

Chapter 2

From Newton to Euler and Navier-Stokes

2.1 The Continuum Description

In ordinary dynamics, as first formulated by Isaac Newton, the motion of a single particle of mass m , following an orbit $\mathbf{x}(t)$ under the influence of some force $\mathbf{F}(\mathbf{x}, t)$, is described by two simple equations of motion, the first linking the acceleration $\mathbf{a} = d\mathbf{V}/dt$ to the force \mathbf{F} and the second linking the velocity \mathbf{V} to the position change:

$$\frac{d\mathbf{x}}{dt} = \mathbf{V}, \quad m \frac{d\mathbf{V}}{dt} = \mathbf{F}(\mathbf{x}, t). \quad (2.1.1)$$

A fluid (or gas) consists in principle of a large number of particles (ions, atoms or molecules), each of which individually satisfies equations like (2.1.1). However, the expression for the force \mathbf{F} is horribly complicated in this case. It has to take account of *all* the interactions between the individual particles that depends sensitively on the ever-changing position of each particle.

Consider for instance the case of a gas consisting of identical particles with mass m that interact through mutual forces, such as the gravitational interaction or the Coulomb force between charged particles. If we number the particles using greek indices (e.g. α, β, \dots), the force equation for the α -th particle looks like

$$m \frac{d\mathbf{V}_\alpha}{dt} = \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta}(\mathbf{x}_\alpha, \mathbf{x}_\beta). \quad (2.1.2)$$

Here $\mathbf{F}_{\alpha\beta}(\mathbf{x}_\alpha, \mathbf{x}_\beta)$ is the force on particle α exerted by particle β , and the summation over β enumerates all possible interactions.

Note that this force generally depends on the continuously changing position of each particle! For instance: in a gas of electrons with mass m_e and charge $-e$ the electron-electron force is the repulsive electrostatic Coulomb force:

$$\mathbf{F}_{\alpha\beta}(\mathbf{x}_\alpha, \mathbf{x}_\beta) = \frac{e^2 (\mathbf{x}_\alpha - \mathbf{x}_\beta)}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3}. \quad (2.1.3)$$

Let us the ionized plasma in the Solar Corona as an example: one cubic centimeter of Coronal gas contains about 10^6 electrons. This means that one would have to calculate $\sim 10^{12}$ interactions to describe the dynamics of all electrons in this volume, and another $\sim 10^{12}$ interactions for the protons that are also present as the gas is electrically neutral, and consists mostly of hydrogen. This is clearly an impractical approach.

The power of the fluid description lies in the fact that it dispenses with a detailed consideration of the constituent individual particles in some small (infinitesimal) volume \mathcal{V} , and replaces the massive point particles contained in that volume by a smeared-out 'smooth' distribution of mass with the same total mass Δm . To that end one introduces, at each position \mathbf{x} and time t , a local mass density $\rho(\mathbf{x}, t)$, which is formally defined as:

$$\rho(\mathbf{x}, t) = \frac{\text{total mass in a small volume centered at } (\mathbf{x}, t)}{\text{volume}} = \lim_{\Delta \mathcal{V} \rightarrow 0} \frac{\Delta m}{\Delta \mathcal{V}}. \quad (2.1.4)$$

It also defines an average velocity $\mathbf{V}(\mathbf{x}, t)$ that is essentially the center-of-mass velocity of the collection of those particles residing inside the small volume $\Delta \mathcal{V}$ at a *given* position \mathbf{x} at time t :

$$\mathbf{V}(\mathbf{x}, t) = \frac{1}{\Delta m} \sum_{\Delta \mathcal{V}} m_\alpha \mathbf{v}_\alpha. \quad (2.1.5)$$

Here $\Delta m = \sum_{\Delta \mathcal{V}} m_\alpha$ is the total mass of the particles residing in the volume-element $\Delta \mathcal{V}$ and \mathbf{v}_α is the velocity of particle α .

This *continuum description* of a fluid leads to an equation analogous to (2.1.1). For a fluid with mass *density* ρ , subject to forces with a force *density* (the net force per unit volume) \mathbf{f} , the equation of motion reads

$$\rho \frac{d\mathbf{V}}{dt} = \mathbf{f}(\mathbf{x}, t). \quad (2.1.6)$$

This deceptively simple-looking equation of motion hides two technical difficulties. The first and obvious difficulty is the definition of the precise form of the force density \mathbf{f} . We will consider that question in more detail below. The second and less obvious (but mathematically rather more intricate) problem is the correct interpretation of the time-derivative d/dt .

In Newtonian dynamics this problem never explicitly arises: there it is immediately obvious that one has to evaluate the force \mathbf{F} that appears in the equation of motion (2.1.1) at the current time and the current position $\mathbf{x}(t)$ of the particle along its orbit. Therefore, if the force depends explicitly on both position and time the Newtonian force equation should be written more precisely as:

$$m \frac{d\mathbf{V}}{dt} = \mathbf{F}(\mathbf{x}(t), t) \quad (2.1.7)$$

That immediately implies that the acceleration at any time t is also only well-defined at the particle position.

I will now show that this interpretation essentially still holds in fluid mechanics, but that the fact that we are using a continuum description (rather than following a single particle) complicates things: in principle a fluid or gas can fill the whole space (or some limited volume, such as a fluid container). There simply is no single particle that defines where to look at a given time! This implies that the velocity \mathbf{V} has to be interpreted as a *distribution* of velocities over space: a *velocity field* $\mathbf{V}(\mathbf{x}, t)$, which changes with time *and* position. Therefore, fluid dynamics is formally a *field theory*, and the fluid velocity is characterized by a vector field! As we will see, in order to fully describe a simple fluid we need additional fields, such as the mass density field $\rho(\mathbf{x}, t)$, the pressure field $P(\mathbf{x}, t)$ and the gravitational potential $\Phi(\mathbf{x}, t)$, all *scalar fields*.

In three different (but equivalent) notations¹ the velocity vector can be represented as:

$$\mathbf{V}(\mathbf{x}, t) = (V_x, V_y, V_z) = V_x \hat{\mathbf{x}} + V_y \hat{\mathbf{y}} + V_z \hat{\mathbf{z}}. \quad (2.1.8)$$

The magnitude and direction of the vector \mathbf{V} is determined by the three functions $V_x(\mathbf{x}, t)$, $V_y(\mathbf{x}, t)$ and $V_z(\mathbf{x}, t)$, which are the three components of the velocity vector at each point in space-time.

Here we use a Cartesian (rectangular) coordinate system with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$, but any other properly defined coordinate system, such as cylindrical or spherical coordinates, will do equally well.²

The velocity \mathbf{V} in fluid or gas dynamics has been defined as the *local average* over some small volume of the velocities of the constituent particles at position \mathbf{x} and point in time t , c.f. Eq. (2.1.5). Elementary considerations from statistical physics tell you that the velocity \mathbf{v} of an *individual particle* in that volume is never precisely equal to the average velocity \mathbf{V} : thermal motion of the particles is superposed on the

¹Throughout this book I will use two different notations for unit vectors, whichever is more convenient in the context of the expressions: I will either write $\hat{\mathbf{x}}$ or $\hat{\mathbf{e}}_x$ for the unit vector in the x -direction, and similar expressions for the unit vectors in the y - and z -directions of a Cartesian coordinate system. If I do not specify the coordinate system used I will simply write $\hat{\mathbf{e}}_i$ for the i -th unit vector, where $i = 1, 2, 3$.

²In this context it is important to realize that a vector like \mathbf{V} has a mathematically well-defined meaning, *independent* of the coordinate system that is used to represent this vector! It is simply an arrow with a certain length and orientation. Coordinate systems *represent* a vector in terms of components, and changing the coordinate system only changes the components (representation), but **not** the vector itself! This implies an important property of all proper physical theories: they are **covariant**, meaning that physical laws should not depend on the choice of coordinate systems. When written in *vector language*, or more generally in *tensor language*, a physical law (equation of motion, conservation law, ...) always looks the same. We will occasionally use this principle in these notes, for instance by doing intermediate steps in a complicated calculation in the most convenient set of coordinates for that particular problem, and then writing the end result in vector form. That is generally valid in *any* coordinate system.

average motion. This leads to a range of possible velocities for an individual particle. As we will see below, the influence of the deviations from the average velocity, the so-called *velocity dispersion*, is taken into account by introducing the *fluid pressure* and the associated pressure force. The velocity dispersion also defines the thermal energy per particle. This is the mean kinetic energy per particle measured by someone who moves with the flow with (local) average velocity \mathbf{V} .

2.2 Eulerian and Lagrangian Time Derivatives

Let us consider the precise interpretation of the time derivative d/dt . As already discussed above, the interpretation of the time derivative d/dt is obvious in the case of Newtonian mechanics: it is the change in time, as measured by a hypothetical observer that moves with the particle along its orbit. The same interpretation should hold for the time-derivative d/dt in fluid mechanics. It is the time derivative seen by an observer moving along with the flow. Therefore, in the world of fluid/gas dynamics d/dt is usually called the *comoving* or *Lagrangian* time-derivative.

This is *not* the only time-derivative one can think of in a fluid description, where all physical quantities are *fields* that depend both on position \mathbf{x} and time t , which are now *independent variables*.³ Let us assume that some quantity $Q(\mathbf{x}, t)$ is measured by two observers. $Q(\mathbf{x}, t)$ stands for any field used in the fluid description. The stationary observer is at some fixed position \mathbf{x} in space, while the second observer moves with the fluid at the (local) velocity $\mathbf{V}(\mathbf{x}, t)$. We will call this second observer the *comoving observer*. We calculate the change in Q in a small time interval Δt , as seen by these two observers, evaluated while sitting at/passing the same position \mathbf{x} .

The first (stationary) observer measures a change

$$\begin{aligned}\delta Q &= Q(t + \Delta t, \mathbf{x}) - Q(t, \mathbf{x}) \\ &\approx \left(\frac{\partial Q}{\partial t} \right) \Delta t,\end{aligned}\tag{2.2.1}$$

assuming $\Delta t \ll t$. This is a straightforward application of the definition for the partial time derivative of $Q(\mathbf{x}, t)$, called the *Eulerian time derivative* in fluid mechanics.

The change seen in the same time interval by the comoving observer is influenced by his position shift. For small Δt this shift amounts to

$$\Delta \mathbf{x} = \mathbf{V} \Delta t = (V_x \Delta t, V_y \Delta t, V_z \Delta t).\tag{2.2.2}$$

³This situation is different from single-particle dynamics where \mathbf{x} is a dependent variable that depends on time!

The comoving observer by definition follows the trajectory taken by the local flow, so it stands to reason to define the change ΔQ that he measures in a small time interval Δt as

$$\Delta Q \equiv \left(\frac{dQ}{dt} \right) \Delta t. \quad (2.2.3)$$

Evaluating ΔQ using (2.2.2):

$$\begin{aligned} \Delta Q &= Q(t + \Delta t, \mathbf{x} + \Delta \mathbf{x}) - Q(t, \mathbf{x}) \\ &\approx \frac{\partial Q}{\partial t} \Delta t + (\Delta \mathbf{x} \cdot \nabla) Q \\ &= \left[\frac{\partial Q}{\partial t} + (\mathbf{V} \cdot \nabla) Q \right] \Delta t \end{aligned} \quad (2.2.4)$$

This leads to

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + (\mathbf{V} \cdot \nabla) Q. \quad (2.2.5)$$

Here $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is the gradient operator, and the short-hand notation $\mathbf{V} \cdot \nabla$ is defined in Cartesian coordinates as

$$\mathbf{V} \cdot \nabla \equiv V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} + V_z \frac{\partial}{\partial z}. \quad (2.2.6)$$

In other coordinates it can take a more complicated form, see the Appendix.

This derivation shows that the *Eulerian* time derivative $\partial/\partial t$, as measured by the first observer at a fixed position, and the comoving (or *Lagrangian*) time derivative, as measured by the second observer moving with velocity \mathbf{V} , are related by:

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla)}. \quad (2.2.7)$$

This last relation is written in vector form, and therefore valid in *any* coordinate system! This means that the equation of motion for a fluid, which involves the comoving time derivative, should be written as:

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = \mathbf{f}. \quad (2.2.8)$$

This form of the equation of motion for a fluid explicitly shows the reason why fluid dynamics is more difficult than the Newtonian dynamics of a single particle. The term $(\mathbf{V} \cdot \nabla) \mathbf{V}$ is formally quadratic in \mathbf{V} and thus introduces non-linearity into the equation of motion. This is the price one has to pay for having to deal with a velocity *field* $\mathbf{V}(\mathbf{x}, t)$ where position \mathbf{x} and time t are both to be considered as independent variables.

2.2.1 Eulerian and Lagrangian Change

By combining relations (2.2.1) and (2.2.4) one can also see that the Eulerian change δQ , as measured at a fixed position, and the Lagrangian change ΔQ of some quantity Q , given a small position shift $\Delta \mathbf{x}$, are related by:

$$\Delta Q = \delta Q + (\Delta \mathbf{x} \cdot \nabla) Q. \quad (2.2.9)$$

This relation is valid for small $\Delta \mathbf{x}$ regardless the precise nature of Q (scalar function, vector, tensor, ...). This relation will be an important ingredient in the theory of small-amplitude waves that is treated in Chap. 7.

Finally, the following bears repeating: even though I have used Cartesian coordinates x , y and z in the derivations, the final expressions (2.2.7), (2.2.8) and (2.2.9) are written in vector form and (in this form) are generally valid, regardless the choice of coordinates one ultimately uses to represent these equations in terms of the vector components of $\Delta \mathbf{x}$, \mathbf{f} or \mathbf{V} .

2.3 Pressure of an Isotropic Gas

The precise form of the force density \mathbf{f} of course depends on the circumstances. Generally speaking, it consists of contributions that are *internal* to the fluid, such as the pressure force or the force due to internal friction, and forces that are applied by external sources, for example the gravitational pull of the Earth on its atmosphere.

The most important internal force density of a gas or fluid is the pressure force.⁴ The pressure force takes account of the spread of velocities of the constituent particles around the mean velocity \mathbf{V} . This velocity spread means that the exact momentum of an individual particle, and the mean momentum of the fluid differ. This momentum difference, or more precisely the associated flux of momentum, ultimately leads to a macroscopic force when one averages over all particles. The spread in velocities is due to the thermal motion of the particles. The precise derivation of fluid pressure in terms of the microscopic physics of the constituent particles follows from kinetic gas theory. It is possible, however, to give an approximate derivation of the pressure force that gives important insight into its nature.

Consider a collection of particles of identical mass m in some local volume \mathcal{V} . The individual velocity of particle α ⁵ is given by

$$\mathbf{v}_\alpha = \mathbf{V}(\mathbf{x}, t) + \boldsymbol{\sigma}_\alpha(\mathbf{x}, t). \quad (2.3.1)$$

⁴I will follow the general convention to speak of ‘forces’ even though, technically speaking, one should speak of force densities.

⁵Greek indices are used to distinguish particles.

Here the velocity has been written as the sum of the *mean* velocity \mathbf{V} of the whole set of particles, and the deviation $\boldsymbol{\sigma}_\alpha$ from the mean of particle α . If there are in total N particles in the volume this definition implies, using a notation $\overline{\quad}$ for the average,

$$\bar{\mathbf{v}} \equiv \frac{1}{N} \sum_{\alpha=1}^N \mathbf{v}_\alpha = \mathbf{V}(\mathbf{x}, t) \quad (2.3.2)$$

and

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{v}} - \mathbf{V} = \mathbf{0}. \quad (2.3.3)$$

Here I have used that \mathbf{V} already is an average, and must therefore satisfy $\overline{\mathbf{V}} = \mathbf{V}$.

Let us write down the equation of motion of each particle. We do this in the ‘fluid mechanics’ form⁶:

$$m \frac{d\mathbf{u}_\alpha}{dt} = m \left[\frac{\partial \mathbf{u}_\alpha}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha \right] = \mathbf{F}_\alpha \quad (2.3.4)$$

Substituting Eq. (2.3.1) for \mathbf{u}_α and summing over all N particles, using definition (2.3.2) for the average, yields an average equation of motion:

$$Nm \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} + \overline{(\boldsymbol{\sigma} \cdot \nabla) \boldsymbol{\sigma}} \right] = N \bar{\mathbf{F}}. \quad (2.3.5)$$

One sees that the only term involving the deviations from the mean velocity that survives this averaging procedure is a term that is *quadratic* in $\boldsymbol{\sigma}$:

$$Nm \overline{(\boldsymbol{\sigma} \cdot \nabla) \boldsymbol{\sigma}}. \quad (2.3.6)$$

This term will in general *not* vanish. All terms that are linear in $\boldsymbol{\sigma}$ are averaged out because of (2.3.3). This procedure assumes implicitly that the averaging process is not influenced by the action of time- and space derivatives.

The mean number density (number of particles per unit volume) equals $n = N/\mathcal{V}$, while the external (mean) force density is $\mathbf{f}_{\text{ext}} = N\bar{\mathbf{F}}/\mathcal{V}$. Dividing (2.3.5) by \mathcal{V} and re-ordering terms one can write:

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = -\rho \overline{(\boldsymbol{\sigma} \cdot \nabla) \boldsymbol{\sigma}} + \mathbf{f}_{\text{ext}}, \quad (2.3.7)$$

with $\rho \equiv nm$ the mass density. One sees that the effect of the random thermal motion leads to a force term that is quadratic in $\boldsymbol{\sigma}$. In the next Section we will evaluate this term for a gas with an isotropic distribution of the random velocities: the typical case

⁶Those of you who are uncomfortable with this step may assume that the fluid is composed of ‘subfluids’ that consist of all particles that happen to have the same total velocity \mathbf{v} .

for a mono-atomic gas in thermodynamic equilibrium in absence of other forces, and a good approximation in many other circumstances.

2.3.1 The Stress Tensor due to Thermal Motion

We now use a result from tensor analysis. For a good introduction see the book by Arfken and Weber [2], Chaps. 1 and 2. Other useful references are [47], Chap. 10 and [10], Chap. 11. Additional information can be found in Appendix A.

Consider the *dyadic tensor* $\mathbf{T} \equiv \mathbf{A} \otimes \mathbf{B}$, which is obtained from the *direct product* of two vectors $\mathbf{A} = A_i \mathbf{e}_i$ and $\mathbf{B} = B_j \mathbf{e}_j$:

$$\mathbf{A} \otimes \mathbf{B} \equiv A_i B_j \mathbf{e}_i \otimes \mathbf{e}_j. \quad (2.3.8)$$

The \mathbf{e}_i with $i = 1, 2, 3$ are the three unit vectors employed in the coordinate system. For instance, in a standard Cartesian coordinate system one has $\mathbf{e}_1 \equiv \hat{\mathbf{x}} = (1, 0, 0)$, $\mathbf{e}_2 \equiv \hat{\mathbf{y}} = (0, 1, 0)$ and $\mathbf{e}_3 \equiv \hat{\mathbf{z}} = (0, 0, 1)$. A vector \mathbf{A} can be represented as a column vector,

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix},$$

and the direct product $\mathbf{T} = \mathbf{A} \otimes \mathbf{B}$ as a 3×3 matrix with components $T_{ij} = A_i B_j$:

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}.$$

In (2.3.8) we employ the *Einstein summation convention* where one sums over all *repeated* indices, in this case over $i = 1, 2, 3$ and $j = 1, 2, 3$. One can show that the following relation holds generally if one takes the *divergence* of such a dyadic tensor, This mathematical operation yields a *vector* in a manner that, for now, may be employed as recipe:

$$\nabla \cdot (\mathbf{A} \otimes \mathbf{B}) = (\nabla \cdot \mathbf{A}) \mathbf{B} + (\mathbf{A} \cdot \nabla) \mathbf{B}. \quad (2.3.9)$$

Here we use the divergence of the vector \mathbf{A} :

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (2.3.10)$$

The operator $(\mathbf{A} \cdot \nabla) \mathbf{B}$ is defined as⁷

$$(\mathbf{A} \cdot \nabla) \mathbf{B} = \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) \mathbf{B}. \quad (2.3.11)$$

Relation (2.3.9) is essentially the product rule for differentiation. As an example, using some of the formal definitions found in Appendix A and employing Cartesian coordinates (x, y, z) , we can calculate the x -component of the vector $(\nabla \cdot (\mathbf{A} \otimes \mathbf{B}))$:

$$\begin{aligned} [\nabla \cdot (\mathbf{A} \otimes \mathbf{B})]_x &= \frac{\partial}{\partial x} (A_x B_x) + \frac{\partial}{\partial y} (A_y B_x) + \frac{\partial}{\partial z} (A_z B_x) \\ &= \frac{\partial A_x}{\partial x} B_x + A_x \frac{\partial B_x}{\partial x} + \frac{\partial A_y}{\partial y} B_x + A_y \frac{\partial B_x}{\partial y} + \frac{\partial A_z}{\partial z} B_x + A_z \frac{\partial B_x}{\partial z} \\ &= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) B_x + \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) B_x \\ &= [(\nabla \cdot \mathbf{A}) \mathbf{B} + (\mathbf{A} \cdot \nabla) \mathbf{B}]_x. \end{aligned} \quad (2.3.12)$$

Relation (2.3.9) allows us to write:

$$(\rho \boldsymbol{\sigma} \cdot \nabla) \boldsymbol{\sigma} = \nabla \cdot (\rho \boldsymbol{\sigma} \otimes \boldsymbol{\sigma}) - (\nabla \cdot (\rho \boldsymbol{\sigma})) \boldsymbol{\sigma}. \quad (2.3.13)$$

This expression involves the dyadic tensor

$$\mathbf{T} = \rho \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \quad (2.3.14)$$

This tensor has a simple physical interpretation as the so-called *stress tensor* that is associated with the thermal motion of the particles in the fluid or gas: the (i, j) component is

$$T_{ij} = (\rho \sigma_i) \sigma_j = (\text{momentum density in the } i\text{-direction}) \times (\text{velocity in } j\text{-direction}).$$

Physically, it gives the amount of i momentum that is transported across a unit surface per unit time, where the normal to said surface is along the j -direction. Since there are three spatial directions, there are three momentum components that can be transported, and there are three independent ways to orient a unit surface. So one needs in total nine quantities to fully specify the momentum transport. Each index i and j can independently take the values 1, 2 and 3, so there are indeed $3 \times 3 = 9$ components of the tensor T_{ij} . This partially justifies the use of the rank 2 tensor

⁷These last two definitions are *only* valid in Cartesian (rectangular) coordinates. More detailed expressions, valid for general (curvilinear) coordinate systems, can be found in Appendix A.

\mathbf{T} to describe the momentum transport due to thermal motions: it has the required number of degrees of freedom.⁸ If one associates 1 with the x -direction, 2 with the y direction and 3 with the z -direction in a Cartesian coordinate grid one can represent the tensor \mathbf{T} as a 3×3 matrix:

$$\mathbf{T} = \begin{pmatrix} \rho \sigma_x^2 & \rho \sigma_x \sigma_y & \rho \sigma_x \sigma_z \\ \rho \sigma_y \sigma_x & \rho \sigma_y^2 & \rho \sigma_y \sigma_z \\ \rho \sigma_z \sigma_x & \rho \sigma_z \sigma_y & \rho \sigma_z^2 \end{pmatrix}. \quad (2.3.15)$$

However, we will see that things simplify considerably in an isotropic fluid or gas so that, after averaging over all possible orientations of the vector σ , the stress tensor has only three non-vanishing diagonal components.

2.3.2 The Case of an Isotropic Fluid or Gas in Equilibrium

If the detailed microscopic physics is in equilibrium, and if there is no preferred direction so that the fluid is *isotropic*, the second term on the right hand side of Eq. (2.3.13) vanishes upon averaging over all possible directions of σ . Another way to see this is to realize that an isotropic system looks the same if it is rotated over an arbitrary angle in an arbitrary direction. The quantity $\nabla \cdot (\rho \sigma)$ is a scalar, and therefore has a value that is not influenced by any rotation of the system. On the other hand, σ is a vector which *does* feel the effect of a rotation. Therefore, in order for the system to be invariant under rotations the second term on the right hand side of Eq. (2.3.13) must vanish identically. For a gas or fluid where the molecular velocities are distributed isotropically (see Fig. 2.1) we therefore have:

$$\rho \overline{(\sigma \cdot \nabla)} \sigma = \nabla \cdot (\rho \overline{\sigma \otimes \sigma}). \quad (2.3.16)$$

The assumption of isotropy also implies that the following relations must be valid:

$$\overline{\sigma_x^2} = \overline{\sigma_y^2} = \overline{\sigma_z^2} = \frac{1}{3} \overline{\sigma^2}. \quad (2.3.17)$$

More importantly, it also implies that the cross-correlation unequal velocity components vanishes:

$$\overline{\sigma_x \sigma_y} = \overline{\sigma_x \sigma_z} = \overline{\sigma_y \sigma_z} = \dots = 0. \quad (2.3.18)$$

⁸More important is the fact that \mathbf{T} behaves in the right way under general coordinate transformations, a subject we will not get into here but that can be found in any textbook on tensor analysis, see for instance [2, 10, 47].

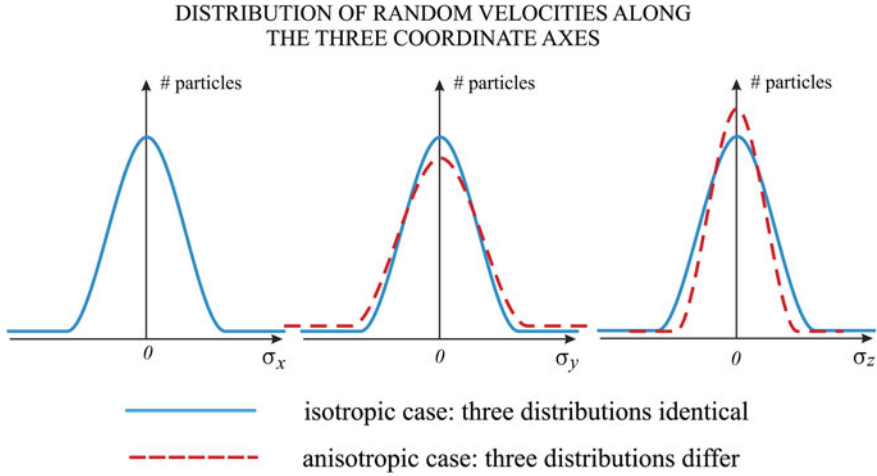


Fig. 2.1 An illustration of the meaning of the assumption of isotropy for the thermal velocities of the gas. The row of three figures give the measured distributions of the thermal velocity components σ_x , σ_y and σ_z along the three coordinate axes. These curves are the probability distribution functions (PDFs) for the three components. The PDFs are symmetric with respect to $\sigma_{x,y,z} = 0$ so that the average velocity satisfies $\overline{\sigma_i} = 0$ for $i = x, y, z$. The width of a PDF determines $\overline{\sigma_i^2}$. In an isotropic fluid the three PDFs are identical to each other

The first relationship says that all three coordinate directions on average contribute equally to $\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$. The second relation follows from the fact that products like $\sigma_x \sigma_y$ change if one rotates the coordinate system, but that such a rotation can have no effect if the *physics* is isotropic. This is only possible if all these cross-terms vanish identically.

To see this explicitly how this comes about, consider a rotation of the coordinate system in the $x - y$ plane over an angle θ . The new unit vectors are

$$\hat{e}_1 = \cos \theta \hat{x} + \sin \theta \hat{y}, \quad \hat{e}_2 = -\sin \theta \hat{x} + \cos \theta \hat{y}, \quad \hat{e}_3 = \hat{z}. \quad (2.3.19)$$

The new components of any vector \mathbf{A} follow from the projection of that vector on the unit vectors, which can be expressed as a scalar product:

$$A_i = \mathbf{A} \cdot \hat{e}_i. \quad (2.3.20)$$

Using (2.3.19) one calculates the velocity components in the rotated coordinate system:

$$\sigma_1 = \sigma_x \cos \theta + \sigma_y \sin \theta, \quad \sigma_2 = -\sigma_x \sin \theta + \sigma_y \cos \theta. \quad (2.3.21)$$

From this one immediately finds:

$$\begin{aligned}\sigma_1^2 &= \sigma_x^2 \cos^2 \theta + \sigma_y^2 \sin^2 \theta + 2\sigma_x \sigma_y \cos \theta \sin \theta. \\ \sigma_2^2 &= \sigma_x^2 \sin^2 \theta + \sigma_y^2 \cos^2 \theta - 2\sigma_x \sigma_y \cos \theta \sin \theta, \\ \sigma_1 \sigma_2 &= (\sigma_y^2 - \sigma_x^2) \sin \theta \cos \theta + \sigma_x \sigma_y (\cos^2 \theta - \sin^2 \theta).\end{aligned}\tag{2.3.22}$$

If one now averages over an isotropic distribution of velocities one should find that the averages do not change. To an observer rotating with the coordinate system the gas has rotated over an angle $-\theta$. However, an isotropic velocity distribution has (by definition) the same statistical properties when rotated over *any angle*, and should therefore have the same statistics regardless the value of θ . This means that one must have:

$$\begin{aligned}\overline{\sigma_1^2} &= \overline{\sigma_2^2} = \overline{\sigma_x^2} = \overline{\sigma_y^2}, \\ \overline{\sigma_1 \sigma_2} &= \overline{\sigma_x \sigma_y}.\end{aligned}\tag{2.3.23}$$

With $\overline{\sigma_x^2} = \overline{\sigma_y^2} = \overline{\sigma^2}/3$ one immediately finds that $\overline{\sigma_1^2} = \overline{\sigma_2^2} = \overline{\sigma^2}/3$, for any θ , provided that the cross term satisfies $\overline{\sigma_x \sigma_y} = 0$, simply because $\sin^2 \theta + \cos^2 \theta = 1$. In that case one also immediately finds $\overline{\sigma_1 \sigma_2} = \overline{\sigma_x \sigma_y} = 0$. The set of rules (2.3.17) and (2.3.18) is the only set of rules that is consistent with an isotropic distribution of thermal velocities.

These two sets of relations, (2.3.17) and (2.3.18), can be summarized in a single equation using the *Kronecker symbol* δ_{ij} , which has the properties $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$:

$$\overline{\sigma_i \sigma_j} = \frac{1}{3} \overline{\sigma^2} \delta_{ij}.\tag{2.3.24}$$

The above representation of the dyadic tensor $\rho \overline{\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}}$ together with (2.3.24) means that one can write:

$$\rho \overline{\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}} = \rho \begin{pmatrix} \frac{1}{3} \overline{\sigma^2} & 0 & 0 \\ 0 & \frac{1}{3} \overline{\sigma^2} & 0 \\ 0 & 0 & \frac{1}{3} \overline{\sigma^2} \end{pmatrix} = \frac{\rho \overline{\sigma^2}}{3} \mathbf{I}.\tag{2.3.25}$$

Here $\mathbf{I} \equiv \text{diag}(1, 1, 1)$ is the 3×3 unit tensor. In component notation one has $I_{ij} = \delta_{ij}$. Defining the *scalar pressure* P as

$$P = \frac{1}{3} \rho \overline{\sigma^2}, \quad (2.3.26)$$

one can write the pressure force density due to the thermal motion as:

$$\rho \overline{(\boldsymbol{\sigma} \cdot \nabla) \boldsymbol{\sigma}} = \nabla \cdot (\rho \overline{\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}}) = \nabla \cdot (P \mathbf{I}). \quad (2.3.27)$$

The definition for the divergence of a *rank 2 tensor* \mathbf{T} (which can be represented by a 3×3 matrix with components T_{ij}) in cartesian coordinates corresponds to a vector, with components

$$\nabla \cdot \mathbf{T} = \begin{pmatrix} \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \\ \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} \\ \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \end{pmatrix}. \quad (2.3.28)$$

If one substitutes expression (2.3.25), written in the form

$$\rho \overline{\sigma_i \sigma_j} = P \delta_{ij}, \quad (2.3.29)$$

into this definition one finds:

$$\nabla \cdot (P \mathbf{I}) = \begin{pmatrix} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial z} \end{pmatrix} = \nabla P. \quad (2.3.30)$$

This calculation leads to the following conclusion: in an isotropic gas the pressure force involves the gradient of the scalar pressure P . This means that the equation of motion (2.3.7) for a frictionless fluid or gas can be written as

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = -\nabla P + \mathbf{f}_{\text{ext}}. \quad (2.3.31)$$

Here \mathbf{f}_{ext} is the force density applied externally to the fluid. Note that the pressure force is formally equal to

$$\mathbf{f}_P = -\nabla P = -\nabla \cdot \mathbf{T}, \quad (2.3.32)$$

with $\mathbf{T} = \text{diag}(P, P, P)$ the stress tensor associated with the pressure.

2.4 The Euler and the Navier-Stokes Equations

If there are no external forces such as gravity, the equation of motion for a frictionless fluid is known as the *Euler equation*:

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = -\nabla P. \quad (2.4.1)$$

Friction (called *viscosity* in fluid/gas dynamics) will be treated in more detail later. In its simplest form the equation of motion for a viscous fluid is the *Navier-Stokes equation*, included here for completeness' sake:

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = -\nabla P + \eta \left[\nabla^2 \mathbf{V} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{V}) \right]. \quad (2.4.2)$$

The coefficient η in the last term on the right-hand side of the Navier-Stokes equation is the *shear viscosity coefficient*. It determines the strength of viscous effects: internal friction in the fluid or gas.

For now it is sufficient to note that viscosity, like pressure, arises from thermal motion, specifically from the fact that (elastic) collisions between atoms or molecules leads to an exchange of momentum. The viscous force density $\eta [\nabla^2 \mathbf{V} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{V})]$ is the macroscopic manifestation of this momentum exchange in the many particle-particle collisions that occur each second, which fluid mechanics (by construction) can not describe in detail. If particles typically travel a linear distance ℓ between collisions before colliding with another particle, and have a typical thermal velocity σ , the viscosity coefficient equals

$$\eta = \frac{1}{3} \rho \sigma \ell. \quad (2.4.3)$$

Equations (2.4.1) and (2.4.2) respectively form the basis of ideal and viscous fluid- or gas dynamics.

2.5 Pressure, Temperature and the Internal Energy

In the Sect. 2.3 we have learned that the thermal motion of the particles, which is the source of the velocity dispersion around the mean velocity \mathbf{V} , leads to a force proportional to the gradient in the pressure P . The minus sign in this pressure force $-\nabla P$ can be understood intuitively: material tends to move away from a region of high pressure or is sucked into a region of low pressure, as any meteorologist will tell you.

Thermodynamics (see for instance [18, 29]) tells us that the energy of a system in thermal equilibrium at temperature T is $\frac{1}{2}k_b T$ per degree of freedom, with k_b Boltzmann's constant. In the case of an isotropic gas in three dimensions, consisting of point particles with no *internal* degrees of freedom, this means

$$\frac{1}{2}m\overline{\sigma_x^2} = \frac{1}{2}m\overline{\sigma_y^2} = \frac{1}{2}m\overline{\sigma_z^2} = \frac{1}{2}k_b T, \quad (2.5.1)$$

or equivalently

$$\overline{\sigma_x^2} = \overline{\sigma_y^2} = \overline{\sigma_z^2} = v_{\text{th}}^2 \text{ with } v_{\text{th}} = \sqrt{k_b T/m}. \quad (2.5.2)$$

This thermodynamic relationship implies that the pressure is related to the number density n (or mass density $\rho = nm$) and temperature T by the ideal gas law:

$$P(\rho, T) = nk_b T = \frac{\rho \mathcal{R} T}{\mu}. \quad (2.5.3)$$

In this expression $\mathcal{R} = k_b/m_H$ is the universal gas constant, and $\mu = m/m_H$ is the mass of the particles, expressed in units of the mass of the hydrogen atom, m_H . The thermal energy density of the gas is the kinetic energy per unit volume that is associated the thermal velocity:

$$\frac{1}{2}nm(\overline{\sigma_x^2} + \overline{\sigma_y^2} + \overline{\sigma_z^2}) = \frac{3}{2}nk_b T = \frac{3}{2}\frac{\rho \mathcal{R} T}{\mu}. \quad (2.5.4)$$

One often uses the *specific energy*, which is the energy per unit mass. If the mass density equals ρ a unit mass occupies a volume⁹ $\overline{V} = 1/\rho$, a quantity known as the *specific volume*.

The specific energy e is therefore:

$$e = \frac{3}{2}\frac{\rho \mathcal{R} T}{\mu}\overline{V} = \frac{3\mathcal{R} T}{2\mu} = \frac{3P}{2\rho}. \quad (2.5.5)$$

⁹Simply from: $\rho\overline{V} = 1$ in the mass units used.

2.6 Gravity and Self-gravity

In single particle dynamics, a gravitational field with a potential $\Phi(\mathbf{x}, t)$ and associated gravitational acceleration $\mathbf{g} = -\nabla\Phi$ leads to a gravitational force on a particle with mass m equal to

$$\mathbf{F}_{\text{gr}} = m \mathbf{g} = -m \nabla\Phi. \quad (2.6.1)$$

Using the same analogy as was used to find the inertial forces on a fluid, the gravitational action on a fluid due to a gravitational field with potential $\Phi(\mathbf{x}, t)$ must be described by using a force density that is the product of the mass density ρ and the gravitational acceleration \mathbf{g} :

$$\mathbf{f}_{\text{gr}} = \rho \mathbf{g} = -\rho \nabla\Phi. \quad (2.6.2)$$

We can represent the gradient of the gravitational potential $\Phi(\mathbf{x}, t)$ as a column vector in cartesian coordinates:

$$\mathbf{g}(\mathbf{x}, t) = -\nabla\Phi(\mathbf{x}, t) = - \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \\ \frac{\partial\Phi}{\partial z} \end{pmatrix}. \quad (2.6.3)$$

If gravity is the only additional force working on the fluid, the equation of motion becomes:

$$\rho \left[\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = -\nabla P - \rho \nabla\Phi. \quad (2.6.4)$$

In astrophysical applications, one has to deal with the case of *self-gravitation*, where the mass of the fluid generates (part of) the gravitational field. In that case we must add Poisson's equation to the system of equations:

$$\nabla^2\Phi(\mathbf{x}, t) = 4\pi G \rho(\mathbf{x}, t). \quad (2.6.5)$$

Poisson's equation relates the gravitational potential $\Phi(\mathbf{x}, t)$ to the mass distribution $\rho(\mathbf{x}, t)$ that acts as a source of gravity. This equation is solved formally by:

$$\Phi(\mathbf{x}, t) = - \int d^3\mathbf{x}' \frac{G \rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}. \quad (2.6.6)$$

Note that Newtons potential works *instantaneously*, and is therefore only valid for 'slowly varying' gravitational fields. Here 'slow' is defined with respect to the light

travel time across the system one is considering. To properly describe the effects of a time-varying gravitational field one has to turn to General Relativity, where the action of gravity is described by tensor fields rather than by a scalar potential Φ . In particular, it is *not* correct to replace Newtons potential by a *retarded* potential to take account of relativistic effects, such as the light travel-time between the mass that is the source of the potential, and the position where one tries to determine the value of the gravitational potential. Such a procedure works for electromagnetism, where one uses retarded potentials to describe electromagnetic fields, see for instance [22], Chap. 14. It does not for gravity as described by the theory of General Relativity. Although such a theory can be formulated it's predictions do not agree with experiments. A discussion of such seemingly obvious but wrong approaches to relativistic gravity can be found in the famous book of Misner, Thorne and Wheeler [32], Chap. 7 and in the book by Zee [51], Chapter IX.5.

2.7 Mass Conservation and the Continuity Equation

In order to solve the equation of motion we need to know how the fluid mass density $\rho(\mathbf{x}, t)$ behaves. It is a *dynamical* quantity that changes in response to the flow. If flow lines locally converge mass is concentrated in that region of space, and the density increases. Conversely: if flow lines locally diverge, the mass density will decrease in that region as time progresses.

Consider a droplet of fluid at position \mathbf{x} with infinitesimal volume $\Delta\mathcal{V}$ and mass $\Delta M = \rho \Delta\mathcal{V}$. Due to the motion of the fluid the droplet will be deformed, as a simple observation of the behavior of milk added to a stirred cup of coffee will immediately show. However, as long as there are no processes that can create particles (e.g. pair creation by high-energy photons) or destroy them, the mass of the droplet is conserved, regardless how complicated its shape becomes:

$$\Delta M = \rho \Delta\mathcal{V} = \text{constant}. \quad (2.7.1)$$

This means that in principle it is sufficient to calculate the change in the droplet volume $\Delta\mathcal{V}$. In order to properly calculate the deformation of a small volume-element in a flow, we must first consider the concept of *material curves*: curves connecting points where each individual point is carried along passively by the flow with a velocity equal to the speed of the flow, see the Fig. 2.2.

2.7.1 Equation of Motion for a Material Curve

Take a *material curve* $X(\ell)$, with ℓ measuring the length along the curve. By definition each point along a material curve is carried along passively by the flow. This means

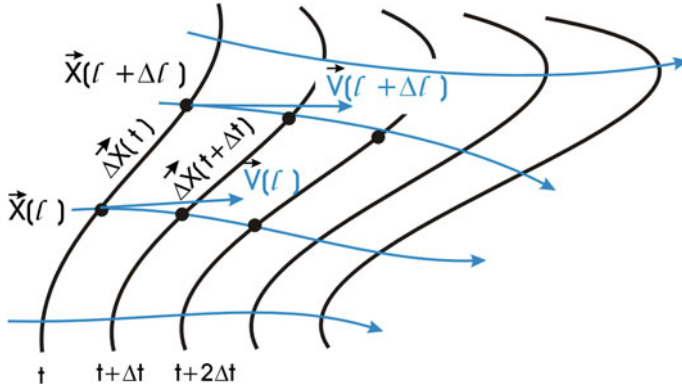


Fig. 2.2 Material lines are carried passively by the flow. The figure shows how a given material line is deformed as time progresses.

that the velocity at the position $X(\ell)$ along the curve is always equal to the local fluid velocity:

$$\frac{dX}{dt} = V(x = X, t). \quad (2.7.2)$$

Consider a infinitesimally small section of the curve with length $\Delta\ell$, located between ℓ and $\ell + \Delta\ell$. For $\Delta\ell \rightarrow 0$ the section of curve can be approximated by the tangent vector

$$\Delta X = X(\ell + \Delta\ell) - X(\ell) \approx \frac{\partial X}{\partial \ell} \Delta\ell. \quad (2.7.3)$$

The vector ΔX changes in time according to

$$\frac{d(\Delta X)}{dt} = V(X(\ell) + \Delta X, t) - V(X(\ell), t). \quad (2.7.4)$$

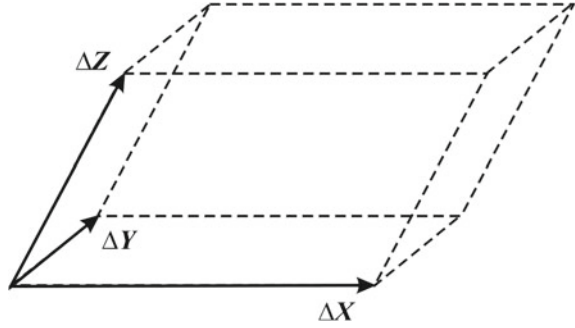
In the limit $|\Delta X| \rightarrow 0$ one can write:

$$\boxed{\frac{d(\Delta X)}{dt} = (\Delta X \cdot \nabla) V.} \quad (2.7.5)$$

2.7.2 Material Volumes

Any small volume in a flow can be defined by three infinitesimal (tangent) vectors ΔX , ΔY and ΔZ . These vectors need not be orthogonal (see Fig. 2.3), and form the ‘edges’ of the infinitesimal volume. If one takes these three edges to be sections of material curves, the entire infinitesimal volume moves with the flow: it is a *material*

Fig. 2.3 The volume defined by three arbitrary vectors $\Delta\mathbf{X}$, $\Delta\mathbf{Y}$ and $\Delta\mathbf{Z}$



volume.¹⁰ This means that a material volume contains a *fixed* amount of mass: no material can flow *across* a material curve, and therefore the mass flux across the outer surfaces of the volume, which are defined by material curves, also vanishes: no mass can flow in or out. The vectors $\Delta\mathbf{X}$, $\Delta\mathbf{Y}$ and $\Delta\mathbf{Z}$ are carried passively by the flow and, as a result, are stretched and rotated according to Eq. (2.7.5). According to the results of vector algebra (e.g. [2], Sect. 1.4) the *oriented* volume spanned by these three infinitesimal vectors equals

$$\Delta\mathcal{V} = \Delta\mathbf{X} \cdot (\Delta\mathbf{Y} \times \Delta\mathbf{Z}) = \begin{vmatrix} \Delta X_x & \Delta X_y & \Delta X_z \\ \Delta Y_x & \Delta Y_y & \Delta Y_z \\ \Delta Z_x & \Delta Z_y & \Delta Z_z \end{vmatrix}. \quad (2.7.6)$$

Taking the time-derivative d/dt of this definition, the product rule for differentiation gives:

$$\frac{d\Delta\mathcal{V}}{dt} = \frac{d\Delta\mathbf{X}}{dt} \cdot (\Delta\mathbf{Y} \times \Delta\mathbf{Z}) + \Delta\mathbf{X} \cdot \left(\frac{d\Delta\mathbf{Y}}{dt} \times \Delta\mathbf{Z} + \Delta\mathbf{Y} \times \frac{d\Delta\mathbf{Z}}{dt} \right). \quad (2.7.7)$$

Now using the result (2.7.5) for material curves one finds:

$$\begin{aligned} \frac{d\Delta\mathcal{V}}{dt} &= [(\Delta\mathbf{X} \cdot \nabla)\mathbf{V}] \cdot (\Delta\mathbf{Y} \times \Delta\mathbf{Z}) \\ &\quad + [(\Delta\mathbf{Y} \cdot \nabla)\mathbf{V}] \cdot (\Delta\mathbf{Z} \times \Delta\mathbf{X}) \\ &\quad + [(\Delta\mathbf{Z} \cdot \nabla)\mathbf{V}] \cdot (\Delta\mathbf{X} \times \Delta\mathbf{Y}) \end{aligned} \quad (2.7.8)$$

¹⁰Two material vectors can be used to define a *material surface*, for instance: $\Delta\mathbf{O} = \Delta\mathbf{X} \times \Delta\mathbf{Y}$. We will have use for this later.

Here I have used the cyclic permutation rule:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}). \quad (2.7.9)$$

Result (2.7.8) is quite general, but rather unwieldy. The algebra can be simplified considerably if one makes a special (and rather obvious) choice for the three vectors $\Delta \mathbf{X}$, $\Delta \mathbf{Y}$ and $\Delta \mathbf{Z}$. Let us take the three infinitesimal vectors to be *orthogonal* and in addition align them with the three coordinate axes of a Cartesian coordinate system:

$$\Delta \mathbf{X} = \begin{pmatrix} \Delta X \\ 0 \\ 0 \end{pmatrix}, \quad \Delta \mathbf{Y} = \begin{pmatrix} 0 \\ \Delta Y \\ 0 \end{pmatrix}, \quad \Delta \mathbf{Z} = \begin{pmatrix} 0 \\ 0 \\ \Delta Z \end{pmatrix}. \quad (2.7.10)$$

It is easily checked that for this particular choice the volume-element (2.7.6) reduces to $\Delta \mathcal{V} = \Delta X \Delta Y \Delta Z$, as should be expected. This assumption simplifies the algebra considerably but, as will be argued below, does not constrain the generality of the final result.

The first term on the right-hand side of (2.7.8) can be written in determinant form as

$$\Delta X \begin{vmatrix} \partial V_x / \partial x & \partial V_y / \partial x & \partial V_z / \partial x \\ 0 & \Delta Y & 0 \\ 0 & 0 & \Delta Z \end{vmatrix} = \Delta X \left(\frac{\partial V_x}{\partial x} \right) \Delta Y \Delta Z. \quad (2.7.11)$$

The remaining two terms can be calculated in a similar fashion, and give $(\partial V_y / \partial y) \Delta X \Delta Y \Delta Z$ and $(\partial V_z / \partial z) \Delta X \Delta Y \Delta Z$. Therefore expression (2.7.8) reduces to the simple form

$$\begin{aligned} \frac{d\Delta \mathcal{V}}{dt} &= \Delta X \Delta Y \Delta Z \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) \\ &= \Delta \mathcal{V} (\nabla \cdot \mathbf{V}). \end{aligned} \quad (2.7.12)$$

Since any volume, regardless its shape, can always be constructed using much smaller rectangular cubes as ‘building blocks’,¹¹ with each small cube individually satisfying relation (2.7.12), this relationship must be true *regardless* the shape of the volume, provided that this total volume remains infinitesimally small.

¹¹The ‘Lego Principle’.

2.7.3 Mass Conservation and the Continuity Equation

Mass conservation, $\rho \Delta \mathcal{V} = \text{constant}$, implies that

$$\frac{d(\rho \Delta \mathcal{V})}{dt} = \Delta \mathcal{V} \left(\frac{d\rho}{dt} \right) + \rho \left(\frac{d\Delta \mathcal{V}}{dt} \right) = 0, \quad (2.7.13)$$

or equivalently

$$\frac{d\rho}{dt} = -\rho \left(\frac{1}{\Delta \mathcal{V}} \frac{d\Delta \mathcal{V}}{dt} \right). \quad (2.7.14)$$

Using the change-of-volume law (2.7.12) together with the now familiar expression for d/dt one finds:

$$\frac{\partial \rho}{\partial t} + (\mathbf{V} \cdot \nabla) \rho = -\rho (\nabla \cdot \mathbf{V}). \quad (2.7.15)$$

Reordering terms in this equation and employing the vector identity¹²

$$\nabla \cdot (f \mathbf{A}) = f (\nabla \cdot \mathbf{A}) + (\mathbf{A} \cdot \nabla) f, \quad (2.7.16)$$

one can write this differential version of the mass conservation law as

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0.} \quad (2.7.17)$$

This equation is known as the **continuity equation**. The Box below gives a simple one-dimensional derivation that uses a different point of view.

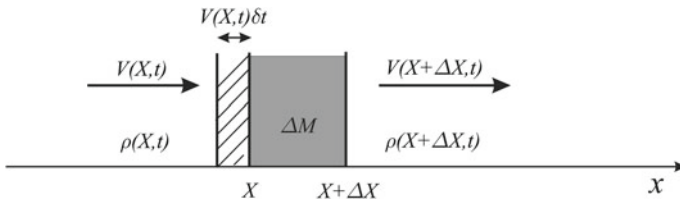


Fig. 2.4 The quantities used for the derivation of the continuity equation for a one-dimensional flow along the x -axis. The flow with velocity $V(x, t)$ is in the direction of positive x , as indicated by the *two arrows*. The density of the fluid is $\rho(x, t)$. We consider the change in the amount of mass ΔM contained between $x = X$ and $x = X + \Delta X$, the *gray box* in the figure. We assume $\Delta X \ll X$ throughout the calculation. The mass contained in the hatched area of width $V \delta t$ is the mass that enters the box in a short time span $\delta t \ll t$

¹²Another consequence of the product rule for differentiation, as is easily checked for Cartesian coordinates!

Continuity Equation for a One-Dimensional Flow

Consider a one-dimensional flow along the x -axis with density $\rho(x, t)$ and velocity $V(x, t)$. The amount of mass that is contained in a fixed infinitesimal one-dimensional box, bounded by $x = X$ and $x = X + \Delta X$ with $\Delta X \ll X$, equals

$$\Delta M(t) = \rho(X, t) \Delta X, \quad (2.7.18)$$

see the Fig. 2.4.

The mass ΔM changes in time since mass flows into or out of the box across its two boundaries. I will assume that $V > 0$ everywhere in and near the box in order to simplify the calculation. In that case the amount of mass that enters the Box by flowing across its edge at $x = X$ in an infinitesimal time span δt is

$$(\Delta M)_{\text{in}} = \rho(X, t) V(X, t) \delta t. \quad (2.7.19)$$

This is the amount of mass contained in the strip of width $\Delta x = V(X, t) \delta t$ along the x -axis. By the same token, the amount of mass that leaves the box in the same time span by flowing across its boundary at $x = X + \Delta X$ is

$$(\Delta M)_{\text{out}} = \rho(X + \Delta X, t) V(X + \Delta X, t) \delta t. \quad (2.7.20)$$

Therefore, the change in the mass contained in the box is

$$\begin{aligned} \delta(\Delta M) &= (\Delta M)_{\text{in}} - (\Delta M)_{\text{out}} \\ &= \{\rho(X, t) V(X, t) - \rho(X + \Delta X, t) V(X + \Delta X, t)\} \delta t. \end{aligned} \quad (2.7.21)$$

Since the walls of the Box are fixed we have

$$\delta(\Delta M) = \delta \rho \Delta X = \left\{ \left(\frac{\partial \rho}{\partial t} \right)_{x=X} \delta t \right\} \Delta X. \quad (2.7.22)$$

Here I have used that δt is infinitesimal.

For small ΔX we have

$$\begin{aligned} \rho(X, t) V(X, t) - \rho(X + \Delta X, t) V(X + \Delta X, t) \\ \simeq - \left(\frac{\partial (\rho V)}{\partial x} \right)_{x=X} \Delta X. \end{aligned} \quad (2.7.23)$$

Substituting these last two relations into (2.7.21) one gets:

$$\left(\frac{\partial \rho}{\partial t}\right) \Delta X \delta t = - \left(\frac{\partial \rho V}{\partial x}\right) \Delta X \delta t, \quad (2.7.24)$$

where it is now understood that all quantities are evaluated at $x = X$. The density must therefore satisfy

$$\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x} (\rho V) \iff \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho V) = 0, \quad (2.7.25)$$

the one-dimensional continuity equation. This argument is easily extended to three dimensions, using a rectangular cube with volume $\Delta \mathcal{V} = \Delta X \Delta Y \Delta Z$, by considering the flow of mass across all six faces of the cube. Such a calculation yields continuity equation (2.7.17).

2.8 The Adiabatic Gas Law

The final missing element in our description of a fluid or gas is a recipe that describes the behavior of the pressure $P = \rho \mathcal{R}T/\mu$. I will limit the discussion here to the special (but important) case of an *adiabatic gas*.

An adiabatic process in thermodynamics is a process where (in a closed system) no energy is added by irreversible heating the system, or extracted by irreversible cooling the system. The *first law of thermodynamics* states that the amount of heat dQ added to a gas in some volume \mathcal{V} is related to the change in the energy dU and/or the volume-change $d\mathcal{V}$ by

$$dQ \equiv T dS = dU + P d\mathcal{V}. \quad (2.8.1)$$

Here S is the *entropy*, T the gas temperature and P the gas pressure. We already calculated the energy *per unit volume* of an ideal gas in thermal equilibrium: it equals $3\rho \mathcal{R}T/2\mu$, see Eq. (2.5.4).

The energy U in a small volume \mathcal{V} is then simply:

$$U = \frac{3}{2} \frac{\rho \mathcal{R}T \mathcal{V}}{\mu}. \quad (2.8.2)$$

The pressure satisfies the ideal gas law (Eq. 2.5.3): $P = \rho \mathcal{R}T/\mu$. An adiabatic process satisfies by definition that no heat is added or subtracted from the system:

$$dQ = T dS = 0. \quad (2.8.3)$$

In that case, the first law of thermodynamics reduces to:

$$d\left(\frac{3\rho\mathcal{R}T\mathcal{V}}{2\mu}\right) + \left(\frac{\rho\mathcal{R}T}{\mu}\right) d\mathcal{V} = 0. \quad (2.8.4)$$

Writing out the first differential, using the product rule $d(fg) = (df)g + f(dg)$, one finds:

$$\left(\frac{5\rho\mathcal{R}T}{2\mu}\right) d\mathcal{V} + \mathcal{V} d\left(\frac{3\rho\mathcal{R}T}{2\mu}\right) = 0. \quad (2.8.5)$$

Using $P = \rho\mathcal{R}T/\mu$ and multiplying by the resulting relation $2/3$ leads to the following relation:

$$\frac{5}{3} P d\mathcal{V} + \mathcal{V} dP = 0. \quad (2.8.6)$$

This can be written as¹³

$$\frac{dP}{P} + \frac{5}{3} \frac{d\mathcal{V}}{\mathcal{V}} = d\{\ln(P \mathcal{V}^{5/3})\} = 0. \quad (2.8.7)$$

This implies that $\ln(P \mathcal{V}^{5/3})$ is constant, which is equivalent with

$$P \times \mathcal{V}^{5/3} = \text{constant}. \quad (2.8.8)$$

As long as the volume \mathcal{V} is small, we can apply this law *locally*. Take an infinitesimal volume \mathcal{V} , containing a fluid of density ρ and pressure P .

As the gas expands (or contracts) the volume changes, and the pressure adjusts according to (2.8.8). The conservation of mass implies that $\rho \mathcal{V} = \text{constant}$. This implies $\mathcal{V} \propto 1/\rho$, and relation (2.8.8) can be rewritten in terms of the density:

$$\boxed{P \rho^{-5/3} = \text{constant}.} \quad (2.8.9)$$

2.8.1 The Polytopic Gas Law, the Specific Heat Coefficients and the Isothermal Gas

Relation (2.8.9) is a special case of a *polytopic gas law*, which generally takes the form

$$P = \text{constant} \times \rho^\gamma. \quad (2.8.10)$$

¹³Here we use that $dx/x = d \ln x$ for any variable x , and $\ln(xy) = \ln x + \ln y$.

The value of the exponent γ ($= 5/3$ for an ideal classical gas) depends on the circumstances. For an ideal gas γ is related to the ratio of *specific heat* at constant volume, c_v , and at constant pressure, c_p : $\gamma = c_p/c_v$ as we will now prove. Let us introduce the *specific volume* \bar{V} , the volume containing a *unit* mass:

$$\bar{V} \equiv \frac{1}{\rho}. \quad (2.8.11)$$

In terms of this quantity, relation (2.8.1) can be written as

$$dQ = T ds = de + P d\left(\frac{1}{\rho}\right), \quad (2.8.12)$$

by applying the general relation (2.8.1) to the specific volume \bar{V} . Here s is the entropy per unit mass (specific entropy), and e the energy per unit mass (specific energy),

$$e \equiv \frac{3}{2} \frac{\mathcal{R}T}{\mu} = \frac{3}{2} \frac{k_b T}{m}. \quad (2.8.13)$$

The specific heat coefficient at constant volume c_v is defined by the relation

$$(dQ)_{\bar{V} = \text{cnst}} = c_v dT. \quad (2.8.14)$$

It determines the amount of energy needed to raise the temperature of a unit mass of gas by an amount dT , keeping the volume (and, because of mass conservation, the density) constant. Using (2.8.12) with $d(1/\rho) = 0$ this definition implies

$$\boxed{c_v = \frac{\partial e}{\partial T} = \frac{3}{2} \frac{k_b}{m}}. \quad (2.8.15)$$

If one writes (2.8.12) in the form

$$dQ = d\left(e + \frac{P}{\rho}\right) - \frac{dP}{\rho}, \quad (2.8.16)$$

one can define the specific heat coefficient at constant pressure c_p by

$$(dQ)_P = c_p dT. \quad (2.8.17)$$

The coefficient c_p determines the amount of energy needed to raise the temperature of a unit mass by an amount dT while the pressure is kept constant so that $dP = 0$. This means that the gas is allowed to expand if it is heated, or will contract as it cools.

Definition (2.8.17) implies

$$c_p = \frac{\partial(e + P/\rho)}{\partial T} = \frac{5}{2} \frac{k_b}{m}. \quad (2.8.18)$$

One must have $c_p > c_v$ because now part of the energy supplied goes into the work done by the gas during the expansion rather than into heat, and more energy is required for a given temperature change dT . From (2.8.15) and (2.8.17) one immediately finds

$$c_p - c_v = \frac{k_b}{m} = \frac{\mathcal{R}}{\mu}. \quad (2.8.19)$$

The first law of thermodynamics can be rewritten in terms of c_p and c_v .

Using relations (2.8.15) to (2.8.19) one finds:

$$\begin{aligned} dQ &= c_v dT + \left(\frac{\rho \mathcal{R} T}{\mu} \right) d \left(\frac{1}{\rho} \right) \\ &= c_v dT - \left(\frac{\mathcal{R} T}{\rho \mu} \right) d\rho \\ &= c_v T \left[\frac{dT}{T} - \left(\frac{c_p}{c_v} - 1 \right) \frac{d\rho}{\rho} \right]. \end{aligned} \quad (2.8.20)$$

Putting $dQ = 0$ as required for an adiabatic process, the resulting equation is solved by

$$\ln T - (\gamma - 1) \ln \rho = \text{constant}, \quad (2.8.21)$$

with

$$\gamma \equiv \frac{c_p}{c_v} = \frac{5}{3}. \quad (2.8.22)$$

Relation (2.8.21) is equivalent with $\ln(T \rho^{-(\gamma-1)}) = \text{constant}$, and leads to the adiabatic temperature-density relation:

$$T \rho^{-(\gamma-1)} = \text{constant}. \quad (2.8.23)$$

It is easily checked that this relation is equivalent with the adiabatic gas law: since $P = \rho \mathcal{R} T / \mu \propto \rho T$ relation (2.8.23) implies $P \propto \rho^\gamma$. This proves the relationship between the index γ in the polytropic gas law (2.8.10) and the specific heat ratio in the case of an adiabatic gas.

As a by-product of this derivation we can calculate the specific entropy s for an ideal gas directly from Eq. (2.8.20). Using $dQ = T ds$ and (2.8.22) one has:

$$T ds = c_v T \left[\frac{dT}{T} - (\gamma - 1) \frac{d\rho}{\rho} \right]. \quad (2.8.24)$$

Dividing out the common factor T , the resulting equation can be written as:

$$d[s - c_v \ln T + (\gamma - 1)c_v \ln \rho] = 0. \quad (2.8.25)$$

This relation can be immediately integrated to

$$s = c_v \ln \left(\frac{T}{\rho^{\gamma-1}} \right) + \text{constant}. \quad (2.8.26)$$

An alternative expression for s follows from the ideal gas law $P = \rho \mathcal{R}T/\mu$:

$$s = c_v \ln (P \rho^{-\gamma}) + \text{constant}. \quad (2.8.27)$$

A special case, often used as a useful approximation in astrophysical models, is the assumption of an **isothermal gas** that satisfies the relation

$$T = \text{constant}. \quad (2.8.28)$$

In that case the pressure $P = \rho \mathcal{R}T/\mu$ is directly proportional to the density. The polytropic index γ in (2.8.10) takes the special value

$$\gamma_{\text{iso}} = 1. \quad (2.8.29)$$

This value for γ is consistent with the temperature-density law (2.8.23): for $\gamma = 1$ it reduces to $T = \text{constant}$.

Note that for an ideal gas an isothermal state of the gas can only be maintained if there exists some mechanism which acts as a ‘thermostat’ that keeps the temperature constant by supplying (extracting) exactly the right amount of energy to the gas if it expands (contracts). A gas embedded in a strong black-body radiation field¹⁴ of fixed temperature often behaves in this manner. The radiation acts as a heat reservoir with such a large heat capacity so that any changes in the internal energy of the gas are immediately compensated by the radiation: if the gas is colder than the radiation field it absorbs radiation until the temperatures equilibrate. Conversely: if the gas is hotter it emits radiation until the temperatures of gas and radiation are equal. These processes force the gas to remain in temperature equilibrium with the radiation field.

2.9 Application: The Isothermal Sphere and Globular Clusters

As a first (astronomical) application we will consider a simple model for a spherically-symmetric, self-gravitating stellar system: the *isothermal sphere*. The isothermal

¹⁴Black Body Radiation has a unique distribution of photon energies. When expressed in terms of the frequency ν (photon energy is $\varepsilon = h\nu$ with h Planck’s constant) this distribution is $n(\nu) = (8\pi\nu^2/c^3)[\exp(h\nu/k_b T) - 1]^{-1}$, where $n(\nu)$ is the number of photons per unit volume and unit frequency so that the number density of photons in a frequency interval $d\nu$ equals $dn_{\text{phot}} = n(\nu) d\nu$.



Fig. 2.5 The globular cluster NGC 5139

sphere is a crude model for a globular cluster, for the quasi-spherical central region (‘bulge’) of a disk galaxy, or for the nucleus of an elliptical galaxy (Fig. 2.5).

Consider a large number of stars with a density distribution that only depends on the distance r from the center of the sphere. If all the stars have a mass m_* and the number density at radius r equals $n(r)$, the mass density equals

$$\rho(r) = n(r)m_*. \quad (2.9.1)$$

If the number of stars is large enough we can describe it as a ‘gas’ of stars with a ‘temperature’ T , which is determined by the orbital velocity dispersion according Eq. (2.5.1):

$$\overline{\sigma_x^2} = \overline{\sigma_y^2} = \overline{\sigma_z^2} \equiv \tilde{\sigma}^2 = \frac{k_b T}{m_*}. \quad (2.9.2)$$

This definition implies that $\sigma^2 = 3\tilde{\sigma}^2$. Typically, a globular cluster contains 100,000 stars and has a mass between 10^4 and $10^6 M_\odot$, with an average mass of $10^5 M_\odot$.

The velocity dispersion of the stars in a globular cluster can be measured by looking at the Doppler broadening of the absorption lines in the spectrum of an entire globular cluster: one observes the ‘average’ spectrum of a large number of

stars, with a velocity dispersion $\tilde{\sigma}$ along the line-of-sight. This leads to a line-width $\Delta\lambda$ in the integrated spectrum of the whole cluster given by

$$\frac{\Delta\lambda}{\lambda} \simeq \frac{\tilde{\sigma}}{c}. \quad (2.9.3)$$

In the isothermal sphere model the cluster is treated as a self-gravitating ball of gas. The pressure of this gas, where the stars play the role of ‘molecules’, equals

$$P(r) = n(r)k_b T = \rho(r)\tilde{\sigma}^2. \quad (2.9.4)$$

The isothermal assumption means that the temperature, and therefore $\tilde{\sigma}$, does not depend on the radius r . All other quantities are assumed to depend only on the radial coordinate r , the distance to the center of the globular cluster.

The consequences of the isothermal sphere model were first investigated exhaustively by Chandrasekhar [11]. A good modern account of this (and related) models can be found in the book by Binney and Tremaine [6].

Since there is only velocity dispersion, and no *bulk* motion of the stars we have $\mathbf{V} = 0$, and the equation of motion becomes the equation for *hydrostatic equilibrium*, where the gravitational force in the radial direction is balanced by the radial pressure gradient:

$$\frac{dP}{dr} = \tilde{\sigma}^2 \left(\frac{d\rho}{dr} \right) = -\rho \frac{G M(r)}{r^2}. \quad (2.9.5)$$

Here we use the fact that for a spherically symmetric mass distribution the gravitational acceleration at some radius r depends only on the amount of mass $M(r)$ contained *within* that radius. Because of this symmetry, the mass outside r does not exert a net force. The amount of mass contained in a spherical shell between r and $r + dr$ equals

$$dM = 4\pi r^2 \rho(r) dr. \quad (2.9.6)$$

The mass contained within a radius r is given by an integral over mass shells:

$$M(r) = \int_0^r dr' 4\pi r'^2 \rho(r'). \quad (2.9.7)$$

The gravitational potential $\Phi(r)$ is defined by the equation

$$g_r = -\frac{G M(r)}{r^2} = -\frac{d\Phi}{dr}. \quad (2.9.8)$$

The isothermal assumption, together with $P(r) = \rho(r)\tilde{\sigma}^2$, implies that the equation of hydrostatic equilibrium (2.9.5) can be written as

$$\tilde{\sigma}^2 \left(\frac{1}{\rho} \frac{d\rho}{dr} \right) = -\frac{d\Phi}{dr}. \quad (2.9.9)$$

This equation has a formal solution $\ln \rho = -\Phi/\tilde{\sigma}^2 + \text{constant}$, or equivalently:

$$\boxed{\rho(r) = \rho_0 e^{-\Phi(r)/\tilde{\sigma}^2}}. \quad (2.9.10)$$

Here ρ_0 is the mass density at $r = 0$, assuming that $\Phi(0) = 0$. This expression gives the density as a function of the gravitational potential, and is known in the context of meteorology as the *barometric height formula*, see the Box below.

The Barometric Height Formula

Consider a static isothermal atmosphere in a constant gravitational field, with gravitational acceleration $\mathbf{g} = -g \hat{\mathbf{z}}$. The pressure force balances gravity:

$$\frac{dP}{dz} = -\rho g. \quad (2.9.11)$$

Using $P = \rho \mathcal{R}T/\mu$ with $T = \text{constant}$, this equation can be written as:

$$\frac{d\rho}{dz} = -\frac{\rho}{\mathcal{H}}, \quad (2.9.12)$$

with

$$\mathcal{H} = \frac{\mathcal{R}T}{\mu g} \quad (2.9.13)$$

the *isothermal scale height*. The solution is simple:

$$\rho(z) = \rho_0 \exp(-z/\mathcal{H}), \quad (2.9.14)$$

with $\rho_0 \equiv \rho(z = 0)$. The density and pressure fall off exponentially with increasing height. The gravitational potential in this case equals

$$\Phi(z) = gz, \quad (2.9.15)$$

where I have chosen $\Phi(0) = 0$. Such a choice is always possible as the potential is determined up to a global constant. This allows us to write the expression for $\rho(z)$ as

$$\rho(z) = \rho_0 \exp(-\mu\Phi(z)/\mathcal{R}T). \quad (2.9.16)$$

The thermal velocity of the gas equals

$$\sigma = \sqrt{\frac{\mathcal{R}T}{\mu}}, \quad (2.9.17)$$

so this is equivalent with

$$\rho(z) = \rho_0 \exp(-\Phi(z)/\sigma^2). \quad (2.9.18)$$

This is exactly the same expression as we derived for the density law in an isothermal sphere.

The potential $\Phi(r)$ must be calculated by solving Poisson's equation for the gravitational field of the cluster. Because of the use of the radial coordinate r it takes the form

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho(r) = 4\pi G \rho_0 e^{-\Phi(r)/\tilde{\sigma}^2}. \quad (2.9.19)$$

One can introduce the following *dimensionless* variables for the radial distance and gravitational potential:

$$\xi = \frac{r}{r_K}, \quad \Psi = \frac{\Phi}{\tilde{\sigma}^2} = \frac{m_* \Phi}{k_b T}. \quad (2.9.20)$$

The radius r_K is a normalizing length scale, the so-called *King radius*. It is defined in terms of the central density ρ_0 and the velocity dispersion $\tilde{\sigma}$ of the cluster:

$$r_K = \left(\frac{\tilde{\sigma}^2}{4\pi G \rho_0} \right)^{1/2} = \left(\frac{k_b T}{4\pi G m_* \rho_0} \right)^{1/2}. \quad (2.9.21)$$

In terms of these variables Poisson's equation takes the following simple form:

$$\boxed{\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\Psi}{d\xi} \right) = e^{-\Psi}.} \quad (2.9.22)$$

This dimensionless form of Poisson's equation displays **no** explicit information about the properties of the cluster. In particular all reference to the central density ρ_0 and the velocity dispersion $\tilde{\sigma}$ has disappeared. The interpretation of this result is as follows. All isothermal spheres are *self-similar*. If one plots the density relative to the central density $\rho(r)/\rho_0$ as a function of the dimensionless radius $\xi = r/r_K$, all globular clusters that behave as an isothermal sphere have exactly the same density profile!

One must solve this equation using two physically motivated boundary conditions:

$$\Psi(\xi = 0) = 0, \quad \left(\frac{d\Psi}{d\xi} \right)_{\xi=0} = 0. \quad (2.9.23)$$

The first boundary condition corresponds to our earlier assumption that $\Phi(0) = 0$, and is not special as the gravitational potential Φ is determined *up to a constant*: this choice is always possible. The second condition is a consequence of the symmetry of the problem: at the center of the sphere *all* the mass is at larger radii, and there can be no net gravitational force: $g_r(0) = -(d\Phi/dr)_{r=0} = 0$.

Unfortunately, there is no analytical solution of this equation for these boundary conditions in closed form. We therefore have to resort to considering the solution near the center ($\xi = 0$) and far from the center ($\xi \gg 1$).

Near $\xi = 0$ one can solve by a power series, using the fact that for $\Psi \ll 1$ the exponential can be expanded:

$$e^{-\Psi} = 1 - \Psi + \frac{1}{2}\Psi^2 + \dots \quad (2.9.24)$$

Assuming a solution of the form

$$\Psi(\xi) = a_1 \xi^2 + a_2 \xi^4 + \dots, \quad (2.9.25)$$

and using the above expansion of the exponential e^Ψ , one determines the coefficients $a_1, a_2 \dots$ by equating powers of ξ on both sides of Eq. (2.9.22). One finds:

$$\Psi(\xi) \simeq \frac{\xi^2}{6} - \frac{\xi^4}{120} + \dots \quad (\text{for } \xi \ll 1). \quad (2.9.26)$$

The corresponding density follows from $\rho = \rho_0 e^{-\Psi}$, using the expansion for $\exp(-\Psi)$ once again:

$$\rho(\xi) \simeq \rho_0 \left(1 - \frac{\xi^2}{6} + \frac{\xi^4}{45} + \dots \right). \quad (2.9.27)$$

For large values of ξ , the solution goes asymptotically to

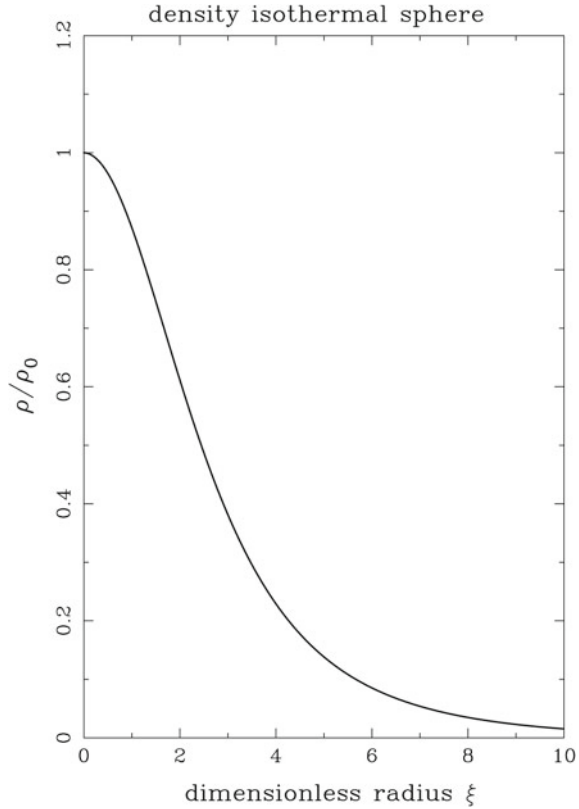
$$\Psi(\xi) \simeq \log \left(\frac{\xi^2}{2} \right) \quad (\text{for } \xi \gg 1). \quad (2.9.28)$$

The density for large values of $\xi = r/r_K$ is therefore

$$\rho(\xi) \approx \rho_0 \left(\frac{2}{\xi^2} \right). \quad (2.9.29)$$

Expressing the density in terms of the radius, this solution is known as the *singular isothermal sphere* solution as the density goes to infinity at $r = 0$:

Fig. 2.6 The mass density in an isothermal sphere relative to the central density ρ_0 as a function of the dimensionless radius $\xi = r/r_K$. The density profile of all globular clusters in hydrostatic equilibrium look the same if one scales the radius in terms of the King radius $r_K = \sqrt{\tilde{\sigma}^2/4\pi G\rho_0}$, and the density with the central density ρ_0



$$\rho(r) = \frac{\tilde{\sigma}^2}{2\pi G r^2}. \quad (2.9.30)$$

The singular isothermal sphere is in fact the only *analytic* solution known to the isothermal sphere equation, as can be checked by substitution. Note that the density in this solution depends only on the velocity dispersion and radius, but is independent of the central density ρ_0 . It can be shown that *any* solution of the isothermal sphere equation takes this form asymptotically at large radii: $r \gg r_K$. The full solution for the density of an isothermal sphere is plotted in the Fig. 2.6.

The density in a singular isothermal sphere decays with radius as $\rho(r) \propto r^{-2}$, which means that the mass within a sphere of radius r grows for large radii as $M(r) \propto r$:

$$M(r) = \int_0^r dr' 4\pi r'^2 \rho(r') \longrightarrow 8\pi\rho_0 r_K^2 r \quad \text{for } r \gg r_K. \quad (2.9.31)$$

Such behavior is clearly unacceptable as a description for a real globular cluster: the mass of an isothermal sphere grows without bound as $r \rightarrow \infty$. This means that

the isothermal sphere can only be an approximate model which fails at large radii because important physical effects are neglected by the model. In this particular case we have neglected tidal effects on the globular cluster due to the Galaxy. These will be considered next.

2.9.1 The Tidal Radius

Observations show that clusters have a well-defined edge beyond which the stellar density rapidly goes to zero. The relatively sharp edge of globular clusters can be explained if one takes account of *tidal forces*: the variation of the gravitational acceleration of the Galaxy across the globular cluster. For a full discussion see: Spitzer [46].

If the cluster is located at a distance R from the galactic center, the gravitational acceleration of the galaxy has a magnitude

$$g_{\text{Gal}}(R) \sim \frac{GM_{\text{Gal}}}{R^2}, \quad (2.9.32)$$

with M_{gal} the mass of the Galaxy. If the radius of the cluster is r_t the variation across the cluster of this acceleration is typically $g_{\text{Gal}}(R + r_t) - g_{\text{Gal}}(R) \simeq r_t(dg_{\text{Gal}}/dR)$. This is essentially the difference between the strength of the Galactic gravitational force at the center, and at the outer edge of the globular cluster. Therefore, the typical magnitude of the tidal acceleration for $r_t \ll R$ is

$$g_t \approx \left| r_t \frac{\partial}{\partial R} \left(\frac{GM_{\text{gal}}}{R^2} \right) \right| = \frac{2GM_{\text{gal}} r_t}{R^3}. \quad (2.9.33)$$

The value of r_t , the so-called *tidal radius*, can be estimated by equating the tidal acceleration to the gravitational pull due to the cluster itself: around r_t tidal forces are just able to pull stars from the cluster, so the tidal acceleration and the acceleration due to the self-gravity of the cluster should nearly balance. If the cluster mass is M_c this balance reads:

$$\frac{GM_c}{r_t^2} \approx \frac{2GM_{\text{gal}} r_t}{R^3}, \quad (2.9.34)$$

or equivalently:

$$\frac{r_t}{R} \approx \left(\frac{M_c}{2M_{\text{gal}}} \right)^{1/3}. \quad (2.9.35)$$

This defines the maximum size of the cluster where the stars in the clusters are still marginally bound by the gravitational pull of the cluster mass.

If one uses estimate (2.9.31) for the mass contained within a radius r_t ,

$$M_c \approx 8\pi\rho_0 r_K^2 r_t, \quad (2.9.36)$$

one finds from (2.9.33):

$$r_t = \left(\frac{4\pi\rho_0 R^3}{M_{\text{gal}}} \right)^{1/2} r_K = \left(\frac{\tilde{\sigma}^2 R^3}{GM_{\text{gal}}} \right)^{1/2}. \quad (2.9.37)$$

Using typical values for the distance, observed velocity dispersion and central mass density of globular clusters and for the mass of our Galaxy,

$$\tilde{\sigma} \simeq 5 \text{ km/s}, \quad \rho_0 \simeq 10^4 M_\odot \text{ pc}^{-3}, \quad R \simeq 10 \text{ kpc}, \quad M_{\text{gal}} \simeq 10^{11} M_\odot,$$

one finds a tidal radius equal to

$$r_t \approx 200 \left(\frac{\tilde{\sigma}}{5 \text{ km/s}} \right) \left(\frac{R}{10 \text{ kpc}} \right)^{3/2} \text{ pc}.$$

The tidal radius is much larger than the King radius, which equals for typical parameters

$$r_K \approx 0.2 \left(\frac{\tilde{\sigma}}{5 \text{ km/s}} \right) \left(\frac{\rho_0}{10^4 M_\odot \text{ pc}^{-3}} \right)^{-1/2} \text{ pc}.$$

That gives an *a posteriori* justification for our use of the asymptotic formula (2.9.36) for the cluster mass.

The King radius yields a good estimate for the size of the dense central core of the cluster: the density in an isothermal sphere drops to $\frac{1}{2}\rho_0$ at $r \sim 3r_K \sim 0.6 \text{ pc}$. These estimates determine the typical mass of a globular cluster, from (2.9.36):

$$M_c \sim \frac{2\tilde{\sigma}^2}{G} \left(\frac{\tilde{\sigma}^2 R^3}{GM_{\text{gal}}} \right)^{1/2} \approx 2.5 \times 10^6 \left(\frac{\tilde{\sigma}}{5 \text{ km/s}} \right)^3 \left(\frac{R}{10 \text{ kpc}} \right)^{3/2} M_\odot.$$

This estimate compares well with the masses of globular clusters that are inferred from observations.

2.10 Application 2: Dark Matter Halos

The singular isothermal sphere is often used as a simple model for the mass distribution in the *dark matter halo* that is believed to be present around many galaxies and clusters. This dark (i.e. non-luminous) halo is believed to consist of *Dark Matter*, probably a massive, electrically neutral and weakly interacting fundamental particle that is outside the Standard Model of particle physics. See [36, 5] for the observational and theoretical background of Dark Matter. A good general introduction to

modern cosmology is the book by Ryden [43]. It is now commonly believed that Dark Matter contains about 70% of all mass in the universe.

The existence of dark matter was first noted by the Swiss astronomer Bernard Zwicky in 1942. In his observations of one of the close, rich clusters of Galaxies, the *Coma Cluster*, he found that the individual galaxies were moving so fast that the cluster could not be gravitationally bound by the mass associated with visible matter. It should have flown apart long ago. He postulated that there was an unseen mass present whose gravitational pull is able to confine the cluster, keeping it from flying apart. Although Zwicky's suggestion was initially ridiculed, Dark Matter is now an essential ingredient in modern cosmological models.

Some of the most persuasive evidence for dark matter comes from the *rotation curves* of disk galaxies (spiral galaxies). There one measures the rotation speed V_{rot} of hydrogen clouds around the galactic center as a function of the distance to the center. Assuming a circular orbit of radius R in the plane of the galaxy this rotation speed is of order

$$V_{\text{rot}} \sim \sqrt{\frac{GM(<R)}{R}}. \quad (2.10.1)$$

Here $M(< R)$ is the mass contained *within* the orbit. The gravitational pull of all mass outside the orbit approximately cancels.¹⁵

On the basis of relation (2.10.1) one expects that the rotation speed decays as

$$V_{\text{rot}} \sim \sqrt{\frac{GM_{\text{gal}}}{R}} \propto R^{-1/2} \quad (2.10.2)$$

in the outer reaches of the galaxy where almost all of the visible mass is inside the radius R . The observations show something different: rather than the velocity law (2.10.2) one finds for large radii:

$$V_{\text{rot}} \sim \text{constant}. \quad (2.10.3)$$

An example of such a rotation curve is shown in the Fig. 2.7.

Using relation (2.10.1) this behaviour implies

$$M(< R) \propto R. \quad (2.10.4)$$

This is exactly the behavior of an isothermal sphere at large radii, see Eq. (2.9.31). The observations suggest that each galaxy is sitting inside an invisible dark matter sphere, the *dark halo*, with an extent considerably larger than the size of the visible galaxy. Apparently this dark matter halo obeys the density law of an isothermal sphere at sufficiently large radius.

¹⁵This cancellation is *exact* if the mass is distributed spherically: Newton's shell theorem.

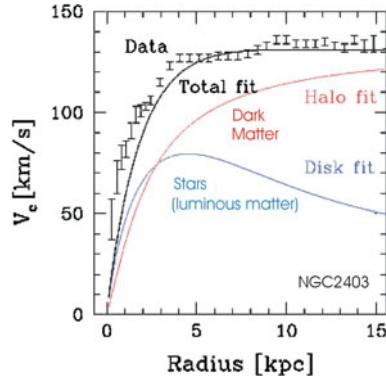


Fig. 2.7 The rotation curve of a spiral galaxy. Note the almost constant rotational speed at large radii. The *blue* and the *red* curve give the rotational speed expected from the visible (luminous) matter alone, and the rotation velocity due to the extra mass of the Dark Halo needed to explain the observations. Note that the net rotation speed depends on the total mass, which is dominated by the halo mass at large radii.

From solution (2.9.31) with $r \Rightarrow R$ and definition (2.9.21) of the King radius r_K we can get the Keplerian rotation speed of a test particle moving on a circular orbit in an isothermal sphere under the influence of gravity for $R \gg r_K$:

$$V_{\text{rot}} \sim \sqrt{8\pi G \rho_0 r_K^2} = \sqrt{2} \tilde{\sigma} = \sqrt{\frac{2}{3}} \sigma. \quad (2.10.5)$$

It is almost equal to the thermal velocity of the particles that make up the isothermal sphere: $\sqrt{2/3} \simeq 0.82$.

Gas Dynamics

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