Chapter 2
Connections on Lie Groupoids and Lie Algebroids

In this chapter we shall describe several different notions of connection. As well as introducing connections on Lie groupoids (path connections) and on Lie algebroids (infinitesimal connections) we shall see how these ideas are related to the classical concepts of covariant derivatives and, more generally, connections on vector bundles.

2.1 Path Connections on Lie Groupoids

There are several related concepts of connection in the general theory of fibre bundles. One such concept involves the lifting of a curve in the base manifold M to 'horizontal' curves in E, in a way that is consistent with the group action on the fibres of the bundle. The various lifts can then be used to determine diffeomorphisms between the fibres at different points on the base curve. It is possible to describe a similar notion for a locally trivial Lie groupoid, and we shall see in the next chapter, when we consider groupoids of fibre morphisms, how the two concepts are related.

The lifting operations we are going to consider will have the important property that the lift must be invariant under reparametrization. This property allows us to start by considering the lift of a curve defined on the specific interval $[0, 1]$, rather than on an arbitrary interval. We shall then show how this is sufficient for us to specify the lift of a curve defined on the whole of $\mathbb{R}$, or indeed, on an arbitrary nonempty open interval. We will also be able to specify the lift of a vector field by lifting its flow.

We recall our convention that a curve need be only continuous and piecewise smooth, rather than smooth on the whole of its domain.

Let $\mathcal{G}$ be a locally trivial Lie groupoid with manifold M of identities. A *path connection* on $\mathcal{G}$ is a map Γ taking each curve $c : [0, 1] \to M$ to a hboxcurve $c^\Gamma : [0, 1] \to \mathcal{G}$, satisfying the following conditions (these are slightly different from those given in [30], but are more directly suited to our purposes):

- $c^\Gamma(0) = 1_{c(0)}$;
- $\alpha c^\Gamma(t) = c(0)$ and $\beta c^\Gamma(t) = c(t)$ for all $t \in [0, 1]$;

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- if $[a, b] \subset [0, 1]$ and $\chi : [0, 1] \to [a, b]$ is a (smooth) diffeomorphism then $c^\Gamma \circ \chi = r_\varphi \circ (c \circ \chi)^\Gamma$ where $\varphi = c^\Gamma \chi(0)$;
- if c is smooth at $t \in [0, 1]$ with tangent vector $\dot{c}(t) \in T_{c(t)}M$ then c^Γ is smooth at t with tangent vector $\dot{c}^\Gamma(t) \in T_{c^\Gamma(t)}\mathcal{G}$;
- if $c_1(0) = c_2(0) = x$, say, and if $\dot{c}_1(0) = \dot{c}_2(0) \in T_xM$, then $\dot{c}_1^\Gamma(0) = \dot{c}_2^\Gamma(0) \in T_{1_x}\mathcal{G}$;
- the correspondence $T_xM \to T_{1_x}\mathcal{G}$ given by $\dot{c}(0) \mapsto \dot{c}^\Gamma(0)$ is linear, and determines a smooth map $\gamma : TM \to T\mathcal{G}$.

We may now use the path connection to lift curves defined on $\mathbb{R}$. The idea is to imagine the complete curve as a sequence of short curves defined on intervals $[n, n+1]$,[1] and to translate the curves in the sequence so that they are defined on $[0, 1]$. We shall, in stages, prove an appropriate reparametrization property for the lifted curve on $\mathbb{R}$. So, given a curve $c : \mathbb{R} \to M$, we define for each $n \in \mathbb{Z}$ the curve $c_n : [0, 1] \to M$ by $c_n(t) = c(n + t)$, and then define the map $c^\Gamma : \mathbb{R} \to \mathcal{G}$ by

$$c^\Gamma(t) = \left(c_n^\Gamma(t - n)\right) \cdot \varphi_n$$

where $t \in [n, n + 1)$ and where

$$\varphi_n = \begin{cases} 1_{c(0)} & (n = 0) \\ c_{n-1}^\Gamma(1) \cdot \varphi_{n-1} & (n > 0) \\ \left(c_n^\Gamma(1)\right)^{-1} \cdot \varphi_{n+1} & (n < 0). \end{cases}$$

We observe that all the groupoid products are defined, and that $\varphi_n = c_{n-1}^\Gamma(1) \cdot \varphi_{n-1}$ for any $n \in \mathbb{Z}$.

Lemma 2.1.1 *The map $c^\Gamma : \mathbb{R} \to \mathcal{G}$ is continuous and satisfies the conditions that $c^\Gamma(0) = 1_{c(0)}$, and that $\alpha c^\Gamma(t) = c(0)$ and $\beta c^\Gamma(t) = c(t)$ for all $t \in \mathbb{R}$.*

Proof It is immediate that c^Γ is continuous at all points except, possibly, integer points $n \in \mathbb{R}$. It is also immediate that $\lim_{t \to n+} c^\Gamma(t) = c^\Gamma(n)$; to demonstrate continuity from the left we note that

$$\lim_{t \to n-} c^\Gamma(t) = \lim_{t \to n-} c_{n-1}^\Gamma(t - (n - 1)) \cdot \varphi_{n-1} = c_{n-1}^\Gamma(1) \cdot \varphi_{n-1}$$

using the continuity of c_{n-1}^Γ on I, whereas

$$c^\Gamma(n) = c_n^\Gamma(0) \cdot \varphi_n = 1_{c_n(0)} \cdot \varphi_n = \varphi_n = c_{n-1}^\Gamma(1) \cdot \varphi_{n-1}.$$

We also see that

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[1] We have temporarily suspended our convention that $n = \dim M$, and for the rest of this section we shall use n as an arbitrary integer.

$$c^\Gamma(0) = c_0^\Gamma(0) \cdot \varphi_0 = 1_{c_0(0)} \cdot 1_{c(0)} = 1_{c(0)},$$

that

$$\alpha c^\Gamma(t) = \alpha(\varphi_n) = \alpha(1_{c(0)}) = c(0)$$

and that

$$\beta c^\Gamma(t) = \beta c_n^\Gamma(t-n) = c_n(t-n) = c(t). \qquad \square$$

We have not yet shown that c^Γ is piecewise smooth, as in principle it could fail to be smooth at infinitely many integral values of t, although we shall see in due course that this cannot happen. We can, though, use the continuity of c^Γ to see that $\varphi_n = c^\Gamma(n)$, because

$$c^\Gamma(n) = \lim_{\varepsilon\to 0} c^\Gamma(n-\varepsilon) = \lim_{\varepsilon\to 0} c_{n-1}^\Gamma(1-\varepsilon)\cdot\varphi_{n-1} = c_{n-1}^\Gamma(1)\cdot\varphi_{n-1} = \varphi_n.$$

We now establish the reparametrization property in detail.

Lemma 2.1.2 *If $b > 0$ and $\chi : [0,1] \to [0,b]$ is an increasing diffeomorphism then $c^\Gamma \circ \chi = (c\circ\chi)^\Gamma$ as maps $I \to \mathcal{G}$.*

Proof Let n be the largest integer not greater than b. If $n = 0$ then $[0,b] \subset [0,1]$ so that the result is an immediate consequence of the reparametrization property of c^Γ; we may therefore assume that $n \geq 1$.

Put $a_k = \chi^{-1}(k)$ for $k = 0, 1, \dots, n$. Let $\theta_k : [0,1] \to [a_k, a_k + 1] \subset I$ be given by $\theta_k(s) = \chi^{-1}(s+k)$, so that θ_k is a diffeomorphism; as $\chi(\theta_k(s)) = s + k$, we see that $c\big(\chi(\theta_k(s))\big) = c(s+k) = c_k(s)$ for $s \in I$, so that

$$c\circ\chi\circ\theta_k = c_k.$$

We also know, using the reparametrization property of the curve $(c\circ\chi)^\Gamma : [0,1] \to \mathcal{G}$, that

$$(c\circ\chi)^\Gamma\theta_k(s) = (c\circ\chi\circ\theta_k)^\Gamma(s)\cdot(c\circ\chi)^\Gamma\theta_k(0) = c_k^\Gamma(s)\cdot(c\circ\chi)^\Gamma(a_k).$$

We now argue recursively on k. Suppose that $(c\circ\chi)^\Gamma(a_k) = c^\Gamma(k)$, so that $(c\circ\chi)^\Gamma(\theta_k(s)) = c_k^\Gamma(s)\cdot c^\Gamma(k)$ for $s \in [0,1]$; then, putting $t = \theta_k(s) \in [a_k, a_{k+1}]$ so that $s = \chi(t) - k$, we see that

$$(c\circ\chi)^\Gamma(t) = c_k^\Gamma(\chi(t)-k)\cdot c^\Gamma(k) = c^\Gamma\chi(t)$$

using the definition of c^Γ for values $\chi(t) \in [n, n+1)$ and the continuity of c^Γ at $n+1$. To justify the recursive step we put $t = a_{k+1}$ to see that $(c\circ\chi)^\Gamma(a_{k+1}) =$

$c^\Gamma\chi(a_{k+1}) = c^\Gamma(k+1)$, and we note that $a_0 = 0$ so that $(c\circ\chi)^\Gamma(a_0) = 1_{c(0)} = c^\Gamma(0)$ to start the recursion.

We have shown, therefore, that $c^\Gamma\chi(t) = (c\circ\chi)^\Gamma(t)$ for all $t \in [0, a_n]$; a slightly modified version of the same argument shows that the result also holds where $t \in [a_n, 1]$ and $\chi(t) \in [n, b]$, so we conclude that $c^\Gamma \circ \chi = (c\circ\chi)^\Gamma$. □

Corollary 2.1.3 *If* $b > 0$ *and* $\chi : [0, b] \to [0, 1]$ *is an increasing diffeomorphism then* $c^\Gamma \circ \chi = (c\circ\chi)^\Gamma$ *as maps* $[0, b] \to \mathcal{G}$.

Proof As $\chi^{-1} : [0, 1] \to [0, b]$ we have

$$(c\circ\chi)^\Gamma \circ \chi^{-1} = (c\circ\chi\circ\chi^{-1})^\Gamma = c^\Gamma$$

so that

$$(c\circ\chi)^\Gamma = c^\Gamma \circ \chi.$$

□

Now for any $b \in \mathbb{R}$ let $\mathsf{t}_b : \mathbb{R} \to \mathbb{R}$ denote the translation map $\mathsf{t}_b(t) = b + t$, for any $b > 0$ let $\mathbf{s}_b : \mathbb{R} \to \mathbb{R}$ denote the scaling map $\mathbf{s}_b(t) = bt$, and let $\mathbf{r} : \mathbb{R} \to \mathbb{R}$ denote the reflection map $\mathbf{r}(t) = -t$.

Lemma 2.1.4 *If* $k \in \mathbb{Z}$ *then*

$$c^\Gamma \circ \mathsf{t}_k = r_{c^\Gamma(k)} \circ (c\circ\mathsf{t}_k)^\Gamma.$$

Proof If $k = 0$ there is nothing to prove; we consider the case $k > 0$. Take $t \in [n, n+1)$ so that $\mathsf{t}_k(t) \in [k+n, k+n+1)$, and note that $c\circ\mathsf{t}_k = c_k$. Now for any $j \in \mathbb{Z}$ we have

$$(c\circ\mathsf{t}_k)_j(t) = c_j(t+k) = c(t+j+k) = c_{j+k}(t)$$

so that $(c\circ\mathsf{t}_k)_j = c_{j+k}$. Thus from the definition,

$$\begin{aligned}(c\circ\mathsf{t}_k)^\Gamma(t) &= (c\circ\mathsf{t}_k)^\Gamma_n(t-n)\cdot(c\circ\mathsf{t}_k)^\Gamma_{n-1}(1)\cdot\ldots\cdot(c\circ\mathsf{t}_k)^\Gamma(1)\\ &= c^\Gamma_{k+n}(t-1)\cdot c^\Gamma_{k+n-1}(1)\cdot\ldots\cdot c^\Gamma_k(1)\end{aligned}$$

if $n \geq 0$, with a similar (but ascending rather than descending) formula when $n < 0$. Similarly, when $k + n \geq 0$ we have

$$\begin{aligned}c^\Gamma(\mathsf{t}_k(t)) &= c^\Gamma_{k+n}(\mathsf{t}_k(t)-k-n)\cdot c^\Gamma_{k+n-1}(1)\cdot\ldots\cdot c^\Gamma(1)\\ &= c^\Gamma_{k+n}(t-n)\cdot c^\Gamma_{k+n-1}(1)\cdot\ldots\cdot c^\Gamma(1)\end{aligned}$$

so that in this case

$$c^\Gamma(\mathsf{t}_k(t)) = (c\circ\mathsf{t}_k)^\Gamma(t)\cdot c^\Gamma_{k-1}(1)\cdot\ldots\cdot c^\Gamma(1) = (c\circ\mathsf{t}_k)^\Gamma(t)\cdot c^\Gamma(k).$$

The same result holds, with appropriate modifications to the details of the calculations, for all $k \in \mathbb{Z}$ and all $n \in Z$ because the recurrence formula $\varphi_j = c^{\Gamma}_{j-1}(1) \cdot \varphi_{j-1}$ (or $\varphi_j = (c \circ \mathsf{t}_k)^{\Gamma}_{j-1}(1) \cdot \varphi_{j-1}$) holds for any $j \in \mathbb{Z}$. □

Lemma 2.1.5 *For the reflection map* $\mathbf{r} : [-1, 1] \to [-1, 1]$ *we have*

$$(c \circ \mathbf{r})^{\Gamma} = c^{\Gamma} \circ \mathbf{r}.$$

Proof Let $\chi : [0, 1] \to [0, 1]$ be the diffeomorphism $\chi(t) = 1 - t$, so that $\mathbf{r} : [0, 1] \to [-1, 0]$ satisfies $\mathbf{r} = \mathsf{t}_{-1} \circ \chi$. Then

$$\begin{aligned} c^{\Gamma} \circ \mathbf{r} &= c^{\Gamma} \circ \mathsf{t}_{-1} \circ \chi \\ &= r_{c^{\Gamma}(-1)} \circ (c \circ \mathsf{t}_{-1})^{\Gamma} \circ \chi \\ &= r_{c^{\Gamma}(-1)} \circ r_{(c \circ \mathsf{t}_{-1})^{\Gamma}(1)} \circ (c \circ \mathsf{t}_{-1} \circ \chi)^{\Gamma} \\ &= r_{c^{\Gamma}(-1)} \circ r_{(c \circ \mathsf{t}_{-1})^{\Gamma}(1)} \circ (c \circ \mathbf{r})^{\Gamma}; \end{aligned}$$

but

$$(c \circ \mathsf{t}_{-1})^{\Gamma}(1) \cdot c^{\Gamma}(-1) = c^{\Gamma}(\mathsf{t}_{-1}(1)) = c^{\Gamma}(0) = 1_{c(0)}.$$

We then see that $\mathbf{r}^{-1} : [-1, 0] \to [0, 1]$ also satisfies $(c \circ \mathbf{r})^{\Gamma} = c^{\Gamma} \circ \mathbf{r}$, showing that the relationship holds for all $t \in [-1, 1]$. □

Lemma 2.1.6 *If* $b \in (0, 1)$ *then*

$$c^{\Gamma} \circ \mathsf{t}_b = r_{c^{\Gamma}(b)} \circ (c \circ \mathsf{t}_b)^{\Gamma}.$$

Proof We first consider the case where $t \in [n, n + 1 - b]$ for some $n \in \mathbb{Z}$, so that $\mathsf{t}_b(t) \in [n + b, n + 1]$. We note that

$$\mathsf{t}_n \mathbf{s}_b \mathsf{t}_1 \mathbf{s}_{b^{-1}} \mathsf{t}_{-n}(t) = b(b^{-1}(t - n) + 1) + n = t + b = \mathsf{t}_b(t)$$

where

$$[n, n+1-b] \xrightarrow{\mathsf{t}_{-n}} [0, 1-b] \xrightarrow{\mathbf{s}_{b-1}} [0, b^{-1}-1] \xrightarrow{\mathsf{t}_1} [1, b^{-1}] \xrightarrow{\mathbf{s}_b} [b, 1] \xrightarrow{\mathsf{t}_n} [n+b, n+1],$$

so that

$$\begin{aligned} c^{\Gamma} \circ \mathsf{t}_b &= c^{\Gamma} \circ \mathsf{t}_n \circ \mathbf{s}_b \circ \mathsf{t}_1 \circ \mathbf{s}_{b^{-1}} \circ \mathsf{t}_{-n} \qquad (1) \\ &= r_{c^{\Gamma}(n)} \circ (c \circ \mathsf{t}_n)^{\Gamma} \circ \mathbf{s}_b \circ \mathsf{t}_1 \circ \mathbf{s}_{b^{-1}} \circ \mathsf{t}_{-n} \\ &= r_{c^{\Gamma}(n)} \circ (c \circ \mathsf{t}_n \circ \mathbf{s}_b)^{\Gamma} \circ \mathsf{t}_1 \circ \mathbf{s}_{b^{-1}} \circ \mathsf{t}_{-n} \\ &= r_{c^{\Gamma}(n)} \circ r_{(c \circ \mathsf{t}_n \circ \mathbf{s}_b)^{\Gamma}(1)} \circ (c \circ \mathsf{t}_n \circ \mathbf{s}_b \circ \mathsf{t}_1)^{\Gamma} \circ \mathbf{s}_{b^{-1}} \circ \mathsf{t}_{-n} \end{aligned}$$

$$
\begin{aligned}
&= r_{c^\Gamma(n)} \circ r_{(c \circ \mathsf{t}_n \circ \mathbf{s}_b)^\Gamma(1)} \circ (c \circ \mathsf{t}_n \circ \mathbf{s}_b \circ \mathsf{t}_1 \circ \mathbf{s}_{b^{-1}})^\Gamma \circ \mathsf{t}_{-n} \qquad (2)\\
&= r_{c^\Gamma(n)} \circ r_{(c \circ \mathsf{t}_n \circ \mathbf{s}_b)^\Gamma(1)} \circ r_{(c \circ \mathsf{t}_n \circ \mathbf{s}_b \circ \mathsf{t}_1 \circ \mathbf{s}_{b^{-1}})^\Gamma(-n)} \circ (c \circ \mathsf{t}_n \circ \mathbf{s}_b \circ \mathsf{t}_1 \circ \mathbf{s}_{b^{-1}} \circ \mathsf{t}_{-n})^\Gamma\\
&= r_{c^\Gamma(n)} \circ r_{(c \circ \mathsf{t}_n \circ \mathbf{s}_b)^\Gamma(1)} \circ r_{(c \circ \mathsf{t}_n \circ \mathbf{s}_b \circ \mathsf{t}_1 \circ \mathbf{s}_{b^{-1}})^\Gamma(-n)} \circ (c \circ \mathsf{t}_b)^\Gamma
\end{aligned}
$$

where step (1) is justified by Corollary 2.1.3 with $\chi = \mathbf{s}_b$ because $\mathsf{t}_1 \mathbf{s}_{b^{-1}} \mathsf{t}_{-n}(t) \in [1, b^{-1}] \subset [0, b^{-1}]$, and step (2) is justified by Lemma 2.1.2 with $\chi = \mathbf{s}_{b^{-1}}$ because $\mathsf{t}_{-n}(t) \in [0, 1-b] \subset [0, 1]$. We now note that

$$
\begin{aligned}
(c \circ \mathsf{t}_n \circ \mathbf{s}_b)^\Gamma(1) \cdot c^\Gamma(n) &= (c \circ \mathsf{t}_n)^\Gamma \mathbf{s}_b(1) \cdot c^\Gamma(n)\\
&= (c \circ \mathsf{t}_n)^\Gamma(b) \cdot c^\Gamma(n)\\
&= c^\Gamma \mathsf{t}_n(b)\\
&= c^\Gamma(n+b)
\end{aligned}
$$

and $\mathsf{t}_n \circ \mathbf{s}_b \circ \mathsf{t}_1 \circ \mathbf{s}_{b^{-1}} = \mathsf{t}_{n+b}$ so that

$$
\begin{aligned}
(c \circ \mathsf{t}_n \circ \mathbf{s}_b \circ \mathsf{t}_1 \circ \mathbf{s}_{b^{-1}})^\Gamma(-n) \cdot (c \circ \mathsf{t}_n \circ \mathbf{s}_b)^\Gamma(1) \cdot c^\Gamma(n) &= (c \circ \mathsf{t}_{n+b})^\Gamma(-n) \cdot c^\Gamma(n+b)\\
&= c^\Gamma \mathsf{t}_{n+b}(-n)\\
&= c^\Gamma(b)
\end{aligned}
$$

from which we see in this case that $c^\Gamma \circ \mathsf{t}_b = r_{c^\Gamma(b)} \circ (c \circ \mathsf{t}_b)^\Gamma$.

Suppose, instead, that $t \in [n+1-b, n+1]$ for some $n \in \mathbb{Z}$, so that $\mathsf{t}_b(t) \in [n+1, n+1+b]$. The argument used above fails at step (1) because the image of $\mathbf{s}_b$ is not contained in I so that we cannot appeal to Corollary 2.1.3. Instead we note that

$$
\mathsf{t}_{n+1} \mathbf{s}_b \mathbf{r} \mathsf{t}_{-1} \mathbf{s}_{b^{-1}} \mathsf{t}_1 \mathbf{r} \mathsf{t}_{-n}(t) = n + 1 - b(b^{-1}(1 - (t - n)) - 1) = t + b = \mathsf{t}_b(t),
$$

where

$$
\begin{aligned}
&[n+1-b, n+1] \xrightarrow{\mathsf{t}_{-n}} [1-b, 1] \xrightarrow{\mathbf{r}} [-1, b-1] \xrightarrow{\mathsf{t}_1} [0, b] \xrightarrow{\mathbf{s}_{b^{-1}}}\\
&\xrightarrow{\mathbf{s}_{b^{-1}}} [0, 1] \xrightarrow{\mathsf{t}_{-1}} [-1, 0] \xrightarrow{\mathbf{r}} [0, 1] \xrightarrow{\mathbf{s}_b} [0, b] \xrightarrow{\mathsf{t}_{n+1}} [n+1, n+1+b],
\end{aligned}
$$

and a similar but slightly longer argument, now in addition using Lemma 2.1.5, shows that once again $c^\Gamma \circ \mathsf{t}_b = r_{c^\Gamma(b)} \circ (c \circ \mathsf{t}_b)^\Gamma$. □

Corollary 2.1.7 *If $a \in \mathbb{R}$ then*

$$
c^\Gamma \circ \mathsf{t}_a = r_{c^\Gamma(a)} \circ (c \circ \mathsf{t}_a)^\Gamma.
$$

Proof If $a \in \mathbb{Z}$ then the result has already been established; so let $a = k + b$ where $b \in (0, 1)$ and $k \in \mathbb{Z}$. Then

$$c^\Gamma \circ \mathsf{t}_a = c^\Gamma \circ \mathsf{t}_b \circ \mathsf{t}_k = r_{c^\Gamma(b)} \circ (c \circ \mathsf{t}_b)^\Gamma \circ \mathsf{t}_k = r_{c^\Gamma(b)} \circ r_{(c\circ\mathsf{t}_b)^\Gamma(k)} \circ (c \circ \mathsf{t}_b \circ \mathsf{t}_k)^\Gamma$$

and

$$c \circ \mathsf{t}_b \circ \mathsf{t}_k = c \circ \mathsf{t}_{k+b} = c \circ \mathsf{t}_a,$$
$$(c \circ \mathsf{t}_b)^\Gamma(k) \cdot c^\Gamma(b) = c^\Gamma \circ \mathsf{t}_b(k) = c^\Gamma(k+b) = c^\Gamma(a). \qquad \square$$

We therefore obtain the general reparametrization result.

Proposition 2.1.8 *If* $[a, b] \subset \mathbb{R}$ *and if* $\chi : [0, 1] \to [a, b]$ *is a diffeomorphism then*

$$c^\Gamma \circ \chi = r_\varphi \circ (c \circ \chi)^\Gamma$$

where $\varphi = c^\Gamma(\chi(0))$.

Proof Suppose first that χ is an increasing diffeomorphism, so that $\chi(0) = a$. Put $\chi_0 = \mathsf{t}_{-a} \circ \chi$, so that $\chi_0 : [0, 1] \to [0, b-a]$ is again an increasing diffeomorphism. Considering the curve $c_a = c \circ \mathsf{t}_a : \mathbb{R} \to \mathcal{G}$ we have

$$c^\Gamma \circ \mathsf{t}_a = r_{c^\Gamma(a)} \circ c_a^\Gamma, \qquad c_a^\Gamma \circ \chi_0 = (c_a \circ \chi_0)^\Gamma$$

and therefore

$$c^\Gamma \circ \chi = c^\Gamma \circ \mathsf{t}_a \circ \chi_0 = r_{c^\Gamma(a)} \circ c_a^\Gamma \circ \chi_0 = r_{c^\Gamma(a)} \circ (c_a \circ \chi_0)^\Gamma = r_{c^\Gamma(a)} \circ (c \circ \chi)^\Gamma.$$

If instead χ is a decreasing diffeomorphism then $\chi \circ \mathbf{r}$ is an increasing diffeomorphism, so that

$$c^\Gamma \circ \chi \circ \mathbf{r} = r_{c^\Gamma(\chi(\mathbf{r}(0)))} \circ (c \circ \chi \circ \mathbf{r})^\Gamma = r_{c^\Gamma\chi(0)} \circ (c \circ \chi)^\Gamma \circ \mathbf{r}$$

and the result follows. $\square$

Corollary 2.1.9 *If* c *is smooth at* $t \in \mathbb{R}$ *then so is* c^Γ*; thus* c^Γ *is piecewise smooth.*

Proof If $t \notin \mathbb{Z}$ and c is smooth at t then c^Γ is smooth at t by construction, because the curve c_n is smooth at $t - n$, its lift c_n^Γ is smooth at t_n, and groupoid multiplication is smooth.

Consider $n \in \mathbb{Z}$ and put $a = n - 1$, $b = n + 1$. Define the diffeomorphism $\chi : [0, 1] \to [a, b]$ by $\chi(t) = 2t + n - 1$, so that

$$c^\Gamma = r_{c^\Gamma(n-1)} \circ (c \circ \chi)^\Gamma \circ \chi^{-1}.$$

Suppose that c is smooth at n, so that $c \circ \chi$ is smooth at $\frac{1}{2}$ and therefore $(c \circ \chi)^\Gamma$ is smooth at $\frac{1}{2}$. As χ^{-1} is smooth at n and groupoid multiplication is smooth, it follows that c^Γ is smooth at n. It follows that there are only finitely many values of t for which

c^Γ is not smooth. At the points where c^Γ is not smooth, all left and right derivatives must nevertheless exist because they do for the curves c_n and c_n^Γ; thus c^Γ is piecewise smooth. □

It is clear that a similar approach can be used to give the lift of a curve defined on an arbitrary open interval containing zero. We may then define the lift of a vector field by lifting the curves in its flow. This is not completely straightforward, as the integral curve through $x \in M$ will lift to give a curve through $1_x \in \mathcal{G}$, resulting in a vector field along 1_M; we may, however, extend this to a vector field on the whole of $\mathcal{G}$ by right translation.

Proposition 2.1.10 *If a vector field X on M has a global flow $\psi : M \times \mathbb{R} \to M$ then the map $\psi^\Gamma : \mathcal{G} \times \mathbb{R} \to \mathcal{G}$ defined by*

$$\psi^\Gamma(\varphi, t) = \psi^\Gamma_\varphi(t) = r_\varphi(\psi_{\beta(\varphi)})^\Gamma(t),$$

where $\psi_{\beta(\varphi)}$ is the curve in M given by $\psi_{\beta(\varphi)}(t) = \psi(\beta(\varphi), t)$, is the flow of a vector field X^Γ on $\mathcal{G}$. The same formula, where the map ψ^Γ is defined on a proper subset of $\mathcal{G} \times \mathbb{R}$, may be used where X does not have a global flow.

Proof Fix $s \in \mathbb{R}$. If $t \in \mathbb{R}$ then

$$\psi_{\beta(\varphi)}\mathsf{t}_s(t) = \psi_{\beta(\varphi)}(s+t) = \psi_{s+t}\beta(\varphi) = \psi_t\psi_s\beta(\varphi) = \psi_t\psi_{\beta(\varphi)}(s) = \psi_{\psi_{\beta(\varphi)}(s)}(t)$$

so that $\psi_{\beta(\varphi)} \circ \mathsf{t}_s = \psi_{\psi_{\beta(\varphi)}(s)}$. Thus

$$\left(\psi_{\beta(\varphi)}\right)^\Gamma \circ \mathsf{t}_s = r_{(\psi_{\beta(\varphi)})^\Gamma(s)} \circ \left(\psi_{\beta(\varphi)} \circ \mathsf{t}_s\right)^\Gamma = r_{(\psi_{\beta(\varphi)})^\Gamma(s)} \circ \left(\psi_{\psi_{\beta(\varphi)}(s)}\right)^\Gamma$$

using Lemma 2.1.7, so that

$$\begin{aligned}
\psi^\Gamma_{\psi^\Gamma_\varphi(s)} &= \psi^\Gamma_{(r_\varphi \circ (\psi_{\beta(\varphi)})^\Gamma)(s)} \\
&= \psi^\Gamma_{(\psi_{\beta(\varphi)})^\Gamma(s)\cdot\varphi} \\
&= r_{(\psi_{\beta(\varphi)})^\Gamma(s)\cdot\varphi} \circ \left(\psi_{\beta((\psi_{\beta(\varphi)})^\Gamma(s)\cdot\varphi)}\right)^\Gamma \\
&= r_\varphi \circ r_{(\psi_{\beta(\varphi)})^\Gamma(s)} \circ \left(\psi_{\psi_{\beta(\varphi)}(s)}\right)^\Gamma \\
&= r_\varphi \circ \left(\psi_{\beta(\varphi)}\right)^\Gamma \circ \mathsf{t}_s \\
&= \psi^\Gamma_\varphi \circ \mathsf{t}_s
\end{aligned}$$

and we see, putting $\psi^\Gamma_t(\varphi) = \psi^\Gamma(\varphi, t)$, that

$$\psi^\Gamma_t\psi^\Gamma_s(\varphi) = \psi^\Gamma_t\psi^\Gamma_\varphi(s) = \psi^\Gamma_{\psi^\Gamma_\varphi(s)}(t) = \psi^\Gamma_\varphi \circ \mathsf{t}_s(t) = \psi^\Gamma_\varphi(s+t) = \psi^\Gamma_{s+t}(\varphi).$$

As in addition

$$\psi_0^\Gamma(\varphi) = \psi_\varphi^\Gamma(0) = r_\varphi\big((\psi_{\beta(\varphi)})^\Gamma(0)\big) = r_\varphi(1_{\psi_{\beta(\varphi)}(0)}) = r_\varphi(1_{\beta(\varphi)}) = \varphi,$$

we see that ψ_t^Γ satisfies the pseudogroup property for a flow. As each curve ψ_x is smooth, it follows that each curve $\psi_\varphi^\Gamma = r_\varphi \circ (\psi_{\beta(\varphi)})^\Gamma$ is smooth and hence defines a tangent vector

$$X_\varphi^\Gamma = \dot{\psi}_\varphi^\Gamma(0) = r_{\varphi *}(\dot{\psi}_{\beta(\varphi)})^\Gamma(0) \in T_\varphi \mathcal{G}.$$

We must now show that the resulting vector field X^Γ on $\mathcal{G}$ given by $\varphi \mapsto X_\varphi^\Gamma$ is smooth. If $\varphi = 1_x$ for $x \in M$ then

$$X_{1_x}^\Gamma = (\dot{\psi}_x)^\Gamma(0) = \gamma\dot{\psi}_x(0) = \gamma(X_x)$$

where $\gamma : TM \to T\mathcal{G}$ is the smooth map associated with Γ. Thus in general

$$X_\varphi^\Gamma = r_{\varphi *}\gamma(X_{\beta(\varphi)}),$$

showing that $X^\Gamma : \mathcal{G} \to T\mathcal{G}$ is a smooth map.

The argument where X does not have a global flow is similar, taking account of the domains of the maps involved. □

2.2 Infinitesimal Connections on Lie Algebroids

A path connection on $\mathcal{G}$ is the groupoid version of a fibre bundle connection on $E \to M$ given by horizontal lifts of curves. The latter may also, though, be given in infinitesimal terms as a type (1, 1) tensor field on E with the property that it maps lifts of tangent vectors in TM in a well defined way to horizontal tangent vectors in TE (that is, to vectors tangent to horizontal curves in E). This infinitesimal approach permits a definition of the curvature of the connection, as the extent to which the induced mapping from vector fields on M to vector fields on E fails to preserve the Lie bracket. We apply a similar approach to path connections on Lie groupoids, although here it is sufficient to define the infinitesimal version on the associated Lie algebroid.

We have, in fact, already seen the infinitesimal version of a path connection on a locally trivial Lie groupoid: it is the map $\gamma : TM \to T\mathcal{G}$ associated with a path connection Γ.

Lemma 2.2.1 *The map γ associated with the path connection Γ on $\mathcal{G}$ satisfies $\alpha_* \circ \gamma = 0$ and $\beta_* \circ \gamma = \mathrm{id}_{TM}$.*

Proof Take $v \in T_xM$ with $c(0) = x, \dot{c}(0) = v$. From $\alpha c^\Gamma(t) = c(0)$ and $\beta c^\Gamma(t) = c(t)$ we see that $\alpha_*\dot{c}^\Gamma(0) = 0$ and $\beta_*\dot{c}^\Gamma(0) = \dot{c}(0)$. □

Corollary 2.2.2 *The map γ takes its values in the Lie algebroid $A\mathcal{G} = \ker \alpha_*|_{1_M}$.* □

We shall therefore say that an *infinitesimal connection* on a general transitive Lie algebroid $\mathcal{A}$, not necessarily arising from a Lie groupoid, is a vector bundle morphism $\gamma : TM \to \mathcal{A}$ that is also a section of the anchor map $\rho : \mathcal{A} \to TM$. Thus an infinitesimal connection is a particular type of algebroid-valued 1-form on M.

We have mentioned that a transitive Lie algebroid may be written as a term in a short exact sequence

$$0 \to \mathcal{K} \xrightarrow{j} \mathcal{A} \xrightarrow{\rho} TM \to 0$$

where j is the inclusion map. We may therefore define, for any infinitesimal connection γ, an associated map $\omega : \mathcal{A} \to \mathcal{K}$, its *kernel projection*, which is the unique vector bundle morphism satisfying $\omega \circ j = \mathrm{id}_{\mathcal{K}}$ and $j \circ \omega + \gamma \circ \rho = \mathrm{id}_{\mathcal{A}}$. (In [30] the kernel projection is called the *reform* of the infinitesimal connection.) We also note the following result.

Lemma 2.2.3 *If $\gamma : TM \to \mathcal{A}$ is an infinitesimal connection and ξ is any section of $\mathcal{A}$ then the Lie derivative $\mathcal{L}_\xi \gamma$ takes its values in the kernel $\mathcal{K}$.*

Proof This follows from Lemma 1.6.1 because

$$\rho \circ \mathcal{L}_\xi \gamma = \mathcal{L}_{\rho\circ\xi}(\rho \circ \gamma) = \mathcal{L}_{\rho\circ\xi}\, \mathrm{id}_{TM} = 0.$$ □

We may now, for any infinitesimal connection $\gamma : TM \to \mathcal{A}$, define its *curvature* $R^\gamma : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \sec(\mathcal{A})$ by

$$R^\gamma(X, Y) = \gamma \circ [X, Y] - [\![\gamma \circ X, \gamma \circ Y]\!],$$

so that R^γ measures the failure of γ to be a morphism of Lie algebroids. If $R^\gamma = 0$ the infinitesimal connection γ is said to be *flat*.

Lemma 2.2.4 *The curvature R^γ is tensorial, so that it may be represented by a map $TM \times_M TM \to \mathcal{A}$ (and therefore may be regarded as an $\mathcal{A}$-valued 2-form on M); in fact it takes its values in the kernel bundle $\mathcal{K}$.*

Proof The tensorial property follows from

$$\begin{aligned} R^\gamma(X, fY) &= \gamma \circ [X, fY] - [\![\gamma \circ X, \gamma \circ fY]\!] \\ &= \gamma \circ \big(f[X, Y] + (Xf)Y\big) - [\![\gamma \circ X, f(\gamma \circ Y)]\!] \\ &= \gamma \circ f[X, Y] + (Xf)(\gamma \circ Y) - f[\![\gamma \circ X, (\gamma \circ Y)]\!] - \big((\rho \circ \gamma \circ X)f\big)(\gamma \circ Y) \\ &= \gamma \circ f[X, Y] - f[\![\gamma \circ X, (\gamma \circ Y)]\!] \\ &= fR^\gamma(X, Y). \end{aligned}$$

As R^γ is obviously skew-symmetric, it is an $\mathcal{A}$-valued 2-form.

From the properties of the anchor map we also see that

$$\rho \circ [\![\gamma \circ X, \gamma \circ Y]\!] = [\rho \circ \gamma \circ X, \rho \circ \gamma \circ Y] = [X, Y]$$

and $\rho \circ \gamma \circ [X, Y] = [X, Y]$, so that $\rho \circ R^\gamma(X, Y) = 0$, showing that R^γ is in fact a $\mathcal{K}$-valued 2-form. □

Lemma 2.2.5 *The curvature R^γ is given in terms of the kernel projection ω of γ by*

$$R^\gamma(\rho \circ \xi, \rho \circ \eta) = -\omega \circ [\![\xi, \eta]\!] + [\![\xi, \omega \circ \eta]\!] + [\![\omega \circ \xi, \eta]\!] - [\![\omega \circ \xi, \omega \circ \eta]\!];$$

in addition the relationship

$$\omega \circ [\![\xi, \eta]\!] - [\![\omega \circ \xi, \omega \circ \eta]\!] = R^\gamma(\rho \circ \xi, \rho \circ \eta) + \mathcal{L}_\xi \gamma \circ (\rho \circ \eta) - \mathcal{L}_\eta \gamma \circ (\rho \circ \xi)$$

holds.

We appeal here to the definition of the Lie derivative by a section ξ of $\mathcal{A}$ given in Sect. 1.6: note that γ is a vector bundle morphism $TM \to \mathcal{A}$, that is, it is an example—indeed the prime example—of the first case discussed in that section.

Proof As $[\rho \circ \xi, \rho \circ \eta] = \rho \circ [\![\xi, \eta]\!]$, the first formula follows from

$$\begin{aligned} R^\gamma(\rho \circ \xi, \rho \circ \eta) &= \gamma \circ \rho \circ [\![\xi, \eta]\!] - [\![\gamma \circ \rho \circ \xi, \gamma \circ \rho \circ \eta]\!] \\ &= [\![\xi, \eta]\!] - \omega \circ [\![\xi, \eta]\!] - [\![\xi - \omega \circ \xi, \eta - \omega \circ \eta]\!] \\ &= -\omega \circ [\![\xi, \eta]\!] + [\![\xi, \omega \circ \eta]\!] + [\![\omega \circ \xi, \eta]\!] - [\![\omega \circ \xi, \omega \circ \eta]\!]. \end{aligned}$$

Rearranging, we obtain

$$[\![\omega \circ \xi, \omega \circ \eta]\!] + \omega \circ [\![\xi, \eta]\!] = [\![\xi, \omega \circ \eta]\!] + [\![\omega \circ \xi, \eta]\!] - R^\gamma(\rho \circ \xi, \rho \circ \eta)$$

so that the second formula follows from

$$\begin{aligned} -[\![\omega \circ \xi, \omega \circ \eta]\!] + \omega \circ [\![\xi, \eta]\!] &= -[\![\xi, \omega \circ \eta]\!] - [\![\omega \circ \xi, \eta]\!] + 2\omega \circ [\![\xi, \eta]\!] + R^\gamma(\rho \circ \xi, \rho \circ \eta) \\ &= [\![\xi, \gamma \circ \rho \circ \eta]\!] + [\![\gamma \circ \rho \circ \xi, \eta]\!] \\ &\quad - 2\gamma \circ \rho \circ [\![\xi, \eta]\!] + R^\gamma(\rho \circ \xi, \rho \circ \eta) \\ &= \mathcal{L}_\xi \gamma \circ (\rho \circ \eta) - \mathcal{L}_\eta \gamma \circ (\rho \circ \xi) + R^\gamma(\rho \circ \xi, \rho \circ \eta). \end{aligned}$$ □

As a tensorial map $TM \times_M TM \to \mathcal{K} \subset \mathcal{A}$ the curvature R^γ has a Lie derivative by any section ξ of $\mathcal{A}$ given by

$$(\mathcal{L}_\xi R^\gamma)(X, Y) = [\![\xi, R^\gamma(X, Y)]\!] - R^\gamma([\rho \circ \xi, X], Y) - R^\gamma(X, [\rho \circ \xi, Y]).$$

Corollary 2.2.6 *The Lie derivative of the curvature may be written as*

$$(\mathcal{L}_\xi R^\gamma)(X, Y) = \mathcal{L}_\xi\gamma \circ [X, Y] - [\![\mathcal{L}_\xi\gamma \circ X, \gamma \circ Y]\!] - [\![\gamma \circ X, \mathcal{L}_\xi\gamma \circ Y]\!].$$

Proof This follows from

$$\begin{aligned}
[\![\xi, R^\gamma(X, Y)]\!] &= [\![\xi, \gamma \circ [X, Y]]\!] - [\![\xi, [\![\gamma \circ X, \gamma \circ Y]\!]]\!] \\
R^\gamma([\rho \circ \xi, X], Y) &= \gamma \circ [[\rho \circ \xi, X], Y] - [\![\gamma \circ [\rho \circ \xi, X], \gamma \circ Y]\!] \\
R^\gamma(X, [\rho \circ \xi, Y]) &= \gamma \circ [X, [\rho \circ \xi, Y]] - [\![\gamma \circ X, \gamma \circ [\rho \circ \xi, Y]]\!]
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_\xi\gamma \circ [X, Y] &= [\![\xi, \gamma \circ [X, Y]]\!] - \gamma \circ [\rho \circ \xi, [X, Y]] \\
[\![\mathcal{L}_\xi\gamma \circ X, \gamma \circ Y]\!] &= [\![[\![\xi, \gamma \circ X]\!], \gamma \circ Y]\!] - [\![\gamma \circ [\rho \circ \xi, X], \gamma \circ Y]\!] \\
[\![\gamma \circ X, \mathcal{L}_\xi\gamma \circ Y]\!] &= [\![\gamma \circ X, [\![\xi, \gamma \circ Y]\!]]\!] - [\![\gamma \circ X, \gamma \circ [\rho \circ \xi, Y]]\!].
\end{aligned}$$

□

All the above applies to infinitesimal connections defined on general transitive Lie algebroids: but if a Lie algebroid is obtained from a locally trivial Lie groupoid with a path connection, we may consider the map γ associated with the path connection as an infinitesimal connection on the Lie algebroid. This correspondence between path connections and infinitesimal connections is a bijection.

Theorem 2.2.7 *If Γ is a path connection on a locally trivial Lie groupoid $\mathcal{G}$ over M then the associated map $\gamma : TM \to A\mathcal{G}$ is an infinitesimal connection on $A\mathcal{G}$. Furthermore, every infinitesimal connection on $A\mathcal{G}$ arises in this way from a unique path connection Γ on $\mathcal{G}$.*

Proof By definition γ is a smooth vector bundle morphism projecting to the identity on M, and from Lemma 2.2.1 it is a section of $\rho = \beta_*$; it is therefore an infinitesimal connection.

Conversely, suppose $\gamma : TM \to A\mathcal{G}$ is an infinitesimal connection. Let $c : [0, 1] \to M$ be a curve; we wish to construct a lifted curve $c^\Gamma : [0, 1] \to \mathcal{G}$ satisfying the conditions for the lift by a path connection and such that $\gamma\big(\dot{c}(0)\big) = \dot{c}^\Gamma(0)$. As $\mathcal{G}$ is locally trivial, for each $t \in [0, 1]$ there is an open neighbourhood U_t of $c(t)$ and a trivialization $\mathrm{T}_t : U_t \times G \times U_t \to \mathcal{G}_{U_t}$ where $G = \mathcal{G}_{c(0)}$ (Lemma 1.1.1). By compactness we may choose finitely many of these neighbourhoods, $U_1, U_2, \ldots, U_m$, covering $c\big([0, 1]\big)$. As in the proof of Lemma 1.1.1 we may assume that each $c\big([0, 1]\big) \cap U_i$ is connected and that the U_i are indexed so that $c(0) \in U_1$ and $c(1) \in U_m$, and so that we may find $c(t_i) \in U_i \cap U_{i+1}$ for $1 \le i \le m - 1$ and $0 < t_1 < \cdots < t_{m-1} < 1$; put $t_0 = 0$ and $t_m = 1$.

If c is not smooth on any interval $[t_i, t_{i+1}]$, further subdivide that interval, so that the result is a sequence of subintervals where c is smooth on each subinterval, and where the image of each subinterval lies completely within one of the trivialization neighbourhoods U. We shall demonstrate the existence of a unique lifted curve segment c^Γ satisfying the condition $\gamma\big(\dot{c}(0)\big) = \dot{c}^\Gamma(0)$ and defined on the first of these intervals $[0, a]$ where $a \leq t_1$; this will be the first segment of a lifted curve defined on the whole of $[0, 1]$. Subsequent segments may be defined recursively in a similar way, using the reparametrization property.

Put $U = U_1$ and $\mathrm{T} = \mathrm{T}_1$ so that $c\big([0, a]\big) \subset U$. For each $t \in [0, a]$ we have a tangent vector $\dot{c}(t) \in T_{c(t)}U$, and hence an element $\gamma(\dot{c}(t))$ of the Lie algebroid fibre $A_{c(t)}\mathcal{G}_U \subset T_{1_{c(t)}}\mathcal{G}_U$. Using the trivialization we may write

$$\mathrm{T}_*^{-1}\big(T_{1_{c(t)}}\mathcal{G}_U\big) = T_{c(t)}U \oplus \mathfrak{g}_{\mathsf{R}} \oplus T_{c(t)}U$$

where $\mathfrak{g}_{\mathsf{R}} = T_{1_{\mathcal{G}_{c(0)}}}\mathcal{G}_{c(0)}$ is the opposite Lie algebra of $G = \mathcal{G}_{c(0)}$, and hence we may write, explicitly,

$$\mathrm{T}_*^{-1}\big(A_{c(t)}\mathcal{G}_U\big) = T_{c(t)}U \oplus \mathfrak{g}_{\mathsf{R}} \oplus \{0_{T_{c(t)}U}\}.$$

We may therefore, using $\beta_*\gamma\dot{c}(t) = \rho\gamma\dot{c}(t) = \dot{c}(t)$, put

$$\mathrm{T}_*^{-1}\gamma\dot{c}(t) = \big(\dot{c}(t), v(t), 0_{T_{c(t)}U}\big)$$

where $v : [0, a] \to \mathfrak{g}_{\mathsf{R}}$ is a curve in $\mathfrak{g}_{\mathsf{R}}$.

We now observe that there is a unique curve $g : [0, a] \to G$ with $\dot{g}(t) = r_{g(t)*}v(t)$ and $g(0) = 1_G$, so that v is the right Darboux derivative of g; as the interval $[0, a]$ is one-dimensional, questions of monodromy do not arise and g may be defined on the whole of $[0, a]$. We may therefore define a lifted curve $c^\Gamma : [0, a] \to \mathcal{G}_U$ by

$$c^\Gamma(t) = \mathrm{T}\big(c(t), g(t), c(0)\big).$$

It is clear that $\alpha c^\Gamma(t) = c(0)$ and $\beta c^\Gamma(t) = c(t)$ for all $t \in [0, a]$; we also see that

$$\begin{aligned}
\mathrm{T}_*^{-1}\dot{c}^\Gamma(t) &= \big(\dot{c}(t), \dot{g}(t), 0_{T_{c(0)}U}\big) \\
&= \big(\dot{c}(t), r_{g(t)}v(t), 0_{T_{c(0)}U}\big) \\
&= r_{(c(t),g(t),c(0))*}\big(\dot{c}(t), v(t), 0_{T_{c(t)}U}\big) \\
&= r_{\mathrm{T}^{-1}(c^\Gamma(t))*}\mathrm{T}_*^{-1}\gamma\dot{c}(t) \\
&= \mathrm{T}_*^{-1}r_{c^\Gamma(t)*}\gamma\dot{c}(t)
\end{aligned} \tag{1a}$$

so that

$$\dot{c}^\Gamma(t) = r_{c^\Gamma(t)*}\gamma\dot{c}(t)$$

for all $t \in [0, a]$, and that in particular $\dot c^{\Gamma}(0) = \gamma\dot c(0)$. The reparametrization property of c^{Γ} follows from the uniqueness of the solution curve g.

We need to see that this construction does not depend upon the particular choice of local trivializations for $\mathcal{G}$, and it is sufficient to demonstrate this for an alternative trivialization $\tilde{\mathrm{T}} : U \times G \times U \to \mathcal{G}_U$. We may therefore write

$$\tilde{\mathrm{T}}_*^{-1}\gamma\dot c(t) = \big(\dot c(t), \tilde v(t), 0_{T_{c(t)}U}\big)$$

where $\tilde v : [0, a] \to \mathfrak{g}_{\mathsf{R}}$ is another curve in $\mathfrak{g}_{\mathsf{R}}$, and therefore obtain another curve $\tilde g : [0, a] \to G$ with $\dot{\tilde g}(t) = r_{\tilde g(t)*}\tilde v(t)$ and $\tilde g(0) = 1_G$; define $\tilde c^{\Gamma} : [0, a] \to \mathcal{G}_U$ by

$$\tilde c^{\Gamma}(t) = \tilde{\mathrm{T}}\big(c(t), \tilde g(t), c(0)\big),$$

so that as before we will have

$$\dot{\tilde c}^{\Gamma}(t) = r_{\tilde c^{\Gamma}(t)*}\gamma\dot c(t)$$

and therefore

$$r^{-1}_{\tilde c^{\Gamma}(t)*}\dot{\tilde c}^{\Gamma}(t) = r^{-1}_{c^{\Gamma}(t)*}\dot c^{\Gamma}(t). \tag{2}$$

Now consider the curve $\mathrm{T}^{-1} \circ \tilde c^{\Gamma}$ in $\mathrm{T}^{-1}(\mathcal{G}_U)$; this will satisfy

$$\mathrm{T}^{-1}\tilde c^{\Gamma}(t) = \big(c(t), \hat g(t), c(0)\big)$$

for some curve $\hat g$ in G with $\hat g(0) = 1_G$, so that

$$\mathrm{T}_*^{-1}\dot{\tilde c}^{\Gamma}(t) = \big(\dot c(t), \dot{\hat g}(t), 0_{T_{c(0)}U}\big). \tag{1b}$$

It follows from (1a), (1b) and (2) that

$$r^{-1}_{\tilde c^{\Gamma}(t)*}\mathrm{T}_*\big(\dot c(t), \dot{\hat g}(t), 0_{T_{c(0)}U}\big) = r^{-1}_{c^{\Gamma}(t)*}\mathrm{T}_*\big(\dot c(t), \dot g(t), 0_{T_{c(0)}U}\big)$$

and therefore that

$$r^{-1}_{\hat g(t)*}\dot{\hat g}(t) = r^{-1}_{g(t)*}\dot g(t) = v(t),$$

so that $\hat g = g$ by uniqueness; thus $\tilde c^{\Gamma} = c^{\Gamma}$.

Using this procedure for each of the successive segments we may, for any curve $c : [0, 1] \to M$, construct a lifted curve $c^{\Gamma} : [0, 1] \to \mathcal{G}$ satisfying the required properties, and in this way we define the path connection Γ whose infinitesimal connection is γ. □

2.3 Connections and Projectability

In Sect. 1.4 we defined full morphisms of Lie groupoids and of Lie algebroids, and indicated that these would be appropriate types of morphism to use when considering whether objects were ‘projectable’. We now use these morphisms to investigate the projectability of path connections on Lie groupoids, and of infinitesimal connections on Lie algebroids.

Let $\mathcal{G}$, $\mathcal{H}$ be Lie groupoids with manifolds M, N of identities, and suppose that $(\mathcal{P}, p)$ is a full morphism from $\mathcal{G}$ to $\mathcal{H}$. We shall say that the path connection Γ on $\mathcal{G}$ is *projectable to* $\mathcal{H}$ if, whenever $c_1, c_2 : [0, 1] \to M$ are curves satisfying $p \circ c_1 = p \circ c_2$, then the lifted curves $c_1^\Gamma, c_2^\Gamma : [0, 1] \to \mathcal{G}$ satisfy $\mathcal{P} \circ c_1^\Gamma = \mathcal{P} \circ c_2^\Gamma$.

Now let $\varkappa : [0, 1] \to N$ be a curve, and suppose initially that there is a neighbourhood $U \subset N$ of $\varkappa(0)$ and a local section $\varsigma : U \to M$ such that $\varkappa(t) \in U$ for all $t \in [0, 1]$. The composite $\varsigma \circ \varkappa$ is then a curve in M satisfying $p \circ (\varsigma \circ \varkappa) = \varkappa$ with a lift $(\varsigma \circ \varkappa)^\Gamma$ in $\mathcal{G}$. Define the lifted curve $\varkappa^{\Gamma^\mathcal{P}} : [0, 1] \to \mathcal{H}$ by $\varkappa^{\Gamma^\mathcal{P}} = \mathcal{P} \circ (\varsigma \circ \varkappa)^\Gamma$; by projectability this does not depend on the choice of local section ς. If there is no such neighbourhood U then split $\varkappa$ into finitely many segments and carry out the procedure described above for each segment individually, combining the results to obtain the lifted curve $\varkappa^{\Gamma^\mathcal{P}}$.

Proposition 2.3.1 *If the path connection Γ on $\mathcal{G}$ is projectable to $\mathcal{H}$ then the lifting operation $\Gamma^\mathcal{P}$ is a well-defined path connection on $\mathcal{H}$, and for any curve $c : [0, 1] \to M$ we have $(p \circ c)^{\Gamma^\mathcal{P}} = \mathcal{P} \circ c^\Gamma$.*

Proof We first describe the lifting procedure in detail for arbitrary curves in N.

Let $\varkappa : [0, 1] \to N$ be a curve. For each $t \in [0, 1]$ let U_t be a neighbourhood of $\varkappa(t)$ which is the domain of a section $\varsigma_t : U_t \to M$; by compactness we may choose finitely many of these neighbourhoods, $U_1, U_2, \ldots, U_m$, covering $\varkappa\bigl([0, 1]\bigr)$, with corresponding local sections ς_i. As in the proof of Lemma 1.1.1 we may assume that each $\varkappa\bigl([0, 1]\bigr) \cap U_i$ is connected and that the U_i are indexed so that $\varkappa(0) \in U_1$ and $\varkappa(1) \in U_m$, and so that we may find $\varkappa(t_i) \in U_i \cap U_{i+1}$ for $1 \le i \le m - 1$ and $0 < t_1 < \cdots < t_{m-1} < 1$; put $t_0 = 0$ and $t_m = 1$. Define curves $\varkappa_i : [0, 1] \to N$, $1 \le i \le m$, by

$$\varkappa_i(t) = \varkappa\bigl(tt_i + (1 - t)t_{i-1}\bigr),$$

so that $\varkappa_i(t) \in U_i$ for all $t \in [0, 1]$; define the lifted curves $\varkappa_i^{\Gamma^\mathcal{P}}$ by $\varkappa_i^{\Gamma^\mathcal{P}} = \mathcal{P} \circ (\varsigma_i \circ \varkappa_i)^\Gamma$. Finally, noting that

$$\alpha_\mathcal{H} \varkappa_i^{\Gamma^\mathcal{P}}(t) = \varkappa_i(0) = \varkappa(t_{i-1}) = \varkappa_{i-1}(1) = \beta_\mathcal{H} \varkappa_{i-1}^{\Gamma^\mathcal{P}}(1)$$

for $2 \le i \le m$ and any $t \in [0, 1]$, define the lifted curve $\varkappa^{\Gamma^\mathcal{P}}$ by

$$\varkappa^{\Gamma^{\mathcal{P}}}(t) = \begin{cases} \varkappa_1^{\Gamma^{\mathcal{P}}}\left(\dfrac{t-t_0}{t_1-t_0}\right) = \varkappa_1^{\Gamma^{\mathcal{P}}}\left(\dfrac{t}{t_1}\right) & t_0 \le t \le t_1 \\ \varkappa_i^{\Gamma^{\mathcal{P}}}\left(\dfrac{t-t_{i-1}}{t_i-t_{i-1}}\right) \cdot \varkappa_{i-1}^{\Gamma^{\mathcal{P}}}(1) & t_{i-1} \le t \le t_i, \quad 2 \le i \le m. \end{cases}$$

We must check that $\varkappa^{\Gamma^{\mathcal{P}}}$ does not depend on the choices made. Suppose that $t_1 < 1$, and that $\hat{U}_1$ is some other neighbourhood of $\varkappa(0)$ with local section $\hat{\varsigma}_1 : \hat{U}_1 : M$ where $\varkappa(t) \in \hat{U}_1$ for $0 \le t \le \hat{t}_1$ where $t_1 < \hat{t}_1$. Let $\varkappa_1, \hat{\varkappa}_1 : [0,1] \to N$ be the corresponding initial curve segments, so that $\varkappa_1(t) = t_1 t$ and $\hat{\varkappa}_1(t) = \hat{t}_1 t$. Then, for $0 \le t \le t_1$, we have

$$\varkappa^{\Gamma^{\mathcal{P}}}(t) = \varkappa_1^{\Gamma^{\mathcal{P}}}\left(\frac{t}{t_1}\right) = \mathcal{P}(\varsigma_1 \circ \varkappa_1)^{\Gamma}\left(\frac{t}{t_1}\right) = \mathcal{P}(\hat{\varsigma}_1 \circ \varkappa_1)^{\Gamma}\left(\frac{t}{t_1}\right)$$

by projectability of Γ. If we now let $\chi : [0,1] \to [0, t_1/\hat{t}_1]$ be given by $\chi(t) = t_1 t/\hat{t}_1$ then χ is a diffeomorphism with the property that $\varkappa_1 = \hat{\varkappa}_1 \circ \chi$, so that

$$\varkappa^{\Gamma^{\mathcal{P}}}(t) = \mathcal{P}(\hat{\varsigma}_1 \circ \varkappa_1)^{\Gamma}\left(\frac{t}{t_1}\right) = \mathcal{P}(\hat{\varsigma}_1 \circ \hat{\varkappa}_1 \circ \chi)^{\Gamma}\left(\frac{t}{t_1}\right) = \mathcal{P}(\hat{\varsigma}_1 \circ \hat{\varkappa}_1)^{\Gamma}\chi\left(\frac{t}{t_1}\right)$$

by the reparametrization property of Γ, using the fact that $(\hat{\varsigma}_1 \circ \hat{\varkappa}_1)^{\Gamma}\chi(0) = (\hat{\varsigma}_1 \circ \hat{\varkappa}_1)^{\Gamma}(0) = 1_{\hat{\varsigma}_1 \hat{\varkappa}_1(0)}$. But

$$\mathcal{P}(\hat{\varsigma}_1 \circ \hat{\varkappa}_1)^{\Gamma}\chi\left(\frac{t}{t_1}\right) = \mathcal{P}(\hat{\varsigma}_1 \circ \hat{\varkappa}_1)^{\Gamma}\left(\frac{t}{\hat{t}_1}\right),$$

and this is the first segment of the curve that would be obtained by lifting with the local section $\hat{\varsigma}_1$ rather than the local section ς_1. A similar argument may be applied recursively to show that the whole lifted curve $\varkappa^{\Gamma^{\mathcal{P}}}$ is independent of the particular choice of partition $[0 = t_0, t_1, t_2, \ldots, t_{m-1}, t_m = 1]$ and of the particular choice of local sections ς_i.

We now check that the lifted curve $\varkappa^{\Gamma^{\mathcal{P}}}$ does indeed satisfy the conditions for a path connection.

First, we see that

$$\varkappa^{\Gamma^{\mathcal{P}}}(0) = \varkappa_1^{\Gamma^{\mathcal{P}}}(0) = \mathcal{P}\big((\varsigma_1 \circ \varkappa_1)^{\Gamma}(0)\big) = \mathcal{P}\big(1_{\varsigma_1 \varkappa_1(0)}\big) = 1_{p\varsigma_1 \varkappa_1(0)} = 1_{\varkappa_1(0)} = 1_{\varkappa(0)}$$

and that

$$\begin{aligned} \alpha_{\mathcal{H}} \varkappa^{\Gamma^{\mathcal{P}}}(t) &= \alpha_{\mathcal{H}}\left(\varkappa_i^{\Gamma^{\mathcal{P}}}\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right) \cdot \varkappa_{i-1}^{\Gamma^{\mathcal{P}}}(1) \cdot \varkappa_{i-1}^{\Gamma^{\mathcal{P}}}(1) \cdot \ldots \cdot \varkappa_1^{\Gamma^{\mathcal{P}}}(1)\right) = \alpha_{\mathcal{H}} \varkappa_1^{\Gamma^{\mathcal{P}}}(1) \\ &= \alpha_{\mathcal{H}} \mathcal{P}(\varsigma_1 \circ \varkappa_1)^{\Gamma}(0) = p\alpha_{\mathcal{G}}(\varsigma_1 \circ \varkappa_1)^{\Gamma}(0) = p(\varsigma_1 \circ \varkappa_1)(0) = \varkappa_1(0) = \varkappa(0) \end{aligned}$$

(with the appropriate modification when $0 \le t \le t_1$) whereas

$$\begin{aligned} \beta_{\mathcal{H}}\varkappa^{\Gamma^{\mathcal{P}}}(t) &= \beta_{\mathcal{H}}\left(\varkappa_i^{\Gamma^{\mathcal{P}}}\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right)\cdot\varkappa_{i-1}^{\Gamma^{\mathcal{P}}}(1)\cdot\varkappa_{i-1}^{\Gamma^{\mathcal{P}}}(1)\cdot\ldots\cdot\varkappa_1^{\Gamma^{\mathcal{P}}}(1)\right)\\ &= \beta_{\mathcal{H}}\varkappa_i^{\Gamma^{\mathcal{P}}}\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right) = \beta_{\mathcal{H}}\mathcal{P}(\varsigma_i\circ\varkappa_i)^{\Gamma}\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right)\\ &= p\beta_{\mathcal{G}}(\varsigma_i\circ\varkappa_i)^{\Gamma}\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right) = p\varsigma_i\varkappa_i\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right) = \varkappa_i\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right) = \varkappa(t). \end{aligned}$$

The reparametrization condition $\varkappa^{\Gamma^{\mathcal{P}}}\circ\chi = r_{\varphi}\circ(\varkappa\circ\chi)^{\Gamma^{\mathcal{P}}}$, where $\varphi = \varkappa^{\Gamma^{\mathcal{P}}}\chi(0)$, is obtained from the reparametrization condition for Γ using an argument similar to that given when showing that $\Gamma^{\mathcal{P}}$ is well defined.

Now let $c : [0, 1] \to M$ be a curve, and suppose that there is a local section $\varsigma : U \to M$ of p such that $pc(t) \in U$ for all $t \in [0, 1]$. By definition $(p\circ c)^{\Gamma^{\mathcal{P}}} = \mathcal{P}\circ(\varsigma\circ p\circ c)^{\Gamma}$; but the two curves $\varsigma\circ p\circ c$ and c satisfy the condition $p\circ(\varsigma\circ p\circ c) = p\circ c$, so that $(\varsigma\circ p\circ c)^{\Gamma} = c^{\Gamma}$ by projectability. We therefore see that $(p\circ c)^{\Gamma^{\mathcal{P}}} = \mathcal{P}\circ c^{\Gamma}$. By reparametrization the same result holds if it is necessary to subdivide c into several segments.

Next, suppose that $\varkappa$ is smooth at $t \in [0, 1]$, and arrange the subdivision of $[0, 1]$ such that $t_{i-1} < t < t_i$ for some $1 \le i \le m$; put $\bar{t} = tt_i + (1-t)t_{i-1}$. Then $\varkappa_i$ is smooth at $\bar{t}$, as is $\varsigma_i\circ\varkappa_i$. So $(\varsigma_i\circ\varkappa_i)^{\Gamma}$ is smooth at $\bar{t}$, and then $\varkappa^{\Gamma^{\mathcal{P}}} = \mathcal{P}\circ(\varsigma_i\circ\varkappa_i)^{\Gamma}$ is also smooth at $\bar{t}$, showing that $\varkappa^{\Gamma^{\mathcal{P}}}$ is smooth at t.

Now suppose that two curves $\varkappa_1, \varkappa_2 : [0, 1] \to N$ satisfy $\varkappa_1(0) = \varkappa_2(0)$ and $\dot{\varkappa}_1(0) = \dot{\varkappa}_2(0)$, and that there is a local section $\varsigma : U \to M$ such that $\varkappa_1\big([0, 1]\big), \varkappa_2\big([0, 1]\big) \subset U$. Put $c_1 = \varsigma\circ\varkappa_1$ and $c_2 = \varsigma\circ\varkappa_2$, so that $c_1(0) = \varsigma\varkappa_1(0) = \varsigma\varkappa_2(0) = c_2(0)$ and $\dot{c}_1(0) = \varsigma_*\dot{\varkappa}_1(0) = \varsigma_*\dot{\varkappa}_2(0) = \dot{c}_1(0)$; thus

$$\dot{\varkappa}_1^{\Gamma^{\mathcal{P}}}(0) = \mathcal{P}_*\dot{c}_1^{\Gamma}(0) = \mathcal{P}_*\dot{c}_2^{\Gamma}(0) = \dot{\varkappa}_2^{\Gamma^{\mathcal{P}}}(0).$$

Clearly the same result will hold if the domain of ς is simply a neighbourhood of $\varkappa_1(0) = \varkappa_2(0)$.

Now let $\varkappa : [0, 1] \to N$ satisfy $\varkappa(0) = y$, $\dot{\varkappa}(0) = w \in T_yN$ and put $\gamma^{\mathcal{P}}(w) = \dot{\varkappa}^{\Gamma^{\mathcal{P}}}(0)$, so that $\gamma^{\mathcal{P}}$ is a well-defined map $T_yN \to T_{1_y}\mathcal{H}$. If $c : [0, 1] \to M$ satisfies $p\circ c = \varkappa$ in some neighbourhood of zero then

$$\left.\frac{d(p\circ c)^{\Gamma^{\mathcal{P}}}}{dt}\right|_{t=0} = \left.\frac{d(\mathcal{P}\circ c^{\Gamma})}{dt}\right|_{t=0} = \mathcal{P}_*\left.\frac{dc^{\Gamma}}{dt}\right|_{t=0}$$

so that $\gamma^{\mathcal{P}}\circ p_* = \mathcal{P}_*\circ\gamma$, where $\gamma(v) = \dot{c}^{\Gamma}(0)$, $v = \dot{c}(0)$.

Finally let U be a neighbourhood of y and let $\varsigma : U \to X$ be a local section of p, so that $p_*\circ\varsigma_* = \mathrm{id}_{TU}$. Then

$$\gamma^{\mathcal{P}}|_{TU} = \gamma^{\mathcal{P}}\circ p_*\circ\varsigma_* = \mathcal{P}_*\circ\gamma\circ\varsigma_*$$

so that $\gamma^{\mathcal{P}}$ is a smooth fibre-linear map $TN \to T\mathcal{H}$. ☐

We now consider the projectability of infinitesimal connections on Lie algebroids. Let $\mathcal{A} \to M$ and $\mathcal{B} \to N$ be Lie algebroids, and let (P, p) be a full Lie algebroid morphism from $\mathcal{A}$ to $\mathcal{B}$. We shall say that the infinitesimal connection γ is *projectable to* $\mathcal{B}$ if, whenever $v_1, v_2 \in TM$ with $p_*(v_1) = p_*(v_2)$, then $P\gamma(v_1) = P\gamma(v_2)$. If γ is projectable, define its projection $\gamma^P : TN \to \mathcal{B}$ by, for $w \in TN$, choosing $v \in TM$ with $p_*(v) = w$ and setting $\gamma^P(w) = P\gamma(v)$.

Proposition 2.3.2 *The map* γ^P *is an infinitesimal connection on* $\mathcal{B}$.

Proof First, take $w_1, w_2 \in T_yN$ for some $y \in N$, and choose $x \in M$ such that $p(x) = y$. As p is a submersion we may find $v_1, v_2 \in T_xM$ such that $p_*(v_1) = w_1$ and $p_*(v_2) = w_2$. Then $\gamma^P(w_1) = P\gamma(v_1)$ and $\gamma^P(w_2) = P\gamma(v_2)$, so that

$$\tau_{\mathcal{B}}\gamma^P(w_1) = \tau_{\mathcal{B}}P\gamma(v_1) = p\tau_{\mathcal{A}}\gamma(v_1) = p(x) = p\tau_{\mathcal{A}}\gamma(v_2) = \tau_{\mathcal{B}}P\gamma(v_2) = \tau_{\mathcal{B}}\gamma^P(w_1),$$

showing that γ^P is well defined. To see that it is fibred over the identity on N, take $w \in T_yN$ and as before choose $x \in M$ such that $p(x) = y$. Taking $v \in T_xM$ such that $p_*(v) = w$ we see that

$$\tau_B\gamma^P(w) = \tau_B P\gamma(v) = p\tau_A\gamma(v) = p(x) = y.$$

To show that γ^P is smooth at $w \in T_yN$, choose a neighbourhood U of y and a local section $\varsigma : U \to M$ of p, so that $\varsigma_* : TU \to TM$ satisfies $p_* \circ \varsigma_* = (\mathrm{id}_U)_* = \mathrm{id}_{TU}$. Then $\gamma^P\big|_{TU} = P \circ \gamma \circ \varsigma_*$, showing that γ^P is smooth.

It also follows from this that, restricted to the fibre T_yN, $\gamma^P\big|_y = P|_{\gamma\varsigma(y)} \circ \gamma|_{\varsigma(y)} \circ \varsigma_{*y}$ is a composition of linear maps and so is linear. Furthermore, the rank of each map in the composition does not depend on the choice of fibre to which it is restricted, so that the composite map γ^P has constant rank on TU. Although the map ς is only a local section of the surjective submersion p, different maps ς_* will all have the same constant rank $\dim M$ (even if M is not connected), and as γ and P have global constant rank it follows that γ^P has constant rank throughout TM and is therefore a vector bundle morphism.

Finally, for any $w \in TN$ take $v \in TM$ with $w = p_*(v)$ so that

$$\rho_B\gamma^P(w) = \rho_B P\gamma(v) = p_*\rho_A\gamma(v) = p_*(v) = w;$$

thus γ^P is a section of the anchor map $\rho_{\mathcal{B}}$, confirming that γ^P is indeed an infinitesimal connection. □

Lemma 2.3.3 *If* γ *is a projectable infinitesimal connection and* X *is a projectable vector field on* M *then the section* $\gamma \circ X$ *of* $\mathcal{A}$ *is projectable; in addition* $(\gamma \circ X)^P = \gamma^P \circ X^p$.

Proof If $x_1, x_2 \in M$ satisfy $p(x_1) = p(x_2)$ then, because $p_* X_{x_1} = p_* X_{x_2}$ by projectability of X, we see that $P\gamma(X_{x_1})) = P\gamma(X_{x_2})$ by projectability of γ, so that the section $\gamma \circ X$ is projectable.

Now let $y \in N$, and let $x \in M$ satisfy $p(x) = y$. By definition the section $(\gamma \circ X)^P$ of $\mathcal{B}$ satisfies

$$(\gamma \circ X)^P(y) = P\big((\gamma \circ X)(x)\big) = P\gamma(X_x);$$

but the section $\gamma^P \circ X^p$ satisfies

$$(\gamma^P \circ X^p)(y) = \gamma^P(X^p_y) = \gamma^P p_*(X_x) = P\gamma(X_x)$$

so $(\gamma \circ X)^P = \gamma^P \circ X^p$. ☐

Corollary 2.3.4 *The curvature R^{γ^P} of the projected infinitesimal connection γ^P satisfies*

$$R^{\gamma^P}(X^p, Y^p) = \big(R^\gamma(X, Y)\big)^P.$$

Proof A straightforward calculation gives

$$\begin{aligned} R^{\gamma^P}(X^p, Y^p) &= \gamma^P \circ [X^p, Y^p] - [\![\gamma^P \circ X^p, \gamma^P \circ Y^p]\!] \\ &= \gamma^P \circ [X, Y]^p - [\![\gamma^P \circ X^p, \gamma^P \circ Y^p]\!] \\ &= \big(\gamma \circ [X, Y]\big)^P - [\![(\gamma \circ X)^P, (\gamma \circ Y)^P]\!] \\ &= \big(\gamma \circ [X, Y]\big)^P - [\![\gamma \circ X, \gamma \circ Y]\!]^P \\ &= \big(R^\gamma(X, Y)\big)^P. \end{aligned}$$

☐

2.4 The Kernel Derivative of an Infinitesimal Connection

In this and the following sections we turn our attention to various notions of covariant derivative associated with an infinitesimal connection on a Lie algebroid.

Let $\gamma : TM \to \mathcal{A}$ be an infinitesimal connection. We define an operator on sections κ of the kernel $\mathcal{K} \to M$, the *kernel derivative*, by $\nabla^\gamma_X \kappa = [\![\gamma \circ X, \kappa]\!]$.

Lemma 2.4.1 *The operator ∇^γ is a covariant derivative.*

Proof The operator is clearly $\mathbb{R}$-linear in both variables, and

$$\nabla^\gamma_{fX}\kappa = [\![\gamma \circ (fX), \kappa]\!] = [\![f(\gamma \circ X), \kappa]\!] = f[\![\gamma \circ X, \kappa]\!] - \big((\rho \circ \kappa)f\big)(\gamma \circ X) = f\nabla^\gamma_X \kappa$$

whereas

$$\nabla^\gamma_X(f\kappa) = [\![\gamma \circ X, f\kappa]\!] = f[\![\gamma \circ X, \kappa]\!] + \big((\rho \circ \gamma \circ X)f\big)\kappa = f\nabla^\gamma_X\kappa + (Xf)\kappa. \quad \square$$

Note that the formula $[\![\gamma \circ X, \xi]\!]$ does not define a covariant derivative on sections ξ of $\mathcal{A}$ because the condition $\rho \circ \xi = 0$ is needed to ensure that $\nabla^\gamma_{fX} = f\nabla^\gamma_X$.

Lemma 2.4.2 *The kernel derivative* ∇^γ *is a derivation of the Lie algebra bracket* $\{\cdot, \cdot\}$ *on the fibres of* $\mathcal{K}$*:*

$$\nabla^\gamma_X\{\kappa, \lambda\} = \{\nabla^\gamma_X\kappa, \lambda\} + \{\kappa, \nabla^\gamma_X\lambda\}.$$

Proof The Lie algebra bracket is the restriction to $\mathcal{K}$ of the Lie algebroid bracket on sections of $\mathcal{A}$, so that this is just the Jacobi identity again:

$$\begin{aligned}\nabla^\gamma_X\{\kappa, \lambda\} &= [\![\gamma \circ X, \{\kappa, \lambda\}]\!] = [\![\gamma \circ X, [\![\kappa, \lambda]\!]]\!] = [\![[\![\gamma \circ X, \kappa]\!], \lambda]\!] + [\![\kappa, [\![\gamma \circ X, \lambda]\!]]\!] \\ &= [\![\nabla^\gamma_X\kappa, \lambda]\!] + [\![\kappa, \nabla^\gamma_X\lambda]\!] = \{\nabla^\gamma_X\kappa, \lambda\} + \{\kappa, \nabla^\gamma_X\lambda\}. \quad \square\end{aligned}$$

We may apply the kernel derivative to the curvature of γ.

Lemma 2.4.3 *The curvature* R^γ *of the infinitesimal connection satisfies the identity*

$$\oint\big(\nabla^\gamma_X(R^\gamma(Y, Z)) + R^\gamma(X, [Y, Z])\big) = 0$$

where $\oint$ *indicates the cyclic sum over* X*,* Y *and* Z*.*

Proof We have

$$\begin{aligned}\nabla^\gamma_X(R^\gamma(Y, Z)) &= [\![\gamma \circ X, \gamma \circ [Y, Z]]\!] - [\![\gamma \circ X, [\![\gamma \circ Y, \gamma \circ Z]\!]]\!] \\ &= \gamma \circ [X, [Y, Z]] - R^\gamma(X, [Y, Z]) - [\![\gamma \circ X, [\![\gamma \circ Y, \gamma \circ Z]\!]]\!].\end{aligned}$$

The result now follows by the Jacobi identity for both brackets. $\square$

This single identity generalizes the two Bianchi identities of standard connection theory, and in fact incorporates them both as we shall show later: it is therefore called the *Bianchi identity*.

Lemma 2.4.4 *The curvature* R^∇ *of* ∇^γ *is related to the curvature* R^γ *of* γ *as follows:*

$$R^\nabla(X, Y)\kappa = [\![\kappa, R^\gamma(X, Y)]\!] = \{\kappa, R^\gamma(X, Y)\}.$$

Proof

$$\begin{aligned} R^{\nabla}(X, Y)\kappa &= \nabla^{\gamma}_X \nabla^{\gamma}_Y \kappa - \nabla^{\gamma}_Y \nabla^{\gamma}_X \kappa - \nabla^{\gamma}_{[X,Y]}\kappa \\ &= [\![\gamma \circ X, [\![\gamma \circ Y, \kappa]\!]]\!] - [\![\gamma \circ Y, [\![\gamma \circ X, \kappa]\!]]\!] - [\![\gamma \circ [X, Y], \kappa]\!] \\ &= [\![\kappa, \gamma \circ [X, Y]]\!] - [\![\kappa, [\![\gamma \circ X, \gamma \circ Y]\!]]\!] \\ &= [\![\kappa, R^{\gamma}(X, Y)]\!] = \{\kappa, R^{\gamma}(X, Y)\}. \end{aligned}$$ □

We may use the kernel derivative to define a 'differential' on $\mathcal{K}$-valued p-forms on M. Writing $\sec\big(\mathcal{K} \otimes \bigwedge^p(T^*M)\big)$ for the $C^{\infty}(M)$-module of $\mathcal{K}$-valued p-forms on M, an infinitesimal connection γ defines a map of modules

$$d_{\nabla^{\gamma}} : \sec\big(\mathcal{K} \otimes \textstyle\bigwedge^p(T^*M)\big) \to \sec\big(\mathcal{K} \otimes \bigwedge^{p+1}(T^*M)\big),$$

which we call the exterior kernel differential, as follows:

$$\begin{aligned} d_{\nabla^{\gamma}} Q(X_1, X_2, \ldots, X_{p+1}) &= \sum_{r=1}^{p+1} (-1)^{r+1} \nabla^{\gamma}_{X_r}(Q(X_1, \ldots \widehat{X_r} \ldots, X_{p+1})) \\ &\quad + \sum_{1 \le r, s \le p+1} (-1)^{r+s} Q([X_r, X_s], X_1, \ldots \widehat{X_r} \ldots \widehat{X_s} \ldots, X_{p+1}) \\ &= \sum_{r=1}^{p+1} (-1)^{r+1} [\![\gamma \circ X_r, Q(X_1, \ldots \widehat{X_r} \ldots, X_{p+1})]\!] \\ &\quad + \sum_{1 \le r, s \le p+1} (-1)^{r+s} Q([X_r, X_s], X_1, \ldots \widehat{X_r} \ldots \widehat{X_s} \ldots, X_{p+1}). \end{aligned}$$

The formula has obvious similarities with a well-known one for the exterior derivative of a p-form, and the fact that the exterior kernel differential is a module map can be deduced in the same way as the fact that the exterior derivative has that property can. But $d_{\nabla^{\gamma}}$ is not in general a coboundary operator of course.

Lemma 2.4.5 *The Bianchi identity for γ, namely*

$$\oint \big(\nabla^{\gamma}_X(R^{\gamma}(Y, Z)) + R^{\gamma}(X, [Y, Z])\big) = 0,$$

may be expressed using the exterior kernel differential as

$$d_{\nabla^{\gamma}} R^{\gamma} = 0.$$ □

2.5 Covariant Algebroid Derivatives

We now wish to generalize the idea of a covariant derivative, replacing vector fields by sections of a Lie algebroid $\mathcal{A} \to M$. The operators we describe may be defined on any vector bundle over M, and are independent of any infinitesimal connection on $\mathcal{A}$.

As motivation, consider a vector bundle $\pi : E \to M$ with a covariant derivative operator ∇ giving, for every section σ of E and any vector field X on M, a new section $\nabla_X\sigma$ of E. As every Lie algebroid $\mathcal{A} \to M$ supports an anchor map $\rho : \mathcal{A} \to M$, we may also define the derivative of a section of E by a section ξ of $\mathcal{A}$ to be $\nabla_{\rho\circ\xi}\sigma$, its derivative by the vector field $\rho \circ \xi$. This observation suggests that, given a vector bundle and a Lie algebroid over the same base manifold M, we could define a more general type of covariant derivative operator. So a *covariant algebroid derivative on E*, or more specifically an *$\mathcal{A}$-derivative on E*, will be defined to be an $\mathbb{R}$-linear map D from $\sec(\mathcal{A})$ to $\mathbb{R}$-linear operators $\sec(E) \to \sec(E)$, $\xi \mapsto D_\xi$, such that for $f \in C^\infty(M)$ and $\sigma \in \sec(E)$

$$\begin{aligned} D_{f\xi}\sigma &= fD_\xi\sigma \\ D_\xi(f\sigma) &= fD_\xi\sigma + (\rho\circ\xi)(f)\sigma. \end{aligned}$$

Thus D behaves much like an ordinary covariant derivative operator arising from a linear connection; and indeed a 'TM-derivative' on E is just a covariant derivative in the ordinary sense.

As in the case of an ordinary covariant derivative operator, the first of these conditions ensures that $(D_\xi\sigma)(x)$ depends only on the value of ξ at x, so one may also think of D as defining, for each $\boldsymbol{a} \in \mathcal{A}_x$, an $\mathbb{R}$-linear operator $D_{\boldsymbol{a}}$ from local sections of E defined near x to E_x. From the second condition it follows that if $\rho(\boldsymbol{a}) = 0$ then $D_{\boldsymbol{a}}\sigma$ depends only on the value of σ at x, and $D_{\boldsymbol{a}}$ then defines a linear map of E_x to itself. Likewise, if D and $\hat{D}$ are two $\mathcal{A}$-derivatives on the same vector bundle E then for each $\xi \in \sec(\mathcal{A})$, $\hat{D}_\xi - D_\xi$ is a vector bundle morphism of E.

There is an obvious generalization of the concept of curvature to $\mathcal{A}$-derivatives. For any $\mathcal{A}$-derivative D we set

$$C(\xi,\eta)\sigma = D_\xi(D_\eta\sigma) - D_\eta(D_\xi\sigma) - D_{[\![\xi,\eta]\!]}\sigma.$$

Then $C(\xi,\eta)\sigma \in \sec(E)$; it depends $C^\infty(M)$-linearly on all of its arguments, and it is skew-symmetric in the first two. For each $\xi, \eta \in \sec(\mathcal{A})$, $C(\xi,\eta)$ is a vector bundle morphism of E. The object C so defined is called the *curvature* of D. An $\mathcal{A}$-derivative whose curvature vanishes is said to be *flat*.

For $\mathcal{A}$-derivatives on $\mathcal{A}$ we can also define the *torsion* T of D,

$$T(\xi,\eta) = D_\xi\eta - D_\eta\xi - [\![\xi,\eta]\!];$$

$T(\xi,\eta) \in \sec(\mathcal{A})$, and depends $C^\infty(M)$-linearly and skew-symmetrically on its arguments. Notice that if D is an $\mathcal{A}$-derivative on $\mathcal{A}$ so is D^* defined by

$$D^*_\xi \eta = D_\eta \xi + [\![\xi,\eta]\!];$$

D^* is called the $\mathcal{A}$-derivative on $\mathcal{A}$ *dual* to D. Then

$$T(\xi,\eta) = D_\xi \eta - D^*_\xi \eta.$$

For an $\mathcal{A}$-derivative on $\mathcal{A}$, the operator D extends in the obvious way to tensor-type objects, such as its torsion T, so that

$$D_\xi T(\eta,\zeta) = D_\xi(T(\eta,\zeta)) - T(D_\xi\eta,\zeta) - T(\eta,D_\xi\zeta).$$

Moreover, the curvature and torsion of an $\mathcal{A}$-derivative on $\mathcal{A}$ satisfy *Bianchi identities*, which may be derived in exactly the same way as those for an ordinary covariant derivative on TM: in particular

$$\oint \big(C(\xi,\eta)\zeta + D_\xi T(\eta,\zeta) + T(T(\xi,\eta),\zeta)\big) = 0$$

(cyclic sum)—the first Bianchi identity for D.

2.6 Representations and Semidirect Sums

Let $\mathcal{A} \to M$ be a Lie algebroid, and let D be an $\mathcal{A}$-derivative on $E \to M$. We have a particular interest in $\mathcal{A}$-derivatives with vanishing curvature: flat $\mathcal{A}$-derivatives. A flat $\mathcal{A}$-derivative D satisfies

$$D_{[\![\xi,\eta]\!]} = D_\xi \circ D_\eta - D_\eta \circ D_\xi = [D_\xi, D_\eta]$$

for all $\xi,\eta \in \sec(\mathcal{A})$ and therefore defines a homomorphism from $\sec(\mathcal{A})$ with the Lie algebroid bracket to the space of linear operators on $\sec(E)$ with the commutator bracket; it is therefore called a *representation* of $\mathcal{A}$ on E. A flat $\mathcal{A}$-derivative on $\mathcal{A}$ itself is called a *self-representation* of $\mathcal{A}$.

If $\mathcal{A}$ is transitive and $\mathcal{K}$ is its kernel then for any $\kappa \in \sec(\mathcal{K})$ and $\xi \in \sec(\mathcal{A})$, $[\![\xi,\kappa]\!] \in \sec(\mathcal{K})$; and furthermore if we set $D_\xi\kappa = [\![\xi,\kappa]\!]$ then D is an $\mathcal{A}$-derivative on $\mathcal{K}$, which by the Jacobi identity is flat. The corresponding representation is called the *canonical representation* of $\mathcal{A}$. The kernel $\mathcal{K}$ is a Lie algebra bundle with Lie algebra bracket which is just the restriction of the Lie algebroid bracket. Now for the canonical representation

$$D_\xi[\![\kappa,\lambda]\!] = [\![D_\xi\kappa,\lambda]\!] + [\![\kappa,D_\xi\lambda]\!]$$

for all $\kappa, \lambda \in \mathrm{sec}(\mathcal{K})$, again by the Jacobi identity, which is to say that D is a derivation of the Lie algebra bracket on $\mathcal{K}$. We may regard the canonical representation as a template for the kernel derivatives of infinitesimal connections on $\mathcal{A}$, because for any such connection γ we have

$$\nabla^{\gamma}_{X}\kappa = [\![\gamma \circ X, \kappa]\!] = D_{\gamma \circ X}\kappa.$$

We have already observed that the kernel derivative of an infinitesimal connection does not extend to the whole of $\mathcal{A}$, and for the same reason $\eta \mapsto [\![\xi, \eta]\!]$ does not define an $\mathcal{A}$-derivative on $\mathcal{A}$: there is no canonical self-representation of $\mathcal{A}$. Self-representations of a Lie algebroid which extend the canonical representation, that is, which reduce to it when restricted to acting on $\mathcal{K}$, are therefore of particular interest, as we shall see in Chap. 7.

Now suppose we have a short exact sequence of vector bundles over M,

$$0 \to \mathcal{B} \to \mathcal{V} \xrightarrow{P} \mathcal{A} \to 0,$$

where $\mathcal{A}$ is a Lie algebroid with bracket $[\![\cdot, \cdot]\!]$ and $\mathcal{B}$ is a bundle of Lie algebras with bracket $\{\cdot, \cdot\}$, and suppose also that there is a representation D of $\mathcal{A}$ on $\mathcal{B}$ satisfying the derivation property

$$D_{\xi}\{\kappa, \lambda\} = \{D_{\xi}\kappa, \lambda\} + \{\kappa, D_{\xi}\lambda\}$$

where ξ is a section of $\mathcal{A}$ and κ, λ are sections of $\mathcal{B}$. If it is the case that the short exact sequence of sections

$$0 \to \mathrm{sec}(\mathcal{B}) \to \mathrm{sec}(\mathcal{V}) \to \mathrm{sec}(\mathcal{A}) \to 0$$

splits, so that we can write $\mathrm{sec}(\mathcal{V}) = \mathrm{sec}(\mathcal{A}) \oplus \mathrm{sec}(\mathcal{B})$, then we can define a bracket on sections of $\mathcal{V}$ by the formula

$$[\![(\xi, \kappa), (\eta, \lambda)]\!]^{D} = \Big([\![\xi, \eta]\!], D_{\xi}\lambda - D_{\eta}\kappa + \{\kappa, \lambda\}\Big)$$

Proposition 2.6.1 *The bracket $[\![\cdot, \cdot]\!]^{D}$ is a Lie bracket under which $\mathcal{V}$ becomes a Lie algebroid with anchor map $\rho \circ P$ where ρ is the anchor of $\mathcal{A}$. If $\mathcal{A} = TM$ then $\mathcal{B}$ is the kernel of $\mathcal{V}$.*

Proof The bracket $[\![\cdot, \cdot]\!]^{D}$ is the standard *semidirect sum bracket* on $\mathrm{sec}(\mathcal{V}) = \mathrm{sec}(\mathcal{A}) \oplus \mathrm{sec}(\mathcal{B})$; skew-symmetry and $\mathbb{R}$-bilinearity are obvious, and the Jacobi identity is a straightforward calculation using the facts that D is flat and that D_{ξ} is a derivation. By construction

$$P \circ [\![(\xi, \kappa), (\eta, \lambda)]\!]^{D} = P\Big([\![\xi, \eta]\!], D_{\xi}\lambda - D_{\eta}\kappa + \{\kappa, \lambda\}\Big) = [\![\xi, \eta]\!] = [\![P \circ (\xi, \kappa), P \circ (\eta, \lambda)]\!],$$

so that

$$\rho \circ P \circ [\![(\xi, \kappa), (\eta, \lambda)]\!]^D = [\rho \circ P \circ (\xi, \kappa), \rho \circ P \circ (\eta, \lambda)].$$

Finally if $f \in C^\infty(M)$ then

$$\begin{aligned}
[\![(\xi, \kappa), f(\eta, \lambda)]\!]^D &= \Big([\![\xi, f\eta]\!], D_\xi(f\lambda) - D_{f\eta}\kappa + \{\kappa, f\lambda\}\Big) \\
&= \Big(f[\![\xi, \eta]\!] + \big((\rho \circ \xi)f\big)\eta, fD_\xi\lambda + \big((\rho \circ \xi)f + f\{\kappa, \lambda\}\big)\lambda - fD_\eta\kappa\Big) \\
&= f\Big([\![\xi, \eta]\!], D_\xi\lambda - D_\eta\kappa + \{\kappa, \lambda\}\Big) + \big((\rho \circ \xi)f\big)(\eta, \lambda) \\
&= f[\![(\xi, \kappa), (\eta, \lambda)]\!]^D + \big((\rho \circ P \circ (\xi, \kappa))f\big)(\eta, \lambda).
\end{aligned}$$

It is clear that the restriction of $[\![\cdot, \cdot]\!]^D$ to $\mathcal{B}$ is $\{\cdot, \cdot\}$, so if $\mathcal{A} = TM$ then it is immediate that $\mathcal{B}$ is the kernel of $\mathcal{V}$. □

Corollary 2.6.2 *Suppose that $\mathcal{K} \to M$ is a Lie algebra bundle, with bracket $\{\cdot, \cdot\}$, which as a vector bundle is equipped with a flat TM-connection whose covariant derivative operator ∇ is a derivation of $\{\cdot, \cdot\}$. Then the vector bundle $TM \oplus_M \mathcal{K}$ can be given the structure of a transitive Lie algebroid by defining the bracket of sections by*

$$[\![(X, \kappa), (Y, \lambda)]\!] = \Big([X, Y], \nabla_X\lambda - \nabla_Y\kappa + \{\kappa, \lambda\}\Big)$$

and the anchor by $\rho \circ (X, \kappa) = X$. Furthermore, $X \mapsto (X, 0)$ defines an infinitesimal connection γ, such that $R^\gamma = 0$. □

We have already seen an example of the construction given in the above corollary, namely the Lie algebroid of the trivial Lie groupoid $M \times G \times M$ described in Proposition 1.3.2, where the covariant derivative $\nabla_X\kappa$ of the section κ of the trivial Lie algebra bundle $M \times \mathfrak{g}$ is taken as the Lie derivative of the corresponding $\mathfrak{g}$-valued function $\bar{\kappa}$, namely

$$\nabla_X\kappa = \nabla_X(\mathrm{id}, \bar{\kappa}) = \big(\mathrm{id}, X(\bar{\kappa})\big).$$

An example of the construction in the substantive proposition, where the short exact sequence of vector bundles does not have a canonical splitting but the induced sequence of sections does, comes when we consider the jet bundle of a Lie algebroid $\tau : \mathcal{A} \to M$.

As τ is by definition a vector bundle, its first jet bundle $\tau_1 : J^1\tau \to M$ is also a vector bundle, with the vector space structure on its fibres inherited from the vector space structure on the set of global sections of τ. It is a standard construction that $J^1\tau$ is part of a short exact sequence of vector bundles

$$0 \to \mathcal{A} \otimes T^*M \to J^1\tau \to \mathcal{A} \to 0$$

where the map $\tau_{1,0} : J^1\tau \to \mathcal{A}$ is given by $\tau_{1,0}(j^1_x\xi) = \xi(x)$; the injection $\mathcal{A} \otimes T^*M \to J^1\tau$ arises by observing that the map of sections $\xi \otimes df \mapsto j^1(f\xi) - f(j^1\xi)$, where

$f \in C^\infty(M)$, gives a well-defined pointwise map. A splitting of this sequence amounts to a linear connection on the vector bundle τ, as we shall see in the next section; but the sequence of sections *does* have a canonical splitting $j^1 : \sec(\mathcal{A}) \to \sec(J^1\tau)$, so that any section of $\tau_1 : J^1\tau \to M$ may be written uniquely as a sum $j^1\xi + Q$ where ξ is a section of $\mathcal{A}$ and Q is an $\mathcal{A}$-valued 1-form on M, regarded as a section of $J^1\tau$.

We first define a bracket on the sections of $\mathcal{A} \otimes_M T^*M$, regarded as vector bundle morphisms $TM \to \mathcal{A}$, by

$$\{P, Q\} = Q \circ \rho \circ P - P \circ \rho \circ Q.$$

It is easy to check that this bracket is well-defined pointwise, and defines the structure of a Lie algebra bundle on $\mathcal{A} \otimes T^*M$. If, for instance, P and Q are both infinitesimal connections on $\mathcal{A}$ (that is, $\rho \circ P = \rho \circ Q = \mathrm{id}_M$) then $\{P, Q\}$ is just the difference $Q - P$.

We next define a family of endomorphisms of $\sec(\mathcal{A} \otimes T^*M)$, parametrized by sections of $\mathcal{A}$, by setting

$$D_\xi(\eta \otimes \theta) = [\![\xi, \eta]\!] \otimes \theta + \eta \otimes \mathcal{L}_{\rho\circ\xi}\theta$$

where ξ, η are sections of $\mathcal{A}$ and θ is a 1-form on M, and extending by $\mathbb{R}$-linearity. Each D_ξ is a derivation, and it may easily be shown that D is a representation of $\mathcal{A}$ on $\mathcal{A} \otimes T^*M$, so we conclude that the bracket $[\![\cdot, \cdot]\!]^D$ together with the map $\rho \circ \tau_{1,0} : J^1\tau \to TM$ provide a Lie algebroid structure on $J^1\tau$.

2.7 Linearizing a Connection

We conclude this chapter with some remarks about general connections on vector bundles, with no particular reference to Lie algebroids.

One way of describing a general connection on the vector bundle $\pi : E \to M$ is by specifying a smooth complement H to the vertical sub-bundle $V\pi \subset TE$; this complement may be described equivalently as the image of a horizontal lift operator, whereby a vector field X on M is mapped to a vector field X^{h} on E. The direct sum decomposition $TE = H \oplus V\pi$ gives rise to a projection operator P_H which, as mentioned in the introduction to Sect. 2.2, may be regarded as a type (1, 1) tensor field on E.

There is no reason why a connection in this general form should be linear (and therefore correspond to a covariant derivative operator). It is, however, always possible to represent such a connection by a map between vector bundles, and hence obtain a derived linear connection on a pullback bundle by differentiation, using jet bundles. If ξ is a section of $\pi : E \to M$ then its jet $j^1_x\xi$ at the point x may be identified with the tangent map $\xi_{*x} : T_xM \to T_{\xi(x)}E$; using this identification we may construct a canonical tensor field h along the map $\pi_{1,0} : J^1\pi \to E$ by setting $\mathrm{h}_{j^1_x\xi}(v) = (\xi_x \circ \pi)_* v$ for any $v \in T_{\xi(x)}E$. A connection may then be described by a section $\sigma : E \to J^1\pi$, so that $\mathrm{h} \circ \sigma$ is a genuine tensor field on E (rather than along $\pi_{1,0}$) giving rise to the decomposition $TE = H \oplus V\pi$.

We are now able to say that the connection is a linear connection if the map σ is linear: that is, if it provides a splitting of the short exact sequence

$$0 \to E \otimes T^*M \to J^1\pi \to E \to 0,$$

as we mentioned earlier. Recalling that the corresponding exact sequence of spaces of sections

$$0 \to \sec(E \otimes T^*M) \to \sec(J^1\pi) \to \sec(E) \to 0$$

has the canonical splitting $\xi \mapsto j^1\xi$, we may define a covariant differential

$$\nabla : \sec(E) \to \sec(E \otimes T^*M), \qquad \nabla\xi = j^1\xi - \sigma \circ \xi$$

and hence, for any vector field X on M, a covariant derivative ∇_X acting on sections of $E \to M$.

If the connection σ is not linear, we proceed in the following way.

For any $x \in M$ let σ_x denote the restriction of σ to the vector space E_x, so that $\sigma_x : E_x \to J^1_x\pi$. The derivative σ'_x is a map $E_x \times E_x \to J^1_x\pi$, linear in the second variable, so we may construct the fibre derivative

$$\mathcal{F}\sigma : \pi^*E \to J^1\pi, \qquad \mathcal{F}\sigma(y, z) = \sigma'_{\pi(y)}(y, z)$$

where $\pi^*(\pi) : \pi^*(E) = E \times_M E \to M$ is the pullback bundle. There is also a canonical inclusion $J^1\pi \subset J^1\big(\pi^*(\pi)\big)$ as a vector sub-bundle, given by $j^1_x\xi \mapsto j^1_x(\xi, 0_\pi)$ where $0_\pi : M \to E$ is the zero section, and in this way $\mathcal{F}\sigma$ becomes a linear connection on the pullback bundle $\pi^*(\pi)$.

This construction of a derived linear connection will be used in Chap. 12, and it will be convenient to describe it using coordinates. If (x^i, y^α) are fibred coordinates on E then the horizontal lift $X \mapsto X^{\mathsf{h}}$ is given by

$$\frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial x^i} - \Gamma^\alpha_i(x, y)\frac{\partial}{\partial y^\alpha}$$

(the minus sign is conventional) and the projection operator P_H is given by

$$dx^i \otimes \left(\frac{\partial}{\partial x^i} - \Gamma^\alpha_i(x, y)\frac{\partial}{\partial y^\alpha}\right).$$

For the jet bundle description we let y^α_i be the jet coordinates, so that

$$\mathrm{h} = dx^i \otimes \left(\frac{\partial}{\partial x^i} + y_i^\alpha \frac{\partial}{\partial y^\alpha} \right);$$

then $\Gamma_i^\alpha = -\sigma_i^\alpha = -y_i^\alpha \circ \sigma$. If in addition $(x^i; y^\alpha, z^\alpha)$ are fibred coordinates on $\pi^*(E)$ then

$$y_i^\alpha \circ \mathcal{F}\sigma = -\frac{\partial \Gamma_i^\alpha}{\partial y^j} z^j, \qquad z_i^\alpha \circ \mathcal{F}\sigma = 0.$$

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