2. Metrizable Compact Spaces

Our theme is the connections between the topology of X and the abstract properties of $C(X)$. In this chapter we consider a simple, natural category of compact spaces: the metrizable ones.

We show that for a compact Hausdorff space X, metrizability is reflected in separability of $C(X)$. We consider two special spaces, the Hilbert cube and the Cantor set. We show that they are generic in the following sense: The metrizable compact Hausdorff spaces are precisely the closed subspaces of the Hilbert cube, and the quotients of the Cantor set.

2.1 The most accessible topological spaces are the metrizable ones. We open this chapter by stating some standard results about metrizable compact spaces.

(1) We have already mentioned: *In a metrizable compact space, every sequence has a convergent subsequence.*

(2) *Every metrizable compact space is separable* (i.e. has a countable dense subset).

(3) *On a compact metric space every continuous function is uniformly continuous.* Here a comment is in order.

A function f on a set X is, be definition, uniformly continuous under a metric d on X if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x, y \in X, d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Uniform continuity is not a *topological* property: A function may be uniformly continuous under a metric d_1 and not under a metric d_2 that determines the same topology. However, this cannot happen if the topology is compact.

2.2 Theorem *Let $(X_1, d_1), (X_2, d_2), \ldots$ be metric spaces and put $X \coloneqq \prod_n X_n$. For $x \in X$ and $n \in \mathbb{N}$ let x_n be the n-th coordinate of x, so that $x = (x_n)_{n \in \mathbb{N}}$. Define*

$$d(x,y) \coloneqq \sup_n \left(d_n(x_n, y_n) \wedge n^{-1}\right) \quad (x, y \in X).$$

Then d is a metric on X. The topology it determines is the product topology, i.e., if $(x_\alpha)_{\alpha \in A}$ is a net in X and $x \in X$, then

$$d(x_\alpha, x) \xrightarrow{\alpha} 0 \quad \Leftrightarrow \quad d_n\left(x_{\alpha,n}, x_n\right) \xrightarrow{\alpha} 0 \quad (n \in \mathbb{N}).$$

In particular, the coordinate maps $X \longrightarrow X_n$ are continuous.

Proof (I) First, a preparatory observation: If $s, t, u, c \in [0, \infty)$ and $u \leq s + t$ then

$$u \wedge c \leq s \wedge c + t \wedge c.$$

(The right hand member is $s + t$ or it is at least c.)

(II) Consequently, if $x, y, z \in X$, then for every n

$$d_n(x_n, y_n) \wedge n^{-1} \leq d_n(x_n, y_n) \wedge n^{-1} + d_n(y_n, z_n) \wedge n^{-1} \leq d(x,y) + d(y,z).$$

By taking the supremum over n one sees that d satisfies the triangle inequality, and actually is a metric.

(III) Let $(x_\alpha)_{\alpha \in A}$ be a net in X and let $x \in X$.

- If $d(x_\alpha, x) \xrightarrow{\alpha} 0$: Take n in $\mathbb{N}$; we prove $d_n\left(x_{\alpha,n}, x_n\right) \xrightarrow{\alpha} 0$.

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For sufficiently large α we have

$$n^{-1} > d(x_\alpha, x) \geq d_n(x_{\alpha,n}, x_n) \wedge n^{-1};$$

then $d(x_\alpha, x)$ cannot be strictly less than $d_n(x_{\alpha,n}, x_n)$. Thus, $d_n(x_{\alpha,n}, x_n) \leq d(x_\alpha, x)$ if α is large enough, and $d_n(x_{\alpha,n}, x_n) \xrightarrow{\alpha} 0$.

- If $d_n(x_{\alpha,n}, x_n) \xrightarrow{\alpha} 0$ for every n: Let $\varepsilon > 0$. Choose N in $\mathbb{N}$ with $N^{-1} < \varepsilon$. For sufficiently large α,

$$d_n(x_{\alpha,n}, x_n) \leq \varepsilon \quad \mathit{for}\ n = 1, \dots, N.$$

For such α it follows from the definition of d that $d(x_\alpha, x) \leq \varepsilon$. ■

Theorem 2.2 is especially of interest in conjunction with Tychonoff's Theorem 1.22:

2.3 Theorem *A Cartesian product of countably many metrizable compact spaces is metrizable and compact.*

2.4 Examples (**1**) The metrizable compact space $[0,1]^{\mathbb{N}}$ is called the *Hilbert cube*. (The same name is sometimes given to $[-1,1]^{\mathbb{N}}$. Topologically, these spaces are indistinguishable.)

In Theorem 2.8 we prove that every metrizable compact space is homeomorphic with a subspace of $[0,1]^{\mathbb{N}}$.

(**2**) Another special metrizable compact space is $\{0,1\}^{\mathbb{N}}$.

The elements of $\{0,1\}^{\mathbb{N}}$ are the indicators of the subsets of $\mathbb{N}$. With the natural metric on $\{0,1\}$ (the distance between 0 and 1 being 1), Theorem 2.2 yields the metric d on $\{0,1\}^{\mathbb{N}}$:

$$d(\mathbb{1}_A, \mathbb{1}_B) = \sup_n \ |\mathbb{1}_A(n) - \mathbb{1}_B(n)| \wedge n^{-1}.$$

The following is a different "incarnation" of $\{0,1\}^{\mathbb{N}}$. Let $\mathcal{P}(\mathbb{N})$ be the collection of all subsets of $\mathbb{N}$. The formula $A \longmapsto \mathbb{1}_A$ determines a bijection $\mathcal{P}(\mathbb{N}) \longrightarrow \{0,1\}^{\mathbb{N}}$. For $A, B \subset \mathbb{N}$ we have $|\mathbb{1}_A - \mathbb{1}_B| = \mathbb{1}_{A \triangle B}$ where

$$A \triangle B = (A \cup B) \backslash (A \cap B).$$

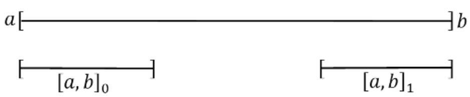
Consequently, we can make a metric δ on $\mathcal{P}(\mathbb{N})$ by

$$\delta(A, B) := \sup\nolimits_{n \in A \triangle B} n^{-1}$$

(the right hand member being 0 if $A \triangle B = \emptyset$); this metric renders $\mathcal{P}(\mathbb{N})$ a compact space and the map $A \longmapsto \mathbb{1}_A$ an isometry.

(**3**) For every bounded closed interval $[a, b]$ we consider two subintervals

$$[a, b]_0 := \left[a, a + \tfrac{b-a}{3}\right] \text{ and } [a, b]_1 := \left[b - \tfrac{b-a}{3}, b\right].$$



Iteration of this construction yields intervals $[a,b]_{00}$, $[a,b]_{001}$, etc.

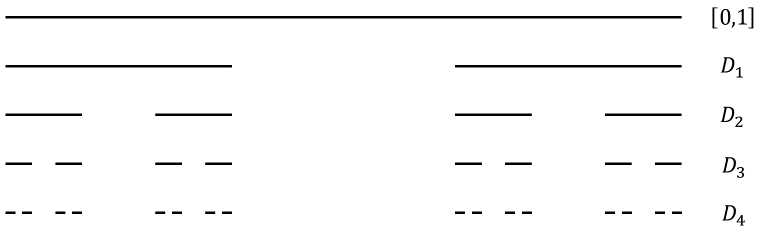
Starting with $I := [0,1]$ we obtain the intervals

$$I_{i_1 \dots i_n} \quad (n \in \mathbb{N};\ i_1, \dots, i_n \in \{0,1\}\ ).$$

Define subsets $D_0,\ D_1, D_2, \dots$ of $[0,1]$ by

$$D_0 := [0,1],\ \ D_n := \textstyle\bigcup_{i_1,\dots,i_n} I_{i_1 \dots i_n} \quad (n \in \mathbb{N}\ ).$$

Then $D_0 \supset\ D_1 \supset D_2 \supset \cdots$. Each D_n is a union of 2^n bounded closed intervals and hence is compact.



The intersection of all these D_n,

$$\mathbb{D} := \textstyle\bigcap_n D_n,$$

is known as the *Cantor set;* it is a compact subset of $[0,1]$.

The Cantor set is homeomorphic to $\{0,1\}^{\mathbb{N}}$. More explicitly: *The function*

$$f\colon x \longmapsto 2 \textstyle\sum_{n=1}^{\infty} x_n 3^{-n} \quad (x \in \{0,1\}^{\mathbb{N}})$$

is a homeomorphism of $\{0,1\}^{\mathbb{N}}$ *onto* $\mathbb{D}$. Proof: The coordinate functions $x \longmapsto x_n$ are continuous. Hence, f, being the uniform limit of the functions

$$x \longmapsto 2 \textstyle\sum_{n=1}^{N} x_n 3^{-n}\ ,$$

is continuous. If $n \in \mathbb{N}$ and $i_1, \dots, i_n \in \{0,1\}$, then $f(\,i_1, \dots, i_n, 0,0,0, \dots)$ is the left end point of the interval $I_{i_1 \dots i_n}$. (Proof by induction.) From this it follows that, if $x = (i_1, i_2, \dots) \in \{0,1\}^{\mathbb{N}}$, then $f(x)$ is the unique element of $\bigcap_n I_{i_1 \dots i_n}$.

Then it is not hard to see that f is a bijection $\{0,1\}^{\mathbb{N}} \longrightarrow \mathbb{D}$. By 1.28(3) f must then be a homeomorphism. (In particular, there exists a bijection $\mathcal{P}(\mathbb{N}) \longrightarrow \mathbb{D}$, so $\mathbb{D}$ is not countable.)

For more about $\mathbb{D}$, see Exercise 2.12.

Among the metrizable compact spaces the Hilbert cube and the Cantor set take special positions. In Theorem 2.7 we will see that every metrizable compact space is homeomorphic to a subspace of the Hilbert cube and to a quotient of the Cantor set. We need some preparation. First, a lemma showing a connection between the two.

2.5 Lemma *There exists a continuous surjection* $\mathbb{D} \longrightarrow [0,1]^{\mathbb{N}}$.

Proof As $\mathbb{D}$ is homeomorphic with $\{0,1\}^{\mathbb{N}}$ we are done if we can find a continuous surjection $\omega\colon \{0,1\}^{\mathbb{N}} \longrightarrow [0,1]^{\mathbb{N}}$.

Choose a bijection $f\colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$. The formula

$$\big(\omega(s)\big)_n := \textstyle\sum_i s_{f(i,n)}\, 2^{-i} \qquad (s \in \{0,1\}^{\mathbb{N}}, n \in \mathbb{N})$$

defines a map $\omega: \{0,1\}^{\mathbb{N}} \longrightarrow [0,1]^{\mathbb{N}}$.

- (ω is surjective:) Let $x_1, x_2, \ldots \in [0,1]$; we make an s in $\{0,1\}^{\mathbb{N}}$ with $\big(\omega(s)\big)_n = x_n$ for each n. Every element of $[0,1]$ can be written as $\sum_{i=1}^{\infty} \xi_i\, 2^{-i}$ with suitable $\xi_1, \xi_2, \ldots$ in $\{0,1\}$. Consequently, for each n there is a sequence $\left(\xi_i^{(n)}\right)_{i\in\mathbb{N}}$ of elements of $\{0,1\}$ with $x_n = \sum_i \xi_i^{(n)}\, 2^{-i}$. Define $s \in \{0,1\}^{\mathbb{N}}$ by

$$s_{f(i,n)} = \xi_i^{(n)} \qquad (i, n \in \mathbb{N}).$$

Then $\big(\omega(s)\big)_n = x_n \quad (n \in \mathbb{N})$.

- (ω is continuous:) For each i and n in $\mathbb{N}$ the (coordinate) function $s \longmapsto s_{f(i,n)}$ is continuous $\{0,1\}^{\mathbb{N}} \longrightarrow \mathbb{R}$. By uniform convergence, for each n the function $s \longmapsto \sum_i s_{f(i,n)} 2^{-i}$ is continuous, i.e. the function $s \longmapsto \big(\omega(s)\big)_n$ is continuous. But that means precisely that ω is continuous (1.20(2)). ■

2.6 Lemma *Let Y be a nonempty closed subset of $\mathbb{D}$. Then there exists a continuous map $\varphi: \mathbb{D} \longrightarrow Y$ with $\varphi(y) = y \ (y \in Y)$.*

Proof (in the language of 2.4(3)) In this proof, $[0,1]$ and the intervals $I_{i_1 \ldots i_n}$ are called "blocks". A block is said to be "useful" if it contains an element of Y. For every $x \in \mathbb{D}$ the blocks that contain x form a decreasing sequence

$$[0,1] \supset I_{i_1} \supset I_{i_1 i_2} \supset \cdots \qquad (*)$$

whose intersection is $\{x\}$.

If B is a block we let B^* be the smallest useful block that has B as a subset. (Such a B^* exists.) If $x \in \mathbb{D}\backslash Y$ the smallest useful block containing x is denoted J_x.

For every useful block B we arbitrarily choose a point $\alpha(B)$ in $B \cap Y$. Now we define $\varphi: \mathbb{D} \longrightarrow Y$ by

$$\begin{cases} \varphi(x) := x \ \text{ if } x \in Y \\ \varphi(x) := \alpha(J_x) \ \text{ if } x \in \mathbb{D}\backslash Y. \end{cases}$$

All we have to prove is that φ is continuous. Actually, we show that

$$|\varphi(x) - \varphi(y)| \le 3|x - y| \qquad (x, y \in \mathbb{D}).$$

Let $x, y \in \mathbb{D}$. In proving the above inequality we may assume $x \neq y$ and $|x - y| < 3^{-1}$. Then there exist a unique N in $\mathbb{N}$ with

$$3^{-N-1} \le |x - y| < 3^{-N}$$

and a unique block B of length 3^{-N} with $x, y \in B$. We distinguish two situations.

(I) (B is useful.) If $x \in Y$, then $\varphi(x) = x$; otherwise $J_x \subset B$ and $\varphi(x) = \alpha(J_x) \in J_x \subset B$. In either case, $\varphi(x) \in B$. By the same token, $\varphi(y) \in B$. But B is an interval with length 3^{-N}, so $|\varphi(x) - \varphi(y)| \le 3^{-N} \le 3 \cdot 3^{-N-1} \le 3|x - y|$.

(II) (B is not useful.) As $x \in \mathbb{D}\backslash Y$, a glance at $(*)$, above, shows the useful blocks having B as a subset are precisely the useful blocks having x as an element. Consequently, B^* is J_x and $\varphi(x) = \alpha(J_x) = \alpha(B^*)$. Similarly, $\varphi(y) = \alpha(B^*)$, so $|\varphi(x) - \varphi(y)| = 0 \le 3|x - y|$. ■

In the following theorem we turn to the theme of the book: The connection between properties of X and those of $C(X)$. Here we view $C(X)$ as a normed vector space under the sup-norm. (See "Conventions and Notations".)

2.7 Theorem *Let X be a nonempty compact Hausdorff space. Then the following statements are equivalent.*

(α) *X is metrizable.*

(β) *X is homeomorphic to a subspace of $[0,1]^{\mathbb{N}}$.*

(γ) *There exists a continuous surjection $\mathbb{D} \longrightarrow X$.*

(δ) *$C(X)$ is separable relative to the topology induced by the norm $\|.\|_\infty$.*

(ε) *There is a countable collection of continuous functions on X that separates the points of X.*

Proof of the implications (α) ⇒ (ε) ⇒ (β) ⇒ (α) and (β) ⇒ (γ) ⇒ (δ) ⇒ (ε).

(α) ⇒ (ε) Choose a metric d for X. For $a \in X$ let a^* be the function $x \longmapsto d(a,x)$ $(x \in X)$. By 2.1(2), X has a countable dense subset A. Then $\{a^*: a \in A\}$ is a countable collection of continuous functions; we claim that it separates the points of X. Indeed, let $x, y \in X$ be so that $a^*(x) = a^*(y)$ for all $a \in A$. There is a sequence $(a_n)_n$ in A with $a_n \longrightarrow x$. For every n we have $d(a_n, x) = d(a_n, y)$, so $a_n \longrightarrow y$. Then $x = y$.

(ε) ⇒ (β) It follows from (ε) that there is a sequence $(f_n)_n$ of continuous functions $X \longrightarrow [0,1]$ that separates the points of X. Then the map

$$f: x \longmapsto (f_1(x), f_2(x), \dots) \quad (x \in X)$$

is a continuous injection $X \longrightarrow [0,1]^{\mathbb{N}}$. According to 1.28(3) $f(X)$ is homeomorphic to X.

(β) ⇒ (α) By theorem 2.2, $[0,1]^{\mathbb{N}}$ is metrizable. Then so is every subspace of $[0,1]^{\mathbb{N}}$.

(β) ⇒ (γ) We may as well assume that X actually *is* a subspace of $[0,1]^{\mathbb{N}}$. Then, by compactness, X is closed.

By Lemma 2.5 there exists a continuous surjection $\omega: \mathbb{D} \longrightarrow [0,1]^{\mathbb{N}}$. Then $\omega^{-1}(X)$ is a nonempty closed subset of $\mathbb{D}$, and Lemma 2.6 provides a continuous surjection $\varphi: \mathbb{D} \longrightarrow \omega^{-1}(X)$. The composition $\omega \circ \varphi$ is a continuous surjection $\mathbb{D} \longrightarrow X$.

(γ) ⇒ (δ) Choose a continuous surjection $\tau: \mathbb{D} \longrightarrow X$. This τ induces a map $T: C(X) \longrightarrow C(\mathbb{D})$ by

$$T(f) := f \circ \tau \quad \big(f \in C(X)\big).$$

The map T is linear and thanks to the surjectivity of τ it satisfies

$$\|T(f)\|_\infty = \|f\|_\infty \quad \big(f \in C(X)\big).$$

Consequently, T is a isometry relative to the metrics determined by the sup-norms. Hence, we only have to prove that $C(\mathbb{D})$ is separable. We use the notations of 2.4(3).

For $n \in \mathbb{N}$, let C_n be the space of all functions on D_n that are constant on each of the intervals $I_{i_1 \dots i_n}$. There are precisely 2^n such intervals, so C_n is a 2^n-dimensional vector space. If we put $C_n(\mathbb{D}) := \{g|\mathbb{D} : g \in C_n\}$, then $C_n(\mathbb{D})$ is a finite dimensional, and in particular separable, linear subspace of $C(\mathbb{D})$. We prove that $\bigcup_n C_n(\mathbb{D})$ is dense in $C(\mathbb{D})$.

Let $f \in C(\mathbb{D})$, $\varepsilon > 0$; we look for a g in $\bigcup_n C_n(\mathbb{D})$ with $\|f - g\|_\infty \le \varepsilon$. By the uniform continuity of f there is an n with

$$x, y \in \mathbb{D}, \qquad |x-y| \le 3^{-n} \quad \Longrightarrow \quad |f(x)-f(y)| \le \varepsilon.$$

For $x \in \mathbb{D}$ (and this n) let x^* be the left end point of the interval $I_{i_1 \dots i_n}$ that contains x; then $x^* \in \mathbb{D}$ and $|x - x^*| \le 3^{-n}$, so

$$|f(x) - f(x^*)| \le \varepsilon.$$

Consequently, if we define a function g on $\mathbb{D}$ by

$$g(x) \coloneqq f(x^*) \qquad (x \in \mathbb{D})$$

then $|f(x) - g(x)| \le \varepsilon$ for all x in $\mathbb{D}$, so $\|f - g\|_\infty \le \varepsilon$. Moreover, $g \in C_n(\mathbb{D})$.

(δ) ⇒ (ε) Let G be a countable dense set in $C(X)$. If $x, y \in X$ and $x \ne y$, there exists a continuous function f on X with $f(x) = 1$, $f(y) = 0$ (Urysohn, 1.28(5)). Choosing g in G with $\|f - g\|_\infty < ½$ we have $g(x) \ne g(y)$. ■

2.8 Corollary *Let X be a compact Hausdorff space and let Y be a Hausdorff space such that there exists a continuous surjection $X \longrightarrow Y$.* (Then Y is compact; see 1.28(1).) *If X is metrizable, then so is Y.* ■

2.9 Exercise (referring to 2.2) Let A be an uncountable set. Prove that the spaces $\{0,1\}^A$ and $[0,1]^A$ under the product topology are not metrizable. (In a metrizable space, a singleton subset is an intersection of countably many open sets.)

2.10 Exercise (referring to 2.2) Reconsidering the proof one sees that in the definition of d the term "n^{-1}" may be replaced by "a_n" where $(a_n)_n$ is any sequence in $(0, \infty)$ that converges to 0. Other options are available.

In the situation of 2.2, define

$$\delta(x, y) \coloneqq \textstyle\sum_n d_n(x_n, y_n) \wedge 2^{-n} \qquad\qquad (x, y \in X).$$

Show that δ is a metric determining the product topology. (The inequality in (I) of the proof of 2.2 may be useful.)

2.11 Exercise (Refers to 2.4(2).) On $\mathcal{P}(\mathbb{N})$ we impose the topology induced by δ.

(1) Let $(A_n)_n$ be a sequence in $\mathcal{P}(\mathbb{N})$ with $A_1 \subset A_2 \subset \cdots$ and $A \coloneqq \bigcup_n A_n$. Prove $A_n \longrightarrow A$.

(2) Let $(A_n)_n$ be a disjoint sequence in $\mathcal{P}(\mathbb{N})$. Prove $A_n \longrightarrow \emptyset$.

(3) Let $P \in \mathcal{P}(\mathbb{N})$. Prove that the maps $A \longmapsto A \cap P$ and $A \longmapsto A \cup P$ of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$ are continuous.

(4) Let $f\colon \mathbb{N} \longrightarrow \mathbb{N}$. Prove that the map $A \longmapsto f^{-1}(A)$ is continuous. (Use convergence, not the metric.)

2.12 Exercise (Refers to 2.4(3))

(1) Show that ⅓ and ¼ lie in $\mathbb{D}$.

(2) $\mathbb{D}_n$ contains no interval with a length exceeding 3^{-n}. It follows that (the closed set) $\mathbb{D}$ contains no interval at all.

Extra: The Cantor Set and Variations

The Cantor set $\mathbb{D}$ was invented in 1874 by Henry John Stephen Smith and independently by Georg Cantor in 1883 who used it to make an injective map of $\mathcal{P}(\mathbb{N})$ into $\mathbb{R}$, showing that $\mathbb{R}$ is not countable. It has many curious properties.

2E.1 Exercise 2.12(2) shows that $\mathbb{D}$ contains no intervals: $\mathbb{D}$ is "tenuous", so to speak. On the other hand, the set

$$\mathbb{D} + \mathbb{D} := \{x + y : x, y \in \mathbb{D}\}$$

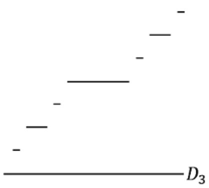
covers all of [0,2]. To see this, take z in [0,2]. Every element of [0,1] can be written as

$$0.s_1 s_2 s_3 \ldots \quad \text{or} \quad \textstyle\sum_{n=1}^{\infty} s_n \cdot 10^{-n}$$

for certain $s_1, s_2, \ldots \in \{0,1,2,\ldots,9\}$. Similarly, we have $\frac{1}{2}z = \sum_{n=1}^{\infty} z_n \cdot 3^{-n}$ with suitable $z_1, z_2, \ldots \in \{0,1,2\}$. For each n, choose x_n and y_n in $\{0,1\}$ such that $z_n = x_n + y_n$. Putting $x := 2\sum x_n \cdot 3^{-n}$ and $y := 2\sum y_n \cdot 3^{-n}$ we get $x + y = z$, whereas $x, y \in \mathbb{D}$ according to 2.6.

2E.2 Related to the Cantor set is the *Cantor function*.

Imagine constructing $\mathbb{D}$ by successively making the sets $D_1, D_2, \ldots$. While this is going on we make a function u on [0,1] as follows. In the first step the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ is omitted from [0,1]; we give u the value $\frac{1}{2}$ at all points of $\left(\frac{1}{3}, \frac{2}{3}\right)$. In the second step the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ are omitted; we give u the value $\frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9}\right)$ and the value $\frac{3}{4}$ on $\left(\frac{7}{9}, \frac{8}{9}\right)$. We continue in this way.



We obtain an increasing function, defined on $[0,1]\backslash\mathbb{D}$. On each of the intervals omitted in the construction of $\mathbb{D}$ this function is constant, its value being a rational number whose denominator is a power of 2. It is not difficult to extend this function to an increasing surjection $[0,1] \longrightarrow \mathbb{D}$, the Cantor function. (It's graph is known as the *Devil's Staircase*.) It is continuous and maps $\mathbb{D}$ *onto* [0,1].

2E.3 Let $r_1, r_2, \ldots \in (0,1)$. At the n^{th} stage of the construction of $\mathbb{D}$ we had 2^n disjoint intervals at our disposal and from each we omitted the middle third part. Now we can carry out a similar construction but omitting not a fraction $\frac{1}{3}$, but a fraction r_1 in the first step, a fraction r_2 in the second, and so on. We obtain sets $[0,1] = E_0 \supset E_1 \supset E_2 \supset \cdots$. Each

E_n is a union of 2^n closed intervals of the same length, l_n, say; then $l_0 = 1$, $l_1 = \frac{1}{2}(1 - r_1), l_2 = \frac{1}{2}(1 - r_1)l_1, \ldots.$

This construction leads to a set $E \coloneqq \bigcap_n E_n$ that is homeomorphic with $\mathbb{D}$ but may have different geometrical properties. For instance, its Lebesgue measure $\lambda(E)$ is

$$\prod_{n=1}^{\infty} r_n$$

which is 0 for the Cantor set but positive if $\sum r_n < \infty$.

We can also make a "distorted" Cantor function, a function u: $[0,1] \longrightarrow [0,1]$ that is continuous, increasing and maps E onto $[0,1]$. Trivially this function is differentiable at every point of $[0,1]\setminus E$; less trivially, at almost every point of E (in the Lebesgue sense) it is differentiable and its derivative has the value $\lambda(E)^{-1}$.

2E.4 Theorem 2.7 says something of $\mathbb{D}$ as a topological space. For another topological property of $\mathbb{D}$ we need a technical term: A topological space X is said to be "zerodimensional" if the subsets of X that are both closed and open form a base for the topology. Thus, zerodimensionality of X is equivalent to the following:

If $U \subset X$ is open and $a \in U$, then there is a
set W, closed and open, with $a \in W \subset U$.

$\mathbb{D}$ is zerodimensional. Indeed, it is not hard to see that a subset of $\mathbb{R}$ is zerodimensional if and only if it contains no intervals.

The following theorem obtains: *Every metrizable compact space that is zerodimensional and has no isolated points is homeomorphic with* $\mathbb{D}$.

Historical references for the Cantor set

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