

Using the estimates one proves convergence of the series.

♡

Remark that if H is a polynomial at most of second order in x, $i\frac{d}{dx}$ one has

$$(Op^w(a))(t) = Op^w(a(z(t))) \tag{78}$$

where ψ_H^t is the classical solution of Hamilton's equations. Indeed in this case one has

$$\frac{i}{\hbar}[\hat{H}, Op(b)] = Op(h, b) \tag{79}$$

In particular if $W(z)$ is an element of Weyl's algebra

$$W(z)Op(b)W(-z) = Op(b_z) \qquad b_z(z') = b(z' - z) \tag{80}$$

(this is a corollary of Eherenfest theorem). Relation (80) does not hold in general if H is not a polynomial of order ≤ 2.

Still, under the assumptions of Theorem 10 a relation of type (80) holds in the limit $\hbar \to 0$ in a weak sense, i.e. as an identity for the matrix elements between *semiclassical states* (e.g. coherent states). We have remarked this in our analysis of the semiclassical limit in volume I of these Lecture Notes.

Theorem 10 can be extended to Hamiltonians which are not in $O(2)$ (for example to Hamiltonians of type $H = \frac{p^2}{2} + V(q)$ with V bounded below) if the classical hamiltonian flow is defined globally in time.

Weyl quantization can be extended to distributions in $\mathcal{S}'$; in this case the operator $\hat{A}$ is bounded from $\mathcal{S}$ to $\mathcal{S}'$ and the correspondence it induces is a bijection.

This follows from an analogue of a Theorem of L.Schwartz which states that every bilinear map from $\mathcal{S}(X)$ to $L^2(X)$ continuous in the $L^2(X)$ topology can be extended as a continuous map from $\mathcal{S}(X)$ to $\mathcal{S}'(X)$.

A way to achieve this extension exploits the properties of Weyl symbol $Op^W(\Pi_{u,v})$ of the rank-one operator $\Pi_{u,v}$ defined, for $u,\ v \in \mathcal{S}(X)$, by

$$\Pi_{u,v}\psi = (\psi, u)v \tag{81}$$

One has then

$$(Op^w(a)u, v) = (2\pi\hbar)^{-1} \int a(x, \xi)\pi_{u,v}(x, \xi)dxd\xi \tag{82}$$

since by definition

$$< Op^w(a)u, v >= Tr(\Pi_{u,v}Op^w(a)) = \int \Pi_{u,v}(x, \xi)A(x,\ \xi)dx\, d\xi \tag{83}$$

The function $\pi_{u,v}(x, \xi)$ is the Wigner function of the pair $u,\ v$. Remark that

$$(Op^w(a)u, v) = (2\pi\ \hbar)^{-n} \int a(z)\pi_{u,v}(z)dz \tag{84}$$

The definition of pseudo-differential operator on a Hilbert space $\mathcal{H}$ can be extended to the case in which the symbol $a(q, p)$ is itself an operator on a Hilbert space $\mathcal{K}$.

A typical case, occasionally used in information theory, is the one in which the *phase space* is substituted with the (linear) space of the (Hilbert) space of Hilbert–Schmidt operators with the commutator as symplectic form.

This linear space is itself a Hilbert space with scalar product $< A, B >= Tr(A^B)$ (in information theory the Hilbert space $\mathcal{K}$ on which the Hilbert–Schmidt operators act is usually chosen to be finite-dimensional).

Another case, which has gained relevance in the Mathematics of Solid State Physics, is the treatment of adiabatic perturbation theory *through the Weyl formalism* [10].

This procedure is useful e.g. in the study of the dynamics of the atoms in crystals but also in the study of a system composed of N nuclei of mass m_N with charge Z and of NZ electrons of mass m_e.

In the latter case one chooses the ratio $\epsilon \equiv \frac{m_e}{m_N}$ as small parameter in a multi-scale approach. We shall come back to this problem in Lecture 5.

7 Berezin Quantization

A quantization which associates to a positive function a positive operator in the Berezin quantization defined by means of *coherent states* i.e. substituting the Wigner function with its Husimi transform.

This quantization does not preserve polynomial relations, the product rules are more complicated than in Weyl quantization and the equivalent to Eherenfest theorem does not hold.

Recall that a coherent state "centered in the point" $(y,\ \eta)$ of phase space is by definition

$$\phi_{y,\eta} \equiv e^{\frac{i}{\hbar}(\eta,x)+i(y\ D_x)}\phi_0(x) \tag{85}$$

where $\phi_0(x)$ is the ground state of the harmonic oscillator for a system with d degrees of freedom.

$$\phi_0 \equiv (\pi\ \hbar)^{-n} e^{-\frac{|x|^2}{2\,\hbar}} \tag{86}$$

Definition 8 The *Berezin quantization* of the classic observable a is the map $a \to Op^B(a)$ given by

$$Op^B_\hbar(a)\phi \equiv (2\pi\ \hbar)^{-d} \int\int a(y,\eta)(\psi, \bar{\phi}_{y,\eta})\phi_{y,\eta} dy\ d\eta \tag{87}$$

$\diamond$

One can prove, either directly or through its relation with Weyl quantization to construct $Op^B_\hbar(a)$, that the Berezin quantization has the following properties:
(1) If $a \geq 0$ then $Op^B_\hbar(a) \geq 0$
(2) The Weyl symbol a^B of the operator $Op^B_\hbar(a)$ is

$$a^B(x,\xi) = (\pi\,\hbar)^{-d} \int\int a(y,\eta) e^{-\frac{1}{\hbar}[(x-y)^2+(\xi-\eta)^2]} dy\, d\eta \tag{88}$$

(3) For every $a \in O(0)$ (bounded with all its derivatives) one has

$$||Op^B_\hbar(a) - Op^w_\hbar(a)|| = O(\hbar) \tag{89}$$

We have noticed that the Berezin quantization is dual to the operation that associates to a vector ψ in the Hilbert space $\mathcal{H}$ a *positive measure* μ_ψ in phase space, the *Husimi measure*.

On the contrary Weyl's quantization is dual to the operation which associates to ψ the Wigner function W_ψ, which is real but not positive in general.

We recall the

Definition 9 (*Husimi measure*) The Husimi's measure μ_ϕ associated to the vector ϕ is defined by

$$d\mu_\psi = \tilde{\rho}(q,p)dq\,dp \qquad \tilde{\rho}(q,p) \equiv |(\phi_{q,p},\psi)|^2 \tag{90}$$

From this one derives that Husimi measure is a positive Radon measure. Its relation with Berezin quantization is given by

$$\int a\, d\mu_\psi = (Op^B_\hbar(a)\psi,\psi), \quad a \in \mathcal{S} \tag{91}$$

$\diamond$

Although it gives *a map between positive functions and positive operators* the Berezin quantization is less suitable for a description of the evolution of quantum observables. In particular Eherenfest's and Egorov's theorems do no hold and the semiclassical propagation theorem has a more complicated form.

The same is true for the formula that gives the Berezin symbol of an operator which is the product of two operators $Op^B_\hbar(a)Op^B_\hbar(b)$ where $a,\ b$ are functions on phase space.

Berezin representation is connected the Bargman–Segal representation of the Weyl system (in the same way as Weyl representation has its origin in the Weyl–Schroedinger representation).

Recall that the Bargman–Segal representation is set in the space of function over C^d which are holomorphic in the sector $Imz_k \geq 0,\ k = 1,\ldots,d$ and square integrable with respect to the gaussian probability measure

$$d\mu_r(z) = \left(\frac{r}{\pi}\right)^d e^{-r|z|^2} dz, \quad r > 0 \tag{92}$$

We shall denote this space $\mathcal{H}_r$. In the formulation of the semiclassical limit the parameter r plays the role h^{-1}.

8 Toeplitz Operators

In the Berezin representation an important is played by the *Toeplitz operators*.

For $g \in L^2(d\mu_r)$ the Toeplitz operator $T_g^{(r)}$ is defined on a dense subspace of $\mathcal{H}_r$ by

$$(T_g^{(r)} f)(z) = \int g(w) f(w) e^{i(z.w)} d\mu_r(w) \tag{93}$$

In part I of these Lecture Notes we introduced the *reproducing kernel* $e^{i(z.w)}$ within the discussion of the Bargman–Segal representation.

Remark that if $g f \in L^2(d\mu_r)$ then $T_g^{(r)} f \in \mathcal{H}_r$. The map $g \to T_g^{(r)}$ (Berezin quantization) is a complete strict deformation (the deformation parameter is r^{-1}).

Under the Bargman–Segal isometry $B_r : L^2(R^n, dx) \to \mathcal{H}_r$ the Weyl–Schroedinger representation is mapped onto the Bargman–Segal complex representation and the quantized operators $\hat{z}_k$ are mapped into Toeplitz operators.

For these Toeplitz operators are valid the same "deformation estimates" which hold in Berezin quantization (and are useful when studying the semiclassical limit)

$$\| T_f^{(r)} T_g^{(r)} - T_{fg}^{(r)} + \frac{1}{r} T^{(r)}_{\sum_j (\frac{\partial f}{\partial z_j} \frac{\partial g}{\partial \bar{z}_j})} \|_r \leq C(f, g) r^{-2} \tag{94}$$

The interested reader can consult [6, 7] for the Berezin quantization and its relation with Toeplitz operators.

Let us remark that Berezin quantization is rarely used in non relativistic Quantum Mechanics to describe the dynamics of particles; as mentioned, Eherenfest's and Egorov's Theorems do not hold and the formulation of theorems about semiclassical evolution is less simple.

On the contrary Berezin quantization is much used in Relativistic Quantum Field Theory (under the name of *Wick quantization*) since it leads naturally to the definition of *vacuum state Ω* (in the case of finite number of degrees of freedom in the Schrödinger representation it is constant function) as the state which is *annihilated* by $\frac{\partial}{\partial z_k}$, $\forall k$) and of the $\sum_k z_k \partial z_k$ (*number operator*).

In the infinite dimensional case it is useful to choose as *vacuum* a gaussian state or equivalently use instead of the Lebesgue measure a Gaussian measure (which is defined in R^∞)

In turn this permits the definition of *normal ordered polynomials* (or *Wick ordered polynomials*) in the variables z_h, $\frac{\partial}{\partial z_k}$; the normal order is defined by the prescription that all the operators $\frac{\partial}{\partial z_k}$ stand to the right of all operators z_k.

The Berezin quantization is much used in Quantum Optics where the coherent states play a dominant role (coherent states are in some way "classical states" of the quantized electromagnetic field).

9 Kohn–Nirenberg Quantization

For completeness we describe briefly the quantization prescription of Kohn-Nirenberg [7], often introduced in the study of inhomogeneous elliptic equation and of the regularity of their solutions. It is seldom used in Quantum Mechanics.

By definition

$$(\sigma_{K.N.}(D,x)f)(x) \equiv \int \sigma(\xi,x)e^{i(x-y)}f(y)dyd\xi \\ \times \int\int \hat{\sigma}(p,q)(e^{i(q,x)}e^{i(p,D)}f)(x)dp\,dq \tag{95}$$

In the particular case $\sigma(\xi,x) = \sum a_k(x)\xi^k$ one has

$$\sigma_{K.N}(D,x) = \sum_k a_k(x)D^k \tag{96}$$

In the Kohn–Nirenberg quantization the relation between an operator and its *symbol* is

$$Op^{KN}(a)\phi(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{d}{2}} \int_{R^d} e^{\frac{i}{\hbar}px} a^{K,N}(x,p)\hat{\phi}(p)dp \tag{97}$$

where we have indicated with $\hat{\phi}$ the Fourier transform of ϕ.

This is the definition of pseudo-differential operator that is found in most books of Partial Differential Equations. In this theory one proves that the pseudo-differential operators are singled out by the fact that they satisfy the weak maximum principle.

In general if the K.N. symbol $a^{K,N}(q,p)$ is real the operator $Op^{K,N}(a)$ is not (essentially) self-adjoint. The quantization of Kohn–Nirenberg is usually employed in micro-local analysis and also in the time-frequency analysis since in these fields it leads to simpler formulations [2, 9].

In general this quantization is most useful when considering equations in which the differential operators appear as low order polynomials (usually second or fourth); in this case it is not interesting to study operators of the form $L(x, \nabla)$ for a generic smooth function L.

If the K.N. operators do not depend polynomially in the differential operators, their reduction to spectral subspaces is not easy. For this reason Weyl quantization is preferred in solid state physics when one wants to analyze operators which refer to *Bloch bands*.

10 Shubin Quantization

The quantizations of Weyl and of Berezin are particular cases of a more general form of quantization, parametrized by a parameter $\tau \in [0, 1]$, as pointed out by Shubin [4].

In this more general form to the function $a \in \mathcal{S}(R^{2d})$ one associates the continuous family of operators $Op^{S,\tau}(a)$ on $\mathcal{S}(R^d)$ defined by

$$(Op^{S,\tau}(a)\phi)(x) = \left(\frac{1}{2\pi\hbar}\right)^d \int\int a((\tau x + (1-\tau)y), \xi) f(y) e^{\frac{i}{\hbar}(x-y,\xi)} dy d\xi \quad (98)$$

It is easy to verify that the choice $\tau = \frac{1}{2}$ corresponds to Weyl's quantization, $\tau = 0$ to Kohn–Niremberg's and $\tau = 1$ to Berezin's. Notice that Eherenfest theorem holds only for $\tau = \frac{1}{2}$.

It is easy to verify that only for $\tau = \frac{1}{2}$ the relation between the operator and its symbol is *covariant* under linear symplectic transformations. In general if $s \in Sp(2d, R)$ is a linear symplectic transformation, there exists a unitary operator S such that

$$S^{-1}(s) Op^w(a) S(s) = Op^w(a \circ s) \quad (99)$$

$S(s)$ belongs to a representation of the metaplectic group generated by quadratic form in the canonical variables.

For all values of the parameter τ one has

$$\mathcal{F} Op^{S,\tau} \mathcal{F}^{-1} = Op^{S,1-\tau}(a \circ J^{-1}) \quad (100)$$

where J is the standard symplectic matrix and $\mathcal{F}$ denoted Fourier transform.

One can consider also *Wigner functions* associated to Shubin's τ-quantization. In particular

$$W_\tau(\phi, \psi)(x \cdot p) = \left(\frac{1}{2\pi\hbar}\right)^d \int_{R^d} e^{-\frac{i}{\hbar}py} \phi(x + \tau y) \psi(x - (1-\tau)y) dy \quad (101)$$

Independently of the value of the parameter τ one has

$$\int_{R^d} W_\tau(x, p) dp = |\phi(x)|^2, \qquad \int_{R^d} W_\tau(x, p) dx = |\hat{\phi}(p)|^2 \quad (102)$$

The relation between W_τ and $Op_\tau(a)$ is

$$(Op_\tau(a)\psi, \phi)_{L^2} = (a, W_\tau(\psi, \phi)) \quad (103)$$

For all τ

$$\Pi_\phi(x, y)) = [2\pi\hbar]^{-d} W_\tau \phi(x, p) \tag{104}$$

where Π_ϕ is the projection operator on the vector ϕ.

11 Born–Jordarn Quantization

We end this lecture with the quantization introduced by Born and Jordan [11] to give a prescription for associating operators to functions over classical phase space of the form $\sum_k f_k(x) P_k(p)$ where $f_k(x),\ x \in R^d$ are sufficiently regular function and $P_k(p)$ are polynomials in the momenta $\{p_j\}$.

Notice that all Hamiltonians introduced in non relativistic Quantum Mechanics have this structure. The correspondence proposed by Born and Jordan is

$$f(x)p_j^n \to \frac{1}{n+1} \sum_{k=0}^{n} \hat{p}_j^{n-k} f(\hat{(x)}) \hat{p}_j^k \tag{105}$$

where $\hat{x}_j$ (in the Schroedinger representation) is multiplication by x_j and $\hat{p}_j = -i\hbar \frac{\partial}{\partial x_{j.}}$

For comparison, Weyl quantization corresponds to the prescription

$$f(x)p_j^k \to \frac{1}{2^k} \sum_{m=0}^{k} \frac{m!}{m!(k-m)!} \hat{p}_j^{n-m} f(\hat{x}) \hat{p}_j^m \tag{106}$$

One has

$$Op^{BJ}(a) = \left(\frac{1}{2\pi}\right)^d \int Op_\tau(a) d\tau \tag{107}$$

Weyl's prescription *coincides with that of Born and Jordan* if the monomial is of rank at most two in the momentum.

Therefore the quantization of Born and Jordan coincides with Weyl's for Hamiltonians that are of polynomial type in the position coordinates (the Hamiltonians that are of common use in Quantum Mechanics).

Also for the quantization of the magnetic hamiltonian the B–J quantization coincides with the Weyl quantization.

One can verify that the symbol of $a_{B,J}$ of operator A in Born–Jordan quantization is given by

$$a_W = \left(\frac{1}{2\pi}\right)^d a * \mathcal{F}_\sigma \Theta \tag{108}$$

where $\mathcal{F}_\sigma$ is the symplectic Fourier transform and the function Θ is given by

$$\Theta(z) = \frac{sin\frac{px}{\hbar}}{\frac{px}{\hbar}} \tag{109}$$

This implies that the symbols a_w and $a_{B,J}$ are related by

$$a_w = \left(\frac{1}{2\pi}\right)^d a_{B,J} * \mathcal{F}_\sigma \Theta \tag{110}$$

therefore $a_{B,J}$ is not determined by a_w.

Through (110) one can define the equivalent of the Wigner function (Born–Jordan functions) in phase space. They are not positive but the negative part for elementary elements is somewhat reduced.

The quantization of Born and Jordan is related to the Shubin quantization by the formula

$$Op^{BJ}(a)\phi = \left(\frac{1}{2\pi\hbar}\right)^d \int_0^1 Op^{S,\tau}(a)\phi d\tau \tag{111}$$

on a suitable domain (in general the operator one obtains is unbounded). From the relation between $Op^{S,\tau}(a)$ and $Op_{1-\tau}(\bar{a})$ one derives

$$Op^{BJ}(a)^* = Op^{BJ}(\bar{a}) \tag{112}$$

Therefore the operator $Op^{BJ}(a)$ is formally self-adjoint if and only if a is a real function.

The relation between a *magnetic Born–Jordan quantization* and the quantization given by the magnetic Weyl algebra is still unexplored.

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