

Chapter 2

Noether's First Theorem

We here are concerned with Lagrangian theory on fibre bundles (Chap. 1). In this case, Noether's first theorem (Theorem 2.7) and Noether's direct second theorem (Theorem 2.6) are corollaries of the global variational formula (1.36).

2.1 Lagrangian Symmetries

Noether's theorems deal with infinitesimal transformations of Lagrangian systems.

Definition 2.1 Given a Lagrangian system $(\mathcal{O}_\infty^* Y, L)$ (Definition 1.5), its *infinitesimal transformations* are defined to be contact derivations of a real ring $\mathcal{O}_\infty^0 Y$ [59, 61].

The derivation $\vartheta \in \mathfrak{D}\mathcal{O}_\infty^0 Y$ (1.16) is termed *contact* if the Lie derivative \mathbf{L}_ϑ (1.18) along ϑ preserves an ideal of contact forms of a DGA $\mathcal{O}_\infty^* Y$, i.e., the Lie derivative \mathbf{L}_ϑ of a contact form is a contact form.

Theorem 2.1 *The derivation ϑ (1.16) is contact if and only if it takes a form*

$$\vartheta = v^\lambda \partial_\lambda + v^i \partial_i + \sum_{0 < |\Lambda|} [d_\Lambda(v^i - y_\mu^i v^\mu) + y_{\mu+\Lambda}^i v^\mu] \partial_i^\Lambda. \quad (2.1)$$

Proof The expression (2.1) results from a direct computation similar to that of the first part of Bäcklund's theorem [76]. One can then justify that local functions (2.1) satisfy the transformation law (1.17). \square

Comparing the expressions (2.1) and (B.81) enables one to regard the contact derivation ϑ (2.1) as the infinite order jet prolongation

$$\vartheta = J^\infty v \quad (2.2)$$

of its restriction

$$v = v^\lambda \partial_\lambda + v^i \partial_i \quad (2.3)$$

to a ring $C^\infty(Y)$. Since coefficients v^λ and v^i of v (2.3) generally depend on jet coordinates y_Λ^i of bounded jet order $0 < |\Lambda| \leq N$, one calls v (2.3) the *generalized vector field*. It can be represented as a section of the pull-back bundle

$$J^N Y \times_Y TY \rightarrow J^N Y.$$

Let ϑ and ϑ' be contact derivations (2.1) whose restrictions to a real ring $C^\infty(Y)$ are generalized vector fields v and v' , respectively. Certainly, their Lie bracket $[\vartheta, \vartheta']$ is a contact derivation. Its restriction to $C^\infty(Y)$ reads

$$[v, v']_J = [\vartheta(v^\lambda) - \vartheta'(v^\lambda)]\partial_\lambda + [\vartheta(v^i) - \vartheta'(v^i)]\partial_i. \quad (2.4)$$

We call it the *bracket of generalized vector fields* $[v, v']_J$. It obeys a relation

$$[J^\infty v, J^\infty v'] = J^\infty([v, v']_J). \quad (2.5)$$

If v and v' are vector fields on Y , their bracket (2.4) is the familiar Lie one.

The contact derivation ϑ (2.1) is said to be *projectable*, if the generalized vector field v (2.3) projects onto a vector field $v^\lambda \partial_\lambda$ on X , i.e., its components ϑ^λ depend only on coordinates on X . In particular, it is readily observed that the bracket (2.4) of projectable generalized vector fields also is projectable.

Every contact derivation ϑ (2.1) admits the canonical splitting

$$\vartheta = \vartheta_H + \vartheta_V = v^\lambda d_\lambda + J^\infty v_V = v^\lambda d_\lambda + [v_V^i \partial_i + \sum_{0 < |\Lambda|} d_\Lambda v_V^i \partial_i^\Lambda], \quad (2.6)$$

$$v = v_H + v_V = v^\lambda d_\lambda + (v^i - y_\mu^i v^\mu) \partial_i = v^\lambda d_\lambda + v_V^i \partial_i, \quad (2.7)$$

into the horizontal and vertical parts ϑ_H and ϑ_V , respectively [59].

Theorem 2.2 *Any vertical contact derivation*

$$\vartheta = J^\infty v = v^i \partial_i + \sum_{0 < |\Lambda|} d_\Lambda v^i \partial_i^\Lambda, \quad v = v^i \partial_i,$$

obeys the relations

$$\vartheta \rfloor d_H \phi = -d_H(\vartheta \rfloor \phi), \quad (2.8)$$

$$\mathbf{L}_\vartheta(d_H \phi) = d_H(\mathbf{L}_\vartheta \phi), \quad \phi \in \mathcal{O}_\infty^* Y. \quad (2.9)$$

Proof It is easily justified that, if ϕ and ϕ' satisfy the relation (2.8), then $\phi \wedge \phi'$ does well. Then it suffices to prove the relation (2.8) when ϕ is a function and $\phi = \theta_\Lambda^i$.

The result follows from the equalities

$$\begin{aligned}\vartheta \lrcorner \theta_\Lambda^i &= v_\Lambda^i, \quad d_H(v_\Lambda^i) = v_{\lambda+\Lambda}^i dx^\lambda, \quad d_H \theta_\lambda^i = dx^\lambda \wedge \theta_{\lambda+\Lambda}^i, \\ d_\lambda \circ v_\Lambda^i \partial_i^A &= v_\Lambda^i \partial_i^A \circ d_\lambda.\end{aligned}$$

The relation (2.9) is a corollary of the equality (2.8). \square

Given a Lagrangian system $(\mathcal{O}_\infty^* Y, L)$, let us consider the Lie derivative $\mathbf{L}_\vartheta L$ (1.18) of its Lagrangian L along the contact derivation ϑ (1.9). The global decomposition (1.36) results in the following corresponding splitting of $\mathbf{L}_\vartheta L$.

Theorem 2.3 *Given a Lagrangian $L \in \mathcal{O}_\infty^{0,n} Y$, its Lie derivative $\mathbf{L}_\vartheta L$ along the contact derivation ϑ (2.6) fulfils the first variational formula*

$$\mathbf{L}_\vartheta L = v_V \lrcorner \delta L + d_H(h_0(\vartheta \lrcorner \mathcal{E}_L)) + \mathcal{L}d_V(v_H \lrcorner \omega), \quad (2.10)$$

where \mathcal{E}_L is the Lepage equivalent (1.37) of L and h_0 is the horizontal projection (1.13).

Proof The formula (2.10) comes from the variational formula (1.36) and the relations (2.8)–(2.9) as follows:

$$\begin{aligned}\mathbf{L}_\vartheta L &= \vartheta \lrcorner dL + d(\vartheta \lrcorner L) = [\vartheta_V \lrcorner dL - d_V \mathcal{L} \wedge v_H \lrcorner \omega] + [d_H(v_H \lrcorner L) + \\ &\quad d_V(\mathcal{L} v_H \lrcorner \omega)] = \vartheta_V \lrcorner dL + d_H(v_H \lrcorner L) + \mathcal{L}d_V(v_H \lrcorner \omega) = \\ &\quad v_V \lrcorner \delta L - \vartheta_V \lrcorner d_H \mathcal{E}_L + d_H(v_H \lrcorner L) + \mathcal{L}d_V(v_H \lrcorner \omega) = \\ &\quad v_V \lrcorner \delta L + d_H(\vartheta_V \lrcorner \mathcal{E}_L + v_H \lrcorner L) + \mathcal{L}d_V(v_H \lrcorner \omega),\end{aligned}$$

where

$$\vartheta_V \lrcorner \mathcal{E}_L = h_0(\vartheta_V \lrcorner \mathcal{E}_L), \quad v_H \lrcorner L = h_0(v_H \lrcorner \mathcal{E}_L)$$

since $\mathcal{E}_L - L$ is a one-contact form and $v_H = \vartheta_H$. \square

Definition 2.2 The generalized vector field v (2.7) on Y is called the *Lagrangian symmetry* (or, shortly, the *symmetry*) of a Lagrangian L if the Lie derivative $\mathbf{L}_{J^\infty v} L$ of L along the contact derivation $J^\infty v$ (2.2) is d_H -exact, i.e.,

$$\mathbf{L}_{J^\infty v} L = d_H \sigma, \quad (2.11)$$

where σ is a horizontal $(n-1)$ -form.

Remark 2.1 Certainly, the jet prolongation $J^\infty v$ of a generalized vector field v in the expression (2.11) always is of finite jet order because a Lagrangian L is of finite order (Sect. 3.3). Therefore, we further use the notation $J^* v$ for the finite order jet prolongation of v whose order however is not specified.

Theorem 2.4 A glance at the expression (2.10) shows the following.

- (i) A generalized vector field v is a Lagrangian symmetry only if it is projectable.
- (ii) Any projectable generalized vector field is a symmetry of a variationally trivial Lagrangian.
- (iii) A projectable generalized vector field v is a Lagrangian symmetry if and only if its vertical part v_V (2.7) is well.
- (iv) A projectable generalized vector field v is a symmetry if and only if the density $v_V \rfloor \delta L$ is d_H -exact.

Remark 2.2 In accordance with the standard terminology, symmetries represented by generalized vector fields (2.3) are called *generalized symmetries* because they depend on derivatives of variables. Generalized symmetries of differential equations and Lagrangian systems have been intensively investigated [25, 40, 59, 76, 85, 108]. Accordingly, by symmetries one means only those represented by vector fields $v = u$ on Y . We agree to call them *classical symmetries*.

Remark 2.3 Owing to the relation (2.5), the bracket (2.4) of Lagrangian symmetries is a Lagrangian symmetry. It follows that symmetries constitute a real Lie algebra \mathcal{G}_L with respect to this bracket, and their jet prolongation $v \rightarrow J^*v$ (2.2) provides a monomorphism of this Lie algebra to the Lie algebra $\mathfrak{d}\mathcal{O}_\infty^0 Y$ (1.16).

Remark 2.4 Let v be a classical symmetry of a Lagrangian L , i.e., it is a vector field on Y . Then the relation

$$\mathbf{L}_{J^*v} \mathcal{E}_L = \delta(\mathbf{L}_{J^*v} L) \quad (2.12)$$

holds [53, 108]. It follows that v also is a symmetry of the Euler–Lagrange operator \mathcal{E}_L of L (Definition 4.5), i.e., $\mathbf{L}_{J^*v} \mathcal{E}_L = 0$, and as a consequence it is an infinitesimal symmetry of the Euler–Lagrange equation \mathfrak{E}_L (1.34) (Definition 4.4). However, the equality (2.12) fails to be true in the case of generalized symmetries.

2.2 Gauge Symmetries: Noether's Direct Second Theorem

As was mentioned above, the notion of gauge symmetries comes from Yang–Mills gauge theory on principal bundles (Sect. 8.2). It is generalized to Lagrangian theory on an arbitrary fibre bundle $Y \rightarrow X$ as follows [9, 10].

Definition 2.3 Let $E \rightarrow X$ be a vector bundle and $E(X)$ a $C^\infty(X)$ -module of sections of $E \rightarrow X$. Let ζ be a linear differential operator on $E(X)$ (Definition A.3) with values into a vector space \mathcal{G}_L of symmetries of a Lagrangian L (Remark 2.3). Elements

$$u_\xi = \zeta(\xi) \quad (2.13)$$

of $\text{Im } \zeta$ are termed the *gauge symmetries* of a Lagrangian L parameterized by sections ξ of $E \rightarrow X$. The latter are regarded as the *gauge parameters*.

Remark 2.5 A differential operator ζ in Definition 2.3 takes its values into a vector space \mathcal{G}_L as a subspace of a $C^\infty(X)$ -module $\mathfrak{D}\mathcal{O}_\infty^0 Y$ seen as a real vector space. The differential operator ζ is assumed to be at least of first order (Remark 2.7).

Equivalently, the gauge symmetry (2.13) is given by a section $\tilde{\zeta}$ of a fibre bundle

$$(J^r Y \times_Y J^m E) \times_Y TY \rightarrow J^r Y \times_Y J^m E$$

(Definition B.14) such that $u_\xi = \zeta(\xi) = \tilde{\zeta} \circ \xi$ for any section ξ of $E \rightarrow X$. Hence, it is a generalized vector field u_ζ on a bundle product $Y \times_X E$ represented by a section of the pull-back bundle

$$J^k(Y \times_X E) \times_Y T(Y \times_X E) \rightarrow J^k(Y \times_X E), \quad k = \max(r, m),$$

which lives in $TY \subset T(Y \times_X E)$. This generalized vector field yields the contact derivation $J^\infty u_\zeta$ (2.2) of a real ring $\mathcal{O}_\infty^0[Y \times_X E]$ which obeys the following condition.

- Given a Lagrangian $L \in \mathcal{O}_\infty^{0,n} E \subset \mathcal{O}_\infty^{0,n}[Y \times_X E]$, let us consider its Lie derivative

$$\mathbf{L}_{J^* u_\zeta} L = J^\infty u_\zeta \rfloor dL + d(J^* u_\zeta \rfloor L) \quad (2.14)$$

where d is the exterior differential on $\mathcal{O}_\infty^*[Y \times_X E]$. Then, for any section ξ of $E \rightarrow X$, the pull-back $\xi^* \mathbf{L}_{J^* u_\zeta} L$ is d_H -exact.

It follows from the first variational formula (2.10) for the Lie derivative (2.14) that the above mentioned condition holds only if u_ζ is projected onto a generalized vector field on Y and, in this case, if and only if the density $(u_\zeta)_V \rfloor \mathcal{E}_L$ is d_H -exact. Thus, we come to the following equivalent definition of gauge symmetries.

Definition 2.4 Let $E \rightarrow X$ be a vector bundle. A gauge symmetry of a Lagrangian L parameterized by sections ξ of $E \rightarrow X$ is defined as a generalized vector field u on $Y \times_X E$ such that:

- (i) a contact derivation $\vartheta = J^\infty u$ of a ring $\mathcal{O}_\infty^0[Y \times_X E]$ vanishes on a subring $\mathcal{O}_\infty^0 E$,
- (ii) a generalized vector field u is linear in coordinates χ_Λ^a on $J^\infty E$, and it is projected onto a generalized vector field on E , i.e., it takes a form

$$u = \left(\sum_{0 \leq |\Lambda| \leq m} u_a^{\lambda \Lambda} (x^\mu) \chi_\Lambda^a \right) \partial_\lambda + \left(\sum_{0 \leq |\Lambda| \leq m} u_a^{i \Lambda} (x^\mu, y_\Sigma^j) \chi_\Lambda^a \right) \partial_i, \quad (2.15)$$

- (iii) the vertical part of u (2.15) obeys the equality

$$u_V \rfloor \delta L = d_H \sigma. \quad (2.16)$$

Theorem 2.5 *By virtue of item (iii) of Definition 2.4, u (2.15) is a gauge symmetry if and only if its vertical part is so.*

Gauge symmetries possess the following particular properties.

(i) Let $E' \rightarrow X$ be another vector bundle and ζ' a linear $E(X)$ -valued differential operator on a $C^\infty(X)$ -module $E'(X)$ of sections of $E' \rightarrow X$. Then $u_{\zeta'(\xi')} = (\zeta \circ \zeta')(\xi')$ also is a gauge symmetry of L parameterized by sections ξ' of $E' \rightarrow X$. It factorizes through the gauge symmetries u_χ (2.13).

(ii) The conserved symmetry current \mathcal{J}_u (2.21) associated to a gauge symmetry in accordance with Noether's first theorem (Theorem 2.7) is reduced to a superpotential (Theorem 2.8).

(iii) *Noether's direct second theorem* (Theorem 2.6) associates to a gauge symmetry of a Lagrangian L the Noether identities (NI) of its Euler–Lagrange operator.

Theorem 2.6 *Let u (2.15) be a gauge symmetry of a Lagrangian L , then its Euler–Lagrange operator δL obeys the NI (2.17).*

Proof The density (2.16) is variationally trivial and, therefore, its variational derivatives with respect to variables χ^a vanish, i.e.,

$$\mathcal{E}_a = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_\Lambda [(u_a^{i\Lambda} - y_\lambda^i u_a^{\lambda\Lambda}) \mathcal{E}_i] = \sum_{0 \leq |\Lambda|} \eta(u_a^i - y_\lambda^i u_a^\lambda)^\Lambda d_\Lambda \mathcal{E}_i = 0 \quad (2.17)$$

(see Remark 7.6 for the notation). In accordance with Definition D.1, the equalities (2.17) are the NI for the Euler–Lagrange operator δL . \square

Remark 2.6 If the gauge symmetry u (2.15) is of second jet order in gauge parameters, i.e.,

$$u_V = (u_a^i \chi^a + u_a^{i\mu} \chi_\mu^a + u_a^{i\nu\mu} \chi_{\nu\mu}^a) \partial_i, \quad (2.18)$$

the corresponding NI (2.17) take a form

$$u_a^i \mathcal{E}_i - d_\mu (u_a^{i\mu} \mathcal{E}_i) + d_{\nu\mu} (u_a^{i\nu\mu} \mathcal{E}_i) = 0. \quad (2.19)$$

Let us note that the NI (2.17) need not be independent (Sect. 6.2).

Remark 2.7 A glance at the expression (2.19) shows that, if a gauge symmetry is independent of derivatives of gauge parameters (i.e., a differential operator ζ in Definition 2.3 is of zero order), then all variational derivatives of a Lagrangian equals zero, i.e., this Lagrangian is variationally trivial. Therefore, such gauge symmetries usually are not considered.

Remark 2.8 The notion of gauge symmetries can be generalized as follows. Let a differential operator ζ in Definition 2.3 need not be linear. Then elements of $\text{Im } \zeta$ are called the *generalized gauge symmetry*. However, Noether's direct second Theorem 2.6 is not relevant to generalized gauge symmetries because, in this case, an Euler–Lagrange operator satisfies the identities depending on gauge parameters.

It follows from Noether's direct second Theorem 2.6 that gauge symmetries of Lagrangian field theory characterize its degeneracy. A problem is that any Lagrangian possesses gauge symmetries and, therefore, one must separate them into the trivial and nontrivial ones. Moreover, gauge symmetries can be reducible, i.e., $\text{Ker } \zeta \neq 0$. Another problem is that gauge symmetries need not form an algebra [48, 60, 63]. The Lie bracket $[u_\phi, u_{\phi'}]$ of gauge symmetries $u_\phi, u_{\phi'} \in \text{Im } \zeta$ is a symmetry, but it need not belong to $\text{Im } \zeta$. To solve these problems, we follow a different definition of gauge symmetries (Definitions 7.4–7.5) as those associated to nontrivial NI by means of Noether's inverse second Theorem 7.9. They are parameterized by Grassmann-graded ghosts, but not gauge parameters (Remark 7.7).

2.3 Noether's First Theorem: Conservation Laws

Let $(\mathcal{O}_\infty^* Y, L)$ be a Lagrangian system (Definition 1.5). The following is *Noether's first theorem*.

Theorem 2.7 *Let the generalized vector field v (2.7) be a symmetry of a Lagrangian L (Definition 2.2), i.e., let it obey the equality (2.11). Then the first variational formula (2.10) restricted to the Euler–Lagrange equation \mathfrak{E}_L (1.34) takes a form of the weak conservation law on-shell*

$$0 \approx -d_H(-h_0(\vartheta \rfloor \mathcal{E}_L) + \sigma) \approx -d_H \mathcal{J}_v \quad (2.20)$$

of a symmetry current

$$\mathcal{J}_v = \mathcal{J}_v^\mu \omega_\mu = -h_0(\vartheta \rfloor \mathcal{E}_L) + \sigma \quad (2.21)$$

along a generalized vector field v . It is called the Lagrangian conservation law.

The weak conservation law (2.20) leads to a *differential conservation law*

$$\partial_\lambda (\mathcal{J}_v^\lambda \circ s) = 0$$

on classical solutions s of the Euler–Lagrange equation (1.35) (Definition B.13). This differential conservation law, in turn, yields an *integral conservation law*

$$\int_{\partial M} s^* \mathcal{J}_v = 0, \quad (2.22)$$

where M is an n -dimensional compact submanifold of X with a boundary ∂M .

Remark 2.9 Of course, the symmetry current \mathcal{J}_v (2.21) is defined with the accuracy to a d_H -closed term. For instance, if we choose a different Lepage equivalent \mathcal{E}_L

(1.37) in the variational formula (1.36), the corresponding symmetry current differs from \mathcal{J}_v (2.21) in a d_H -exact term. This term is independent of a Lagrangian, and it does not contribute to the integral conservation law (2.22).

Obviously, the symmetry current \mathcal{J}_u (2.21) is linear in a generalized vector field v . Therefore, one can consider a superposition of symmetry currents

$$\mathcal{J}_v + \mathcal{J}_{v'} = \mathcal{J}_{v+v'}, \quad \mathcal{J}_{cv} = c \mathcal{J}_v, \quad c \in \mathbb{R},$$

associated to different symmetries v and that of weak conservation laws (2.20).

A symmetry v of a Lagrangian L is called *exact* if the Lie derivative $\mathbf{L}_{J^*v}L$ of L along J^*v vanishes, i.e., $\mathbf{L}_{J^*v}L = 0$.

In this case, the first variational formula (2.10) takes a form

$$0 = v_V \rfloor \delta L + d_H(h_0(J^*v \rfloor \mathcal{E}_L)),$$

and results in the weak conservation law (2.20):

$$0 \approx d_H(h_0(J^*v \rfloor \mathcal{E}_L)) \approx -d_H \mathcal{J}_v, \quad (2.23)$$

of the symmetry current $\mathcal{J}_v = -h_0(J^*v \rfloor \mathcal{E}_L)$.

For instance, let $v = v^i \partial_i$ be a vertical generalized vector field on $Y \rightarrow X$. If it is an exact symmetry of L , the weak conservation law (2.23) takes a form

$$0 \approx -d_H(J^*v \rfloor \mathcal{E}_L). \quad (2.24)$$

Definition 2.5 The equality (2.24) is called the *Noether conservation law* of a *Noether current*

$$\mathcal{J} = -J^*v \rfloor \mathcal{E}_L. \quad (2.25)$$

If a Lagrangian L admits the gauge symmetry u (2.15), the weak conservation law (2.20) of the corresponding symmetry current \mathcal{J}_u (2.21) holds. We call it the *gauge conservation law*. Because gauge symmetries depend on parameter variables and their jets, all gauge conservation laws possess the following peculiarity.

Theorem 2.8 *If u (2.15) is a gauge symmetry of a Lagrangian L , the corresponding conserved symmetry current \mathcal{J}_u (2.21) along u takes a form*

$$\mathcal{J}_u = W + d_H U = (W^\mu + d_v U^{v\mu}) \omega_\mu, \quad (2.26)$$

where the term W vanishes on-shell, and $U = U^{v\mu} \omega_{v\mu}$ is a horizontal $(n-2)$ -form.

Proof Theorem 2.8 is a particular variant of Theorem 7.11. □

Definition 2.6 A term U in the expression (2.26) is called the *superpotential*.

If a symmetry current admits the decomposition (2.26), one says that it is reduced to a superpotential [39, 61, 66, 128]. If a symmetry current \mathcal{J} reduces to a superpotential, the integral conservation law (2.22) becomes tautological.

Remark 2.10 Theorem 2.8 generalizes the result in [66] for gauge symmetries u whose gauge parameters $\chi^\lambda = u^\lambda$ are components of a projection $u^\lambda \partial_\lambda$ of u onto X .

Remark 2.11 In mechanics on a configuration bundle over \mathbb{R} (Chap. 4), the whole conserved current (2.26) along a gauge symmetry vanishes on-shell (Theorem 4.7).

Sometimes, Theorem 2.8 is called *Noether's third theorem* on a superpotential.



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