

Chapter 2

Differential-Algebraic Equations

In this chapter, we introduce the differential algebraic equations which we abbreviate as DAEs. DAEs arise in a variety of applications such as modelling constrained multibody systems, electrical networks, aerospace engineering, chemical processes, computational fluid dynamics, gas transport networks, see [10–12, 35]. Therefore their analysis and numerical treatment plays an important role in modern mathematics. In many articles, DAEs are also called singular systems [11], descriptor systems [12, 35, 36], generalized state space systems [35], semi-state systems, degenerated systems, constrained systems, implicit systems but in most literatures they are called DAEs [24, 25, 29]. In this book, we shall also call them DAEs.

2.1 What are DAEs?

Consider an explicit ordinary differential equation (ODE),

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad (2.1.1)$$

where $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$ and $\mathbf{x} \in \mathbb{R}^n, \mathbf{f} : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. In general the first order ODE system can be written in implicit form:

$$\mathbf{F}(t, \mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0}. \quad (2.1.2)$$

According to [32], if the Jacobian matrix $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is nonsingular then it is possible to solve (2.1.2) for $\dot{\mathbf{x}}$ in order to obtain an ODE (2.1.1). However, if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is singular, this is no longer possible and the solution $\mathbf{x}$ has to satisfy certain algebraic constraints. Hence, if $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ is singular, Eq. (2.1.2) is referred to as differential algebraic equation (DAE). In modeling, the formulation of pure ODE problems often requires the combination of conservation laws (mass and energy balance), constitutive equations (equations

of state, pressure drops, heat transfer) and design constraints (desired operations). This means that there are some problems where not all the equations in a differential system involve derivatives, thus we can come up with a special case of differential algebraic equations which can be written as:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{y}), \quad (2.1.3a)$$

$$\mathbf{0} = \mathbf{g}(t, \mathbf{x}, \mathbf{y}), \quad (2.1.3b)$$

where $\mathbf{x}$ -differential variables, $\mathbf{y}$ -algebraic variables and (2.1.3b) is a constraint equation. Equation (2.1.3) is a special type of DAE which is commonly called semi-explicit DAE.

2.2 Models for DAEs

According to [12], it is well known from modern control theory that two main mathematical representations for dynamical systems are the transfer matrix representation and the state space representation. The former describes only the input-output property of the system, while the latter gives further insight into the structural property of the system.

2.2.1 State Space Representation

State space representation was developed between the end of the 1950s and the beginning of the 1960s. It has the advantage that, not only it provides an efficient method for control system analysis and synthesis, but also offers a deeper understanding about the various properties of the systems. The state space models of the systems are obtained mainly using the so called state space variable method. To obtain a state model of a practical system, we need to choose some physical variables such as currents and voltages in an electrical network. Then, by the physical relationships among the variables or by some model identification techniques such as modified nodal analysis in network analysis, a set of equations can be established. Naturally, this set of equations are usually differential and/or algebraic equations, which form a mathematical model of the system. By properly defining a state vector $\mathbf{x}(t)$ and an input vector $\mathbf{u}(t)$, which are formed by the physical variables of the system, and an output vector $\mathbf{y}(t)$, whose elements are properly chosen measurable variables of the system, this set of equations can be arranged into two equations: one is the so called **state equation**, which can be written in the following implicit form:

$$\mathbf{f}(\dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}(t), t) = \mathbf{0}, \quad (2.2.1)$$

and the second one is the **output equation**, or the observation equation, which can be in the general form of

$$g(y(t), x(t), u(t), t) = 0, \quad (2.2.2)$$

where f and g are vector functions of appropriate dimensions with respect to $\dot{x}(t)$, $x(t)$, $y(t)$, $u(t)$ and t . According to [12], Eqs. (2.2.1) and (2.2.2) give the state space representation for a general nonlinear dynamical system. We consider a special form of (2.2.1)–(2.2.2):

$$E(t)\dot{x}(t) = F(x(t), u(t), t) \quad (2.2.3a)$$

$$y(t) = K(x(t), u(t), t), \quad (2.2.3b)$$

where $t \geq 0$ is the time variable, F and K are appropriate dimensional vector functions, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the measured output vector. The matrix $E(t)$ must be singular for some $t \geq 0$ for our case since we are considering DAEs. Equation (2.2.3) is the general form for the so called (nonlinear) DAEs. If we consider the case where F and K are linear functions of the vectors $x(t)$ and $u(t)$, the general nonlinear DAE (2.2.3) simplifies to the following form:

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad (2.2.4a)$$

$$y(t) = C^T(t)x(t) + D^T(t)u(t), \quad (2.2.4b)$$

where $E(t), A(t) \in \mathbb{R}^{n \times n}$, $C(t) \in \mathbb{R}^{n \times \ell}$, $D(t) \in \mathbb{R}^{m \times \ell}$, $B(t) \in \mathbb{R}^{n \times m}$ are matrix functions of time t , and they called the coefficients matrices of the system (2.2.4). Equation (2.2.4) describes the so called linear time-varying DAE. If the matrix coefficients are constant, i.e., time independent, the system (2.2.4) is called linear constant-coefficient DAE or linear time-invariant (LTI) DAE, which can be written as:

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (2.2.5a)$$

$$y(t) = C^T x(t) + D^T u(t), \quad (2.2.5b)$$

where $E, A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times \ell}$, $D \in \mathbb{R}^{m \times \ell}$, $B \in \mathbb{R}^{n \times m}$ are constant matrices. For the DAEs (2.2.5), there is also a concept of the **dynamical order**, which is defined as the rank of singular matrix E . Equations (2.2.4) and (2.2.5) are the two basic classes of DAEs. From this point, we restrict ourselves on the DAEs of the form (2.2.5) unless stated otherwise.

2.2.2 Transfer Matrix Representation

In this section, we discuss the transfer matrix representation. This representation is derived from the state space representation using the Laplace transform. The transfer matrix representation is commonly used to validate reduced models in the model order reduction community and is commonly called the transfer function.

Definition 2.2.1 (*Laplace transform*) The Laplace transform of a function $f(t)$ in the time domain is the function $F(s)$ in the frequency domain and it is defined as:

$$\mathcal{L}\{f(t)\} = F(s) := \int_0^\infty e^{-st} f(t) dt, \text{ where } s = \sigma + j\omega \in \mathbb{C}, \text{ with } \sigma, \omega \in \mathbb{R}.$$

We shall restrict ourselves on the transfer matrix representation of the LTI DAEs (3.2.36) and also assume $s = j\omega$, i.e., $\sigma = 0$. Taking the Laplace transform of (3.2.36) and simplifying, we obtain

$$\mathbf{Y}(s) = [\mathbf{C}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}^T] \mathbf{U}(s) + \mathbf{C}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}\mathbf{x}(0), \quad (2.2.6)$$

where $\mathbf{U}(s)$ and $\mathbf{Y}(s)$ are the Laplace transforms of $\mathbf{u}(t)$ and $\mathbf{y}(t)$, respectively. The rational matrix-valued function

$$\mathbf{H}(s) = \mathbf{C}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}^T \in \mathbb{R}^{\ell \times m}, \quad (2.2.7)$$

is called the transfer matrix representation of (2.2.5) or transfer function. Then, $\mathbf{H}(s)$ gives the relation between the Laplace transforms of the input $\mathbf{u}(t)$ and the output $\mathbf{y}(t)$. In other words, $\mathbf{H}(s)$ describes the input-output behavior of (2.2.5) in the frequency domain.

2.3 Linear Constant Coefficient DAEs

In this section, we discuss that analysis of the LTIDAEs. For simplicity, the coefficient matrix $\mathbf{D}$ in (2.2.5) is taken to be the zero matrix unless specified. Thus, (2.2.5) simplifies to:

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (2.3.1a)$$

$$\mathbf{y}(t) = \mathbf{C}^T\mathbf{x}(t), \quad (2.3.1b)$$

where $\mathbf{E} \in \mathbb{R}^{n \times n}$ is singular, $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{n \times \ell}$, and $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector, $\mathbf{y}(t) \in \mathbb{R}^\ell$ is the output vector and $\mathbf{x}_0 \in \mathbb{R}^n$ is the initial value.

2.3.1 Solvability of DAEs

Here, we are interested in the solutions of the homogenous system obtained by setting $\mathbf{u}(t) = 0$, then (2.3.1a) becomes

$$\mathbf{E}\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{0}. \quad (2.3.2)$$

As for the case of ODEs we use the guess solution $\mathbf{x}(t) = \mathbf{x}_0 e^{\lambda t}$. Substituting the guess solution into (2.3.2) leads to $(\lambda \mathbf{E} - \mathbf{A})\mathbf{x}_0 e^{\lambda t} = \mathbf{0}$. We can see that (2.3.2) has a non-trivial solution provided $\lambda \mathbf{E} - \mathbf{A}$ is regular and $\mathbf{x}_0 \neq \mathbf{0}$ satisfies $(\lambda \mathbf{E} - \mathbf{A})\mathbf{x}_0 = \mathbf{0}$. λ and $\mathbf{x}_0$ are called the generalized eigenvalues and eigenvectors, respectively. Thus we say that (2.3.1a) is solvable provided the matrix pencil $\lambda \mathbf{E} - \mathbf{A}$ is regular. We note that $\lambda \mathbf{E} - \mathbf{A}$ can also be written as $(\mathbf{E}, \mathbf{A})$ which is called the matrix pencil or matrix pair. This leads to the following definition.

Definition 2.3.1 ([20]) A matrix pair $(\mathbf{E}, \mathbf{A})$ is called regular if the polynomial $\mathcal{P}(\lambda) = \det(\lambda \mathbf{E} - \mathbf{A})$ is not identically zero, it is called singular otherwise.

A pair $(\mathbf{E}, \mathbf{A})$ with nonsingular $\mathbf{E}$ is always regular, and its polynomial $\mathcal{P}(\lambda)$ is of degree n . In case of singular matrices $\mathbf{E}$, the polynomial is lower. According to [21], regularity of a matrix pair is closely related to the solution behavior of the corresponding DAE. In particular, regularity is necessary and sufficient for the property that for every sufficiently smooth inhomogeneity $\mathbf{u}$ the DAE is solvable and the solution is unique for every consistent initial value. This is well understood, if we consider the Weierstraß-Kronecker canonical form of a given DAE as discussed in Sect. 2.3.3.

Definition 2.3.2 ([34]) A pair $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ is said to be a generalized eigenvalue $\lambda = \frac{\alpha}{\beta}$ of the matrix pencil $\lambda \mathbf{E} - \mathbf{A}$ if $\det(\alpha \mathbf{E} - \beta \mathbf{A}) = 0$. If $\beta \neq 0$, then the pair (α, β) represents a finite eigenvalue $\lambda = \frac{\alpha}{\beta}$ of the matrix pencil $\lambda \mathbf{E} - \mathbf{A}$. But if $\beta = 0$, the pair $(\alpha, 0)$ represents an infinite eigenvalue of $\lambda \mathbf{E} - \mathbf{A}$. Clearly, the pencil $\lambda \mathbf{E} - \mathbf{A}$ has an eigenvalue at infinity if and only if the matrix $\mathbf{E}$ is singular.

The set of all finite eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$ is denoted by $\sigma_f(\mathbf{E}, \mathbf{A})$ while the infinite spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$ is denoted by $\sigma_\infty(\mathbf{E}, \mathbf{A})$. Thus, the set of all generalized eigenvalues (finite and infinite) of the matrix pencil $(\mathbf{E}; \mathbf{A})$ is called the spectrum of $(\mathbf{E}, \mathbf{A})$ and denoted by $\sigma(\mathbf{E}, \mathbf{A}) = \sigma_f(\mathbf{E}, \mathbf{A}) \cup \sigma_\infty(\mathbf{E}, \mathbf{A})$. We note that if $\mathbf{E}$ is nonsingular, then $\sigma(\mathbf{E}, \mathbf{A}) = \sigma_f(\mathbf{E}, \mathbf{A})$ which is equal to the spectrum of $\mathbf{E}^{-1}\mathbf{A}$. This also means that if $\mathbf{E}$ is nonsingular, the homogeneous equation (2.3.2) represents an implicit regular ODE and its fundamental solution system forms an n -dimensional subspace in $\mathcal{C}^1$. But what happens if $\mathbf{E}$ is singular, this is closely related to the notion of the regular matrix pencil $(\mathbf{E}, \mathbf{A})$ [20] as discussed in Sect. 2.3.3.

2.3.2 Stability of DAEs

According to [12], in practice a system should be stable, otherwise it may not work properly or may even be destroyed in practical use. As in the ODE case, when studying the stability of DAEs, we need to only consider the homogenous system (2.3.2).

Definition 2.3.3 ([34]) The DAE (2.3.1) is called asymptotically stable if $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions $\mathbf{x}(t)$ of the homogenous system $\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$.

This leads us to the following theorem that collects equivalent conditions for system (2.3.1) to be asymptotically stable.

Theorem 2.3.1 ([34]) Consider a DAE (2.3.1) with regular matrix pencil $\lambda\mathbf{E} - \mathbf{A}$. The following statements are equivalent.

1. System (2.3.1) is asymptotically stable.
2. All finite eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$ lie in the open left half complex plane, i.e., $\sigma(\mathbf{E}, \mathbf{A}) \subset \mathbb{C}^-$, where $\mathbb{C}^- = \{s \in \mathbb{C}, \operatorname{Re}(s) \leq 0\}$ represents the open left half complex plane.

According to [34], the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ is called c-stable if it is regular and all the finite eigenvalues of $\lambda\mathbf{E} - \mathbf{A}$ have negative real part. We note that, in view of the above theorem, the infinite eigenvalues of the matrix pencil $(\mathbf{E}, \mathbf{A})$ have no effect on the stability of DAEs of the form (2.3.1), since the infinite eigenvalues of $\lambda\mathbf{E} - \mathbf{A}$ do not affect the behavior of the homogenous system at infinity [34]

2.3.3 Weierstraß-Kronecker Canonical Form

In this section, we present the Weierstraß-Kronecker canonical form. This is the most commonly used tool to understand the DAE structure of linear constant-coefficient DAEs [12, 21, 25]. Scaling (2.3.1) by nonsingular matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ and the state variable $\mathbf{x}$ according to $\mathbf{x} = \mathbf{Q}\tilde{\mathbf{x}}$ with a nonsingular matrix $\mathbf{Q} \in \mathbb{R}^{n \times n}$, we obtain

$$\tilde{\mathbf{E}}\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t), \quad (2.3.3a)$$

$$\mathbf{y}(t) = \tilde{\mathbf{C}}^T\tilde{\mathbf{x}}(t), \quad (2.3.3b)$$

where $\tilde{\mathbf{E}} = \mathbf{PEQ}$, $\tilde{\mathbf{A}} = \mathbf{PAQ}$, $\tilde{\mathbf{B}} = \mathbf{PB}$ and $\tilde{\mathbf{C}} = \mathbf{Q}^T\mathbf{C}$, which is again a DAE with constant coefficients. According to [21], the relation $\mathbf{x} = \mathbf{Q}\tilde{\mathbf{x}}$ gives a one-to-one correspondence between the corresponding solution sets. This means that we can consider the transformed problem (2.3.3) instead of (2.3.1) in order to understand the underlying structure of constant coefficients linear DAEs. It can easily be shown that this relation between systems (2.3.1) and (2.3.3) poses reflexivity, transitivity and symmetry, and is thus an equivalence relation.

Theorem 2.3.2 (Weierstraß-Kronecker canonical form [21, 29]) *Let $(\mathbf{E}, \mathbf{A})$ be a regular matrix pencil. Then, we have $(E, A) \sim \left(\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}, \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \right)$, where $\mathbf{J} \in \mathbb{R}^{k \times k}$ for some nonnegative $k \leq n$, is a matrix in Jordan canonical form and $\mathbf{N} \in \mathbb{R}^{(n-k) \times (n-k)}$ is a nilpotent matrix with index $\mu \leq n - k$ also in Jordan canonical form. Moreover, it is allowed that one or the other block is not present.*

The equivalence symbol $\sim$ means that, the regular matrix pencil $\lambda \mathbf{E} - \mathbf{A}$ can be transformed into $\lambda \tilde{\mathbf{E}} - \tilde{\mathbf{A}}$, where $\tilde{\mathbf{E}} = \mathbf{P}\mathbf{E}\mathbf{Q} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}$, $\tilde{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{Q} = \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$, by the use of suitable non-singular matrices $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n \times n}$. There exists a $\mu \in \mathbb{N}$ such that $\mathbf{N}^{\mu-1} \neq \mathbf{0}$ but $\mathbf{N}^\mu = \mathbf{0}$, μ is known as $\mathbf{N}$'s index of nilpotency. μ is the index of the matrix pencil $\lambda \mathbf{E} - \mathbf{A}$ and also the index of the differential algebraic system (2.3.1a). This index concept is commonly called the Kronecker index [12, 21, 29, 33, 34]. The numbers k and $n - k$ corresponds to the number of finite and infinite eigenvalues, respectively, of the spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$. We note that it is possible to have $k = 0$, meaning $\tilde{\mathbf{E}} = \mathbf{N}$, $\tilde{\mathbf{A}} = \mathbf{I}$ this implies that the spectrum of the matrix pencil $\lambda \mathbf{E} - \mathbf{A}$ has only infinite spectrum, i.e $\sigma(E, A) = \sigma_\infty(E, A)$ and, also $k = n$ which yields $\tilde{\mathbf{E}} = \mathbf{I}$, $\tilde{\mathbf{A}} = \mathbf{J}$, this implies that the spectrum of the matrix pencil $\sigma(E, A) = \sigma_f(E, A)$ has only finite spectrum. Assuming the matrix pencil $\lambda \mathbf{E} - \mathbf{A}$ has both the finite and infinite spectrum, then the matrices $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ can be partitioned in blocks $\tilde{\mathbf{B}} = (\tilde{\mathbf{B}}_1^T \ \tilde{\mathbf{B}}_2^T)^T$ and $\tilde{\mathbf{C}} = (\tilde{\mathbf{C}}_1^T \ \tilde{\mathbf{C}}_2^T)^T$, corresponding to the partitions of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{A}}$. Under the coordinate transformation $\tilde{\mathbf{x}} = \mathbf{Q}^{-1}\mathbf{x} = (\tilde{\mathbf{x}}_1^T(t), \tilde{\mathbf{x}}_2^T(t))^T$, system (2.3.3a) can be written as Weierstraß-Kronecker canonical form which leads to an equivalent decoupled system

$$\dot{\tilde{\mathbf{x}}}_1(t) = \mathbf{J}\tilde{\mathbf{x}}_1(t) + \tilde{\mathbf{B}}_1\mathbf{u}(t), \quad (2.3.4a)$$

$$\mathbf{N}\dot{\tilde{\mathbf{x}}}_2(t) = \tilde{\mathbf{x}}_2(t) + \tilde{\mathbf{B}}_2\mathbf{u}(t). \quad (2.3.4b)$$

We observe that (2.3.4a) represents a standard explicit ODE and without loss of generality the solution of (2.3.4b) can be written as

$$\tilde{\mathbf{x}}_2(t) = - \sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 \mathbf{u}^{(i)}(t), \quad (2.3.5)$$

since $\mathbf{N}$ is a nilpotent matrix with index- μ , where $\mathbf{u}^{(i)}(t) = \frac{d^i}{dt} \mathbf{u}(t)$, provided $\mathbf{u}(t)$ is smooth enough, that is at least $\mu - 1$ times differentiable. Equation (2.3.5) shows the dependence of the solution $\mathbf{x}(t)$ of (2.3.1a) on the derivatives of the input function $\mathbf{u}(t)$. We can observe that the higher the index- μ , the more differentiations are involved. It is only in the index-1 case, where we have $\mathbf{N} = \mathbf{0}$, hence $\tilde{\mathbf{x}}_2(t) = \tilde{\mathbf{B}}_2\mathbf{u}(t)$, that no derivatives are involved. According to [20, 25], since numerical differentiations in these circumstances may cause considerable trouble numerically, it is very important to know the index of the DAE as well as details on the structure

responsible for a higher index when modeling and simulating with DAEs in practice. From (2.3.4), we can observe that the number of finite eigenvalues, k and infinite eigenvalues, $n - k$ are equal to the number of differential and algebraic equations, respectively, in a given DAE. Thus, for the case of index-1 DAEs the number of differential equations is equal to the rank of the singular matrix $\mathbf{E}$. The solutions $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ which corresponds to the differential and algebraic part are commonly known as the slow and fast solutions, respectively, see [12, 21, 29, 33, 34]. It can be easily proved that the differential part of (2.3.4) inherits the stability of DAEs of the form (2.3.1), that is, $\sigma_f(\mathbf{E}, \mathbf{A}) = \sigma(\mathbf{J})$.

2.3.3.1 Index Concept of DAEs

An index of a DAE is commonly defined as the measure of the difficulties arising in the theoretical and numerical treatment of a given DAE. According to [21], the motivation to introduce an index is to classify different types of DAEs with respect to the difficulty to solve them analytically as well as numerically. Sometimes the index of a DAE is defined as a measure of how much the DAE deviates from an ODE. In the previous Section, we have defined the index μ as the nilpotency index of a nilpotent matrix $\mathbf{N}$. This index is also known as the Kronecker index of a DAE. Many other index concepts have been introduced, see [8, 29, 32], but in this book we shall restrict ourselves to only three, i.e., Kronecker index, differentiation index and tractability index. We note that all these index concepts coincide for the case of linear DAEs with constant matrices. If we differentiate (2.3.12b) with respect to t we obtain:

$$\dot{\tilde{\mathbf{x}}}_2 = - \sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 \mathbf{u}^{(i+1)}.$$

This means that exactly μ differentiations transform (3.2.48) into a system of explicit ordinary differential equations. Hence, the Kronecker and the differentiation index coincide for LTI DAEs. This type of index is called the differentiation index and it is defined as in Definition 2.3.4. According to [21], the differentiation index was introduced to determine how far the DAE is away from an ODE, for which the analysis and numerical techniques are well established.

Definition 2.3.4 ([32]) The nonlinear DAE, $\mathbf{F}(t, \mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0}$, has differentiation index μ , if μ is the minimal number of differentiations

$$\mathbf{F}(t, \mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0}, \quad \frac{d}{dt} (\mathbf{F}(t, \mathbf{x}, \dot{\mathbf{x}})) = \mathbf{0}, \dots, \quad \frac{d^\mu}{dt^\mu} (\mathbf{F}(t, \mathbf{x}, \dot{\mathbf{x}})) = \mathbf{0}, \quad (2.3.6)$$

such that the Eq. (2.3.6) make it possible to extract an explicit ordinary differentiation system $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ using only algebraic manipulations.

This can be illustrated as follows. Consider a semi-explicit DAE of the form below:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}), \quad (2.3.7a)$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{y}). \quad (2.3.7b)$$

Using chain rule on the constraint equation, we find:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}), \quad (2.3.8a)$$

$$\dot{\mathbf{y}} = -\mathbf{g}_y(\mathbf{x}, \mathbf{y})^{-1} (\mathbf{g}_x(\mathbf{x}, \mathbf{y})\mathbf{f}(\mathbf{x}, \mathbf{y})). \quad (2.3.8b)$$

Thus, the DAE has a differentiation index $\mu = 1$ provided $\det(\mathbf{g}_y) \neq 0$. In the example below, we compare the differential index and the Kronecker index.

Example 2.3.1 Consider a DAE system of the form (2.3.1) with system matrices,

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (2.3.9)$$

The matrix pencil $(\mathbf{E}, \mathbf{A})$ is regular since $\det(\lambda\mathbf{E} - \mathbf{A}) = \lambda^2 + \lambda + 1 \neq 0$. Using

Theorem (2.3.2) we can choose non-singular matrices, $\mathbf{Q} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{P} =$

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ such that $(\mathbf{E}, \mathbf{A}) \sim \left(\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}, \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \right)$, where $\mathbf{J} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ and

$\mathbf{N} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus this DAE system is of Kronecker index 1, since $\mathbf{N}$ is of nilpotent

index 1. Next, we compute the differential index. We need first to rewrite system

(2.3.9) in the semi-explicit form (2.3.7) where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$, $\mathbf{f}(\mathbf{x}, \mathbf{y}) =$

$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$ and $\mathbf{g}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} -x_1 + x_4 \\ x_2 + x_3 + x_4 \end{pmatrix}$. Then $\mathbf{g}_y = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, with $\det(\mathbf{g}_y) = -1 \neq 0$.

Thus the DAE system has differentiation index 1. Hence the Kronecker index and differentiation index coincide, that is, $\mu = \gamma = 1$.

2.3.3.2 Consistent Initial Condition of DAEs

From system (2.3.4), we observe that (2.3.4a) is a linear differential equation which can easily be solved when any initial condition $\tilde{\mathbf{x}}_1(0)$ is applied and its analytic solution can be written as:

$$\tilde{\mathbf{x}}_1(t) = e^{t\mathbf{J}}\tilde{\mathbf{x}}_1(0) + e^{t\mathbf{J}} \int_0^t e^{-\tau\mathbf{J}}\tilde{\mathbf{B}}_1\mathbf{u}(\tau) d\tau. \quad (2.3.10)$$

We observe that the solution (2.3.10) of (2.3.12a) is always unique for any choice of the initial value $\tilde{\mathbf{x}}_1(0)$ while the initial value of (2.3.12b) has to satisfy the hidden constraint,

$$\tilde{\mathbf{x}}_2(0) = - \sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 \mathbf{u}^{(i)}(0). \quad (2.3.11)$$

We can observe that, we have no enough freedom to arbitrary choose the initial values $\tilde{\mathbf{x}}_2(0)$. For example, if $\mu = 1$, we have to choose the initial value such that $\tilde{\mathbf{x}}_2(0) = -\tilde{\mathbf{B}}_2 \mathbf{u}(0)$. For $\mu > 1$, Eq. (2.3.5) is a differentiation problem, thus the initial value $\tilde{\mathbf{x}}_2(0)$ is fixed, and the input function $\mathbf{u}(t)$ has to be at least $\mu - 1$ times differentiable, i.e., $\mathbf{u}(t) \in C^{\mu-1}$. Hence, initial value problems for (2.3.1) lead to unique classical solutions if the initial value $\mathbf{x}(0) = \mathbf{x}_0$ is consistent, that is, $\mathbf{x}(0) = \mathbf{Q}\tilde{\mathbf{x}}(0) = \mathbf{Q} [\tilde{\mathbf{x}}_1(0)^T \tilde{\mathbf{x}}_2(0)^T]^T$, where $\tilde{\mathbf{x}}_1(0)$ is a free parameter while $\tilde{\mathbf{x}}_2(0)$ has to satisfy (2.3.11). Thus, $\mathbf{x}(0)$ must be a consistent initial condition of a given DAE. We note that, if the initial condition $\mathbf{x}_0$ is inconsistent or the input function $\mathbf{u}(t)$ is not sufficiently smooth, then the solution of the DAEs may have impulsive modes, see [12, 33].

2.3.4 Transfer Matrix Representation of the Kronecker Form

Using (2.3.4) and the decomposed output equation, control problem (2.3.1) can also be written in equivalent form

$$\dot{\tilde{\mathbf{x}}}_1(t) = \mathbf{J}\tilde{\mathbf{x}}_1(t) + \tilde{\mathbf{B}}_1 \mathbf{u}(t), \quad (2.3.12a)$$

$$\tilde{\mathbf{x}}_2(t) = - \sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 \mathbf{u}^{(i)}(t), \quad (2.3.12b)$$

$$\mathbf{y}(t) = \tilde{\mathbf{C}}_1^T \tilde{\mathbf{x}}_1(t) + \tilde{\mathbf{C}}_2^T \tilde{\mathbf{x}}_2(t). \quad (2.3.12c)$$

Taking the Laplace transform of (2.3.12), using the fact that

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - \sum_{k=1}^n s^{k-1} f^{(n-k)}(0),$$

and setting $\tilde{\mathbf{x}}_1(0) = 0$ we obtain:

$$\mathbf{Y}(s) = \tilde{\mathbf{H}}(s)\mathbf{U}(s) + \tilde{\mathbf{C}}_2^T \sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 \sum_{k=1}^i s^{k-1} \mathbf{u}^{(i-k)}(0), \quad (2.3.13)$$

where $\tilde{\mathbf{H}}(s) = \mathbf{H}_1(s) + \mathbf{H}_2(s)$, $\mathbf{H}_1(s) = \tilde{\mathbf{C}}_1^T(s\mathbf{I} - \mathbf{J})^{-1}\tilde{\mathbf{B}}_1$ and $\mathbf{H}_2(s) = -\tilde{\mathbf{C}}_2^T \sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 s^i$. It can be easily proved that $\tilde{\mathbf{H}}(s)$ coincides with the conventional definition of the transfer function $\mathbf{H}(s) = \mathbf{C}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, since (2.3.1) and (2.3.12) are equivalent, i.e., $\mathbf{H}(s) = \tilde{\mathbf{H}}(s)$. If let $\mathcal{P}(s) = \tilde{\mathbf{C}}_2^T \sum_{i=0}^{\mu-1} \mathbf{N}^i \tilde{\mathbf{B}}_2 \sum_{k=1}^i s^{k-1} \mathbf{u}^{(i-k)}(0)$, Eq. (2.3.13) can be decomposed as $\mathbf{Y}(s) = \mathbf{Y}_1(s) + \mathbf{Y}_2(s)$, where $\mathbf{Y}_1(s) = \mathbf{H}_1(s)\mathbf{U}(s)$ and $\mathbf{Y}_2(s) = \mathbf{H}_2(s)\mathbf{U}(s) + \mathcal{P}(s)$ represent the input-output function of the differential and algebraic parts, respectively in the frequency domain. We note that the $\mathbf{H}_1(s)$ and $\mathbf{H}_2(s)$ are commonly called the strictly proper and the polynomial parts of $\mathbf{H}(s)$, respectively, see [33]. We observe that the input-output relation of the differential part is given by $\mathbf{Y}_1(s) = \mathbf{H}_1(s)\mathbf{U}(s)$, where $\mathbf{H}_1(s)$ is its transfer matrix representation and it is independent of the index- μ of the DAE (3.2.48) while input-output relation of the algebraic part is given by $\mathbf{Y}_2(s) = \mathbf{H}_2(s)\mathbf{U}(s) + \mathcal{P}(s)$ which depends on the index- μ of the DAE (3.2.48). We note that $\mathcal{P}(s) = 0$ always for the case of index-1 DAEs. According to [24, 29], transforming Eq. (2.3.1) into a Kronecker canonical form is just in theory, but practical implementation may be difficult or impossible. This is due to the fact that computing the Kronecker canonical form in finite precision arithmetic is, in general, an ill-conditioned problem in the sense that small changes in the data may extremely change the Kronecker canonical form [33]. Proper formulations and projector methods attempt to overcome these drawbacks, allowing additionally for an extension of the results to the time varying context [29]. These techniques provide an index characterization in terms of the original problem description. This motivated us to use the projector and matrix chains approach in order to decompose LTI DAEs into differential and algebraic parts as introduced by März [25]. This will be discussed in Chap. 3.

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