Chapter 2
Mappings on Metric Spaces

In process of the development of general topology, because of the participation of various branches of mathematics and the driving of internal problems, a vast diversity of classes of topological spaces were evolved. Detailed analysis and extensive classification of these objects appear in different poses and with different expressions are the necessary conditions for the existence of general topology and form its main internal tasks [31]. At the 1961 Prague Topological Symposium, Alexandroff [3] raised two kinds of questions:

Question 2.0.1 (*[3]*) Which spaces can be represented as images of "nice" (e.g. metric or zero-dimensional, etc.) spaces under "nice" continuous mappings?

Question 2.0.2 (*[3]*) Which spaces can be mapped onto "nice" spaces by "nice" mappings?

The idea of classifying spaces by mappings caused by *Alexandroff's questions* leads to a method of mutual classifications of spaces and mappings. Alexandroff [4] and Arhangel'skiĭ [31] thought that the essence of this method is the following three fundamental problems which are closely tied up with one another:

Question 2.0.3 (*[31]*) Under what circumstances can each space of a given class $\mathscr{A}$ be mapped onto a space of a class $\mathscr{B}$ by means of a mapping belonging to a class $\mathscr{F}$?

Question 2.0.4 (*[31]*) If the class $\mathscr{F}(\mathscr{A})$ of spaces are images of spaces of type $\mathscr{A}$ by mappings of type $\mathscr{F}$, then what internal properties can the spaces belonging to $\mathscr{F}(\mathscr{A})$ have?

Question 2.0.5 (*[31]*) Let $\mathscr{F}(\mathscr{A}, \mathscr{B})$ denote the class of mappings whose domain is a member of the class $\mathscr{A}$ and whose range is a member of the class $\mathscr{B}$. Let $\mathscr{H}$ be some other class of mappings. What are the properties of mappings of the class $\mathscr{F}(\mathscr{A}, \mathscr{B}) \cap \mathscr{H}$?

These general formulations involve, in particular, the following question:

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Question 2.0.6 (*[31]*) What topological properties are preserved by these and other mappings?

The significance of the idea of mutual classifications of spaces and mappings, i.e. *Alexandroff-Arhangel'skiĭ's questions*, is that to reveal the internal properties of various classes of topological spaces by mappings, and to use mappings as a link connecting the multifarious topological spaces in one. Practice shows this principle not only instilled fresh blood to general topology in many classic topics, but also produced many new research directions, and brought the prosperity of general topology from the late 60s to the whole 80s in the 20th century [4, 238]. As a supplement, readers can read the survey papers [268, 298].

Which classes of spaces can be regarded as the "nice" classes of spaces in Alexandroff's questions?

Which classes of mappings can be regarded as the "nice" classes of mappings in Alexandroff's questions?

We start from the class of metrizable spaces, and to show that quotient mappings, pseudo-open mappings, open mappings and closed mappings really satisfy the requirements of being "nice" mappings. In this chapter, we follow the idea of Alexandroff and look for some important intrinsic properties of images or preimages of metric spaces.

All the mappings in Sect. 2.1 are assumed to be continuous mappings, and all the mappings in Sects. 2.2–3.9 are continuous onto mappings.

2.1 Classes of Mappings

In this section, we give definitions for certain classes of mappings and investigate their basic properties. The relationships between different mappings are very extensive. Here we only describe several properties needed in the subsequent sections. The mapping lemma (see Proposition 2.1.12) reflects the relationships between the transformations of mappings.

Definition 2.1.1 Let $f : X \to Y$ be a mapping.

(1) f is called a *quotient mapping* [50] if, for each $U \subset Y$ with $f^{-1}(U) \in \tau(X)$, $U \in \tau(Y)$.
(2) f is called a *pseudo-open mapping* [26] if, for each $y \in Y$ with $f^{-1}(y) \subset V \in \tau(X)$, $y \in f(V)^\circ$.
(3) f is called a *countably bi-quotient mapping* [424] if, for each $y \in Y$ and each countable open family $\mathscr{U}$ in X covering $f^{-1}(y)$, there is $P \in \mathscr{U}^F$ such that $y \in f(P)^\circ$.
(4) f is called an *open mapping* [42] if $f(V) \in \tau(Y)$ whenever $V \in \tau(X)$.
(5) f is called a *closed mapping* [196] if $f(F) \in \tau^c(Y)$ whenever $F \in \tau^c(X)$.

We first point out two basic relationships between mappings. Let $f : X \to Y$ be a mapping. If $A \subset X$, $B \subset Y$, then

(1) $f^{-1}(B) \subset A \Leftrightarrow B \subset Y - f(X - A)$;
(2) $f(A \cap f^{-1}(B)) = f(A) \cap B$.

Proposition 2.1.2 *(1) Every open mapping is a countably bi-quotient mapping, every countably bi-quotient mapping is a pseudo-open mapping and every pseudo-open mapping is a quotient mapping.*
(2) Every closed mapping is a pseudo-open mapping.

Proof We only need to prove that every closed mapping is a pseudo-open mapping and every pseudo-open mapping is a quotient mapping.

Let $f : X \to Y$ be a mapping.

Assume that f is a closed mapping. If $y \in Y$ and $f^{-1}(y) \subset V \in \tau(X)$, then $y \in Y - f(X - V) \subset f(V)$, so $y \in f(V)^{\circ}$. Thus f is a pseudo-open mapping.

Assume that f is a pseudo-open mapping. If $U \subset Y$ and $f^{-1}(U) \in \tau(X)$, then for any $y \in U, f^{-1}(y) \subset f^{-1}(U)$, so $y \in U^{\circ}$, and hence $U \in \tau(Y)$. Thus f is a quotient mapping. ■

There exist examples to show that not every pseudo-open mapping is a composition of an open mapping and a closed mapping [38].

Below we consider mappings with some additional conditions on fibers.

Definition 2.1.3 Let $f : X \to Y$ be a mapping.

(1) f is called a *compact mapping* [471] (resp. a *countably compact mapping*, an *L-mapping*, an *s-mapping*) if, for every $y \in Y, f^{-1}(y)$ is a compact (resp. countably compact, Lindelöf, separable) set in X.
(2) f is called a *peripherally compact mapping* [471] (resp. *peripherally countably compact mapping*, *peripheral L-mapping*) if, for each $y \in Y$, $\partial f^{-1}(y)$ is a compact (resp. countably compact, Lindelöf) set in X.
(3) f is called a *perfect mapping* [471] if f is a closed compact mapping.
(4) f is called a *quasi-perfect mapping* if f is a closed countably compact mapping.

Obviously, every quasi-perfect mapping is a countably bi-quotient mapping.

Definition 2.1.4 Let $f : X \to Y$ be a mapping.

(1) f is called a *compact-covering mapping* [331] if, for every $K \in \mathscr{K}(Y)$, there is $L \in \mathscr{K}(X)$ such that $f(L) = K$.
(2) f is called a *sequence-covering mapping* [424] if, for every $S \in \mathscr{S}(Y)$, there is $L \in \mathscr{K}(X)$ such that $f(L) = S$.
(3) f is called a *sequentially quotient mapping* [64] if, for every $S \in \mathscr{S}(Y)$, there is $L \in \mathscr{S}(X)$ such that $f(L)$ is a subsequence of S.

We should note that in the different literatures, the definitions of mappings crown to "sequence-covering mappings" in the name may be different [167].

Remark Suppose ϕ is a mapping property. An image (resp. a preimage) of a space under a mapping having the property ϕ is briefly called a ϕ image (resp. preimage) of a space. For example, an image of a space under a compact mapping is called a compact image of a space. A ϕ image of a space and the range space is regular (resp. completely regular, normal) is called a regular (resp. completely regular, normal) ϕ image of a space. The definition of a regular (resp. completely regular, normal) ϕ preimage of a space is similar.

In the following, we give characterizations of some classes of mappings.

Proposition 2.1.5 ([26]) *Let $f : X \to Y$ be a mapping. Then f is a pseudo-open mapping if and only if for every $B \subset Y$, f_B is a quotient mapping.*

Proof Suppose f is a pseudo-open mapping. Then for every $B \subset Y$, $f_B : f^{-1}(B) \to B$ is a pseudo-open mapping, and hence f_B is a quotient mapping.

Conversely, for each $y \in Y$ with $f^{-1}(y) \subset V \in \tau(X)$, if $y \notin f(V)^{\circ}$, let $Z = Y - f(V)$; if $y \in \overline{Z} - Z$, let $F = Z \cup \{y\}$, then Z is not closed in F. Since f_F is a quotient mapping, $f^{-1}(Z)$ is not a closed set in $f^{-1}(F)$, so there is $x \in \overline{f^{-1}(Z)} \cap f^{-1}(y)$, and hence $y = f(x) \in f(\overline{f^{-1}(Z)})$, as a consequence, $\overline{Z} \subset f(\overline{f^{-1}(Z)})$. Thus,

$$y \in \overline{Y - f(V)} \subset f(\overline{f^{-1}(Y - f(V))}) \subset f(X - V),$$

it follows that $f^{-1}(y) \cap (X - V) \neq \varnothing$, a contradiction. Therefore f is a pseudo-open mapping. ■

A mapping satisfying the equivalent condition of being a pseudo-open mapping in the above proposition is called a *hereditarily quotient mapping* [26].

Proposition 2.1.6 ([64]) *Let $f : X \to Y$ be a mapping. Then f is a sequentially quotient mapping if and only if for each $F \subset Y$, F is a sequentially closed (resp. sequentially open) set in Y whenever $f^{-1}(F)$ is a sequentially closed (resp. sequentially open) set in X.*

Proof Suppose f is a sequentially quotient mapping. Suppose $F \subset Y$ and $f^{-1}(F)$ is a sequentially closed set in X. If F is not a sequentially closed set in Y, then F contains a sequence $\{y_n\}$ such that $y_n \to y \in Y - F$, and hence there exist a sequence $\{x_i\}$ in X and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $x_i \in f^{-1}(y_{n_i})$ and $x_i \to x \in f^{-1}(y)$. Then $x \in f^{-1}(F)$, and hence $y \in F$, a contradiction.

Conversely, let $\{y_n\}$ be a sequence in Y such that $y_n \to y$. We may assume $y_n \neq y$ for every $n \in \mathbb{N}$. Since $\{y_n : n \in \mathbb{N}\}$ is not a sequentially closed set in Y, $\cup\{f^{-1}(y_n) : n \in \mathbb{N}\}$ is not a sequentially closed set in X. However, $f^{-1}(y) \cup (\cup\{f^{-1}(y_n) : n \in \mathbb{N}\})$ is a sequentially closed set in X, so there is $\{p_k\} \subset \cup\{f^{-1}(y_n) : n \in \mathbb{N}\}$ such that $p_k \to x \in f^{-1}(y)$. Since each $f^{-1}(y_n)$ is a sequentially closed set, there are at most finitely many p_k belong to any fixed $f^{-1}(y_n)$, so we can pick a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ and $p_{k_i} \in f^{-1}(y_{n_i})$. Let $L = \{x\} \cup \{p_{k_i} : i \in \mathbb{N}\}$. Then $L \in \mathscr{S}(X)$ and $f(L)$ contains a subsequence of $\{y\} \cup \{y_n : n \in \mathbb{N}\}$. Thus f is a sequentially quotient mapping. ■

Proposition 2.1.7 ([119]) *Let $f : X \to Y$ be a mapping. Then f is a closed mapping if and only if for each $y \in Y$ with $f^{-1}(y) \subset U \in \tau(X)$, there is $V \in \tau(Y)$ such that $y \in V$ and $f^{-1}(V) \subset U$.*

Proof Suppose f is a closed mapping. For each $y \in Y$ with $f^{-1}(y) \subset U \in \tau(X)$, take $V = Y - f(X - U)$. Then $y \in V \in \tau(Y)$ and $f^{-1}(V) \subset U$.

Conversely, suppose $F \in \tau^c(X)$. Then for any $y \in Y - f(F)$, $f^{-1}(y) \subset X - F \in \tau(X)$ holds, and hence there is $V \in \tau(Y)$ such that $y \in V$ and $f^{-1}(V) \subset X - F$, so $V \cap f(F) = \varnothing$, and it follows that $f(F) \in \tau^c(Y)$. Thus f is a closed mapping. ∎

Proposition 2.1.8 ([32]) *Let $f : X \to Y$ and $g : X \to Z$ be mappings. If f is a perfect mapping, then $f \triangle g$ is a perfect mapping.*

Proof Let $h = f \triangle g$. Then h can be expressed as the composition of the following two mappings:

$$\mathrm{id}_X \triangle g : X \to X \times Z, \quad f \times \mathrm{id}_Z : X \times Z \to Y \times Z.$$

Since id_X separates points from closed sets in X, by the diagonal theorem $\mathrm{id}_X \triangle g$ is a closed embedding, hence a perfect mapping. Because both f and id_Z are perfect mappings, $f \times \mathrm{id}_Z$ is a perfect mapping. So h is a perfect mapping. ∎

Corollary 2.1.9 *Assume that Φ is a closed hereditary and finitely productive topological property. If there exist a one-to-one mapping $f : X \to Y$ and a perfect mapping $g : X \to Z$, where Y and Z have Φ, then X also has Φ.*

Proof Put $h = f \triangle g : X \to Y \times Z$. By Proposition 2.1.8, h is a perfect mapping. Since h is a one-to-one mapping, h is a closed embedding, so X also has Φ. ∎

Proposition 2.1.10 ([70]) *If X is a compact space, then for any space Y, $\pi_1 : Y \times X \to Y$ is a perfect mapping.*

Proof Obviously, π_1 is a compact mapping. Let $F \in \tau^c(Y \times X)$. If $y \notin \pi_1(F)$, then $(\{y\} \times X) \cap F = \varnothing$, so there is an open subset V of Y containing y such that $(V \times X) \cap F = \varnothing$, and hence $V \cap \pi_1(F) = \varnothing$. As a consequence, $\pi_1(F) \in \tau^c(Y)$. Thus π_1 is a closed mapping. ∎

In the second part of this section, we discuss more precise relationships between different classes of mappings under certain conditions. We first introduce two concepts.

Definition 2.1.11 (1) A space X is called a *strongly Fréchet–Urysohn space* [424] if, for every decreasing sequence $\{A_n\}$ of sets in X with $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there is $x_n \in A_n$ for each $n \in \mathbb{N}$ such that $x_n \to x$.

(2) X is called a *k-space* [136] if, for each $A \subset X$, $A \in \tau^c(X)$ provide $K \cap A \in \tau^c(X)$ for any $K \in \mathscr{K}(X)$.

Obviously, first countable space $\Rightarrow$ strongly Fréchet–Urysohn space $\Rightarrow$ Fréchet–Urysohn space $\Rightarrow$ sequential space $\Rightarrow$ k-space. By the definitions, strongly Fréchet–Urysohn spaces are additive and hereditary; k-spaces are additive, open hereditary and closed hereditary.

Proposition 2.1.12 (The mapping lemma) *Let $f : X \to Y$ be a mapping.*

(1) If Y is a k-space and f is a compact-covering mapping, then f is a quotient mapping [331].
(2) If Y is a sequential space and f is a sequentially quotient mapping or a sequence-covering mapping, then f is a quotient mapping [64, 424].
(3) If Y is a Fréchet–Urysohn space and f is a quotient mapping, then f is a pseudo-open mapping [26, 133].
(4) If Y is a strongly Fréchet–Urysohn space and f is a quotient mapping, then f is a countably bi-quotient mapping [336].
(5) If X is a sequential space and f is a quotient mapping, then f is a sequentially quotient mapping [64].

Proof (1) Suppose $F \subset Y$ and $f^{-1}(F) \in \tau^c(X)$. For each $K \in \mathscr{K}(Y)$, there is $L \in \mathscr{K}(X)$ such that $f(L) = K$. Since $f^{-1}(F) \cap L \in \mathscr{K}(X)$, $F \cap K = f(f^{-1}(F) \cap L) \in \mathscr{K}(Y) \subset \tau^c(Y)$, so $F \in \tau^c(Y)$. Thus f is a quotient mapping.

(2) If f is a sequence-covering mapping, then by replacing $\mathscr{K}(Y)$ with $\mathscr{S}(Y)$, the proof for f being a quotient mapping is similar to that in (1).
Let f be a sequentially quotient mapping. If $F \subset Y$ and $f^{-1}(F) \in \tau^c(X)$, then $f^{-1}(F)$ is a sequentially closed set in X. By Proposition 2.1.6, F is a sequentially closed set in Y, so $F \in \tau^c(Y)$, thus f is quotient mapping.

(3) Suppose $y \in Y$ and $f^{-1}(y) \subset V \in \tau(X)$. If $y \in Y - f(V)^\circ$, then $y \in \overline{Y - f(V)}$, thus there is a sequence $\{y_n\} \subset Y - f(V)$ such that $y_n \to y$. Let

$$Z = \{y_n : n \in \mathbb{N}\}, \quad F = f^{-1}(Z).$$

Then $\overline{F} \subset f^{-1}(\overline{Z}) = F \cup f^{-1}(y)$. Since $f^{-1}(y) \subset V$ and $V \cap F = \varnothing$, $f^{-1}(y) \cap \overline{F} = \varnothing$, so $F \in \tau^c(X)$, and hence $f^{-1}(Y - Z) = X - F \in \tau(X)$, thus $Y - Z \in \tau(Y)$, a contradiction. As a consequence, $y \in f(V)^\circ$, so f is a pseudo-open mapping.

(4) If f is not a countably bi-quotient mapping, then there exist $y \in Y$ and a countable family $\{U_i\}_{i\in\mathbb{N}}$ of open subsets in X covering $f^{-1}(y)$ such that for every $n \in \mathbb{N}$, $y \in Y - f(\bigcup_{i\leqslant n} U_i)^\circ = \overline{Y - f(\bigcup_{i\leqslant n} U_i)}$. Since Y is a strongly Fréchet–Urysohn space, there is $y_n \in Y - f(\bigcup_{i\leqslant n} U_i)$ for each $n \in \mathbb{N}$ such that $y_n \to y$. Let $E = \{y_n : n \in \mathbb{N}\}$ and $F = \{y\} \cup E$. By (3) and Proposition 2.1.5, f_F is a quotient mapping. Since E is not a closed subset of F, $f^{-1}(E)$ is not a closed subset of $f^{-1}(F)$, so there is $x \in \overline{f^{-1}(E)} \cap f^{-1}(y)$, and hence there is $m \in \mathbb{N}$ such that $x \in U_m$. It follows that there is $k \geqslant m$ such that $f^{-1}(y_k) \cap U_m \neq \varnothing$, thus $y_k \in f(U_m)$, a contradiction. So f is a countably bi-quotient mapping.

(5) If $F \subset Y$ and $f^{-1}(F)$ is a sequentially closed set in X, then $f^{-1}(F)$ is a closed set in X, so F is a closed set in Y, and hence F is a sequentially closed set in Y. By Proposition 2.1.6, f is a sequentially quotient mapping. ■

Proposition 2.1.13 *Let $f : X \to Y$, where X is of the point G_δ-property. If one of the following is true, then f is a sequentially quotient mapping.*

(1) f is a sequence-covering mapping.
(2) f is a closed mapping and X is a regular space [291].

Proof (1) If $S \in \mathscr{S}(Y)$, then there is $K \in \mathscr{K}(X)$ such that $f(K) = S$. Denote $S = \{y\} \cup \{y_n : n \in \mathbb{N}\}$, where $y_n \to y$. Pick $x_n \in f^{-1}(y_n) \cap K$ for each $n \in \mathbb{N}$. Since X is a space of the point G_δ-property, by Theorem 1.7.7, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x \in f^{-1}(y)$. Thus f is a sequentially quotient mapping.

(2) Let $\{y_n\}$ be any nontrivial sequence converging to y in Y. Pick $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$. Since f is a closed mapping, each subsequence of $\{x_n\}$ has an accumulation point. Let x be an accumulation point of $\{x_n\}$ and let $\{V_n\}$ be a sequence of open neighborhoods of x, such that, $\overline{V}_{n+1} \subset V_n$ and $\{x\} = \bigcap_{n\in\mathbb{N}} V_n$. Take a subsequence $\{t_k\}$ of $\{x_n\}$ such that $t_k \in V_k$. Let p be an accumulation point of any subsequence of $\{t_k\}$. Then $p \in \bigcap_{k\in\mathbb{N}} \overline{V}_k$, so $p = x$, which means that x is the only accumulation point of $\{t_k\}$, and hence $\{t_k\}$ is a convergent subsequence of $\{x_n\}$. Thus f is a sequentially quotient mapping. ■

Lemma 2.1.14 ([330, 336]) *Let $f : X \to Y$ be a closed mapping, where X is a normal space or a countably paracompact space. If Y is a q-space or a strongly Fréchet–Urysohn space, then f is a peripherally countably compact mapping.*

Proof If there is $y \in Y$ such that $\partial f^{-1}(y)$ is not a countably compact set in X, then $\partial f^{-1}(y)$ contains a discrete countable closed subspace $\{x_i : i \in \mathbb{N}\}$. Since X is a normal space or a countably paracompact space, there is a locally finite family $\{V_i : i \in \mathbb{N}\}$ of open subsets in X such that $x_i \in V_i$ (see Theorem A.1.11 in Appendix A).

(1) When Y is a q-space, let g be a q-function on Y. Since $x_i \in \partial f^{-1}(y)$, we can select by the inductive method

$$z_1 \in V_1 \cap f^{-1}(g(1,y)) - f^{-1}(y), \text{ and}$$
$$z_{i+1} \in V_{i+1} \cap f^{-1}(g(i+1,y)) - f^{-1}(\{y, f(z_1), \dots, f(z_i)\}).$$

Since $f(z_i) \in g(i,y)$, $\{f(z_i)\}$ has an accumulation point. But $z_i \in V_i$ and f is a closed mapping, $\{f(z_i)\}$ has no accumulation point, a contradiction.

(2) Suppose Y is a strongly Fréchet–Urysohn space. Since $x_i \in V_i \cap \partial f^{-1}(y)$,

$$y \in \overline{f(V_i) - \{y\}} \subset \overline{f(\bigcup_{j\geqslant i} V_j) - \{y\}}.$$

So there is $y_i \in f(\bigcup_{j\geqslant i} V_j) - \{y\}$ for each $i \in \mathbb{N}$ such that $y_i \to y$. Since $\{V_i : i \in \mathbb{N}\}$ is a locally finite family and f is a closed mapping, there is a subsequence $\{y_{i_k}\}$ of $\{y_i\}$ such that $\{y_{i_k} : k \in \mathbb{N}\}$ is a discrete closed set in Y, a contradiction. ■

Lemma 2.1.15 ([330]) *If $f : X \to Y$ is an onto mapping, then there is a closed subspace Z of X, such that, for each $y \in Y$, $(f_{|Z})^{-1}(y)$ is either a singleton or the nonempty set $\partial f^{-1}(y)$.*

Proof For each $y \in Y$, pick $p_y \in f^{-1}(y)$. Let

$$Z = \cup\{\partial f^{-1}(y) : \partial f^{-1}(y) \neq \varnothing\} \cup \{p_y : \partial f^{-1}(y) = \varnothing\}.$$

Since

$$\begin{aligned} X - Z = \; & \cup \{f^{-1}(y)^{\circ} : \partial f^{-1}(y) \neq \varnothing\} \bigcup \\ & \cup \{f^{-1}(y)^{\circ} - \{p_y\} : \partial f^{-1}(y) = \varnothing\}, \end{aligned}$$

$Z \in \tau^c(X)$. Obviously, for each $y \in Y$, $(f_{|Z})^{-1}(y)$ is either a singleton or the nonempty set $\partial f^{-1}(y)$. ■

Proposition 2.1.16 ([330]) *Let X be a paracompact space. If $f : X \to Y$ is a closed onto mapping, then f is a compact-covering mapping.*

Proof For each nonempty compact set K in Y, by Lemma 2.1.14, f_K is a peripherally compact mapping. By Lemma 2.1.15, there is a closed subspace L of $f^{-1}(K)$ such that $f_{K|L}$ is a perfect onto mapping, so $L \in \mathscr{K}(X)$ and $f(L) = K$. ■

Since every Lindelöf regular space is a paracompact space, by Proposition 2.1.16, every closed onto L-mapping of a regular space is a compact-covering mapping.

Question 2.1.17 (*[341]*) Characterize the class of spaces Y such that every closed onto mapping $f : X \to Y$ is a countably bi-quotient mapping.

Lemma 2.1.14 derives a general question:

Question 2.1.18 (*[460]*) Let $f : X \to Y$ be a closed mapping. Find additional conditions for X or Y, such that, with these conditions, $\partial f^{-1}(y)$ has nice properties for each $y \in Y$.

Theorem 2.2.2, Lemmas 2.7.20 and 2.7.21, Theorems 3.4.16, 3.8.16 and Corollary 3.8.17 etc. are all associated with this problem.

2.2 Perfect Mappings

The importance of perfect mappings in the mapping theory, same as the role of compact spaces in general topology, is without rebuke. The main purpose of this section is to give characterizations of images and preimages of metric spaces under perfect mappings. To this end, we introduce a series of generalized metric spaces, such as p-spaces and so on. We also introduce a metrization theorem of adjunction spaces obtained by Y. Liu and L. Liu [315] and other relevant metrization theorems.

Theorem 2.2.1 ([363, 441]) *Perfect mappings preserve metrizability.*

Proof Let $f : X \to Y$ be a perfect mapping, where X is a metrizable space. In virtue of the Stone theorem, X is a paracompact space, so Y is a paracompact space (see Corollary A.1.3 in Appendix A). To prove that Y is a metrizable space, we only need to prove that Y is a developable space. By Theorem 1.3.5, there is a development $\{\mathscr{U}_n\}$ for X such that $\mathscr{U}_{n+1}$ is a star-refinement of $\mathscr{U}_n$. Then,

(1.1) for each nonempty compact set K in X, $\{\mathrm{st}(K, \mathscr{U}_n)\}_{n\in\mathbb{N}}$ is a neighborhood base of K in X.
For every $y \in Y$ and $n \in \mathbb{N}$, put

$$\begin{aligned} U_{y,n} &= \mathrm{st}(f^{-1}(y), \mathscr{U}_n), \\ W_{y,n} &= Y - f(X - U_{y,n}), \\ V_{y,n} &= f^{-1}(W_{y,n}). \end{aligned}$$

Then $f^{-1}(y) \subset U_{y,n}$. It follows that
(1.2) $y \in W_{y,n}$, and $f^{-1}(y) \subset V_{y,n} \subset U_{y,n}$.
(1.3) $\{W_{y,n}\}_{n\in\mathbb{N}}$ is a local base of y.
Since f is a closed mapping, $W_{y,n}$ is an open set in Y. If W is an open neighborhood of y, then $f^{-1}(y) \subset f^{-1}(W)$. By (1.1), there is $n \in \mathbb{N}$ such that $\mathrm{st}(f^{-1}(y), \mathscr{U}_n) \subset f^{-1}(W)$, i.e. $U_{y,n} \subset f^{-1}(W)$, so $W_{y,n} \subset W$. Thus (1.3) is proved.
By (1.2), $f^{-1}(y) \subset V_{y,n+1}$, and by (1.1), there is $m \geqslant n+1$ such that $U_{y,m} \subset V_{y,n+1}$.
(1.4) If $y \in W_{z,m}$, then $W_{z,m} \subset W_{y,n}$.
Fix $x \in f^{-1}(y)$. Since $f^{-1}(y) \subset V_{z,m} \subset U_{z,m}$, $x \in \mathrm{st}(f^{-1}(z), \mathscr{U}_m)$, so there is $U_x \in \mathscr{U}_m$ such that $x \in U_x$ and

$$\varnothing \neq f^{-1}(z) \cap U_x \subset f^{-1}(z) \cap U_{y,m} \subset f^{-1}(z) \cap V_{y,n+1},$$

and hence $z \in f(V_{y,n+1}) = W_{y,n+1}$. As a consequence, we obtain $(*)$: $f^{-1}(z) \subset V_{y,n+1}$. Below we prove $W_{z,m} \subset W_{y,n}$.
Suppose $t \in W_{z,m}$. Since $f^{-1}(t) \subset U_{z,m}$, $s \in \mathrm{st}(f^{-1}(z), \mathscr{U}_m)$ whenever $s \in f^{-1}(t)$, and hence there is $U_s \in \mathscr{U}_m$ such that $s \in U_s$ and $f^{-1}(z) \cap U_s \neq \varnothing$. Take $s' \in f^{-1}(z) \cap U_s$. By $(*)$, $f^{-1}(z) \subset U_{y,n+1}$, so $s' \in \mathrm{st}(f^{-1}(y), \mathscr{U}_{n+1})$,

and hence there is $U_{s'} \in \mathscr{U}_{n+1}$ such that $s' \in U_{s'}$ and $f^{-1}(y) \cap U_{s'} \neq \varnothing$. It follows that $s' \in U_s \cap U_{s'}$. Take $s'' \in f^{-1}(y) \cap U_{s'}$. Since $\mathscr{U}_m$ refines $\mathscr{U}_{n+1}$, $s, s'' \in U_s \cup U_{s'} \subset \mathrm{st}(U_{s'}, \mathscr{U}_{n+1})$. Because $\mathscr{U}_{n+1}$ is a star-refinement of $\mathscr{U}_n$, there is $V_{s'} \in \mathscr{U}_n$ such that

$$s \in \mathrm{st}(U_{s'}, \mathscr{U}_{n+1}) \subset V_{s'} \subset \mathrm{st}(f^{-1}(y), \mathscr{U}_n) = U_{y,n}.$$

So $f^{-1}(t) \subset U_{y,n}$, and hence $t \in W_{y,n}$. It follows that $W_{z,m} \subset W_{y,n}$.
For each $n \in \mathbb{N}$, let $\mathscr{W}_n = \{W_{y,n}\}_{y \in Y}$. If $y \in Y$ and W is a neighborhood of y, then by (1.3), there is $n \in \mathbb{N}$ such that $W_{y,n} \subset W$. By (1.4), there is $m \in \mathbb{N}$ such that $\mathrm{st}(y, \mathscr{W}_m) \subset W_{y,n}$. Thus $\{\mathrm{st}(y, \mathscr{W}_n)\}_{n \in \mathbb{N}}$ is a local base of y, so $\{\mathscr{W}_n\}$ is a development for Y, and hence Y is a developable space. ■

It should be noticed that when $f : X \to Y$ is a perfect mapping, for a cover $\mathscr{U}$ of X, $\{Y - f(X - U) : U \in \mathscr{U}\}$ may not be a cover of Y. For example, let $X = \mathbb{I} \times \{0, 1\}$. Take X as a subspace of $\mathbb{R}^2$ with the Euclidean topology. Let $A = \{(0, 0), (0, 1)\}$, $Y = X/A$ and let $f : X \to Y$ be the quotient mapping. Then f is a perfect mapping. Take $\mathscr{U} = \{\mathbb{I} \times \{0\}, \mathbb{I} \times \{1\}\}$. Then $\mathscr{U}$ is an open cover of X, and for any $U \in \mathscr{U}$, $f(0, 0) \notin Y - f(X - U)$.

The closed image $\mathbb{S}_1 \times \mathbb{N}/(\{0\} \times \mathbb{N})$ of the metric space $\mathbb{S}_1 \times \mathbb{N}$ is a copy of the sequential fan S_ω. To make a closed image of a metric spaces being metrizable, some additional conditions are necessary.

Theorem 2.2.2 (The Hanai–Morita–Stone theorem) *If $f : X \to Y$ is a closed mapping and X is a metric space, then the following are equivalent:*

(1) Y is a metrizable space.
(2) Y is a strongly Fréchet–Urysohn space [336].
(3) f is a peripherally compact mapping [363, 441].
(4) f is a countably bi-quotient mapping [336].

Proof By Theorem 2.2.1, Lemmas 2.1.14, 2.1.15 and Proposition 2.1.12 (the mapping lemma), we only need to prove that countably bi-quotient mappings preserve the strongly Fréchet–Urysohn property. Let $\{A_n\}$ be a decreasing sequence of sets in Y. If $y \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$, then there is $x \in f^{-1}(y)$ such that $x \in \bigcap_{n \in \mathbb{N}} \overline{f^{-1}(A_n)}$. Because otherwise, $f^{-1}(y) \subset \bigcup_{n \in \mathbb{N}} (X - \overline{f^{-1}(A_n)})$, so there is $m \in \mathbb{N}$ such that $y \in f(X - \overline{f^{-1}(A_m)})^\circ$, and hence $A_m \cap f(X - \overline{f^{-1}(A_m)}) \neq \varnothing$, a contradiction. Thus there is $x_n \in f^{-1}(A_n)$ such that $x_n \to x$, so $f(x_n) \in A_n$ and $f(x_n) \to y$. ■

The above result is a deformation of the Hanai–Morita–Stone theorem. In fact, the Hanai–Morita–Stone theorem can be described as follows: if $f : X \to Y$ is a closed mapping and X is a metric space, then Y is a metric space if and only if Y is a first countable space, if and only if f is a peripherally compact mapping. Its prototype is a result of Vaĭnšteĭn [471]: every closed mapping from a metric space onto a metric space is a peripherally compact mapping. The Hanai–Morita–Stone theorem is also called the *Morita–Hanai–Stone theorem*.

Assume that X and Y are two disjoint spaces. Let A be a closed subset of X and let $f : A \to Y$ be a continuous mapping. The quotient space Z of $X \oplus Y$ obtained by mapping x and $f(x)$ ($\forall x \in A$) to one point is called the *adjunction space* and is often denoted as $X \cup_f Y$. The quotient mapping $p : X \oplus Y \to Z$ is called the *adjunction mapping* [69]. The following metrization theorem of adjunction spaces can be regarded as an application of Theorem 2.2.2.

Recall the concept of a *natural mapping* (or an *obvious mapping*). Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a cover of X. Put $Z = \bigoplus_{\alpha \in \Lambda} X_\alpha$. For each $\alpha \in \Lambda$, denote the homeomorphism from the subspace X_α of Z onto the subspace X_α of X as h_α. Then $h : Z \to X$ is called a natural mapping (or an obvious mapping) if $h_{|X_\alpha} = h_\alpha$ for each $\alpha \in \Lambda$.

Theorem 2.2.3 ([67, 315]) *If X and Y are metric spaces, then $X \cup_f Y$ is a metrizable space if and only if $X \cup_f Y$ is a first countable space.*

Proof Let $Z = X \cup_f Y$. Assume that Z is a first countable space. By the mapping lemma (see Proposition 2.1.12), p is a pseudo-open mapping. Let $X_1 = \mathrm{cl}_X(X - A)$, $Z_1 = \mathrm{cl}_Z p(X - A)$.

(3.1) If B is a closed set of X_1, then $f(B)$ is a closed set of Y.
Suppose $y \in Y - f(B)$. Then there are disjoint open sets U and V in X containing B and $f^{-1}(y)$ respectively. Let $W = V \cup Y$, $G = p(W)^\circ$. Then W is a neighborhood of $p^{-1}(p(y)) = \{y\} \cup f^{-1}(y)$ in $X \oplus Y$, so $p(y) \in G$, and hence $p^{-1}(G) \cap (U - A) = \varnothing$. Since $B \subset \mathrm{cl}_X(U - A)$, $p^{-1}(G) \cap f(B) = \varnothing$. Thus $f(B)$ is a closed set in Y.

(3.2) $p_1 = p_{|X_1} : X_1 \to Z_1$ is a closed mapping.
It is easy to verify that $Z_1 = p(X - A) \cup \{y \in Y : f^{-1}(y) \cap \partial A \neq \varnothing\}$, and hence $p(X_1) = Z_1$. By (3.1) and $p_{1|X-A} = \mathrm{id}_{X-A}$, p_1 is a closed mapping.

(3.3) $X \cup_f Y$ is a metrizable space.
By (3.2) and Theorem 2.2.2, Z_1 is a metrizable closed subspace of $X \cup_f Y$. Since $p(Y)$ is homeomorphic to Y, $p(Y)$ is a metrizable closed subspace of $X \cup_f Y$. Let $q : p(Y) \oplus Z_1 \to X \cup_f Y$ be the natural mapping. Then q is a perfect mapping. By Theorem 2.2.1, $X \cup_f Y$ is a metrizable space. ■

Example 2.2.4 (*[315]*) A non-metrizable adjunction space.

(1) Let $X = \mathbb{I}^2 - \{(0, 0), (1, 0)\}$ and $Y = \mathbb{I}$. Give X and Y the Euclidean subspace topology. Let $A = \{(x, 0) : 0 < x < 1\}$, and define $f : A \to Y$ by $f(x, 0) = x$. Then $f^{-1}(y)$ is a compact set for every $y \in Y$. Since $f(A) = (0, 1)$ is not a closed set in Y, by (3.1) in the proof of Theorem 2.2.3, $X \cup_f Y$ is not a first countable space.

(2) The definitions of X and A are same as that in (1). Take $Y = \{0\}$. Define $f : A \to Y$ by $f(A) = \{0\}$. By using the notions in the proof of Theorem 2.2.3, we have $X_1 = X$, $Z_1 = Z$ and $p_1 : X_1 \to Z_1$ is a closed mapping. Since $\partial p_1^{-1}(0) = A$ is not compact in X_1, Z_1 is not a metrizable space.

Below we introduce a characterization of perfect preimages of metric spaces.

Lemma 2.2.5 ([162, 469]) *If $\{\mathscr{U}_n\}$ be a sequence of open covers of a space X such that $\mathscr{U}_{n+1}$ star-refines $\mathscr{U}_n$, then there is a pseudo-distance d on X such that*

(1) for any $y \in X$, $y \in \bigcap_{n\in\mathbb{N}} st(x, \mathscr{U}_n)$ if and only if $d(x, y) = 0$;
(2) U is an open subset of (X, d) if and only if for each $x \in U$, there is $m \in \mathbb{N}$ such that $st(x, \mathscr{U}_m) \subset U$.

Proof Denote $\mathscr{U}_0 = \{X\}$. For every $x, y \in X$, define

$$D(x, y) = \inf\{2^{-n} : y \in \mathrm{st}(x, \mathscr{U}_n)\},$$
$$d(x, y) = \inf\left\{\sum_{i=0}^{n} D(x_i, x_{i+1}) : n \in \mathbb{N},\ x_i \in X,\ x_0 = x \text{ and } x_{n+1} = y\right\}.$$

Then d is a pseudo-distance on X, and by using the inductive method, we know that, whenever $n \geqslant 2$,

$$D(x, y) \leqslant 2D(x, x_1) + 4\sum_{i=1}^{n-1} D(x_i, x_{i+1}) + 2D(x_n, y).$$

So $D(x, y)/4 \leqslant d(x, y) \leqslant D(x, y)$, and hence (1) holds. Since for every $x \in X$ and $n \in \mathbb{N}$,

$$\mathrm{st}(x, \mathscr{U}_{n+2}) = B(x, 1/2^{n+2}) \subset B(x, 1/2^n) \subset \mathrm{st}(x, \mathscr{U}_n),$$

(2) holds. ■

Theorem 2.2.6 [360] (The Morita theorem) *A space X is a quasi-perfect preimage of a metric space if and only if X is an M-space.*

Proof The necessity. Let $f : X \to Y$ be a quasi-perfect mapping, where Y is a metric space. Then there is a development $\{\mathscr{U}_n\}$ for Y such that $\mathscr{U}_{n+1}$ star-refines $\mathscr{U}_n$. For each $n \in \mathbb{N}$, define $\mathscr{F}_n = f^{-1}(\mathscr{U}_n)$. Then $\mathscr{F}_{n+1}$ star-refines $\mathscr{F}_n$. If $x \in X$ and $\{x_n\}$ is any sequence in X with $x_n \in \mathrm{st}(x, \mathscr{F}_n)$, then $f(x_n) \in \mathrm{st}(f(x), \mathscr{U}_n)$, and hence $f(x_n) \to f(x)$. Since f is a quasi-perfect mapping, $\{x_n\}$ has an accumulation point. Thus $\{\mathscr{F}_n\}$ is an M-sequence in X, and hence X is an M-space.

The sufficiency. Let $\{\mathscr{U}_n\}$ be an M-sequence in X. By Lemma 2.2.5, there is a pseudo-distance d on X such that

(6.1) for each $y \in X$, $y \in \bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n)$ if and only if $d(x, y) = 0$;
(6.2) U is an open set in (X, d) if and only if for each $x \in U$, there is $m \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{U}_m) \subset U$.

Define an equivalence relation "$\sim$" on X as follows: for every $x, y \in X$, $x \sim y$ if and only if $d(x, y) = 0$. Let $Y = X/\sim$ be the quotient set, and define $\rho : Y \times Y \to \mathbb{R}^+$ by $\rho([x], [y]) = d(x, y)$, then (Y, ρ) is a metric space. We prove that $f : X \to Y$ is a quasi-perfect mapping. Since $f^{-1}(B_\rho([x], \varepsilon)) = B_d(x, \varepsilon) \in \tau(X)$ for every $x \in X$ and $\varepsilon > 0$, f is a continuous function.

First, we prove that $\{\mathrm{st}(x, \mathscr{U}_n)\}_{n\in\mathbb{N}}$ is a neighborhood base of the set $\{x\} = \bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n)$ in X. In fact, since $\mathscr{U}_{n+1}$ star-refines $\mathscr{U}_n$, $\mathrm{st}^2(x, \mathscr{U}_{n+1}) \subset \mathrm{st}(x, \mathscr{U}_n)$, so $\overline{\mathrm{st}(x, \mathscr{U}_{n+1})} \subset \mathrm{st}(x, \mathscr{U}_n)$. By the convergence lemma, $\{\mathrm{st}(x, \mathscr{U}_n)\}_{n\in\mathbb{N}}$ is a neighborhood base of the countably compact closed set $[x]$ in X. Now, we can prove f is a quasi-perfect mapping. In fact, on the one hand, $f^{-1}([x]) = [x]$ is a countably compact set in X. On the other hand, if $x \notin f^{-1}(f(H))$ for any $H \in \tau^c(X)$, then $[x] \cap H = \varnothing$, and hence there is $m \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{U}_m) \cap H = \varnothing$. If $y \in \mathrm{st}(x, \mathscr{U}_{m+1})$, then $\mathrm{st}(y, \mathscr{U}_{m+1}) \subset \mathrm{st}(x, \mathscr{U}_m)$, thus $[y] \cap H = \varnothing$, so $\mathrm{st}(x, \mathscr{U}_{m+1}) \cap f^{-1}(f(H)) = \varnothing$, and hence $f^{-1}(f(H)) \in \tau^c(X)$. It follows that $f(H) \in \tau^c(Y)$. Therefore f is a quasi-perfect mapping. ■

By the properties of paracompact spaces, we obtain the following corollaries.

Corollary 2.2.7 ([360]) *A space X is a perfect preimage of a metric space if and only if X is a paracompact M-space.*

Corollary 2.2.8 ([368]) *Every paracompact M-space is a strong Σ-space.*

Proof If X is a paracompact M-space, then there is a metric space Y and a perfect mapping $f : X \to Y$. Let $\mathscr{B}$ be a σ-locally finite base for Y and let $\mathscr{K} = \{f^{-1}(y) : y \in Y\}$. Then $f^{-1}(\mathscr{B})^-$ is a σ-locally finite closed (mod k)-network w.r.t. $\mathscr{K}$ in X. So X is a strong Σ-space. ■

M-spaces and paracompact M-spaces characterize the quasi-perfect preimages and the perfect preimages of metric spaces respectively. These spaces are just the spaces sought in Alexandroff's questions (see Question 2.0.2). In the second part of this section, we give characterizations of these spaces and discuss the metrization problems of them.

Definition 2.2.9 (*[17]*) A space X is called a *submetrizable space* if X is the preimage of a metric space under a one-to-one mapping.

Obviously, X is a submetrizable space if and only if there exists a metrizable topology on X which is coarser than the topology on X. Every submetrizable space is a $\sigma^\sharp$-space with a G_δ^*-diagonal. Submetrizability is additive, hereditary and countably productive.

Proposition 2.2.10 ([320]) *A space X is a submetrizable space if and only if X has a G_δ-diagonal sequence $\{\mathscr{U}_n\}$ such that $\mathscr{U}_{n+1}$ star-refines $\mathscr{U}_n$.*

Proof If X is a submetrizable space, then there exist a metric space Y and a one-to-one mapping $f : X \to Y$. Let $\{\mathscr{F}_n\}$ be a development for Y such that $\mathscr{F}_{n+1}$ star-refines $\mathscr{F}_n$. For each $n \in \mathbb{N}$, define $\mathscr{U}_n = f^{-1}(\mathscr{F}_n)$. Then $\{\mathscr{U}_n\}$ is a G_δ-diagonal sequence in X and $\mathscr{U}_{n+1}$ star-refines $\mathscr{U}_n$.

Conversely, there is a pseudo-distance d on X satisfying the conditions (1) and (2) of Lemma 2.2.5. Since $\{\mathscr{U}_n\}$ is a G_δ-diagonal sequence in X, for every $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$. So d is a distance on X. Because each $B_d(x, \varepsilon)$ is an open set in X, the metrizable topology derived by d is coarser than the topology for X. ■

Corollary 2.2.11 ([65]) *Every paracompact space with a G_δ-diagonal is a submetrizable space.*

Theorem 2.2.12 ([93]) *A space X is a metrizable space if and only if X is an M-space with a G_δ-diagonal.*

Proof We only need to prove the sufficiency. By the Morita theorem and Chaber theorem, there exist a metric space Z and a perfect mapping $g : X \to Z$, and hence X is a paracompact space. By Corollary 2.2.11, there exist a metric space Y and a one-to-one mapping $f : X \to Y$. By Corollary 2.1.9, X is a metrizable space. ■

The above theorem can be restated as follows: a perfect (or quasi-perfect) preimage of a metric space is metrizable if and only if it has a G_δ-diagonal. For the sake of convenience, we say that a topological property Φ satisfies the *perfect preimage G_δ-diagonal theorem* provided that if $f : X \to Y$ is a perfect mapping, X is a regular space with a G_δ-diagonal and Y has the property Φ, then X also has Φ.

Example 2.2.13 A preimage of $\mathbb{I}$ under a finite-to-one closed mapping may have no G_δ-diagonal, the Alexandroff double-arrow space is such an example.

Suppose X is the Alexandroff double-arrow space (see Example 1.8.9 and we still use the notations in Example 1.8.9). Let $f : X \to \mathbb{I}$ be the projection mapping, where $\mathbb{I}$ has the Euclidean topology. Then f is an at most two-to-one closed mapping. It is easy to verify that X has no G_δ-diagonal.

Because of the above example, when discussing the problem whether a perfect preimage of a space in the class $\mathscr{C}$ belongs to $\mathscr{C}$, where $\mathscr{C}$ is a class of generalized metric spaces, generally the condition that "having a G_δ-diagonal" should be attached. This is the origin of "the perfect preimage G_δ-diagonal theorem". Of course, we also can discuss so called "the perfect preimage $\sigma^\sharp$-theorem" and so on. Since the situation is similar, we shall not make the repeat.

In the last part of this section, we give some characterizations of paracompact M-spaces.

Definition 2.2.14 A regular space X is called a *p-space* [25] if there is a sequence $\{\mathscr{U}_n\}$ of families of open sets in βX such that

(1) each $\mathscr{U}_n$ covers X;
(2) for each $x \in X$, $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n) \subset X$.
If we also have
(3) for each $x \in X$, $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n) = \bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x, \mathscr{U}_n)}$,
then X is called a *strict p-space* [31]. The sequence $\{\mathscr{U}_n\}$ above is said to be a *pluming* or a *strict pluming* in X.

Since every locally compact space is an open set in its Čech–Stone compactification, every locally compact space is a p-space. Some internal characterizations of strict p-spaces and p-spaces make them more convenient to be used.

Proposition 2.2.15 ([91]) *A completely regular space X is a strict p-space if and only if there is a strict p-sequence in X, i.e. there is a sequence $\{\mathscr{U}_n\}$ of open covers of X such that*

(1) for each $x \in X$, $C_x = \bigcap_{n\in\mathbb{N}} st(x, \mathscr{U}_n)$ is a compact subset of X;
(2) for each $x \in X$, $\{st(x, \mathscr{U}_n)\}_{n\in\mathbb{N}}$ is a neighborhood base of C_x in X.

Proof The necessity. Let $\{\mathscr{F}_n\}$ be a strict pluming on X. We may assume that $\mathscr{F}_{n+1}$ partially refines $\mathscr{F}_n$. For each $n \in \mathbb{N}$, put $\mathscr{U}_n = \mathscr{F}_{n|X}$. We prove the sequence $\{\mathscr{U}_n\}$ of open covers of X is a strict p-sequence in X.

(1) For each $x \in X$, $C_x = \bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{F}_n) = \bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x, \mathscr{F}_n)}$ is a closed subset of βX contained in X, so C_x is a compact set in X.
(2) For each $x \in X$ and $C_x \subset U \in \tau(X)$, there is $G \in \tau(\beta X)$ such that $G \cap X = U$. Then $\{G\} \cup \{\beta X - \overline{\mathrm{st}(x, \mathscr{F}_n)} : n \in \mathbb{N}\}$ covers the compact space βX, so there is $m \in \mathbb{N}$ such that $\overline{\mathrm{st}(x, \mathscr{F}_m)} \subset G$, and hence $\mathrm{st}(x, \mathscr{U}_m) \subset G$, thus $\{\mathrm{st}(x, \mathscr{U}_n)\}_{n\in\mathbb{N}}$ is a neighborhood base of C_x in X.

The sufficiency. Let $\{\mathscr{U}_n\}$ be a strict p-sequence in X. For each $n \in \mathbb{N}$, take $\mathscr{F}_n \subset \tau(\beta X)$ such that $\mathscr{F}_{n|X} = \mathscr{U}_n$. Then $\mathscr{F}_n$ is a cover of X and for each $x \in X$, $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n) \subset \bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{F}_n) \subset \bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x, \mathscr{F}_n)}$. If $y \in \bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x, \mathscr{F}_n)} - C_x$, pick $W \in \tau(\beta X)$ such that $y \in W$ and $\overline{W} \cap C_x = \varnothing$, then there is $m \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{U}_m) \cap \overline{W} = \varnothing$, so $\overline{\mathrm{st}(x, \mathscr{F}_m)} \cap W = \varnothing$, and hence $y \notin \overline{\mathrm{st}(x, \mathscr{F}_m)}$, a contradiction. Therefore, $\bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x, \mathscr{F}_n)} \subset C_x$. Consequently, $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{F}_n) = \bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x, \mathscr{F}_n)} \subset X$. ■

Thus, every space with a strict p-sequence is a $w\Delta$-space. Obviously, the property of having a strict p-sequence is countably productive [158].

Theorem 2.2.16 ([77]) *A completely regular space X is a p-space if and only if there is a p-sequence in X, i.e. there is a sequence $\{\mathscr{U}_n\}$ of open covers of X such that for every $x \in X$ and $n \in \mathbb{N}$, if $x \in U_n \in \mathscr{U}_n$, then*

(1) $D_x = \bigcap_{n\in\mathbb{N}} \overline{U_n}$ is a compact set in X;
(2) $\{\bigcap_{n\leqslant k} \overline{U_n}\}_{k\in\mathbb{N}}$ is a network of D_x in X.

Proof The necessity. Let $\{\mathscr{P}_n\}$ be a pluming in X. Choose a sequence $\{\mathscr{U}_n\}$ of open covers of X satisfying that for every $n \in \mathbb{N}$ and $U \in \mathscr{U}_n$, there is $P \in \mathscr{P}_n$ such that $\mathrm{cl}_{\beta X} U \subset P$. We prove that $\{\mathscr{U}_n\}$ is a p-sequence in X. For each $x \in X$ and any sequence $\{U_n\}$ of open sets in X with $x \in U_n \in \mathscr{U}_n$, put $D_x = \bigcap_{n\in\mathbb{N}} \overline{U_n}$.

(1) Since $\bigcap_{n\in\mathbb{N}} \mathrm{cl}_{\beta X} U_n \subset \bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{P}_n) \subset X$,

$$D_x = \bigcap_{n\in\mathbb{N}} (X \cap \mathrm{cl}_{\beta X} U_n) = \bigcap_{n\in\mathbb{N}} \mathrm{cl}_{\beta X} U_n$$

is a compact set in X.

(2) Let $D_x \subset U \in \tau(X)$. Take $G \in \tau(\beta X)$ such that $U = G \cap X$. Then $\{G\} \cup \{\beta X - \mathrm{cl}_{\beta X} U_n : n \in \mathbb{N}\}$ covers βX, so there is $k \in \mathbb{N}$ such that $\bigcap_{n\leqslant k} \mathrm{cl}_{\beta X} U_n \subset G$, and hence $\bigcap_{n\leqslant k} \overline{U}_n \subset U$. Thus, $\{\bigcap_{n\leqslant k} \overline{U}_n\}_{k\in\mathbb{N}}$ is a network of D_x in X.

The sufficiency. Let $\{\mathscr{U}_n\}$ be a p-sequence in X. For each $n \in \mathbb{N}$, take $\mathscr{P}_n \subset \tau(\beta X)$ such that $\mathscr{P}_{n|X} = \mathscr{U}_n$. Then $\mathscr{P}_n$ covers X. Let $x \in X$. If $y \in \bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{P}_n) - X$, then there is a sequence $\{P_n\}$ of subsets of βX such that $\{x, y\} \subset P_n \in \mathscr{P}_n$, and hence $\bigcap_{n\in\mathbb{N}} \overline{P_n \cap X}$ is a compact set in X, so there exist $G \in \tau(\beta X)$ and $k \in \mathbb{N}$ such that $\bigcap_{n\leqslant k} \overline{P_n \cap X} \subset G \subset \mathrm{cl}_{\beta X} G \subset \beta X - \{y\}$. Let

$$W = \left(\bigcap_{n\leqslant k} P_n\right) \cap (\beta X - \mathrm{cl}_{\beta X} G).$$

Then $W \cap X = \varnothing$. However, $y \in W \in \tau(\beta X)$, so $W \cap X \neq \varnothing$, a contradiction. Thus $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{P}_n) \subset X$. Therefore, $\{\mathscr{P}_n\}$ is a pluming in X. ■

Obviously, the property of having a p-sequence is countably productive [158].

A space X is called an *isocompact space* [49] if every countably compact closed set in X is a compact set. Every submetacompact space is an isocompact space (see Theorem A.4.2 in Appendix A).

Corollary 2.2.17 *Let X be an isocompact regular space. If X is a $w\Delta$-space, then X has a p-sequence.*

Proof For the $w\Delta$-sequence $\{\mathscr{U}_n\}$ in X, take a sequence $\{\mathscr{V}_n\}$ of open covers of X such that $\overline{\mathscr{V}_n}$ refines $\mathscr{U}_n$. By the convergence lemma, $\{\mathscr{V}_n\}$ is a p-sequence in X. ■

Theorem 2.2.18 ([77, 156]) *For every submetacompact regular space X, the following are equivalent:*

(1) X has a strict p-sequence.
(2) X has a p-sequence.
(3) X is a $w\Delta$-space.

Proof We only need to prove (2) ⇒ (3) ⇒ (1).

(2) ⇒ (3). Let $\{\mathscr{U}_n\}$ be a p-sequence in X, where $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$. For each $n \in \mathbb{N}$, take a θ-sequence $\{\mathscr{V}_{n,j}\}_{j\in\mathbb{N}}$ in X such that $\mathscr{V}_{n,j}$ refines $\mathscr{U}_n \wedge (\bigwedge_{m,i<n} \mathscr{V}_{m,i})$. For each $k \in \mathbb{N}$, let $\mathscr{V}_k = \mathscr{V}_{k,1}$. We prove $\{\mathscr{V}_k\}$ is a $w\Delta$-sequence in X. Suppose $x \in X$ and $\{x_k\}$ is a sequence in X satisfying $x_k \in \mathrm{st}(x, \mathscr{V}_k)$. Then there are strictly increasing

sequences $\{k_i\}$, $\{j_i\}$ in $\mathbb{N}$ such that $k_{i+1} > j_i$ and $1 \leqslant |(\mathscr{V}_{k_i,j_i})_x| < \aleph_0$. For each $n \in \mathbb{N}$, since $\{x_{k_i} : i > n\} \subset \mathrm{st}(x, \mathscr{V}_{k_{n+1}}) \subset \mathrm{st}(x, \mathscr{V}_{k_n,j_n})$ and $\mathscr{V}_{k_n,j_n}$ refines $\mathscr{U}_n$, we may assume that there is $U_n \in \mathscr{U}_n$ such that $\{x_{k_i} : i > n\} \cup \{x\} \subset U_n$, so $\{x_{k_i}\}$ has an accumulation point, and hence $\{x_k\}$ has an accumulation point.

(3) $\Rightarrow$ (1). Let $\{\mathscr{U}_n\}$ be a $w\Delta$-sequence in X. By Lemma 1.4.8, there is a sequence $\{\mathscr{V}_n\}$ of open covers of X such that $\mathscr{V}_{n+1}$ refines $\mathscr{V}_n \wedge \mathscr{U}_n$, and for each $x \in X$, $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{V}_n) = \bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x, \mathscr{V}_n)} \subset \bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n)$. By the convergence lemma, $\{\mathscr{V}_n\}$ is a strict p-sequence in X. ∎

Corollary 2.2.19 *For every paracompact space X, the following are equivalent:*

(1) X is an M-space.
(2) X is a wΔ-space.
(3) X is a p-space.

By Corollaries 2.2.19, 2.2.8 and Theorem 1.5.11, paracompact p-spaces and paracompact M-spaces are countably productive [25, 360].

Corollary 2.2.20 ([25]) *Every cosmic p-space is metrizable.*

Proof Let X be a cosmic p-space. Then X is a paracompact p-space with a G_δ-diagonal. It follows from Corollary 2.2.19 and Theorem 2.2.12 that X is metrizable. ∎

An earlier version of the above corollary is a result proved by Arhangel'skiĭ [21] that every cosmic compact space is metrizable.

A Tychonoff space X is said to be *Čech-complete* if it is a G_δ-set in some Hausdorff compactification of it. It is easy to see that a Tychonoff space X is Čech-complete if and only if it is a G_δ-set in every Hausdorff compactification of it. A space is a Čech-complete paracompact space if and only if it can be mapping by a perfect map onto a complete metric space [135]. The problem of characterizing the perfect preimages of metric spaces was posed by Alexandroff [3]. To solve this problem, Arhangel'skiĭ [25] introduced the concept of p-spaces and proved the following theorem.

Theorem 2.2.21 [25] (The Arhangel'skiĭ theorem) *A space is a paracompact p-space if and only if it can be mapped onto a metrizable space by a perfect mapping.*

Proof Firstly, if X is a Tychonoff space, Y is a p-space and $f : X \to Y$ is a perfect mapping, then X is also a p-space. Indeed, let $\{\mathscr{U}_n\}$ be a pluming of the space Y in the Čech–Stone compactification βY. Since $f : X \to Y$ is perfect, the extension $F : \beta X \to \beta Y$ of the mapping f satisfies the condition $F(\beta X - X) \subset \beta Y - Y$. For each $n \in \mathbb{N}$, let $\mathscr{V}_n = F^{-1}(\mathscr{U}_n)$. Then $\{\mathscr{V}_n\}$ is a pluming of the space X in βX. In fact, for each $x \in X$, $F(\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{V}_n)) \subset \bigcap_{n\in\mathbb{N}} \mathrm{st}(f(x), \mathscr{U}_n) \subset Y$, thus $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{V}_n) \subset X$.

The scheme of the proof of the main part of the theorem is as follows. Let $\{\mathscr{U}_n\}$ be a pluming of a paracompact space X in the Čech–Stone compactification βX. We construct by induction a sequence $\{\mathscr{V}_n\}$ of open covers of X satisfying the conditions: (1) for each $x \in X$ and each $n \in \mathbb{N}$ the closure $\mathrm{cl}_X(\mathrm{st}(x, \mathscr{V}_n))$ is contained in some element of $\mathscr{U}_n$; (2) $\mathscr{V}_{n+1}$ is a refinement of $\mathscr{V}_n$; (3) $\mathscr{V}_{n+1}$ is a star refinement of $\mathscr{V}_n$. Points $x, y \in X$ are said to be equivalent if $y \in \bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n)$. Then the canonical mapping from X to the space Y of equivalent classes corresponding to the above equivalence is the required perfect mapping from X onto the metrizable space Y. ■

This theorem generalizes the characterization of Čech-complete paracompactness given by Frolík [135]. It shows that the p-space property is a "genuine" generalization of Čech-completeness.

Example 2.2.22 (1) There is a locally compact space (hence a p-space) which is not isocompact, for example ω_1.

(2) There is a p-space which is not locally compact, for example the space $\mathbb{P}$ of irrationals.

(3) These is a non-regular developable space without any p-sequence [156].
Let $S = \{(x, y) : x, y \in \mathbb{R}, y > 0\}$, $L = \{(x, 0) : x \in \mathbb{R}\}$ and $X = S \cup L$. Denote the Euclidean subspace topology on X by τ^*. Define the half-disc topology on X as follows [437]:

$$\tau = \{\tau^*\} \cup \{\{x\} \cup (S \cap U) : x \in L,\ x \in U \in \tau^*\}.$$

Then (X, τ) is called a half-disc topological space. It is easy to verify that X is not a regular space, and by using spherical neighborhoods of points in $\mathbb{R}^2$, it is also easy to show X is a developable space.
If X has a p-sequence $\{\mathscr{U}_n\}$, then for $x = (0, 0) \in X$ and each $n \in \mathbb{N}$, fix $U_n \in (\mathscr{U}_n)_x$ and take a neighborhood $\{x\} \cup (S \cap B(x, 2r_n))$ of x contained in U, where $B(x, 2r_n)$ is a spherical neighborhood of x in $\mathbb{R}^2$ and $r_{n+1} < r_n$. Let $x_n = (r_n, 0)$. Then $x_n \in \bigcap_{k\leqslant n} \overline{U}_k$, and hence the sequence $\{x_n\}$ has an accumulation point, which contradicts the fact that $\{x_n : n \in \mathbb{N}\}$ is a discrete closed subset of X. Consequently, X has no p-sequence.

(4) There is a locally compact, submetrizable space which is not a β-space [162].
Let B be the Bernstein set in the real line $\mathbb{R}$ (see Example 1.8.5). Then every uncountable closed set in $\mathbb{R}$ meets both B and $\mathbb{R} - B$. Let $\{B_\alpha : \alpha < 2^\omega\}$ be the family of all the countable sets in B the closures of which in $\mathbb{R}$ are uncountable. For each $\alpha < 2^\omega$, by the transfinite induction, pick $x_\alpha \in \overline{B}_\alpha - (B \cup \{x_\beta : \beta < \alpha\})$ and $x_{\alpha,n} \in B_\alpha$ such that $x_{\alpha,n} \to x_\alpha$. Let $X = B \cup \{x_\alpha : \alpha < 2^\omega\}$. Define a topology for X as follows: each point in B is isolated; the elements of a neighborhood base of x_α have the form $\{x_\alpha\} \cup \{x_{\alpha,n} : n \geqslant m\}$, $m \in \mathbb{N}$.
Obviously, X is a locally compact submetrizable space. Let $H = \{x_\alpha : \alpha < 2^\omega\}$. Then H is a closed set in X. If X is a β-space, then by Theorem 1.7.7, X is a semi-stratifiable space, and hence X has a sequence $\{U_n\}$ of open sets such that $H = \bigcap_{n\in\mathbb{N}} U_n$. If some $B - U_n$ is uncountable, then there is $\alpha < 2^\omega$ such that $B_\alpha \subset B - U_n$, and which contradicts the fact that $x_{\alpha,n} \to x_\alpha \in U_n$, thus every $B - U_n$ is countable. Since B is uncountable, $B \cap (\bigcap_{n\in\mathbb{N}} U_n) \neq \varnothing$, a contradiction.

(5) p-spaces and M-spaces are independent.
The space X in the above (4) is a p-space which is not an M-space. On the other

hand, let X be the *Frolík space* [134], i.e. $\mathbb{N} \subset X \subset \beta\mathbb{N}$ and X is a countably compact subspace of $\beta\mathbb{N}$ with cardinality not greater than $\mathfrak{c}$. Then X is an M-space. Since every compact set in X is a finite set, X is not a k-space, hence X is not a p-space (see Propositions 2.4.10 and 2.4.11).

Question 2.2.23 (*[162]*) Is every normal Moore space submetrizable?

Question 2.2.24 (*[158]*) Is the product of a strict p-space and a $w\Delta$-space a $w\Delta$-space?

2.3 Quotient Mappings

By taking quotient mappings as its core, in this section, we give the characterizations of quotient images, pseudo-open images and countably bi-quotient images of metric spaces, and show that they are just sequential spaces, Fréchet–Urysohn spaces and strongly Fréchet–Urysohn space respectively. As applications, we give the characterizations of quotient images of normally metric spaces and connected metric spaces. This topic relies on the mapping properties of generalized sequentiality properties.

Proposition 2.3.1 *(1) k-spaces and sequential spaces are preserved by quotient mappings [133, 228].*
(2) Fréchet–Urysohn spaces are preserved by pseudo-open mappings [26, 133].
(3) Strongly Fréchet–Urysohn spaces are preserved by countably bi-quotient mappings [424].

Proof Let $f : X \to Y$ be a mapping.

(1) Suppose f is a quotient mapping and X is a k-space. If a subset A of Y satisfies that for each $K \in \mathscr{K}(Y), A \cap K \in \tau^c(Y)$, then for each $L \in \mathscr{K}(X), A \cap f(L) \in \tau^c(Y)$, so $f^{-1}(A) \cap L = f^{-1}(A \cap f(L)) \cap L \in \tau^c(X)$, and hence $f^{-1}(A) \in \tau^c(X)$, thus $A \in \tau^c(Y)$. Consequently, Y is a k-space.
In the above argument, actually we proved the following stronger result: Let A be any subset of Y. If $A \cap f(L) \in \tau^c(Y)$ for each $L \in \mathscr{K}(X)$, then $A \in \tau^c(Y)$. Replace $\mathscr{K}(Y)$ and $\mathscr{K}(X)$ in the above proof with $\mathscr{S}(Y)$ and $\mathscr{S}(X)$ respectively, we obtain the result that sequential spaces are preserved by quotient mappings.
(2) Suppose f is a pseudo-open mapping and X is a Fréchet–Urysohn space. If $y \in \overline{A} \subset Y$ and $f^{-1}(y) \cap \overline{f^{-1}(A)} = \varnothing$, then $y \in f(X - \overline{f^{-1}(A)})^\circ \subset Y - \overline{A}$, a contradiction. So there is $x \in f^{-1}(y) \cap \overline{f^{-1}(A)}$, it follows that there is a sequence $\{x_n\} \subset f^{-1}(A)$ such that $x_n \to x$, thus $\{f(x_n)\} \subset A$ and $f(x_n) \to y$. Consequently, Y is a Fréchet–Urysohn space.
(3) was proved in Theorem 2.2.2. ■

Definition 2.3.2 Let $\mathscr{F}$ be a cover of a space X. We say that X has the *weak topology* with respect to $\mathscr{F}$ [117], or say that X is *determined* by $\mathscr{F}$ [167] provided that for any $A \subset X, A \in \tau^c(X)$ if and only if for every $F \in \mathscr{F}, A \cap F \in \tau^c(F)$.

Obviously, X is a k-space (resp. sequential space) if and only if X has the weak topology with respect to $\mathscr{K}(X)$ (resp. $\mathscr{S}(X)$).

Proposition 2.3.3 ([167]) *Suppose $\mathscr{F}$ is a cover of X and $Z = \bigoplus \mathscr{F}$. Let $f : Z \to X$ be the natural mapping. Then f is a quotient mapping if and only if X has the weak topology with respect to $\mathscr{F}$.*

Proof Suppose f is a quotient mapping. For each $A \subset X$, if $A \cap F \in \tau^c(F)$ whenever $F \in \mathscr{F}$, then $f^{-1}(A) \in \tau^c(Z)$, and hence $A \in \tau^c(X)$, thus X has the weak topology with respect to $\mathscr{F}$. If X has the weak topology with respect to $\mathscr{F}$ and $A \subset X$ satisfies $f^{-1}(A) \in \tau^c(Z)$, then for each $F \in \mathscr{F}$, $A \cap F \in \tau^c(F)$, and hence $A \in \tau^c(X)$, so f is a quotient mapping. ∎

Theorem 2.3.4 ([136]) *For every space X, the following are equivalent:*

(1) X is a k-space.
(2) X is a quotient image of a paracompact locally compact space.
(3) X is a quotient image of a locally compact space.

Proof (1) $\Rightarrow$ (2). If X is a k-space, then X has the weak topology with respect to $\mathscr{K}(X)$. By Proposition 2.3.3, X is a quotient image of the paracompact locally compact space $\bigoplus \mathscr{K}(X)$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). We only need to prove that every locally compact space is a k-space. Suppose Y is a locally compact space and $A \subset Y$. If $A \notin \tau^c(Y)$, then there is $y \in \overline{A} - A$. Let V be an open neighborhood of y such that $\overline{V} \in \mathscr{K}(Y)$. Then $y \in \overline{\overline{V} \cap A} - \overline{V} \cap A$, so $\overline{V} \cap A \notin \tau^c(Y)$, and hence Y is a k-space. ∎

Corollary 2.3.5 *Let X be a k-space. If every compact subset of X is of the point G_δ-property, then X is a sequential space.*

Proof By Theorem 1.7.7, $\bigoplus \mathscr{K}(X)$ is a first countable space. Since X is a quotient image of $\bigoplus \mathscr{K}(X)$, by Proposition 2.3.1, X is a sequential space. ∎

Replace $\mathscr{K}(X)$ in the proof of Theorem 2.3.4 with $\mathscr{S}(X)$, we get some characterizations of quotient images of metric spaces.

Theorem 2.3.6 ([133]) *For every space X, the following are equivalent:*

(1) X is a sequential space.
(2) X is a quotient image of a locally compact metric space.
(3) X is a quotient image of a metric space.

By using the mapping lemma (see Proposition 2.1.12), we obtain the following characterizations of pseudo-open images and countably bi-quotient images of metric spaces.

Theorem 2.3.7 ([26, 133]) *For every space X, the following are equivalent:*

(1) X is a Fréchet–Urysohn space.
(2) X is a pseudo-open image of a locally compact metric space.
(3) X is a pseudo-open image of a metric space.

Theorem 2.3.8 ([424]) *For every space X, the following are equivalent:*

(1) X is a strongly Fréchet–Urysohn space.
(2) X is a countably bi-quotient image of a locally compact metric space.
(3) X is a countably bi-quotient image of a metric space.

In the second part of this section, we investigate quotient images of two special classes of spaces. We first introduce one special class of metric spaces, the quotient images of spaces in this class are metrizable spaces.

Definition 2.3.9 (*[367]*) A metric space (X, d) is call a *normally metric space*, if X^d is compact in X.

The normally metric space X defined by Mrówka [367] is that there is a metric d on X such that for each pair A, B of disjoint closed sets in X, $d(A, B) > 0$. Mrówka proved that this definition is equivalent to Definition 2.3.9. We do not use this fact in this book.

Theorem 2.3.10 ([11, 210]) *For every metric space X, the following are equivalent:*

(1) X is a normally metric space.
(2) Every closed image of X is a metric space.
(3) Every quotient image of X is a metric space.

Proof (1) $\Rightarrow$ (3). Let $f : X \to Y$ be a quotient mapping. Denote a countable dense subset of the compact metric space X^d by $D = \{a_i : i \in \mathbb{N}\}$. Define

$$\mathscr{B} = \{B(a_i, 1/j) : i, j \in \mathbb{N}\},\ \mathscr{U} = \mathscr{B}^F.$$

Then $\mathscr{U}$ is a countable family in X. For each $U \in \tau(X)$, let

$$\begin{aligned} H(U) &= (Y - f(X^d - U)) \cap f(U), \\ G_1(U) &= (X - f^{-1}(f(X^d - U))) \cap (f^{-1}(f(U)) - U), \\ G_2(U) &= (X - f^{-1}(f(X^d - U))) \cap U. \end{aligned}$$

Then $f^{-1}(H(U)) = G_1(U) \cup G_2(U)$. Since $G_1(U) \subset X - X^d$, $G_2(U) \in \tau(X)$ and $H(U) \in \tau(Y)$. Let

$$\mathscr{W} = \{H(U) : U \in \mathscr{U}\} \cup \{\{y\} : y \in Y - f(X^d)\}.$$

To prove Y is a metric space, we show that Y is a paracompact space and $\mathscr{W}$ is a σ-locally finite base for Y.

(10.1) Y is a paracompact space.
For any open cover $\mathscr{V}$ of Y, there is $\mathscr{V}' \in \mathscr{V}^{<\omega}$ such that $f(X^d) \subset \cup\mathscr{V}'$. Then $\mathscr{V}' \cup \{\{y\} : y \in Y - \cup\mathscr{V}'\}$ is a locally finite open refinement of $\mathscr{V}$, and hence Y is a paracompact space.

(10.2) $\mathscr{W}$ is a base for Y.
For every $y \in Y$ and $y \in V \in \tau(Y)$, we may assume that there is $x \in X^d$ such that $y = f(x)$. Then there is $U \in \mathscr{U}$ such that $X^d \cap f^{-1}(f(x)) \subset U \subset f^{-1}(V)$, so $y \in H(U) \subset V$.

(10.3) $\mathscr{W}$ is a σ-locally finite base.
Denote the set of all the finite subfamilies of $\{H(U) : U \in \mathscr{U}\}$ covering $f(X^d)$ by $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$, because $\mathscr{U}$ is countable. For each $i \in \mathbb{N}$, let

$$\mathscr{W}_i = \mathscr{P}_i \cup \{\{y\} : y \in Y - \cup\mathscr{P}_i\}.$$

Then $\mathscr{W} = \bigcup_{i\in\mathbb{N}} \mathscr{W}_i$ and $\mathscr{W}_i$ is locally finite.

(3) $\Rightarrow$ (2) is obvious. Below we prove (2) $\Rightarrow$ (1). If $X^d \notin \mathscr{K}(X)$, then X^d contains a countable discrete closed subspace Z. Let $f : X \to X/Z$ be the quotient mapping. Then f is a closed mapping and $\partial f^{-1}([Z]) = Z$ is not a compact set in X. By Theorem 2.2.2, X/Z is not a metric space, a contradiction. Thus $X^d \in \mathscr{K}(X)$. ∎

Corollary 2.3.11 ([476]) *A space X is a compact metric space if and only if each image of X is a metric space.*

Proof We only need to prove the sufficiency. By Theorem 2.3.10, X is a normally metric space. If X is not a compact space, then X contains a countable discrete closed subspace $Z \subset X - X^d$. Let $Y = X - Z$. Take $\tau_1 = \tau(X)_{|Y}$ and take a non-metrizable Hausdorff topology on Z as τ_2. Since $(Z, \tau_{1|Z})$ is a clopen subspace of X, $\mathrm{id}_X : X \to (Y, \tau_1) \oplus (Z, \tau_2)$ is a continuous mapping. However, $(Y, \tau_1) \oplus (Z, \tau_2)$ is not a metric space, a contradiction. Consequently, X is a compact metric space. ∎

Corollary 2.3.12 ([210]) *For every metric space X, the following are equivalent:*

(1) Every quotient image of X is a metric space.
(2) Every quotient image of X is a first countable space.
(3) Every quotient image of X is a Fréchet–Urysohn space.
(4) Every quotient mapping on X is a countably bi-quotient mapping.
(5) Every quotient mapping on X is a pseudo-open mapping.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3) is obvious. We only need to prove (3) $\Rightarrow$ (1). If $X^d \notin \mathscr{K}(X)$, then X^d contains a countable discrete closed subspace $\{x_n : n \in \omega\}$. Take a discrete family $\{U_n : n \in \omega\}$ of open sets in X such that $x_n \in U_n$. To each $n \in \omega$, take a sequence $\{x_{n,m}\}_m \subset U_n$ such that $x_{n,m} \to x_n$. Let $Y = X - \{x_n : n \in \mathbb{N}\}$. Define a mapping $f : X \to Y$ by

$$f(x) = \begin{cases} x_{0,n}, & x = x_n, n \in \mathbb{N}, \\ x, & x \in Y. \end{cases}$$

Give Y the quotient topology induced by f. Then the Hausdorff space Y contains a copy of Arens space S_2, and hence Y is not a Fréchet–Urysohn space, a contradiction. Consequently, $X^d \in \mathscr{K}(X)$. By Theorem 2.3.10, every quotient image of X is a metric space. ■

Example 2.3.13 (*[272]*) There exist a metric space X and a quotient mapping $f : X \to Y$ such that f is neither a closed mapping nor an open mapping.

Suppose $X = \mathbb{I} \times \omega$ and $\mathscr{B}$ is a countable base of the Euclidean topology for $\mathbb{I}$. Let

$$V(B, m) = B \times (\{0\} \cup \{n \in \mathbb{N} : n \geqslant m\}),\ B \in \mathscr{B},\ m \in \mathbb{N};$$
$$\mathscr{P} = \{\{x\} : x \in \mathbb{I} \times \mathbb{N}\} \cup \{V(B, m) : B \in \mathscr{B},\ m \in \mathbb{N}\}.$$

Give X the topology generated by the base $\mathscr{P}$. Then X is a regular space and $\mathscr{P}$ is a σ-discrete base for X, and hence X is a metrizable space. Since the topology for X^d is the Euclidean subspace topology for $\mathbb{I}$, X^d is a compact set in X, and hence X is a normally metric space. Let $f : X \to \mathbb{I}$ be the projection mapping. Then the quotient mapping f is neither a closed mapping nor an open mapping.

Example 2.3.13 shows that "pseudo-open mapping" and "countably bi-quotient mapping" in Corollary 2.3.12 cannot be strengthened to "closed mapping" and "open mapping" respectively. Example 2.3.14 below shows that "closed mapping" or "quotient mapping" in Theorem 2.3.10 cannot be replaced with "countably bi-quotient mapping" either.

Example 2.3.14 (*[272]*) There is a metric space which is not a normally metric space such that every countably bi-quotient image of this space is a metric space.

For each $n \in \mathbb{N}$, let $I_n = \mathbb{I}$. Put $X = \bigoplus_{n\in\mathbb{N}} I_n$. Then the metric space X is not a normally metric space. Let $f : X \to Y$ be a countably bi-quotient mapping. Then Y is a Lindelöf, locally compact and locally metrizable space, hence a metric space.

Question 2.3.15 Characterize the class of spaces such that every countably bi-quotient image of these spaces is metrizable.

Another special class of spaces is the class of connected sequential spaces. Connectedness is invariant under mappings. Is every connected space an image of a connected metric space [465]?

Definition 2.3.16 (*[120]*) A space X is called an *s-connected space* if X cannot be represented as the union of two disjoint nonempty sequentially open sets. An s-connected space is also called a *sequentially connected space*.

Obviously, every connected sequential space is an s-connected space and every s-connected space is a connected space.

Theorem 2.3.17 ([278]) *For every space X, the following are equivalent:*

(1) X is a sequence-covering image of a connected metric space.
(2) X is an image of a connected metric space.
(3) X is an s-connected space.

Proof (2) $\Rightarrow$ (3). We only need to prove that if $f : M \to X$ is a mapping and M is s-connected, then X is also s-connected. Otherwise, X is the union of two disjoint nonempty sequentially open sets A and B. If $\{a_n\}$ is a sequence in X converging to a point $a \in f^{-1}(A)$, then $f(a_n) \to f(a) \in A$, so $\{f(a_n)\}$ is eventually in A, and hence $\{a_n\}$ is eventually in $f^{-1}(A)$. It follows that $f^{-1}(A)$ is a sequentially open set in M. For the same reason, $f^{-1}(B)$ is also a sequentially open set in M. Thus, M is the union of two disjoint nonempty sequentially open sets, a contradiction.

(3) $\Rightarrow$ (1). Suppose X is an s-connected space. Let $M = \bigoplus \mathscr{S}(X)$ and denote the metric space induced by the topological sum as (M, d). Let $q : (M, d) \to X$ be the natural mapping. Then q is continuous.

Define $\rho : (M \times \mathbb{I})^2 \to \mathbb{R}^+$ as follows:

$$\rho((y_1, t_1), (y_2, t_2)) = \begin{cases} d(y_1, y_2) + t_1 + t_2, & y_1 \neq y_2, \\ |t_2 - t_1|, & y_1 = y_2. \end{cases}$$

It is easy to verify that ρ is a distance on $M \times \mathbb{I}$ and M is homeomorphic to the subspace $M \times \{0\}$ of $(M \times \mathbb{I}, \rho)$.

Define a binary relation R on $M \times \mathbb{I}$ as follows: $(y_1, t_1)R(y_2, t_2)$ if and only if $q(y_1) = q(y_2)$ and $t_1 = t_2 = 1$, or $y_1 = y_2$ and $t_1 = t_2$. Then R is an equivalent relation. Let $Z = (M \times \mathbb{I})/R$, and let $p : M \times \mathbb{I} \to Z$ be the quotient mapping. For every $y \in M,\ n \in \mathbb{N}$, put

$$B_{y,n} = p(\{(y, 1)\} \cup \{(y', t) : q(y') = q(y),\ 1 - 1/n < t < 1\}).$$

Define a topology τ on Z as follows: when $(y, t) \in M \times \mathbb{I}$, if $t \neq 1$, then the neighborhoods of $p(y, t)$ in Z have the same form as the neighborhoods of (y, t) in the product space $M \times \mathbb{I}$; if $t = 1$, each element of a neighborhood base of $p(y, t)$ in Z has the form $B_{y,n},\ n \in \mathbb{N}$. Then (Z, τ) is a regular space and for each $U \in \tau$, $p^{-1}(U)$ is an open set in $(M \times \mathbb{I}, \rho)$. Thus, p is continuous.

(17.1) (Z, τ) is a metrizable space.
Assume that $\mathscr{B}$ is a σ-locally finite base for the metric space $(M \times \mathbb{I}, \rho)$. Let $\mathbb{Q}'$ be the set of all rational numbers in the interval $(1/3, 1)$ of $\mathbb{I}$. Put

$$\begin{aligned} \mathscr{P}_1 &= \{p(B) : B \in \mathscr{B},\ B \subset M \times [0, 1/2)\}, \\ \mathscr{P}_2 &= \{p(\{y\} \times (r_1, r_2)) : y \in M,\ r_1, r_2 \in \mathbb{Q}'\}, \\ \mathscr{P}_3 &= \{B_{y,n} : y \in M,\ n \in \mathbb{N}\}, \\ \mathscr{P} &= \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}_3. \end{aligned}$$

Then $\mathscr{P}$ is a σ-locally finite base for (Z, τ), and hence (Z, τ) is a metrizable space.
Define $h : (M \times \mathbb{I}, \rho) \to X$ and $f : (Z, \tau) \to X$ such that $h(y, t) = q(y)$ and $f \circ p = h$.

(17.2) f is a sequence-covering mapping.
Notice that $p(E \times [0, 1)) \in \tau$ for each open subset E of (M, d) and $p(q^{-1}(x) \times$

$(0, 1]) \in \tau$ for each $x \in X$. If A is an open set in X, then $q^{-1}(A)$ is an open set in (M, d), and hence $f^{-1}(A) = p(q^{-1}(A)\times[0, 1))\cup(\bigcup_{x\in A} p(q^{-1}(x)\times(0, 1])) \in \tau$. Thus f is continuous. Assume that $\{x_n\}$ is a nontrivial sequence in X converging to x. By the structure of q, there is a sequence $\{y_n\}$ in (M, d) converging to y such that $q(y) = x$ and $q(y_n) = x_n$. Since $\rho((y_n, 0), (y, 0)) = d(y_n, y)$, $(y_n, 0) \to (y, 0)$. Because p is continuous, $p(y_n, 0) \to p(y, 0)$. Since $f(p(y_n, 0)) = x_n$, f is a sequence-covering mapping.

(17.3) (Z, τ) is an s-connected space.
Otherwise, Z is the union of two disjoint nonempty sequentially open sets C and D. Obviously, $f^{-1}(x)$ is an s-connected set in Z, so either $f^{-1}(x) \subset C$, or $f^{-1}(x) \subset D$, and hence there exist sets A, B in X such that $X = A \cup B$, $C = \bigcup_{x\in A} f^{-1}(x)$ and $D = \bigcup_{x\in B} f^{-1}(x)$, i.e. $C = f^{-1}(A)$, $D = f^{-1}(B)$. Suppose $\{x_n\}$ is a sequence in X converging to $x \in A$. By (17.2), there is a sequence $\{z_n\}$ in Z converging to some point $z \in f^{-1}(x)$ such that $f(z_n) = x_n$. Then $z \in C$, so $\{z_n\}$ is eventually in C, and hence $\{x_n\}$ is eventually in A. Consequently, A is a sequentially open set in X. For the same reason, B is also a sequentially open set in X, and which contradicts the s-connectivity of X. Thus, Z is an s-connected space. ■

By the mapping lemma (see Proposition 2.1.12), we have the following corollary.

Corollary 2.3.18 ([120, 278]) *A space X is a quotient image (resp. pseudo-open image) of a metric space if and only if X is a connected sequential space (resp. Fréchet–Urysohn space).*

Example 2.3.19 (*[278]*) There is a connected space which is not an s-connected space.

Let $\beta\mathbb{R}$ be the Čech–Stone compactification of $\mathbb{R}$. We first prove that for any $p \in \beta\mathbb{R} - \mathbb{R}$, there does not exist nontrivial sequence in $\mathbb{R}$ converging to p. In fact, if there is a nontrivial sequence $\{x_n\}$ in $\mathbb{R}$ converging to p, take $A = \{x_{2n} : n \in \mathbb{N}\}$, and $B = \{x_{2n-1} : n \in \mathbb{N}\}$, then A, B are disjoint closed sets in $\mathbb{R}$. Since $\mathbb{R}$ is normal, $\mathrm{cl}_{\beta\mathbb{R}}(A) \cap \mathrm{cl}_{\beta\mathbb{R}}(B) = \varnothing$ (see Corollary 3.6.4 of Engelking [119]), and $p \in \mathrm{cl}_{\beta\mathbb{R}}(A) \cap \mathrm{cl}_{\beta\mathbb{R}}(B)$, a contradiction.

Since $\mathbb{R}$ is connected, $\beta\mathbb{R}$ is also connected. On the other hand, $\mathbb{R}$ is both a sequentially open set and a sequentially closed set in $\beta\mathbb{R}$, so $\beta\mathbb{R}$ is not an s-connected space.

By Theorem 2.3.17, the connected space $\beta\mathbb{R}$ is not an image of any connected metric space.

In the last proposition of this section, we give the relationship between Fréchet–Urysohn spaces and strongly Fréchet–Urysohn spaces by means of products.

Proposition 2.3.20 ([336]) *A space X is a strongly Fréchet–Urysohn space if and only if $X \times \mathbb{S}_1$ is a Fréchet–Urysohn space.*

Proof Suppose X is a strongly Fréchet–Urysohn space. For any $A \subset X\times\mathbb{S}_1$, if $p \in \overline{A}$, we may assume $p = (x, 0)$. For each $n \in \mathbb{N}$, define

$$A_n = A \cap (X \times (\mathbb{S}_1 - \{1/m : m < n\})),$$
$$B_n = \pi_1(A_n).$$

Then $\{B_n\}$ is a decreasing sequence of sets in X and $x \in \bigcap_{n\in\mathbb{N}} \overline{B}_n$, so there is $x_n \in B_n$ such that $x_n \to x$, and hence there is $\{z_n\} \subset A$ such that $z_n \to p$. Thus $X \times \mathbb{S}_1$ is a Fréchet–Urysohn space.

Conversely, suppose $X \times \mathbb{S}_1$ is a Fréchet–Urysohn space. For any decreasing sequence $\{A_n\}$ of sets in X, if $x \in \bigcap_{n\in\mathbb{N}} \overline{A}_n$, take $A = \bigcup_{n\in\mathbb{N}}(A_n \times \{1/n\})$, then $(x, 0) \in \overline{A} \subset X \times \mathbb{S}_1$, so there is a sequence $\{z_i\} \subset A$ such that $z_i \to (x, 0)$, and hence there exist a subset $\{n_i : i \in \mathbb{N}\}$ of $\mathbb{N}$ and $x_i \in A_{n_i}$ such that $z_i = (x_i, 1/n_i)$ and $x_i \to x$ when $n_i \to \infty$. Consequently, there is $y_n \in A_n$ such that $y_n \to x$. Thus X is a strongly Fréchet–Urysohn space. ■

2.4 Open Mappings

In this section, we give characterizations of open images of metric spaces and paracompact M-spaces. In 1960, Ponomarev [401] proved that a space X is a first countable space if and only if it is an open image of a metrizable space. The Ponomarev theorem is one of the original motivations for the Alexandroff idea [35], and the particular method of representing a non-metrizable space as an image of a subspace of a Baire's zero-dimensional space created by Ponomarev, referred to as Ponomarev's method, is a remarkable contribution to the mapping theory of metric spaces. In the following sections of this chapter, we introduce this method systematically.

Definition 2.4.1 (*[340]*) A family $\mathscr{B}$ of open sets in a space X is called an *outer base* of a set A in X if for each $x \in A$ and $x \in U \in \tau(X)$, there is $B \in \mathscr{B}$ such that $x \in B \subset U$.

Lemma 2.4.2 (The König lemma) *Let $\{X_i\}$ be a sequence of nonempty finite sets. If for each $n < m$, there is a correspondence $\pi_n^m : X_m \to X_n$ such that $\pi_n^m = \pi_n^k \circ \pi_k^m$ and $\pi_m^m = id_{X_m}$, then there is $(x_i) \in \prod_{i\in\mathbb{N}} X_i$ such that $\pi_n^m(x_m) = x_n$.*

Proof For each $i \in \mathbb{N}$, give X_i the discrete topology. Then X_i is a compact space. Let

$$X = \prod_{i\in\mathbb{N}} X_i,$$
$$Y = \{(x_i) \in X : \pi_n^m(x_m) = x_n,\ n < m\}.$$

Then Y is a closed set in the compact space X. Because in fact, if $y = (y_i) \in X - Y$, then there is $n < m$ such that $\pi_n^m(y_m) \neq y_n$. Let $V = \{(x_i) \in X : x_m = y_m,\ x_n = y_n\}$. Then $y \in V \in \tau(X)$ and $V \cap Y = \varnothing$, so Y is a closed set in X. To complete our proof, we only need to show $Y \neq \varnothing$. For each $m \in \mathbb{N}$, take

$$Y_m = \{(x_i) \in X : \text{ if } n < m, \text{ then } \pi_n^m(x_m) = x_n\}.$$

Then Y_m is a nonempty closed set in X, and hence $\{Y_m\}$ is a sequence of closed sets in X with the finite intersection property, so $Y = \bigcap_{m\in\mathbb{N}} Y_m \neq \varnothing$. ∎

Lemma 2.4.3 ([340]) *Suppose X is a space and $K \in \mathcal{K}(X)$. If K has a countable outer base $\mathcal{U}$ in X, then there is a sequence $\{\mathcal{U}_i\}$ of finite subsets of $\mathcal{U}$ such that*

(1) for each $i \in \mathbb{N}$, $K \subset \cup\mathcal{U}_i$;
(2) if $x \in K$ and $x \in U_i \in \mathcal{U}_i$ for every $i \in \mathbb{N}$, then $\{U_i\}_{i\in\mathbb{N}}$ is a neighborhood base of x;
(3) for every $x \in K$ and $i \in \mathbb{N}$, there is $U_i \in \mathcal{U}_i$ such that $x \in \overline{U_{i+1} \cap K} \subset U_i$.

Proof Let $\{\mathcal{V}_i\}$ be the families of all finite subsets of $\mathcal{U}$ covering K. By the inductive method, we can take a subsequence $\{\mathcal{U}_i\}$ of $\{\mathcal{V}_i\}$ such that for every $i \in \mathbb{N}$, $\mathcal{U}_i$ partially refines $\mathcal{V}_i$ and $\{\overline{U \cap K} : U \in \mathcal{U}_{i+1}\}$ partially refines $\mathcal{U}_i$. Below we verify that $\{\mathcal{U}_i\}$ satisfies conditions (1)–(3).

(1) holds obviously. For each $x \in K$ and $x \in U_i \in \mathcal{U}_i$, let $x \in W \in \tau$. Take $V \in \mathcal{U}$ and $\mathcal{V} \in \mathcal{U}^{<\omega}$ such that $x \in V \subset W$ and $K - V \subset \cup\mathcal{V} \subset X - \{x\}$. Then there is $m \in \mathbb{N}$ such that $\mathcal{V} \cup \{V\} = \mathcal{V}_m$, so $U_m \subset V$, and hence $\{U_i\}_{i\in\mathbb{N}}$ is a neighborhood base of x. To verify (3), define $\mathcal{W}_i = (\mathcal{U}_i)_x$ for every $x \in K$ and $i \in \mathbb{N}$. Then $\mathcal{W}_i$ is finite and satisfies that if $U_{i+1} \in \mathcal{W}_{i+1}$, then there is $U_i \in \mathcal{W}_i$ such that $\overline{U_{i+1} \cap K} \subset U_i$. By the König lemma, there is $U_i \in \mathcal{U}_i$ such that $x \in \overline{U_{i+1} \cap K} \subset U_i$ for each $i \in \mathbb{N}$. ∎

Proposition 2.4.4 ([340]) *Suppose X is a first countable space. If $\mathcal{U}$ is a base for X, then there exist a metrizable space M and an open mapping $f : M \to X$ such that*

(1) if $K \in \mathcal{K}(X)$ and K has a countable outer base $\mathcal{U}_K \subset \mathcal{U}$, then there is $L \in \mathcal{K}(M)$ such that $f(L) = K$;
(2) if $(\mathcal{U})_E$ is countable for some $E \subset X$, then $f^{-1}(E)$ has a countable base.

Proof Denote $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$. For each $i \in \mathbb{N}$, let Λ_i be the set Λ with the discrete topology. Define

$$M = \left\{\beta = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Lambda_i : \{U_{\alpha_i}\} \text{ is a neighborhood base of some point } x(\beta) \text{ in } X\right\}.$$

Then M is a metrizable space. For each $\beta \in M$, $x(\beta)$ is the only point being determined, so we can define a function $f : M \to X$ by $f(\beta) = x(\beta)$.

(4.1) f is continuous.

By the first countability of X, f is an onto mapping. Let $\beta = (\alpha_i) \in M$ and $f(\beta) = x \in U \in \tau(X)$. Then there is $m \in \mathbb{N}$ such that $x \in U_{\alpha_m} \subset U$. Put

$$V = \{\gamma \in M : \pi_m(\gamma) = \alpha_m\}.$$

Then $\beta \in V \in \tau(M)$ and $f(V) \subset U_{\alpha_m} \subset U$. Thus f is continuous.

(4.2) f is an open mapping.
For every $n \in \mathbb{N}$ and $\alpha_i \in \Lambda_i$ ($\forall i \leqslant n$), put

$$B(\alpha_1, \ldots, \alpha_n) = \{\beta \in M : \pi_i(\beta) = \alpha_i, i \leqslant n\}.$$

If $\beta = (\beta_i) \in B(\alpha_1, \ldots, \alpha_n)$, then

$$f(\beta) \in \bigcap_{i \in \mathbb{N}} U_{\beta_i} \subset \bigcap_{i \leqslant n} U_{\alpha_i}.$$

So $f(B(\alpha_1, \ldots, \alpha_n)) \subset \bigcap_{i \leqslant n} U_{\alpha_i}$. If $x \in \bigcap_{i \leqslant n} U_{\alpha_i}$, for each $i > n$, take $\alpha_i \in \Lambda_i$ such that $\{U_{\alpha_i}\}_{i>n}$ is a neighborhood base of x. Let $\beta = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i$. Then $\beta \in B(\alpha_1, \ldots, \alpha_n)$ and $f(\beta) = x$, so $\bigcap_{i \leqslant n} U_{\alpha_i} \subset f(B(\alpha_1, \ldots, \alpha_n))$. Thus,

$$f(B(\alpha_1, \ldots, \alpha_n)) = \bigcap_{i \leqslant n} U_{\alpha_i}.$$

Since $\{B(\alpha_1, \ldots, \alpha_n) : \alpha_i \in \Lambda_i, i \leqslant n\}$ is a base for M, f is an open mapping. By the definition of f, (2) holds. Below we prove (1).

(4.3) If $K \in \mathscr{K}(X)$ and K has a countable outer base $\mathscr{U}_K \subset \mathscr{U}$, then there is a sequence $\{\mathscr{U}_i\}$ of finite subsets of $\mathscr{U}_K$ satisfying all conditions of Lemma 2.4.3.
For each $i \in \mathbb{N}$, there is $\Gamma_i \in \Lambda_i^{<\omega}$ such that $\mathscr{U}_i = \{U_{\alpha_i} : \alpha_i \in \Gamma_i\}$. We may assume that $U_{\alpha_i} \cap K \neq \varnothing$. Let

$$L = \left\{\beta = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Gamma_i : \overline{U_{\alpha_{i+1}} \cap K} \subset U_{\alpha_i} \text{ for each } i \in \mathbb{N}\right\}.$$

Then L is a closed set in $\prod_{i \in \mathbb{N}} \Gamma_i$. Because in fact, if $(\alpha_i) \in \prod_{i \in \mathbb{N}} \Gamma_i - L$, then there is $m \in \mathbb{N}$ such that $\overline{U_{\alpha_{m+1}} \cap K} \not\subset U_{\alpha_m}$. Let $W = \{(\beta_i) \in \prod_{i \in \mathbb{N}} \Gamma_i : \beta_m = \alpha_m\}$. Then $(\alpha_i) \in W \in \tau(\prod_{i \in \mathbb{N}} \Gamma_i)$ and $W \cap L = \varnothing$. Thus L is a compact set in $\prod_{i \in \mathbb{N}} \Gamma_i$. Since $K \cap (\bigcap_{i \in \mathbb{N}} U_{\alpha_i}) \neq \varnothing$ for each $\beta = (\alpha_i) \in L$, we can take $x \in K \cap (\bigcap_{i \in \mathbb{N}} U_{\alpha_i})$. Then $\beta \in M$ and $f(\beta) = x$, so $L \subset M$ and $f(L) \subset K$. By Lemma 2.4.3(3), $K \subset f(L)$. Consequently, $L \in \mathscr{K}(M)$ and $f(L) = K$. ■

Theorem 2.4.5 ([401]) (The Ponomarev theorem) *A space X is a first countable space if and only if X is an open image of a metrizable space.*

Proof The necessity comes from Proposition 2.4.4 and the sufficiency is obtained by the fact that first countability is invariant under open mappings. ■

Theorem 2.4.6 ([58]) *Any countably compact subset of a space X having a quasi-G_δ-diagonal is a compact metrizable G_δ-subset of X.*

Proof Let $\{\mathscr{U}_n\}$ be a quasi-G_δ-diagonal sequence for X, and C be a countably compact subset of X. It is easy to see that $\{\mathscr{U}_{n|C}\}$ is a quasi-G_δ-diagonal sequence for C. By Theorem 1.4.10, C is compact.

(1) Every closed set of C is a G_δ-set in C.
Let M be a closed subset of C, and assume that M has no isolated points. Let I be the set of isolated points of M. When $M - I$ is a G_δ-set, it easily follows that M itself is a G_δ-set. For each $x \in M$, there is a strictly increasing sequence $\{m(i,x)\}_{i\in\mathbb{N}}$ of positive integers such that for all $n \in \mathbb{N}$, $x \in \cup\mathscr{U}_n$ if and only if $n = m(i,x)$ for some $i \leqslant n$. For each $i \in \mathbb{N}$, pick $U(i,x) \in \mathscr{U}_{m(i,x)}$ such that $x \in U(i,x)$.
Let $H(1,x) = U(1,x) \cap C$ for each $x \in M$. Then $\{H(1,x) : x \in M\}$, together with $C - M$, covers C. Let $\mathscr{W}_1$ be a finite subcover of this cover. Note the fact that if $W \in \mathscr{W}_1$ and $W \cap M \neq \varnothing$, then there is some $x \in M$ such that $W \subset U(1,x)$.
Next, for each $x \in M$, by the regularity of C, let $V(2,x)$ be an open neighborhood of x in C such that $\mathrm{cl}_C[V(2,x)] = \overline{V(2,x)}$ is contained in some element of $\mathscr{W}_1$. Let $H(2,x) = V(2,x) \cap U(2,x)$. Then $\{H(2,x) : x \in M\}$, together with $C - M$, covers C. Let $\mathscr{W}_2$ be a finite subcover of this cover.
Continue this process. Then for each $n \in \mathbb{N}$ we obtain a finite cover $\mathscr{W}_n$ of open subsets of C and a family $\{H(n,x) : x \in M\}$ of open sets, such that, the following hold for all $n \in \mathbb{N}$:

(1.1) if $W \in \mathscr{W}_n$ and $W \cap M \neq \varnothing$, then $W \subset H(n,x)$ for some $x \in M$;
(1.2) for all $x \in M$, $x \in H(n,x) \subset \bigcap_{i\leqslant n} U(i,x)$;
(1.3) for all $x \in M$ and all $j \leqslant n$, $\overline{H(n+1,x)}$ is contained in some element of $\mathscr{W}_j$.

Now let $V_n = \mathrm{st}(M, \mathscr{W}_n)$. Clearly each V_n is open in C and $M \subset \bigcap_{n\in\mathbb{N}} V_n$. Let $p \in \bigcap_{n\in\mathbb{N}} V_n - M$. Let $\{W_i\}_{i\leqslant t_1}$ be all elements of $\mathscr{W}_1$ which contain p and intersect M. Such elements of $\mathscr{W}_1$ exist since $p \in V_1$. By (1.1), for each $r \leqslant t_1$, there is some $x_r \in M$ such that $W_r \subset H(1,x_r)$. Let $T_1 = \{x_r : r \leqslant t_1\}$ and $j_1 = 1$.
Now let $j_2 = j_1 + 1$. Let $\{W'_i\}_{i\leqslant t_2}$ be all elements of $\mathscr{W}_{j_2}$ which contain p and intersect M. Such elements exist since $p \in V_{j_2}$. By (1.2), for $r \leqslant t_2$, there is some $x'_r \in M$ such that $W'_r \subset H(j_2, x'_r)$. By (1.3), for each $r \leqslant t_2$, $\overline{H(j_2,x'_r)}$ is contained in some element of $\mathscr{W}_{j_1}$. It easily follows that $\overline{H(j_2,x'_r)} \subset H(j_1,x_s)$ for some $x_s \in T_1$. Let $T_2 = \{x'_r : r \leqslant t_2\}$.
Continuing this process, we obtain a strictly increasing sequence $\{j_n\}_{n\in\mathbb{N}}$ of positive integers and a nonempty finite subset T_n of M for each $n \in \mathbb{N}$, such that, the following hold for all $n \in \mathbb{N}$:

(1.4) if $x \in T_n$, then $p \in H(j_n,x)$;
(1.5) if $x \in T_{n+1}$, then $\overline{H(j_{n+1},x)} \subset H(j_n,y)$ for some $y \in T_n$.

For each $n \in \mathbb{N}$, let $\mathscr{H}_n = \{H(j_n,x) : x \in T_n\}$. Then by Lemma 2.4.2, there is a sequence $\{x_n\}$ in M such that $p \in H(j_n,x_n)$ and $\overline{H(j_{n+1},x_{n+1})} \subset H(j_n,x_n)$

for all $n \in \mathbb{N}$. Since C is compact, $\{x_n\}$ has an accumulation point, say q. Clearly $q \in M$, and it is easy to check that $q \in \overline{H(j_n, x_n)}$, $n \in \mathbb{N}$. (Recall that $x_n \in H(j_n, x_n)$ by (1.2), and that $\overline{H(j_{n+1}, x_{n+1})} \subset H(j_n, x_n)$.) Now $p \notin M$, so $p \neq q$. Hence there is some $n_0 \in \mathbb{N}$ such that $p \notin \mathrm{st}(q, \mathscr{U}_{n_0}) \neq \varnothing$. Choose $n \geqslant n_0$ such that $x_n \in \mathrm{st}(q, \mathscr{U}_{n_0})$. Since $x_n \in \cup\mathscr{U}_{n_0}$, there is some $i_0 \leqslant n_0$ such that $n_0 = m(i_0, x_n)$. Note that $U(i_0, x_n) \in \mathscr{U}_{m(i_0,x_n)} = \mathscr{U}_{n_0}$ and $i_0 \leqslant n \leqslant j_n$. Now $\{p, q\} \subset \overline{H(j_n, x_n)} \subset \bigcap_{i \leqslant j_n} U(i, x_n)$, so both p and q belong to $U(i_0, x_n)$. Thus $p \in \mathrm{st}(q, \mathscr{U}_{n_0})$, a contradiction. Hence $p \in M$, and so the proof that M is a G_δ-set in C is completed.

(2) C is metrizable.
Suppose $\{\mathscr{V}_n\}$ is a quasi-G_δ-diagonal sequence for C. For every $n \in \mathbb{N}$, by (1), there is a sequence $\{C_{n,j}\}$ of closed sets in C such that $\cup\mathscr{V}_n = \bigcup_{j\in\mathbb{N}} C_{n,j}$. Let $\mathscr{V}_{n,j} = \mathscr{V}_n \cup \{C - C_{n,j}\}$ for every $n, j \in \mathbb{N}$. Then $\{\mathscr{V}_{n,j}\}$ is a G_δ-diagonal sequence for C. By Proposition 1.4.9, C has a G_δ^*-diagonal. Assume that $\{\mathscr{G}_n\}$ is a G_δ^*-diagonal sequence for C such that $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$ for each $n \in \mathbb{N}$. For every $x \in O \in \tau(C)$, $\{O\} \cup \{C - \overline{\mathrm{st}(x, \mathscr{G}_n)} : n \in \mathbb{N}\}$ is an open cover of C which contains a finite subcover. So there exists $m \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{G}_m) \subset O$. It follows that $\{\mathscr{G}_n\}$ is a development of C. By the Bing metrization criterion, C is metrizable.

(3) C is a G_δ-subset in X.
We may assume that $\mathscr{U}_1 = \{X\}$. Being a compact and metrizable subset of X, C is hereditarily Lindelöf, so that there is a countable subfamily $\mathscr{U}_n' \subset \mathscr{U}_n$ that covers $C \cap \bigcup \mathscr{U}_n$. Let $\mathscr{U} = \bigcup_{n\in\mathbb{N}} \mathscr{U}_n'$. Because $\mathscr{U}$ is countable, we may index it as $\mathscr{U} = \{U_i : i \in \mathbb{N}\}$. For each $x \in C$ and $n \in \mathbb{N}$, let $U'(x, n) = \bigcap_{i \leqslant n}\{U_i : x \in U_i\}$. Let $G_n = \bigcup_{x\in M} U'(x, n)$. Then $C \subset \bigcap_{n\in\mathbb{N}} G_n$. For a contradiction, suppose that there is some point $z \in \bigcap_{n\in\mathbb{N}} G_n - C$. Choose a point $x_i \in C$ with $z \in U'(x_i, i)$ for each $i \in \mathbb{N}$. Because $x_i \in C$, there is an accumulation point p of the sequence $\{x_i\}$ in C. Because $p \neq z$, we may find $m \in \mathbb{N}$ such that $p \in \mathrm{st}(p, \mathscr{U}_m) \subset X - \{z\}$. Because $p \in C \cap \bigcup \mathscr{U}_m \subset C \cap \bigcup \mathscr{U}_m'$, some $U \in \mathscr{U}_m'$ has $p \in U$. Then U appears somewhere in the listing of $\mathscr{U}$ given above, say $U = U_k$ for some $k \in \mathbb{N}$. Because p is an accumulation point of the sequence $\{x_i\}$ and $p \in U = U_k$, there is some $j > k$ with $x_j \in U_k$. But then we have $z \in U'(x_j, j) \subset U_k \subset \mathrm{st}(p, \mathscr{U}_m) \subset X - \{z\}$ and that is impossible. Hence C is a G_δ-set in X. ■

Question 2.4.7 (*[465]*) Is every first countable connected space an open image of a connected metric space?

In the second part of this section, we introduce open images of paracompact M-spaces.

Proposition 2.4.8 ([335, 475]) *Let $\mathscr{K}$ and $\mathscr{P}$ be covers of a space Y, where the elements of $\mathscr{K}$ are countably compact closed subsets of Y and $\mathscr{P}$ is closed under finite intersections. If for every $y \in P \in \mathscr{P}$, there is $K \in \mathscr{K}$ such that*

(i) $y \in K \subset P$,
(ii) *some countable subfamily of* $\mathscr{P}$ *is a network for* K *in* Y,

then there exist a metrizable space M, *a* σ*-discrete base* $\mathscr{B}$ *of* M *and a subspace* X *of* $Y \times M$ *satisfying the following conditions: let* $f = \pi_{1|X}$ *and* $g = \pi_{2|X}$, *then*

(1) $\mathscr{P} = f(g^{-1}(\mathscr{B}))$;
(2) *if* $\beta \in M$, *then* $f(g^{-1}(\beta)) \in \mathscr{K}$;
(3) g *is a closed mapping;*
(4) *for each* $E \subset Y$, $|\{B \in \mathscr{B} : B \cap g(f^{-1}(E)) \neq \varnothing\}| \leqslant \aleph_0 \cdot |(\mathscr{P})_E|$.

Proof Denote $\mathscr{P} = \{P_\alpha\}_{\alpha \in \Lambda}$. For each $i \in \mathbb{N}$, let Λ_i be the set Λ with the discrete topology. Define

$$M = \Big\{\beta = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i : \{P_{\alpha_i}\} \text{ is a decreasing network of some element } K_\beta \text{ of } \mathscr{K} \text{ in } Y\Big\}.$$

Then M is a metrizable space and for each $\beta \in M$, K_β is the only point being determined. Let

$$X = \{(y, \beta) \in Y \times M : y \in K_\beta\}.$$

For every $n \in \mathbb{N}$ and $\alpha_i \in \Lambda_i$ $(\forall i \leqslant n)$, define

$$B(\alpha_1, \ldots, \alpha_n) = \{\beta \in M : \pi_i(\beta) = \alpha_i, i \leqslant n\},$$
$$\mathscr{B} = \{B(\alpha_1, \ldots, \alpha_n) : \alpha_i \in \Lambda_i, i \leqslant n \in \mathbb{N}\}.$$

Then $\mathscr{B}$ is a σ-discrete base of M.

(1) $f(g^{-1}(B(\alpha_1, \ldots, \alpha_n))) = P_{\alpha_n}$.
Obviously, $f(g^{-1}(B(\alpha_1, \ldots, \alpha_n))) \subset P_{\alpha_n}$. If $y \in P_{\alpha_n}$, then there is $K \in \mathscr{K}$ such that $y \in K \subset P_{\alpha_n}$ and K has a countable network $\mathscr{F} \subset \mathscr{P}$ in Y. We may denote $\mathscr{F} = \{P_{\alpha_i}\}_{i>n}$, where $\alpha_i \in \Lambda_i$ and $P_{\alpha_n} \supset P_{\alpha_i} \supset P_{\alpha_{i+1}}$. Let $\beta = (\alpha_i)$. Then $\beta \in B(\alpha_1, \ldots, \alpha_n)$ and $y \in K = f(g^{-1}(\beta))$, so $P_{\alpha_n} \subset f(g^{-1}(B(\alpha_1, \ldots, \alpha_n)))$. Thus $f(g^{-1}(B(\alpha_1, \ldots, \alpha_n))) = P_{\alpha_n}$.
(2) For each $\beta \in M$, $f(g^{-1}(\beta)) = K_\beta \in \mathscr{K}$ by the definition.
(3) For each closed set C in X, let $\beta = (\alpha_i) \in \overline{g(C)}$. Then for every $n \in \mathbb{N}$, there is $(y_n, \beta^{(n)}) \in (Y \times B(\alpha_1, \ldots, \alpha_n)) \cap C$, so $y_n \in P_{\alpha_n}$, and hence $\{y_n\}$ has an accumulation point, we denote it by y_0. Since $\{\beta^{(n)}\}$ converges to β in M, $(y_0, \beta) \in C$, thus $\beta \in g(C)$. Consequently, g is a closed mapping.
(4) By (1), for each $E \subset Y$, $|\{B \in \mathscr{B} : B \cap g(f^{-1}(E)) \neq \varnothing\}| \leqslant \aleph_0 \cdot |(\mathscr{P}_E)|$. ■

In the above proof of Proposition 2.4.8, by (1), one can see that f is an onto mapping. Moreover, if $\mathscr{P}$ is an open cover of Y, then f is an open mapping. By (2), g is an onto mapping. By (3), g is a quasi-perfect mapping, thus X is an M-space.

Definition 2.4.9 (*[28]*) A space X is called a *pointwise countable type space* (or *space of pointwise countable type*) if for each $x \in X$, there is a compact set K containing x such that K has a countable neighborhood base in X.

Obviously, every first countable space is of pointwise countable type, and every space of pointwise countable type is a q-space.

Proposition 2.4.10 ([28]) *Every regular space with a p-sequence is of pointwise countable type.*

Proof Let $\{\mathscr{U}_n\}$ be a p-sequence in a regular space X. For every $x \in X$ and $n \in \mathbb{N}$, take $U_n \in (\mathscr{U}_n)_x$. By the regularity of X, there is a sequence $\{V_n\}$ of open subsets of X such that $x \in V_{n+1} \subset \overline{V}_{n+1} \subset V_n \subset U_n$ for each $n \in \mathbb{N}$. Let $K = \bigcap_{n\in\mathbb{N}} V_n$. By the convergence lemma, $x \in K \in \mathscr{K}(X)$ and $\{V_n\}_{n\in\mathbb{N}}$ is a countable neighborhood base of K in X. Thus X is of pointwise countable type. ■

Proposition 2.4.11 ([28]) *Every space of pointwise countable type is a k-space.*

Proof Let X be a space of pointwise countable type. Suppose there is a set A which is not closed in X such that $K \cap A \in \tau^c$ for each $K \in \mathscr{K}(X)$. Take $x \in \overline{A} - A$. Then there is a compact set C in X containing x such that C in X has a decreasing neighborhood base $\{U_i\}_{i\in\mathbb{N}}$. Choose an open neighborhood V of x such that $\overline{V} \cap C \cap A = \varnothing$. Let $B = \{x_i : i \in \mathbb{N}\}$, where $x_i \in A \cap V \cap U_i$. Then $B \cup C \in \mathscr{K}(X)$, and hence $B = A \cap (\overline{V} \cap (B \cup C)) \in \mathscr{K}(X)$. Since $C \cap B = \varnothing$, there is $i \in \mathbb{N}$ such that $U_i \cap B = \varnothing$, a contradiction. Consequently, X is a k-space. ■

Lemma 2.4.12 ([475]) *Let X be of pointwise countable type. If $x \in U \in \tau$, then there is $K \in \mathscr{K}(X)$ such that $x \in K \subset U$ and K has a countable neighborhood base in X.*

Proof Choose $L \in \mathscr{K}(X)$ such that $x \in L$ and L has a decreasing neighborhood base $\{U_i\}_{i\in\mathbb{N}}$ in X. By the inductive method, there is a decreasing sequence $\{V_i\}$ of open subsets in X such that $x \in V_i \subset U_i \cap U$ and $\overline{V}_{i+1} \cap L \subset V_i \cap L$. We notice that if V_n has been selected, then by the compactness of L, there is $V_{n+1} \in \tau$ such that $x \in V_{n+1} \subset V_n \cap U_{n+1}$ and $\overline{V}_{n+1} \cap L - V_n = \varnothing$. Define $K = L \cap \bigcap_{i\in\mathbb{N}} V_i$. Then $K \in \mathscr{K}(X)$ and $x \in K \subset U$. If $K \subset W \in \tau$, then $L \subset W \cup \bigcup_{i\in\mathbb{N}}(X - \overline{V}_i)$, so there is $m \in \mathbb{N}$ such that $L \subset W \cup (X - \overline{V}_m)$, and hence there is $k \geqslant m$ such that $U_k \subset W \cup (X - \overline{V}_m)$, thus $V_k \subset U_k \cap \overline{V}_m \subset W$. Therefore, $\{V_i\}_{i\in\mathbb{N}}$ is a neighborhood base of K in X. ■

Theorem 2.4.13 ([475]) *A space X is of pointwise countable type if and only if X is an open image of a paracompact M-space.*

Proof The necessity can be obtained by Lemma 2.4.12, Proposition 2.4.8 and the remark after Proposition 2.4.8. To prove the sufficiency, we only need to show that every paracompact M-space is of pointwise countable type and spaces of pointwise countable type are preserved by open mappings.

Suppose X is a paracompact space and $\{\mathscr{U}_n\}$ is an M-sequence in X. Then for every $x \in X$ and $n \in \mathbb{N}$, $\overline{\mathrm{st}(x, \mathscr{U}_{n+1})} \subset \mathrm{st}(x, \mathscr{U}_n)$. By the convergence lemma, $\{\mathrm{st}(x, \mathscr{U}_n)\}_{n\in\mathbb{N}}$ is a neighborhood base of the compact set $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n)$ containing x in X. So X is of pointwise countable type.

Let $f : X \to Y$ be an open mapping, where X is of pointwise countable type. For each $y \in Y$, pick $x \in f^{-1}(y)$. Then there is a compact set K containing x such that K has a decreasing neighborhood base $\{U_i\}_{i\in\mathbb{N}}$ in X, and hence $\{f(U_i)\}_{i\in\mathbb{N}}$ is a neighborhood base of the compact set $f(K)$ containing y in Y. Thus Y is of pointwise countable type. ■

Arhangel'skiĭ [28] also introduced the concept of countable type spaces: a space X is called a *countable type space* or of *countable type* if, for each compact subset F of X, there is a compact set K in X containing F such that K has a countable neighborhood base in X. Obviously, every countable type space is of pointwise countable type, and countable type spaces are preserved by compact-covering open mappings. Choban [103] proved that every regular space with a p-sequence is of countable type. Wicke [475] proved that a space X is of countable type if and only if X is a compact-covering open image of a paracompact p-space. The Michael line (see Example 1.8.5) is of countable type which is not a p-space.

The property of pointwise countable type, q-space property, sequential space property, k-space property, strongly Fréchet–Urysohn property, Fréchet–Urysohn property, g-first countability and so on are all topological properties weaker than the first countability, and hence these properties are collectively called the *generalized sequentiality properties*.

Example 2.4.14 None of the following spaces is preserved by perfect mappings: first countable spaces, spaces of pointwise countable type, q-spaces or g-first countable spaces. Hence, perfect mappings do not preserve open images of metric spaces or of paracompact M-spaces.

If X is the butterfly space (see Example 1.8.3), then X is a first countable space. Let $K = \mathbb{I} \times \{0\}$. Then K is a compact set in X. Let $f : X \to X/K$ be the quotient mapping. Then f is a perfect mapping and X/K is not a first countable space. Since X/K is a regular Fréchet–Urysohn space of the point G_δ-property, X/K is neither a q-space (see Theorem 1.7.7) nor a g-first countable space (see Corollary 1.6.18).

Example 2.4.15 Some implication relationships do not exist among the generalized sequentiality properties.

(1) There is a g-first countable space which is neither a Fréchet–Urysohn space nor a q-space. The Arens space S_2 (see Example 1.8.6) is such a space.
(2) There is a compact space (hence a space of pointwise countable type) which is not a sequential space. The compactification $\beta\mathbb{N}$ is such a space.
(3) There is a Fréchet–Urysohn space which is not a strongly Fréchet–Urysohn space. The sequential fan S_ω (see Example 1.8.7) is such a space.
(4) There is a q-space which is not a k-space. Let X be the Frolík space [134], i.e. $\mathbb{N} \subset X \subset \beta\mathbb{N}$ and X is a countably compact subspace of $\beta\mathbb{N}$ with cardinality

not greater than $\mathfrak{c}$. Then every compact set in X is a finite set, and hence X is not a k-space.

(5) There is a paracompact and first countable space which is not a p-space. The butterfly space is such a space.

Example 2.4.16 A finite-to-one open image of a metric space may not be metrizable.

Denote the V-space (see Example 1.8.1) by X. For each $r \in \mathbb{R}$, let $X_r = \{(x, y) \in \mathbb{R}^2 : y = |x - r|\}$. Then X_r is a metrizable clopen subspace of X and $\{X_r : r \in \mathbb{R}\}$ is a point-finite open cover of X. Let $M = \bigoplus_{r\in\mathbb{R}} X_r$ and let $f : M \to X$ be the natural mapping. Then M is a metric space and f is a finite-to-one open mapping. We prove that f is a compact-covering mapping. For each compact set K in X, let $K_0 = K \cap (\mathbb{R} \times \{0\})$ and $K_1 = K - \cup\{X_r : r \in \pi_1(K_0)\}$. Since $\mathbb{R} \times \{0\}$ is a discrete closed subspace of X, K_0 is a finite set. Since K_1 is a compact set of the discrete space $X - (\mathbb{R} \times \{0\})$, K_1 is finite. Hence there is a compact set L in M such that $f(L) = K$.

Question 2.4.17 *[411] (R.C. Olson)* Is every quotient L-mapping from a space with a point-countable base onto a space of pointwise countable type a countably bi-quotient mapping?

2.5 Closed Mappings

Seeking characterizations of the closed images of metric spaces is a problem raised by Arhangel'skiĭ [31]. Lašnev [240] studied this problem and gave the first solution, and Foged [130] gave another answer to Arhangel'skiĭ's problem. This section consists of three parts. In the first part, we introduce the characterizations of closed images of metric spaces, which contains the famous Foged theorem. In the second part, we investigate properties of *HCP* families and several metrization theorems are obtained. In the third part, we discuss the countable productivity of closed images of metric spaces.

Definition 2.5.1 (*[427]*) A space X is called a *Lašnev space* if X is a closed image of a metric space.

Obviously, Lašnev spaces are additive and hereditary. By Proposition 2.3.20, $S_\omega \times \mathbb{S}_1$ is not a Lašnev space, and hence Lašnev spaces are not finitely productive.

Definition 2.5.2 Suppose $\mathscr{P}$ is a family of sets in a space X.

(1) $\mathscr{P}$ is said to be *hereditarily closure-preserving* in X if, for every $H(P) \subset P \in \mathscr{P}$, $\{H(P) : P \in \mathscr{P}\}$ is closure-preserving [240].
(2) $\mathscr{P}$ is said to be *weakly hereditarily closure-preserving* if, for every $x(P) \in P \in \mathscr{P}$, the family $\{\{x(P)\} : P \in \mathscr{P}\}$ is closure-preserving, i.e. $\{x(P) : P \in \mathscr{P}\}$ is a discrete closed subspace in X [87].
(3) $\mathscr{P}$ is said to be *countably hereditarily closure-preserving* (resp. *countably weakly hereditarily closure-preserving*) if every countable subfamily of $\mathscr{P}$ is hereditarily closure-preserving (resp. weakly hereditarily closure-preserving) [138].

(4) $\mathscr{P}$ is said to be *CF* if the family $\{K \cap P : P \in \mathscr{P}\}$ is finite for each compact set $K \subset X$ [347].

Hereditarily closure-preserving can be simply expressed as *HCP* [87]. Obviously, locally finite $\Rightarrow$ *HCP* $\Rightarrow$ weakly *HCP* and closure-preserving; compact-finite $\Rightarrow$ *CF*. It is easy to verify that *HCP* families and weakly *HCP* families are preserved by closed mappings.

Proposition 2.5.3 ([257]) *If $\mathscr{P}$ is an HCP family in a regular space X, then $\overline{\mathscr{P}}$ is also an HCP family in X.*

Proof Let $\mathscr{P} = \{P_\alpha\}_{\alpha\in\Lambda}$. If $\overline{\mathscr{P}}$ is not an *HCP* family in X, then for each $\alpha \in \Lambda$, there is $H_\alpha \subset \overline{P_\alpha}$ such that $\bigcup_{\alpha\in\Lambda} \overline{H_\alpha} \notin \tau^c$. Take $x \in \overline{\bigcup_{\alpha\in\Lambda} H_\alpha} - \bigcup_{\alpha\in\Lambda} \overline{H_\alpha}$. For each $\alpha \in \Lambda$, there exist $V_\alpha, U_\alpha \in \tau$ such that $x \in V_\alpha$, $\overline{H_\alpha} \subset U_\alpha$ and $V_\alpha \cap U_\alpha = \varnothing$, and hence $H_\alpha \subset U_\alpha \cap \overline{P_\alpha} \subset \overline{U_\alpha \cap P_\alpha}$. Thus

$$x \in \overline{\cup\{\overline{U_\alpha \cap P_\alpha} : \alpha \in \Lambda\}} = \cup\{\overline{U_\alpha \cap P_\alpha} : \alpha \in \Lambda\}.$$

Hence there is $\beta \in \Lambda$ such that $x \in \overline{U_\beta \cap P_\beta}$, so $U_\beta \cap P_\beta \cap V_\beta \neq \varnothing$, a contradiction. Consequently, $\overline{\mathscr{P}}$ is an *HCP* family in X. ■

Lemma 2.5.4 *Suppose $\mathscr{P}$ is a countably weakly HCP family in a space X. Let*

$$D = \{x \in X : \mathscr{P} \text{ is not point-finite at } x\}.$$

(1) If K is a countably compact subset of X, then there is a finite subset F of K such that $(\mathscr{P})_{K-F}$ is finite [386].
(2) $\{P - D : P \in \mathscr{P}\} \cup \{\{x\} : x \in D\}$ is compact-finite [405].
(3) $\mathscr{P}$ is a CF family [347].

Proof Let K be a nonempty countably compact subset of X. First, $K \cap D$ is a finite set. Because otherwise, there exist an infinite subset $\{x_i : i \in \mathbb{N}\}$ of K and an infinite subfamily $\{P_i : i \in \mathbb{N}\}$ of $\mathscr{P}$ such that $x_i \in P_i$ for each $i \in \mathbb{N}$. It follows that $\{x_i : i \in \mathbb{N}\}$ is a closed discrete subset of K, which contradicts the countable compactness of K. Let $F = K \cap D$. Then F is a finite subset of K. If there is an infinite subfamily $\{Q_i : i \in \mathbb{N}\}$ of $\mathscr{P}$ such that $Q_i \cap (K - F) \neq \varnothing$ for each $i \in \mathbb{N}$, then take a sequence $\{y_i\}$ in K such that $y_i \in Q_i - D$ for each $i \in \mathbb{N}$. Since $\mathscr{P}$ is point-finite at each point y_i and $\mathscr{P}$ is a countably weakly *HCP* family, $\{y_i : i \in \mathbb{N}\}$ is a closed discrete subset of K, a contradiction. So $(\mathscr{P})_{K-F}$ is finite, and hence (1) is proved.

Since $K \cap (P - D) = (K - F) \cap P$, $\{P - D : P \in \mathscr{P}\} \cup \{\{x\} : x \in D\}$ is compact-finite, so (2) is proved, and hence (3) is also proved. ■

Lemma 2.5.5 ([130, 222]) *If $\mathscr{P}$ a countably weakly HCP family in a Fréchet–Urysohn space X, then $\{\cap\mathscr{P}^* : \mathscr{P}^* \in \mathscr{P}^{<\omega}\}$ is an HCP family.*

Proof Otherwise, there exist an index set Λ and $\mathscr{P}_\alpha \in \mathscr{P}^{<\omega}$, $G_\alpha \subset \cap\overline{\mathscr{P}_\alpha}$, such that, $\bigcup_{\alpha\in\Lambda}\overline{G}_\alpha$ is not a closed subset of X. So there is a sequence $\{x_n\} \subset \bigcup_{\alpha\in\Lambda}\overline{G}_\alpha$ such that $x_n \to x \in X - \bigcup_{\alpha\in\Lambda}\overline{G}_\alpha$. For each $n \in \mathbb{N}$, there is $\alpha(n) \in \Lambda$ such that $x_n \in \overline{G}_{\alpha(n)}$. We may assume that all $\alpha(n)$ are different. Since $\mathscr{P}_\alpha$ is finite, there is a subsequence $\{\alpha(n_k)\}$ of $\{\alpha(n)\}$ such that $\overline{\mathscr{P}_{\alpha(n_k)}} - \bigcup_{i<k}\overline{\mathscr{P}_{\alpha(n_i)}} \neq \varnothing$. For each $k \in \mathbb{N}$, pick $P_k \in \mathscr{P}_{\alpha(n_k)}$. Then $x_{n_k} \in \overline{P}_k$. Since X is a Fréchet–Urysohn space, there is a sequence $\{y_{k,m}\}_{m\in\mathbb{N}}$ in P_k converging to x_{n_k}. Then, $x \in \overline{\{y_{k,m} : k, m \in \mathbb{N}\}}$, and hence there is a sequence $\{y_{k_i,m_i}\}_{i\in\mathbb{N}}$ in $\{y_{k,m} : k, m \in \mathbb{N}\}$ converging to x. Since $\mathscr{P}$ is a countably weakly *HCP* family, we may assume $x_{n_k} \notin P_k$, so $\{k_i : i \in \mathbb{N}\}$ is infinite, and hence for each $m \in \mathbb{N}$, $\{P \in \bigcup_{k\in\mathbb{N}}\mathscr{P}_{\alpha(n_k)} : \{y_{k_i,m_i} : i \geqslant m\} \cap P \neq \varnothing\}$ is infinite, a contradiction. ■

Lemma 2.5.6 ([130]) *Suppose there is a k-network $\bigcup_{n\in\mathbb{N}}\mathscr{P}_n$ for a Fréchet–Urysohn space X such that $\mathscr{P}_n \subset \mathscr{P}_{n+1}$. If a sequence Z in X converges to $x \in U - Z$, where $U \in \tau$, then there is $m \in \mathbb{N}$ such that Z is eventually in int$(\cup\{P \in \mathscr{P}_m : P \subset U\})$.*

Proof Define $\mathscr{P}_n^* = \{P \in \mathscr{P}_n : P \subset U\}$ for each $n \in \mathbb{N}$. If the lemma is not true, then we can choose a subsequence $\{z_n\}$ of Z such that $z_n \in U - \mathrm{int}(\cup\mathscr{P}_n^*) \subset \overline{U - \cup\mathscr{P}_n^*}$. So there is a sequence $\{z_{n,k}\}_k \subset U - \cup\mathscr{P}_n^*$ such that $z_{n,k} \to z_n$. Thus $x \in \overline{\{z_{n,k} : n, k \in \mathbb{N}\}}$, and hence there is a subsequence $\{z_{n_j,k_j}\}_j$ such that $z_{n_j,k_j} \to x$, where $n_j < n_{j+1}$. Since $\bigcup_{n\in\mathbb{N}}\mathscr{P}_n$ is a k-network, there is $m \in \mathbb{N}$ such that $\{z_{n_j,k_j}\}$ is eventually in $\cup\mathscr{P}_m^*$, which contradicts the fact that $z_{n_j,k_j} \in U - \cup\mathscr{P}_m^*$ whenever $n_j \geqslant m$. ■

Proposition 2.5.7 *Let $f : X \to Y$ be a compact-covering mapping. If $\mathscr{P}$ is a k-network for X, then $f(\mathscr{P})$ is a k-network for Y.*

Theorem 2.5.8 *For every regular space X, the following are equivalent:*

(1) X is a Lašnev space.
(2) X is a k-space with a σ-HCP k-network and contains no closed copy of S_2.
(3) X is a Fréchet–Urysohn space with a σ-HCP k-network [130].
(4) X is a Fréchet–Urysohn space with a σ-compact-finite k-network [299].
(5) X is a Fréchet–Urysohn space with a σ-CF k-network [347].

Proof (1) ⇒ (2). Let $f : M \to X$ be a closed mapping, where M is a metric space. By Proposition 2.3.1, X is a Fréchet–Urysohn space. Since S_2 is not a Fréchet–Urysohn space (see Example 1.8.6), X contains no closed copy of S_2. Let $\mathscr{P}$ be a σ-locally finite base for M. Since f is a closed mapping and *HCP* families are preserved by closed mappings, by Propositions 2.1.16 and 2.5.7, $f(\mathscr{P})$ is a σ-*HCP* k-network for X.

(2) ⇒ (3) [454]. Obviously, X is of the point G_δ-property. By Corollary 2.3.5, X is a sequential space. If X is not a Fréchet–Urysohn space, then there is a subset A of X such that $\overline{A} \neq \widetilde{A}$, where

$$\widetilde{A} = \{z \in X : \text{ there is a sequence } \{z_n\} \text{ in } A \text{ such that } z_n \to z\}.$$

So $\widetilde{A}$ is not a closed set in X, and hence there is a nontrivial sequence $\{x_n\}$ in $\widetilde{A}$ converging to $x \in \overline{A} - \widetilde{A}$. Take a sequence $\{U_n\}$ of open sets in X such that $\overline{U}_{n+1} \subset U_n$ and $\{x\} = \bigcap_{n\in\mathbb{N}} U_n$. We may assume $x_n \in U_n$. Choose a sequence $\{V_n\}$ of disjoint open sets in X such that $x_n \in V_n$. For each $n \in \mathbb{N}$, there is a sequence $\{x_{n,m}\}_m$ in $A \cap U_n \cap V_n$ converging to x_n. Let

$$M = \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n,m} : n, m \in \mathbb{N}\}.$$

Then M is a closed copy of S_2 in X, a contradiction. Thus X is a Fréchet–Urysohn space.

(3) $\Rightarrow$ (1). Let $\mathscr{P}$ be a σ-*HCP* k-network in the Fréchet–Urysohn space X. By Lemma 2.5.5, we may assume $\mathscr{P} = \bigcup_{n\in\mathbb{N}} \mathscr{P}_n$, where each $\mathscr{P}_n$ is an *HCP* family which is closed under finite intersections and $\mathscr{P}_n \subset \mathscr{P}_{n+1}$. For each $P \in \mathscr{P}_n$, define

$$R_n(P) = P - \operatorname{int}(\cup\{Q \in \mathscr{P}_n : P \not\subset Q\}),$$
$$\mathscr{R}_n = \{R_n(P) : P \in \mathscr{P}_n\}.$$

(8.1) For every $x \in U \in \tau$, let $\{x_n\}$ be a sequence in X converging to $x \in X - \{x_n : n \in \mathbb{N}\}$. If $\{x_n\}$ is eventually in $\operatorname{int}(\cup\{P \in \mathscr{P}_m : P \subset U\})$, then $\{x_n\}$ is eventually in $\operatorname{int}(\cup\mathscr{R}'_m)$ and $\cup\mathscr{R}'_m \subset U$, where $\mathscr{R}'_m = \{R \in \mathscr{R}_m : R \cap \{x_n : n \in \mathbb{N}\}$ is infinite$\}$. In fact, denote

$$L = \{x_n : n \in \mathbb{N}\},$$
$$V = \operatorname{int}(\cup\mathscr{P}_m) - \cup\{Q \in \mathscr{P}_m \cup \mathscr{R}_m : Q \cap L \text{ is a finite set}\}.$$

Then $V \in \tau$. By Lemma 2.5.4, $\{x_n\}$ is eventually in V. If $y \in Q \in \mathscr{P}_m$ for some $y \in V$, then $Q \cap L$ is infinite, and hence $\mathscr{P}_m$ is point-finite at y (by Lemma 2.5.4), so $\cap(\mathscr{P}_m)_y \in \mathscr{P}_m$. Let $P(y) = \cap(\mathscr{P}_m)_y$. Then $y \notin \cup\{Q \in \mathscr{P}_m : P(y) \not\subset Q\}$, so $y \in R_m(P(y))$, thus $R_m(P(y)) \cap L$ is infinite. As a consequence, $R_m(P(y)) \in \mathscr{R}'_m$, therefore $y \in R_m(P(y)) \subset \cup\mathscr{R}'_m$, which shows $V \subset \cup\mathscr{R}'_m$, and hence $\{x_n\}$ is eventually in $\operatorname{int}(\cup\mathscr{R}'_m)$. For each $R_m(P) \in \mathscr{R}'_m$, there is $Q \in \mathscr{P}_m$ such that $R_m(P) \subset P \subset Q \subset U$. Because otherwise,

$$\operatorname{int}(\cup\{Q \in \mathscr{P}_m : Q \subset U\}) \subset \operatorname{int}(\cup\{Q \in \mathscr{P}_m : P \not\subset Q\}) \subset X - R_m(P),$$

then $\{x_n\}$ is eventually in $X - R_m(P)$, so $R_m(P) \cap L$ is a finite set, a contradiction. Consequently, $\cup\mathscr{R}'_m \subset U$.
Now for each $n \in \mathbb{N}$ define $\mathscr{R}^*_n = \mathscr{R}_n \cup \{X - \operatorname{int}(\cup\mathscr{R}_n)\}$ and denote $\mathscr{R}^*_n = \{R_\alpha : \alpha \in \Lambda_n\}$. Give Λ_n the discrete topology. Let

$$M = \left\{\beta = (\alpha_n) \in \prod_{n\in\mathbb{N}} \Lambda_n : \{R_{\alpha_n}\} \text{ is a net at some point } x(\beta) \text{ in } X\right\}.$$

Then M is a metrizable space and $x(\beta)$ is the only point determined by β for each $\beta \in M$. Define $f : M \to X$ by $f(\beta) = x(\beta)$.

(8.2) f is an onto mapping.
For each $x \in X$, if x is isolated in X, then there is $m \in \mathbb{N}$, such that $\{x\} \in \mathscr{P}_m$, so $R_m(\{x\}) = \{x\}$, and hence there is $\beta \in M$ such that $f(\beta) = x$. If x is an accumulation point of X, then there is a sequence $\{x_k\}$ in $X - \{x\}$ converging to x. For each $n \in \mathbb{N}$, pick $\alpha_n \in \Lambda_n$ such that $R_{\alpha_n} \cap \{x_k : k \in \mathbb{N}\}$ is infinite (otherwise, we can pick $\alpha_n \in \Lambda_n$ such that $x \in R_{\alpha_n}$). Then $x \in R_{\alpha_n}$ is always true. By Lemma 2.5.6 and (8.1), $\{R_{\alpha_n}\}_{n\in\mathbb{N}}$ is a net at x. Let $\beta = (\alpha_n)$. Then $\beta \in M$ and $f(\beta) = x$.

(8.3) f is continuous.
Let $U \in \tau(X)$ and $\beta \in f^{-1}(U)$. Denote $\beta = (\alpha_n)$, then there is $m \in \mathbb{N}$ such that $f(\beta) \in R_{\alpha_m} \subset U$, so $\beta \in \{\gamma \in M : \pi_m(\gamma) = \alpha_m\} \subset f^{-1}(U)$, and hence $f^{-1}(U) \in \tau(M)$.

(8.4) f is a closed mapping.
Let $F \in \tau^c(M)$. If $x \in \overline{f(F)} - f(F)$, then there is a sequence $\{x_i\}$ in $f(F)$ converging to x. To each $i \in \mathbb{N}$, take $\beta_i = (\alpha_{i,n}) \in F \cap f^{-1}(x_i)$. Then for each $n \in \mathbb{N}$, $x_i \in R_{\alpha_{i,n}} \in \mathscr{R}_n^*$. By Lemma 2.5.4, there is $m \in \mathbb{N}$ such that $\{R \in \mathscr{R}_1^* : R \cap \{x_i : i \geqslant m\} \neq \varnothing\}$ is finite, so there exist an infinite set I_1 in $\mathbb{N}$ and $\alpha_1 \in \Lambda_1$ such that for each $i \in I_1$, $\alpha_{i,1} = \alpha_1$. By the inductive method, we can choose a decreasing sequence $\{I_n\}$ of infinite sets in $\mathbb{N}$ and $\beta = (\alpha_n) \in \prod_{n\in\mathbb{N}} \Lambda_n$, such that, $\alpha_{i,n} = \alpha_n$ whenever $n \in \mathbb{N}$ and $i \in I_n$. For each $n \in \mathbb{N}$, pick $k(n) \in I_n$ such that $k(n) < k(n+1)$. Then $\beta_{k(n)} \to \beta$, and hence $\beta \in F$. For every $n \in \mathbb{N}$ and $i \in I_n$, we have $x_i \in R_{\alpha_n}$, so $x \in R_{\alpha_n}$. Let $x \in U \in \tau(X)$. Since $x_{k(n)} \to x$, by Lemma 2.5.4 and (8.1), there is $m \in \mathbb{N}$ such that $\{x_{k(n)}\}$ is eventually in $\cup\{R \in \mathscr{R}_m : R \cap \{x_{k(n)} : n \in \mathbb{N}\}$ is infinite $\} \subset U$. If $i \geqslant m$, then $k(i) \in I_i \subset I_m$, so $R_{\alpha_m} = R_{\alpha_{k(i),m}}$, and hence $\{R_{\alpha_n}\}_{n\in\mathbb{N}}$ is a net at x. Thus $x = f(\beta) \in f(F)$, a contradiction. Therefore f is a closed mapping.

In summary, X is a Lašnev space.

(3) $\Rightarrow$ (4). Let $\bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ be a k-network for a Fréchet–Urysohn space X, where each $\mathscr{P}_n$ is *HCP* and $\mathscr{P}_n \subset \mathscr{P}_{n+1}$. For each $n \in \mathbb{N}$, put

$$D_n = \{x \in X : (\mathscr{P}_n)_x \text{ is not finite}\},$$
$$\mathscr{F}_n = \{P - D_n : P \in \mathscr{P}_n\} \cup \{\{x\} : x \in D_n\}.$$

Then, by Lemma 2.5.4, to prove that $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ is a σ-compact-finite k-network for X, we only need to show it is a k-network. For each compact set $K \subset U \in \tau$ in X, there exist $m \in \mathbb{N}$ and $\mathscr{P} \in \mathscr{P}_m^{<\omega}$ such that $K \subset \cup\mathscr{P} \subset U$. Let

$$\mathscr{F} = \{P - D_m : P \in \mathscr{P}\} \cup \{\{x\} : x \in K \cap D_m\}.$$

Then $\mathscr{F} \in \mathscr{F}_m^{<\omega}$ and $K \subset \cup\mathscr{F} \subset U$.

(4) $\Rightarrow$ (5) is obviously. Next, we prove (5) $\Rightarrow$ (3). Let $\bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ be a k-network for a Fréchet–Urysohn space X such that each $\mathscr{P}_n$ is *CF* and closed under finite unions in X. We assume that each $\mathscr{P}_n$ covers X. Put $\mathscr{P}_n = \{P_\lambda : \lambda \in \Lambda_n\}$. We define an equivalence relation $\sim$ in X by

$$x \sim y \text{ if and only if } \{\lambda \in \Lambda_n : x \in P_\lambda\} = \{\lambda \in \Lambda_n : y \in P_\lambda\}.$$

Then X is decomposed by the family $\mathscr{H}_n$ of all equivalence classes. Since $\mathscr{P}_n$ is *CF*, $\mathscr{H}_n$ is a disjoint and compact-finite refinement of $\mathscr{P}_n$. $\mathscr{H}_n$ is weakly *HCP* because X is a k-space, thus it is *HCP* by Lemma 2.5.5. Let $K \subset U$ with K compact and U open in X. There exist $m \in \mathbb{N}$ and $P \in \mathscr{P}_m$ such that $K \subset P \subset U$. Then $\mathscr{H}_{m|K}$ is finite, thus $K \subset \cup\mathscr{H}' \subset P$ for some finite $\mathscr{H}' \subset \mathscr{H}_m$. Therefore $\bigcup_{n\in\mathbb{N}} \mathscr{H}_n$ is a σ-*HCP* k-network for X. This completes the proof. ∎

In the second part of this section, we establish several metrization theorems by using the above characterizations of Lašnev spaces.

Lemma 2.5.9 ([87]) *Suppose $\mathscr{P}$ is a countably weakly HCP family of open sets in a space X and $A \subset X$. If $x \in A^d$ and there is a G_δ-set G in X containing x such that $G \cap (A - \{x\}) = \varnothing$, then $(\mathscr{P})_x$ is finite.*

Proof Otherwise, there is a countable family $\{P_n\}_{n\in\mathbb{N}}$ in $(\mathscr{P})_x$. Let $G = \bigcap_{n\in\mathbb{N}} G_n$, where $G_n \in \tau$. Put

$$H_1 = A \cap P_1 \cap G_1, \;\; H_{n+1} = H_n \cap P_{n+1} \cap G_{n+1}, \;\; n \in \mathbb{N}.$$

Then

$$x \in P_1 \cap G_1 \cap \overline{A - \{x\}} \subset \overline{H_1 - \{x\}} = \overline{\bigcup_{n\in\mathbb{N}} H_n - H_{n+1}} = \bigcup_{n\in\mathbb{N}} \overline{H_n - H_{n+1}},$$

so there is $m \in \mathbb{N}$ such that $x \in \overline{H_m - H_{m+1}}$, and hence $P_{m+1} \cap G_{m+1} \cap (H_m - H_{m+1}) \neq \varnothing$, a contradiction. ∎

Corollary 2.5.10 *Let $\mathscr{P}$ be a countably HCP family of open sets in a space X. If x is an accumulation point in X and $\{x\}$ is a G_δ-set, then $(\mathscr{P})_x$ is finite.*

Corollary 2.5.11 *Let $\bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ be a local base of a point x in a space X such that each $\mathscr{P}_n$ is an HCP family. If x is an accumulation point in X, then $\mathscr{P}_n$ is finite.*

Proof For every $n \in \mathbb{N}$ and $P \in \mathscr{P}_n$, pick $x(P) \in P - \{x\}$ and let $F_n = \{x(P) : P \in \mathscr{P}_n\}$. Then $F_n \in \tau^c$. Put $A = \bigcup_{n\in\mathbb{N}} F_n$ and $G = X - A$. Then $x \in A^d \cap G$, G is a G_δ-set in X and $G \cap (A - \{x\}) = \varnothing$. By Lemma 2.5.9, $\mathscr{P}_n$ is finite. ∎

Lemma 2.5.12 ([87]) *Let $\mathscr{P}$ be a countably weakly HCP family of open sets in a space X. If X is a k-space, then $\cap\mathscr{P} \in \tau$.*

Proof For each $K \in \mathscr{K}(X)$, by Lemma 2.5.4, there is $F \in K^{<\omega}$ such that $(\mathscr{P})_{K-F}$ is a finite set. Then $P \cap K \subset F$ for each $P \in \mathscr{P} - (\mathscr{P})_{K-F}$, and hence $K \cap (\cap\mathscr{P}) \in \tau(K)$. Since X is a k-space, $\cap\mathscr{P} \in \tau(X)$. ∎

Lemma 2.5.13 ([418]) *Let X be a Fréchet–Urysohn space with a CF family $\mathscr{P}$. Then $\cap\mathscr{P}^\circ \subset [\cap\mathscr{P}]^\circ$.*

Proof Suppose there is a point $x \in \cap\mathscr{P}^\circ - [\cap\mathscr{P}]^\circ$. Then $x \in \overline{X - \cap\mathscr{P}}$. Take a sequence $\{x_n\}$ in $X - \cap\mathscr{P}$ converging to x. Let $K = \{x\} \cup \{x_n : n \in \mathbb{N}\}$. Then for each $P \in \mathscr{P}$, $P \cap K$ is an open set in K containing x and the family $\{P \cap K : P \in \mathscr{P}\}$ is finite. Hence $\cap\{P \cap K : P \in \mathscr{P}\}$ is an open set in K containing x. But

$$\cap\{P \cap K : P \in \mathscr{P}\} = (\cap\mathscr{P}) \cap K = \{x\},$$

a contradiction. ■

Definition 2.5.14 Suppose P is a topological property and $\mathscr{B}$ is a family of sets in a space X. We say that $\mathscr{B}$ has the *property P at non-isolated points* if $\mathscr{B}$ has the property P at each non-isolated point of X.

For example, suppose $\mathscr{B}$ is a base for a space X. We say that $\mathscr{B}$ is a σ-locally finite base at non-isolated points [252] if, $\mathscr{B} = \bigcup_{n\in\mathbb{N}} \mathscr{B}_n$ and for each $n \in \mathbb{N}$, $\mathscr{B}_n$ is locally finite at each non-isolated point of X.

Let $\mathscr{B}$ be a family of subsets of a space X. For every $x \in X$, $\mathscr{B}$ is called *HCP* at x if, for any $H(B) \subset B \in \mathscr{B}$, $x \in \overline{\cup\{H(B) : B \in \mathscr{B}\}}$, then $x \in \cup\{\overline{H(B)} : B \in \mathscr{B}\}$. Obviously, $\mathscr{B}$ is *HCP* for X if and only if for every $x \in X$, $\mathscr{B}$ is *HCP* at x.

Theorem 2.5.15 *For every regular space X, the following are equivalent:*

(1) X is a metrizable space.
(2) X has a base which is σ-discrete at non-isolated points [252].
(3) X has a base which is σ-locally finite at non-isolated points [252].
(4) X has a σ-HCP base [87]. (The Burke–Engelking–Lutzer metrization theorem)
(5) X is a k-space with a σ-countably weakly HCP base [87].
(6) X is of the point G_δ-property and has a σ-countably HCP base [211].
(7) X is a strongly Fréchet–Urysohn space with a σ-countably weakly HCP base.
(8) X is a strongly Fréchet–Urysohn space with a σ-compact-finite k-network [299].
(9) X is a k-space with a σ-CF quasi-base [418].

Proof By Theorem 1.3.2, we obtain (1) ⇒ (2) and (9). Obviously (2) ⇒ (3). (8) ⇒ (1) is obtained by Theorems 2.5.8 and 2.2.2. By Corollary 2.5.11, we obtain (4) ⇒ (5) and (6). (6) ⇒ (7) is obtained by Corollary 2.5.10. By Lemma 2.5.5, Theorems 2.5.8 and 2.2.2, we obtain (7) ⇒ (1).

(3) ⇒ (4). It is sufficient to prove that if $\mathscr{B}$ is locally finite at non-isolated points for X, then $\mathscr{B}$ is *HCP* for X. Let $\mathscr{B} = \{B_\alpha : \alpha \in \Gamma\}$. For every $\alpha \in \Gamma$, choose $H_\alpha \subset B_\alpha$. Put $\mathscr{H} = \{H_\alpha\}_{\alpha\in\Gamma}$. If $x \in \overline{\cup\mathscr{H}}$, we can assume that x is a non-isolated point of X, then there exists an open neighborhood $U(x)$ of x such that the subfamily $(\mathscr{H})_{U(x)}$ is finite because $\mathscr{H}$ is locally finite at non-isolated points. Since $\overline{\cup\mathscr{H}} = \overline{\cup(\mathscr{H} - (\mathscr{H})_{U(x)})} \bigcup \overline{\cup(\mathscr{H})_{U(x)}}$ and $U(x) \bigcap \overline{\cup(\mathscr{H} - (\mathscr{H})_{U(x)})} = \varnothing$, we have $x \in \overline{\cup(\mathscr{H})_{U(x)}} \subset \cup\overline{\mathscr{H}}$.

(5) ⇒ (7). Let $\bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ be a base for the k-space X such that each $\mathscr{P}_n$ is a countably weakly *HCP* family. For each $x \in X$, by Lemma 2.5.12, $\{\cap(\mathscr{P}_n)_x\}_{n\in\mathbb{N}}$ is a local base of x, thus X is a strongly Fréchet–Urysohn space.

(9) $\Rightarrow$ (8). Let X be a k-space with a σ-CF quasi-base $\mathscr{B} = \bigcup_{n\in\mathbb{N}} \mathscr{B}_n$. Since every quasi-base is a k-network, we only need to show X is first countable by Theorem 2.5.8.

Since X is a k-space that each compact subset is metrizable, by Corollary 2.3.5, X is sequential.

Now we prove X is Fréchet–Urysohn. Suppose X is not Fréchet–Urysohn. Then there is a subset $A \subset X$ such that the set

$$\tilde{A} = \{x \in X : \text{ there is a sequence } \{x_n\} \subset A \text{ such that } x_n \to x\}$$

is not closed in X. Since X is sequential, and there are a point $x \in X - \tilde{A}$ and a sequence $\{x_n\} \subset \tilde{A} - A$ converging to x. For each $n \in \mathbb{N}$, take a sequence $\{x_{n,m}\}_{m\in\mathbb{N}} \subset A$ converging to x_n. For each $n, m \in \mathbb{N}$, let

$$\mathscr{B}_{n,m} = \{B \in \mathscr{B}_n : \{x\} \cup \{x_n : n \geqslant m\} \subset B^\circ\}.$$

Using the same argument as in Lemma 2.5.13, for each $n, m \in \mathbb{N}$ we can take a function $f_{n,m} : \{k \in \mathbb{N} : k \geqslant m\} \to \mathbb{N}$ satisfying

$$C(f_{n,m}) = \{x_{i,j} : i \geqslant m, j \geqslant f_{n,m}(i)\} \subset \cap\mathscr{B}_{n,m}.$$

Note that every neighborhood of x contains some $C(f_{n,m})$. Indeed, let U be a neighborhood of x. Since $\mathscr{B}$ is a quasi-base for X, there is $B \in \mathscr{B}$ with $x \in B^\circ \subset B \subset U$. Let $B \in \mathscr{B}_{n,m}$ for some $n, m \in \mathbb{N}$. Then $C(f_{n,m}) \subset B$. For each $k \in \mathbb{N}$, let $N_k = \cap\{C(f_{n,m}) : n, m \leqslant k\}$. Obviously, each N_k is nonempty and $N_{k+1} \subset N_k$. Take an arbitrary point $y_k \in N_k$ for each $k \in \mathbb{N}$. Then $\{y_k\}$ is a sequence in A converging to x. This is a contradiction.

Finally, X is first countable. Let $x \in X$. For each $n \in \mathbb{N}$, let $\mathscr{B}'_n = \{B \in \mathscr{B}_n : x \in B^\circ\}$. By Lemma 2.5.13, $x \in [\cap\mathscr{B}'_n]^\circ$. Hence $\{[\cap\mathscr{B}'_n]^\circ : n \in \mathbb{N}\}$ is a countable neighborhood base at x, thus X is first countable. ■

We give some explanations for the conditions of Theorem 2.5.15 by using the following examples.

Example 2.5.16 (*[87]*) There is a non-metrizable regular space which has a σ-weakly HCP base.

Let A be the set of ordinal numbers with the cardinality smaller than $\aleph_\omega$. Let $Z = \{0, 1\}^A$. For every $z \in Z$ and $\alpha \in A$, let $z(\alpha) = \pi_\alpha(z)$. Denote the element of Z all coordinates of which are 0 by s. Suppose $\mathscr{B}$ is an open neighborhood base of s in the product space Z. Let

$$X = \{s\} \cup \{z \in Z : \{\alpha \in A : z(\alpha) = 0\} \in A^{<\omega}\}.$$

Define the following topology for X: $\mathscr{B}_{|X}$ is a neighborhood base of the only accumulation point s of X. Then X is a regular space which is not first countable, and hence X is not a metrizable space. For each $n \in \mathbb{N}$, define

$$\mathscr{P}_{1,n} = \{\{z\} : z \in X - \{s\} \text{ and } |\{\alpha \in A : z(\alpha) = 0\}| = n\}.$$

Then $\mathscr{P}_{1,n}$ is a discrete family of open sets in X. For each $B \in \mathscr{B}$, let

$$\Lambda_B = \{\alpha \in A : \pi_\alpha(B) = \{0\}\}.$$

Then $\Lambda_B \in A^{<\omega}$ and B is determined by Λ_B uniquely. Furthermore, let

$$\mathscr{P}_{2,n} = \{B \cap X : B \in \mathscr{B} \text{ and } \Lambda_B \subset [0, \omega_n]\},$$

where ω_n is the smallest ordinal number of cardinality $\aleph_n$. Then $|\mathscr{P}_{2,n}| \leqslant \aleph_n$. To prove that $\mathscr{P}_{2,n}$ is a weakly *HCP* family, we only need to show that for every $B \cap X \in \mathscr{P}_{2,n}$ and $p(B) \in B \cap X - \{s\}$, we have $s \notin \overline{\{p(B) : B \cap X \in \mathscr{P}_{2,n}\}}$. Let

$$\begin{aligned} \Gamma_B &= \{\alpha \in A : p(B)(\alpha) = 0\}, \\ \Gamma &= \cup\{\Gamma_B : B \cap X \in \mathscr{P}_{2,n}\}. \end{aligned}$$

Then $\Gamma_B \in A^{<\omega}$, and hence $|\Gamma| \leqslant \aleph_n$. Pick $\beta \in A - \Gamma$. Let

$$V = \{z \in Z : z(\beta) = 0\}.$$

Then $s \in V \cap X \in \tau(X)$ and $V \cap \{p(B) : B \cap X \in \mathscr{P}_{2,n}\} = \varnothing$, so

$$s \notin \overline{\{p(B) : B \cap X \in \mathscr{P}_{2,n}\}}.$$

Thus X has a σ-weakly *HCP* base $\bigcup_{n\in\mathbb{N}}(\mathscr{P}_{1,n} \cup \mathscr{P}_{2,n})$.

By Theorem 2.5.15, X is not a k-space. Since $X = \{s\} \cup \bigcup_{n\in\mathbb{N}}(\cup\mathscr{P}_{1,n})$, X is an F_σ-discrete space (a space X is said to be an *F_σ-discrete space* if X is the union of countably many discrete closed subsets of it), and hence X is of the point G_δ-property.

Example 2.5.17 (*[85]*) The property of having a σ-weakly *HCP* base is not preserved by perfect mappings.

Let $\{Z_n\}_{n\in\mathbb{N}}$ be a family of disjoint regular spaces having σ-weakly *HCP* bases (see Example 2.5.16). Take a point $a \notin \bigcup_{n\in\mathbb{N}} Z_n$. Let $X = \{a\} \cup \bigcup_{n\in\mathbb{N}} Z_n$ and give X the following topology: each Z_n is an open subspace of X; each basic neighborhood of a has the form $\{a\} \cup \bigcup_{n\geqslant k} Z_n$, $k \in \mathbb{N}$. Since each Z_n is a clopen subspace of X, it is easy to verify that X is a non-metrizable regular space with a σ-weakly *HCP* base.

For each $n \in \mathbb{N}$, Z_n is not first countable. Let z_n be a point which is not first countable in Z_n. Define $Z = \{a\} \cup \{z_n : n \in \mathbb{N}\}$. Since the sequence $\{z_n\}$ converges to a in X, Z is a compact subset of X. It is easy to verify that the set Z has no countable neighborhood base in X. Take $Y = X/Z$ and let $f : X \to Y$ be the quotient mapping. Then f is a perfect mapping and $f(a)$ has no countable neighborhood base in Y. We prove that the space Y has no σ-weakly *HCP* base. Otherwise, let $\bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ be a base for Y such that each $\mathscr{P}_n$ is weakly *HCP*. Denote $b = f(a)$. Since b has no countable neighborhood base in Y, there is $m \in \mathbb{N}$ such that $\mathscr{P}_m$ is not point-countable at b, so there is an uncountable family $\{V_\alpha : \alpha < \omega_1\} \subset (\mathscr{P}_m)_b$. For each $i < \omega, f^{-1}(V_i)$ is an open neighborhood of a, so there exist a strictly increasing sequence $\{n_i\}$ in $\mathbb{N}$ and a sequence $\{x_i\}$ in X, such that, $x_i \in (Z_{n_i} - \{z_{n_i}\}) \cap f^{-1}(V_i)$ and the sequence $\{x_i\}$ converges to a in X. Let $A = \{f(x_i) : i \in \mathbb{N}\}$. Then $b \in \overline{A}$, and hence there is a subset $\{y_\alpha : \alpha < \omega_1\}$ of Y such that each $y_\alpha \in V_\alpha \cap A - \{y_\beta : \beta < \alpha\} \subset A$, which contradicts the countability of A. So Y has no σ-weakly *HCP* base, which shows the property of having a σ-weakly *HCP* base is not preserved by perfect mappings.

Example 2.5.18 (*[85]*) A regular space with a σ-weakly *HCP* base may not be a meta-Lindelöf space.

Take an uncountable regular cardinal number κ and expand it to a strictly increasing sequence $\{\delta_n\}_{n\in\omega}$ of regular cardinal numbers such that $\delta_0 = \kappa$. Let $Z = \{0, 1\}^\lambda$, where $\lambda = \sup_{n\in\omega} \delta_n$. For each $z \in Z$, denote

$$\operatorname{supp} z = \{t \in \lambda : z(t) = 1\}.$$

Then for each $t \in \lambda$, there is only one $e_t \in Z$ such that $\operatorname{supp} e_t = \{t\}$. Let $E = \{e_t : t \in \delta_0\}$. For each $n \in \omega$, denote

$$J_n = \{z \in Z : |\lambda - \operatorname{supp} z| \leqslant \kappa,\ \lambda - \operatorname{supp} z \subset \delta_n\}.$$

Obviously, $E \cap J_n = \varnothing$. Let $X = E \cup \bigcup_{n\in\omega} J_n$.

Give X the following topology: every point in $\bigcup_{n\in\omega} J_n$ is isolated; a neighborhood base of each point e_t in E is taken as the restriction of a neighborhood base of this point in the product space Z to X. It is obvious that X is a completely regular space. For every $t \in \delta_0$ and $F \in [\lambda - \{t\}]^{<\omega}$, let

$$U(t, F) = \{x \in X : t \in \operatorname{supp} x = \lambda - F\}.$$

Then $U(t, F)$ is a basic neighborhood of e_t in X. For each $n \in \omega$, let

$$\mathscr{B}_n = \{U(t, F) : t \in \delta_0,\ F \in [\delta_n - \{t\}]^{<\omega}\}.$$

(1) For every $t \in \delta_0$ and $\beta \in \lambda - \delta_n$, since $U(t, \{\beta\}) \cap J_n = \varnothing$, e_t is not an accumulation point of J_n, so J_n is a discrete closed subset of X.
(2) Since $|\mathscr{B}_n| = \delta_n$, we can take $\mathscr{B}_n = \{B_\alpha : \alpha \in \delta_n\}$. Let $P = \{p_\alpha : \alpha \in \delta_n\}$, where $p_\alpha \in B_\alpha$. We prove P is a discrete closed subset of X. For each $t \in \delta_0$, since $U(t, \varnothing) \cap E = \{e_t\}$, E is a discrete closed subset of X, so we may assume $E \cap P = \varnothing$. Since $|\bigcup_{\alpha\in\delta_n}(\lambda - \operatorname{supp} p_\alpha)| \leqslant \delta_n$, there is $\gamma \in \lambda - (\delta_0 \cup \bigcup_{\alpha\in\delta_n}(\lambda - \operatorname{supp} p_\alpha))$. Then $U(t, \{\gamma\}) \cap P = \varnothing$, $\forall t \in \delta_0$. Thus, P is a discrete closed subset of X. As a consequence, $\mathscr{B}_n$ is weakly *HCP*.
Define
$$\mathscr{B} = \bigcup_{n\in\omega} \mathscr{B}_n \cup \bigcup_{n\in\omega} \{\{x\} : x \in J_n\}.$$

Obviously, $\mathscr{B}$ is a base for X. By (1) and (2), $\mathscr{B}$ is σ-weakly *HCP*.

(3) For each family of basic open sets in X which has the form $\mathscr{W} = \{U(t, F_t) : t \in \delta_0\}$, there is $p \in X$ such that $\mathrm{ord}(p, \mathscr{W}) = \kappa$.

In fact, recall the Δ-system lemma. A family $\mathscr{D}$ is said to be a Δ-*system* if there is a set R such that $\forall A, B \in \mathscr{D}[(A \neq B) \Rightarrow A \cap B = R]$. The Δ-system lemma [237]: Suppose κ is an uncountable regular cardinal number and X is an infinite set. If $\mathscr{D}$ is a subset of $[X]^{<\omega}$ with the cardinality κ, then $\mathscr{D}$ contains a Δ-system with the cardinality κ.

By the Δ-system lemma, the family $\{\{t\} \cup F_t : t \in \delta_0\}$ of finite sets in λ contains a subfamily $\{\{t\} \cup F_t : t \in \Lambda\}$ with the cardinality κ such that for some $R \subset X$ and every different $t, t' \in \Lambda$, $(\{t\} \cup F_t) \cap (\{t'\} \cup F_{t'}) = R$, so $U(t, F_t) \neq U(t', F_{t'})$. We may assume that $\Lambda \cap R = \varnothing$ and there is $m \in \omega$ such that $\bigcup_{t \in \Lambda} F_t \subset \delta_m$. Take $p \in Z$ such that $p(t) = 1$ and $\mathrm{supp}\, p = \lambda - \bigcup_{t \in \Lambda} F_t$. Since $|\bigcup_{t \in \Lambda} F_t| \leqslant \kappa$, $p \in J_m \cap U(t, F_t)$, $\forall t \in \Lambda$. Therefore, $\mathrm{ord}(p, \mathscr{W}) = \kappa$.

By (3), the open cover $\{U(t, \varnothing) : t \in \delta_0\}$ of X has no point-countable open refinement, so X is not a meta-Lindelöf space.

Example 2.5.19 There is a non-metrizable regular space which has a σ-countably *HCP* base.

In an uncountable set X, fix a point p as the special point of X. Define the following topology on X: for each $F \subset X$, $F \in \tau^c(X)$ if and only if either $p \in F$ or F is countable.

The topological space is called a *Fortissimo space* [264, 437] and denoted by X_p. Every Fortissimo space X_p has the following properties:

(1) X_p is a regular Lindelöf space.
(2) X_p does not have the point G_δ-property.
(3) Any family of subsets of X_p is a countably *HCP* family.
(4) For any uncountable subset A of X_p, $\{\{x\} : x \in A\}$ is not a closure-preserving family of sets.
(5) Each compact set in X_p is a finite set.

We verify the above items one by one as follows:

(1) Obviously, each singleton of X_p is a closed set, and

$$\{\{x\} : x \in X_p - \{p\}\} \cup \{B \subset X_p : p \in B \text{ and } |X_p - B| \leqslant \aleph_0\}$$

is a closed base for X_p, so X_p is a regular space. For any open cover $\mathscr{U}$ of X_p, pick $U \in (\mathscr{U})_p$. Then $|X_p - U| \leqslant \aleph_0$, so $\mathscr{U}$ has a countable subcover.

(2) For any sequence $\{U_n\}$ of open neighborhoods of p, $|X_p - \bigcap_{n \in \mathbb{N}} U_n| \leqslant \aleph_0$, so $\{p\} \neq \bigcap_{n \in \mathbb{N}} U_n$, and hence X_p does not have the point G_δ-property.

(3) Suppose $\mathscr{P}$ is a family of sets in X_p. For any countable subfamily $\{P_n\}_{n \in \mathbb{N}}$ of $\mathscr{P}$ and $Q_n \subset P_n$, let $F = \bigcup_{n \in \mathbb{N}} \overline{Q_n}$. If $p \in F$, then F is a closed set in X_p; if $p \notin F$, then $|\overline{Q_n}| \leqslant \aleph_0$, so $|F| \leqslant \aleph_0$, and hence F is also a closed set in X_p. Thus $\mathscr{P}$ is a countably *HCP* family.

(4) Since A is uncountable, $p \in \overline{A - \{p\}}$, so $A - \{p\}$ is not a closed set in X_p, and hence $\{\{x\} : x \in A\}$ is not a closure-preserving family of sets.
(5) We may assume $p \in K \in \mathscr{K}(X)$. If K is infinite, take an infinite set $\{x_n : n \in \mathbb{N}\}$ in $K - \{p\}$, then the open cover $\{X_p - \{x_n : n \in \mathbb{N}\}\} \cup \{\{x_n\} : n \in \mathbb{N}\}$ of K has no finite subcover, a contradiction. Thus K is a finite set.

Question 2.5.20 (*[310]*) Does every space with a σ-weakly *HCP* base have the point G_δ-property?

Question 2.5.21 (*[312]*) Is every regular k-space with a σ-compact-finite k-network a meta-Lindelöf space?

In the last part of this section, we discuss properties of countable products and perfect preimages of Lašnev spaces. The following example shows that the property of having a σ-*HCP* k-network is not finitely productive.

Example 2.5.22 (*[221]*) $S_{\omega_1} \times \mathbb{S}_1$ has no σ-*HCP* k-network.

Let $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where the common point s of all $X_\alpha = \{s\} \cup \{x_{\alpha,n} : n \in \mathbb{N}\}$ is the only accumulation point in X, and U is an open neighborhood of s if and only if for each $\alpha < \omega_1$, there is $m_\alpha \in \mathbb{N}$ such that

$$\{s\} \cup \{x_{\alpha,n} : n \geqslant m_\alpha, \alpha < \omega_1\} \subset U.$$

Then X is a copy of S_{ω_1}.

Let $\mathscr{F}$ be a σ-*HCP* closed k-network of $S_{\omega_1} \times \mathbb{S}_1$. By transfinite induction, we can choose an uncountable set $\{F_\alpha\}_{\alpha<\omega_1}$ in $\mathscr{F}$ such that $|F_\alpha \cap (\{s\} \times \mathbb{S}_1)| = \aleph_0$. In fact, we first pick $\mathscr{F}' \in \mathscr{F}^{<\omega}$ such that $X_0 \times \mathbb{S}_1 \subset \cup\mathscr{F}'$. Then there exist $F_0 \in \mathscr{F}'$ and $x_0 \in F_0 \cap ((X_0 - \{s\}) \times \mathbb{S}_1)$ such that $|F_0 \cap (\{s\} \times \mathbb{S}_1)| = \aleph_0$. Let $\alpha < \omega_1$ and assume that for each $\beta < \alpha$, we have chosen $F_\beta \in \mathscr{F}$ and $x_\beta \in F_\beta \cap ((X_\beta - \{s\}) \times \mathbb{S}_1)$ such that $|F_\beta \cap (\{s\} \times \mathbb{S}_1)| = \aleph_0$. Since the closed set $\{x_\beta : \beta < \alpha\}$ does not meet the compact set $X_\alpha \times \mathbb{S}_1$ in $S_{\omega_1} \times \mathbb{S}_1$, there is $\mathscr{F}'' \in \mathscr{F}^{<\omega}$ such that

$$X_\alpha \times \mathbb{S}_1 \subset \cup\mathscr{F}'' \subset S_{\omega_1} \times \mathbb{S}_1 - \{x_\beta : \beta < \alpha\},$$

so there exist $F_\alpha \in \mathscr{F}''$ and $x_\alpha \in F_\alpha \cap ((X_\alpha - \{s\}) \times \mathbb{S}_1)$ such that $|F_\alpha \cap (\{s\} \times \mathbb{S}_1)| = \aleph_0$. Since $F_\alpha \cap \{x_\beta : \beta < \alpha\} = \varnothing$, $F_\beta \neq F_\alpha$ whenever $\beta < \alpha$. This completes the inductive construction.

Because of the compactness of $\{s\} \times \mathbb{S}_1$ and Lemma 2.5.4, $\{F \in \mathscr{F} : |F \cap (\{s\} \times \mathbb{S}_1)| = \aleph_0\}$ is countable, a contradiction. So $S_{\omega_1} \times \mathbb{S}_1$ has no σ-*HCP* k-network.

Lemma 2.5.23 ([199]) *Suppose neither X nor Y is a discrete space. If $X \times Y$ is a Lašnev space, then it is a metrizable space.*

Proof Since X is a non-discrete Fréchet–Urysohn space, X contains a closed copy of $\mathbb{S}_1$, so $\mathbb{S}_1 \times Y$ is a Fréchet–Urysohn space. By Proposition 2.3.20 and Theorem 2.5.15, Y is a metrizable space. With the same reason, X is a metrizable space. ■

Theorem 2.5.24 *Suppose $\{X_n\}$ is a sequence of Lašnev spaces.*

(1) *If a subspace Z of $\prod_{n\in\mathbb{N}} X_n$ is a Fréchet–Urysohn space, then Z is a Lašnev space [448].*
(2) *If each $|X_n| \geqslant 2$ and $\prod_{n\in\mathbb{N}} X_n$ is a Fréchet–Urysohn space, then $\prod_{n\in\mathbb{N}} X_n$ is a metrizable space.*

Proof (1) By Theorem 2.5.8, X_n has a k-network $\bigcup_{m\in\mathbb{N}} \mathscr{P}_{n,m}$, where $\mathscr{P}_{n,m}$ is a compact-finite family of sets in X_n and $\mathscr{P}_{n,m} \subset \mathscr{P}_{n,m+1}$. For each $i \in \mathbb{N}$, take

$$\mathscr{P}_i = \left(\prod_{n\leqslant i} \mathscr{P}_{n,i}\right) \times \left\{\prod_{n>i} X_n\right\}.$$

Then $(\bigcup_{i\in\mathbb{N}} \mathscr{P}_i)_{|Z}$ is a σ-compact-finite k-network for the regular Fréchet–Urysohn space Z, so Z is a Lašnev space.
(2) Let

$$X = \prod_{n\in\mathbb{N}} X_{2n},\ Y = \prod_{n\in\mathbb{N}} X_{2n-1}.$$

Then neither X nor Y is a discrete space. Since $X \times Y$ is homeomorphic to $\prod_{n\in\mathbb{N}} X_n$, it is a Lašnev space. By Lemma 2.5.23, $X \times Y$ is a metrizable space, and hence $\prod_{n\in\mathbb{N}} X_n$ is a metrizable space. ∎

Question 2.5.25 (*[448]*) Find characterizations for subspaces of countable products of Lašnev spaces.

The mapping $\pi_1 : S_\omega \times \mathbb{S}_1 \to S_\omega$ is a perfect mapping. Although $S_\omega \times \mathbb{S}_1$ has a G_δ-diagonal, $S_\omega \times \mathbb{S}_1$ has no σ-*HCP* k-network (and hence is not a Lašnev space). Thus, not every space with a σ-*HCP* k-network (or even a Lašnev space) satisfies the perfect preimage G_δ-diagonal theorem. However, we have the following result.

Corollary 2.5.26 ([260]) *Suppose $f : X \to Y$ is a perfect mapping, and Y is a Lašnev space. If X is a Fréchet–Urysohn space with a G_δ-diagonal, then X is a Lašnev space.*

Proof As a perfect preimage of a paracompact space, X is a paracompact space. By Corollary 2.2.11, there is a metric space M and a one-to-one mapping $g : X \to M$. By using Corollary 2.1.9 for the property of "have a σ-compact-finite k-network", we obtain that X has a σ-compact-finite k-network. Thus X is a Lašnev space. ∎

2.6 Compact-Covering Mappings

The purpose of this section is to give a preliminary introduction of characterizations of various compact-covering images of metric spaces obtained by Michael and Nagami [340]. In the next section we will continue to introduce the characterizations of compact-covering s-images of metric spaces.

A mapping $f : X \to Y$ is said to be *almost open* [24] if, for each $y \in Y$, there is $x \in f^{-1}(y)$ such that $f(U)$ is a neighborhood of y in Y whenever U is a neighborhood of x in X. Obviously, each open mapping is an almost open mapping and each almost open mapping is a countably bi-quotient mapping.

Theorem 2.6.1 *(1) A space X is a compact-covering image of a metric space if and only if every compact set in X is metrizable* [340].

(2) A space X is a compact-covering quotient (resp. pseudo-open, countably bi-quotient) image of a metric space if and only if X is a k-space (resp. Fréchet–Urysohn space, strongly Fréchet–Urysohn space) such that every compact set in X is metrizable [340].

(3) X is a compact-covering and almost open image of a metrizable space if and only if X is a first countable space and each compact subset of X is metrizable [496].

Proof (1) Suppose X is the image of a metric space M under a compact-covering mapping f. For each $K \in \mathscr{K}(X)$, there is $L \in \mathscr{K}(M)$ such that $f(L) = K$. Since L is a compact metric space, K is a metrizable space. Conversely, suppose every compact set in X is metrizable. Let $M = \oplus\mathscr{K}(X)$. Then M is a metrizable space and the natural mapping from M onto X is a compact-covering mapping.

(2) The necessity is obtained by Proposition 2.3.1, and the sufficiency is followed from the mapping lemma (see Proposition 2.1.12).
Note that, since a space X is a sequential space if and only if X has the weak topology with respect to $\{K \in \mathscr{K}(X) : K$ is metrizable subspace$\}$, the condition of "a k-space" in (2) is equivalent to that of "a sequential space".

(3) It is easy to verify that the first countability is preserved by almost open mappings. So by (1), we can get the necessity. Assume that X is a first countable space and each compact subset of X is metrizable. By (1) and Theorem 2.4.5, there exist metrizable spaces M_1, M_2, a compact-covering mapping $f : M_1 \to X$ and an open mapping $g : M_2 \to X$. Let $M = M_1 \oplus M_2$ and define $h : M \to X$ such that $h_{|M_1} = f$ and $h_{|M_2} = g$. Then M is a metrizable space and h is a compact-covering and almost open mapping. Thus X is a compact-covering and almost open image of a metrizable space. ■

The following characterization of compact-covering open images of metric spaces relies on the next lemma.

Lemma 2.6.2 ([340]) *Let K be a metrizable compact set in a space X. If there is a countable neighborhood base of K in X, then K has a countable outer base in X.*

Proof Let $\{U_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$ be a countable base of the subspace K and a countable open neighborhood base of the set K in X respectively. Take

$$A = \{(n, m) \in \mathbb{N} \times \mathbb{N} : U_n \supset \overline{U}_m\}.$$

Then for every $(n, m, k) \in A \times \mathbb{N}$, there is $U_{n,m} \in \tau(X)$ such that $\overline{U}_m \subset U_{n,m} \subset \overline{U}_{n,m} \subset X - (K - U_n)$. Let $W(n, m, k) = U_{n,m} \cap V_k$. We denote the family of all the finite intersections of the sets with the form of $W(n, m, k)$ by $\mathscr{H}$. Then $\mathscr{H}$ is countable. We prove that $\mathscr{H}$ is an outer base of K in X. For every $p \in K$ and $p \in U \in \tau(X)$, define

$$B = \{\alpha \in A \times \mathbb{N} : p \in W(\alpha)\};$$
$$H(F) = \bigcap_{\alpha \in F} W(\alpha), \ F \in B^{<\omega}.$$

Assume that there is no $F \in B^{<\omega}$ such that $H(F) \subset U$. Pick $p(F) \in H(F) - U$. Let

$$Q(F) = \{p(F') : F \subset F' \in B^{<\omega}\}.$$

Then $K \cap \overline{Q(F)} \neq \varnothing$. Because otherwise, there exist $k, n, m \in \mathbb{N}$ such that $V_k \cap \overline{Q(F)} = \varnothing$ and $p \in \overline{U_m} \subset U_n$. Denote $\alpha = (n, m, k)$ and $F' = F \cup \{\alpha\}$. Then $\alpha \in B$ and $p(F') \in W(\alpha) \cap Q(F) \subset V_k \cap Q(F) = \varnothing$, a contradiction. Thus, $\{K \cap \overline{Q(F)} : F \in B^{<\omega}\}$ has the finite intersection property, and hence $K \cap (\cap\{\overline{Q(F)} : F \in B^{<\omega}\}) \neq \varnothing$. On the other hand, for each $x \in K - \{p\}$, pick $\alpha \in B$ such that $x \notin \overline{W(\{\alpha\})}$. Then $x \notin \overline{Q(\{\alpha\})}$, so $(K - \{p\}) \cap (\cap\{\overline{Q(F)} : F \in B^{<\omega}\}) = \varnothing$. It follows that $\cap\{K \cap \overline{Q(F)} : F \in B^{<\omega}\} = \{p\} \subset U$. By the compactness of K, there is $F \in B^{<\omega}$ such that $U \cap Q(F) \neq \varnothing$, a contradiction. Consequently, K has a countable outer base in X. ■

By Lemma 2.6.2 and Proposition 2.4.4, we obtain the following theorem.

Theorem 2.6.3 ([340]) *A space X is a compact-covering open image of a metric space if and only if every compact set in X is metrizable and has a countable neighborhood base in X.*

The above theorem can be extended by using weak neighborhood bases instead of neighborhood bases, i.e., we can characterize spaces in which every compact set is metrizable and has a countable neighborhood wake base in X by certain images of metrizable spaces [295].

Example 2.6.4 (*[233]*) There is a closed mapping which is not a sequence-covering mapping.

Denote the Mrówka space by $\psi(\mathbb{N})$ (see Example 1.8.4). Define $f : \psi(\mathbb{N}) \to \mathbb{S}_1$ by $f(\psi(\mathbb{N}) - \mathbb{N}) = \{0\}$ and $f(n) = 1/n, \ n \in \mathbb{N}$. Then f is a closed mapping. In fact, if U is an open set in $\psi(\mathbb{N})$ and $f^{-1}(0) \subset U$, then $\mathbb{N} - U$ is a finite set. Because otherwise, let $\{n_i\}$ be a sequence consisting infinite points in $\mathbb{N} - U$. By the maximality of $\mathscr{A}$, there is $A \in \mathscr{A}$ such that A contains infinitely many points of $\{n_i\}$, and hence for any finite subset F of A, $A - F \not\subset U$, which contradicts the fact that U is a neighborhood of the point A in $\psi(\mathbb{N})$. Let $V = \{0\} \cup \{1/n : n \in \mathbb{N} \cap U\}$. Then V is an open neighborhood of 0 in $\mathbb{S}_1$ and $f^{-1}(V) = U$. However, f is not a sequence-covering mapping. Because otherwise, there is a compact set L in $\psi(\mathbb{N})$ such that $f(L) = \mathbb{S}_1$. So $\mathbb{N}$ is a dense subset in $\psi(\mathbb{N})$ and $\mathbb{N} \subset L$, thus $\psi(\mathbb{N}) = L$, a contradiction.

In the second part of this section, we investigate two classes of special compact-covering mappings and images of metrizable spaces under these mappings.

Definition 2.6.5 Suppose $f : X \to Y$ is a mapping.

(1) f is called a *proper mapping* [71] (or *k-mapping* [172]) if $f^{-1}(K)$ is a compact subset of X whenever K is a compact subset of Y.
(2) f is called a *subproper mapping* [54] if, there is a subset Z of X such that $f(Z) = Y$ and $\overline{Z \cap f^{-1}(K)}$ is a compact subset of X whenever K is a compact subset of Y.
(3) The proper images of metrizable spaces are called *k-metrizable spaces* [54]. The subproper images of metrizable spaces are called *k^*-metrizable spaces* [54].

Obviously, each perfect mapping is a proper mapping, each proper mapping is a subproper mapping and each subproper mapping is a compact-covering mapping. Thus, each metrizable space is a k-metrizable space, and each k-metrizable space is a k^*-metrizable space.

A subset A of a space X is called a *k-closed set* in X if for each $K \in \mathscr{K}(X)$, $A \cap K$ is closed in K. Obviously, each closed set is a k-closed set, and a space X is a k-space if and only if every k-closed subset of X is closed in X.

The *k-coreflection kX* of a space X is the set X with the following topology: a subset A of X is a closed set in kX if and only if A is k-closed in X. It is easy to verify:

(1) The identity $\mathrm{id}_X : kX \to X$ is a mapping.
(2) $\mathscr{K}(X) = \mathscr{K}(kX)$.
(3) kX is a k-space.
(4) X is a k-space if and only if $kX = X$.

Definition 2.6.6 (*[293]*) A sequence $\{\mathscr{U}_n\}$ of covers of a space X is called a *point-star network* for X if $\{\mathrm{st}(x, \mathscr{U}_n)\}_{n\in\mathbb{N}}$ is a net at x in X for each $x \in X$. Let Φ be a family property. If every $\mathscr{U}_n$ in a point-star network $\{\mathscr{U}_n\}$ for X has Φ, then $\{\mathscr{U}_n\}$ is called a *Φ point-star network*.

The concept of point-star networks is a generalization of the concept of weak developments (see Definition 1.6.13). A weak development may be called a *point-star weak base*. We can define a *point-star sequential neighborhood network* in a similar way.

Theorem 2.6.7 *For every space X, the following are equivalent:*

(1) X is a k-metrizable space.
(2) kX is a metrizable space [54].
(3) X has a compact-finite k-closed point-star network [302].

Proof (1) $\Rightarrow$ (3). Let $f : M \to X$ be a proper mapping, where M is a metrizable space. By Theorem 1.3.5, there is a sequence $\{\mathscr{U}_n\}$ of open covers of M such that for each $K \in \mathscr{K}(M)$, $\{\mathrm{st}(K, \mathscr{U}_n)\}_{n\in\mathbb{N}}$ is a neighborhood base of K in M. For each $n \in \mathbb{N}$, let $\mathscr{B}_n$ be a locally finite closed refinement of $\mathscr{U}_n$, because M is paracompact. Let $\mathscr{P}_n = f(\mathscr{B}_n)$. Then $\mathscr{P}_n$ is a compact-finite k-closed cover of X. Because in fact, suppose $H \in \mathscr{K}(X)$. Then $f^{-1}(H) \in \mathscr{K}(M)$, so $(\mathscr{B}_n)_{f^{-1}(H)}$ is finite, and hence

$(\mathscr{P}_n)_H$ is also finite. If $P \in \mathscr{P}_n$, then there is $B \in \mathscr{B}_n$ such that $P = f(B)$, thus $P \cap H = f(B \cap f^{-1}(H))$ is a closed subset of H.

For each $x \in U \in \tau(X)$, since the compact set $f^{-1}(x) \subset f^{-1}(U) \in \tau(M)$, there is $n \in \mathbb{N}$ such that $\mathrm{st}(f^{-1}(x), \mathscr{U}_n) \subset f^{-1}(U)$, so $\mathrm{st}(f^{-1}(x), \mathscr{B}_n) \subset f^{-1}(U)$, and hence $\mathrm{st}(x, \mathscr{P}_n) = f(\mathrm{st}(f^{-1}(x), \mathscr{B}_n)) \subset U$. Therefore, $\{\mathscr{P}_n\}$ is a compact-finite k-closed point-star network for X.

(3) $\Rightarrow$ (2). Suppose $\{\mathscr{P}_n\}$ is a compact-finite k-closed point-star network for X. Let $\mathscr{P} = \bigcup_{n\in\mathbb{N}} \mathscr{P}_n$. If K is a compact subset of X, then $\mathscr{P}_{|K}$ is a countable network for K, so K is metrizable in X, and hence K is also metrizable in kX. For each $n \in \mathbb{N}$, it is obvious that $\mathscr{P}_n$ is a compact-finite closed cover of kX. Since kX is a k-space, $\mathscr{P}_n$ is locally finite. For each $x \in X$, since $x \in X - \cup\{P \in \mathscr{P}_n : x \notin P\} \subset \mathrm{st}(x, \mathscr{P}_n)$, $\mathrm{st}(x, \mathscr{P}_n)$ is a closed neighborhood of x in kX.

We first prove kX is a first countable regular space. We may assume that each $\mathscr{P}_{n+1}$ is a refinement of $\mathscr{P}_n$. Let $x \in U \in \tau(kX)$. Then there is $m \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{P}_m) \subset U$. Because otherwise, for each $n \in \mathbb{N}$, $\mathrm{st}(x, \mathscr{P}_n) \not\subset U$, and hence there exist $P_n \in (\mathscr{P}_n)_x$ and $p_n \in P_n - U$. Since $\{\mathscr{P}_n\}$ is a point-star network for X, the sequence $\{p_n\}$ converges to x in X. Since $(\{x\} \cup \{p_n : n \in \mathbb{N}\}) \cap U$ is an open subset of the subspace $\{x\} \cup \{p_n : n \in \mathbb{N}\}$ of X, $\{p_n\}$ is eventually in U, a contradiction. That shows $x \in \mathrm{int}_{kX}[\mathrm{st}(x, \mathscr{P}_m)] \subset \overline{\mathrm{int}_{kX}[\mathrm{st}(x, \mathscr{P}_m)]} \subset \mathrm{st}(x, \mathscr{P}_m) \subset U$. So kX is a first countable regular space.

Now we prove $\mathscr{P}$ is a k-network for kX. Suppose a sequence $\{x_n\}$ in kX converges to $x \in U \in \tau(kX)$. Then there is $m \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{P}_m) \subset U$. Since $\mathrm{st}(x, \mathscr{P}_m)$ is a neighborhood of x in kX, the sequence $\{x_n\}$ is eventually in $\mathrm{st}(x, \mathscr{P}_m)$. Since $\mathscr{P}_m$ is point-finite, there exist $P \in \mathscr{P}_m$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$. So $\mathscr{P}$ is a cs^*-network for kX. By Proposition 1.6.7, $\mathscr{P}$ is a k-network for kX.

By Theorem 2.5.15, kX is a metrizable space.

(2) $\Rightarrow$ (1). Suppose kX is a metrizable space. Since $\mathrm{id}_X : kX \to X$ is a proper mapping, X is a k-metrizable space. ■

In the following, we give a characterization of k^*-metrizable spaces. Suppose X is a space, denote

$$\mathscr{P}^*(X) = \{A \subset X : A \neq \varnothing\}.$$

For each $x \in X$, let

$$\begin{aligned}\mathrm{Nw}(x) = \{(A_n)_{n\in\omega} \in \mathscr{P}^*(X)^\omega :\ & A_{n+1} \subset A_n,\ \text{and if } U \text{ is a neighborhood} \\ & \text{of } x \text{ in } X,\ \text{then there is } n \in \omega \text{ such that } \overline{A_n} \subset U\}.\end{aligned}$$

If $(A_n)_{n\in\omega} \in \mathrm{Nw}(x)$, for each $n \in \omega$, take $x_n \in A_n$, then the sequence $\{x_n\}$ converges to x in X, so $x \in \bigcap_{n\in\omega} \overline{A_n}$. That shows $\{\overline{A_n}\}_{n\in\omega}$ is a decreasing net at x in X, and hence $\{\{x\} \cup A_n\}_{n\in\omega}$ is also a decreasing net at x in X.

Let $\mathrm{Nw}(X) = \bigcup_{x\in X} \mathrm{Nw}(x)$. Give the power set $\mathscr{P}(X)$ the discrete topology. The set $\mathrm{Nw}(X)$ with the subspace topology of the product space $\mathscr{P}(X)^\omega$ is called the *network hyperspace* on X [54]. Obviously, each network hyperspace is a metrizable space. Since X is a T_2 space, for different points x, y in X, $\mathrm{Nw}(x) \cap \mathrm{Nw}(y) = \varnothing$. Define a function $\pi : \mathrm{Nw}(X) \to X$ as follows: for every $x \in X$ and $(A_n)_{n\in\omega} \in \mathrm{Nw}(x)$,

$$\pi((A_n)_{n\in\omega}) = x, \quad \text{i.e., } \pi^{-1}(x) = \mathrm{Nw}(x).$$

Obviously, $\pi((A_n)_{n\in\omega}) = \bigcap_{n\in\omega} \overline{A}_n$.

Lemma 2.6.8 ([54]) *$\pi : Nw(X) \to X$ is a mapping.*

Proof Obviously, π is an onto function. Let $x \in X$ and $(A_n)_{n\in\omega} \in \mathrm{Nw}(x)$. If U is a neighborhood of x in X, then there is $m \in \omega$ such that $\overline{A}_m \subset U$. Let

$$V = \{(B_n)_{n\in\omega} \in \mathrm{Nw}(X) : B_m = A_m\}.$$

Then V is an open neighborhood of $(A_n)_{n\in\omega}$ in the metrizable space $\mathrm{Nw}(X)$ and $\pi(V) \subset U$. Because in fact, if $(B_n)_{n\in\omega} \in V$, then $\pi((B_n)_{n\in\omega}) = \bigcap_{n\in\omega} \overline{B}_n \subset \overline{B}_m = \overline{A}_m \subset U$, and hence $\pi(V) \subset U$. That proves π is continuous. ■

Theorem 2.6.9 ([54]) *A regular space X is k^*-metrizable if and only if X has a σ-compact-finite k-network.*

Proof The necessity. Suppose X is a k^*-metrizable regular space. Then there exist a metrizable space M and a subproper mapping $f : M \to X$, that is, there is a subset Z of M such that $f(Z) = X$ and $\overline{Z \cap f^{-1}(K)}$ is a compact subset of M whenever K is a compact subset of X. Let $\mathscr{B} = \bigcup_{n\in\mathbb{N}} \mathscr{B}_n$ be a σ-locally finite base for the metrizable space M, where each $\mathscr{B}_n$ is locally finite. We prove that $f(\mathscr{B}_{|Z})$ is a σ-compact-finite k-network for X.

First, for each $n \in \mathbb{N}, f(\mathscr{B}_{n|Z})$ is compact-finite in X. In fact, let $K \in \mathscr{K}(X)$. Then $\overline{Z \cap f^{-1}(K)} \in \mathscr{K}(M)$, so only finitely many elements of $\mathscr{B}_n$ meet with $\overline{Z \cap f^{-1}(K)}$, and hence only finitely many elements of $\mathscr{B}_{n|Z}$ meet with $f^{-1}(K)$. It follows K meet with only finitely many elements of $f(\mathscr{B}_{n|Z})$. Therefore, $f(\mathscr{B}_{|Z})$ is σ-compact-finite in X.

Suppose K is a compact set, U is an open set in X and $K \subset U$. Since $\mathscr{B}$ is a base for the space M and the compact set $\overline{Z \cap f^{-1}(K)} \subset f^{-1}(K) \subset f^{-1}(U)$, there is $\mathscr{B}' \in \mathscr{B}^{<\omega}$ such that $\overline{Z \cap f^{-1}(K)} \subset \cup\mathscr{B}' \subset f^{-1}(U)$, so $f(\mathscr{B}'_{|Z}) \in [f(\mathscr{B}_{|Z})]^{<\omega}$ and $K = f(Z \cap f^{-1}(K)) \subset \cup f(\mathscr{B}'_{|Z}) \subset U$. Thus, $f(\mathscr{B}_{|Z})$ is a k-network for X.

The sufficiency. Suppose X has a σ-compact-finite k-network $\mathscr{P}$. Denote $\mathscr{P} = \bigcup_{k\in\omega} \mathscr{P}_k$, where each $\mathscr{P}_k$ is compact-finite. We may assume that $\mathscr{P}$ is closed under finite intersections and for each $m \in \omega$, $\bigcup_{k\geqslant m} \mathscr{P}_k$ is also a k-network for X. Define a sequence $\{\beta_k\}_{k\in\omega}$ of ordinal numbers as follows: $\beta_0 = 0$; let α_k be the first ordinal number with cardinality $|\mathscr{P}_k|$ and let $\beta_{k+1} = \beta_k + \alpha_k$. Denote

$$\mathscr{P}_k = \{P_\alpha : \beta_k \leqslant \alpha < \beta_{k+1}\}, \ k \in \omega.$$

Let $\beta_\omega = \sup_{k\in\omega} \beta_k$. Then $\mathscr{P} = \{P_\alpha : \alpha < \beta_\omega\}$.

For each $x \in X$, denote $\Lambda(x) = \{\alpha < \beta_\omega : x \in P_\alpha\}$. For each $k \in \omega$, since $\mathscr{P}_k$ is point-finite, $\Lambda(x) \cap [\beta_k, \beta_{k+1})$ is finite, so $\Lambda(x)$ has the order type of ω, and hence there is an order-preserving bijection $\alpha_x : \omega \to \Lambda(x)$. For each $k \in \omega$, let $A_k = \bigcap_{i \leqslant k} P_{\alpha_x(i)}$. If U is a neighborhood of x in X, then by the regularity of X, there is $P \in \mathscr{P}$ such that $x \in P \subset \overline{P} \subset U$. Since $x \in P$, there is $k \in \omega$ such that $P = P_{\alpha_x(k)}$, and then $\overline{A}_k \subset \overline{P} \subset U$. So $(A_k)_{k\in\omega} \in \mathrm{Nw}(x)$. Let $s(x) = (A_k)_{k\in\omega}$ and hence we have defined a function $s : X \to \mathrm{Nw}(X)$ such that $\pi \circ s(x) = x$.

Claim. If K is a compact subset of X, then $\overline{s(K)}$ is a compact subset of $\mathrm{Nw}(X)$.

In fact, since every compact subset of X is metrizable and $\mathrm{Nw}(X)$ is a metrizable space, we only need to prove that if $\{x_n\}$ is a convergent sequence in X, then the sequence $\{s(x_n)\}$ has a convergent subsequence in $\mathrm{Nw}(X)$. Let $K = \{x\} \cup \{x_n : n \in \omega\}$, where $x_n \to x \in X$. Define $\Lambda(K) = \{\alpha < \beta_\omega : K \cap P_\alpha \neq \varnothing\}$. For each $k \in \omega$, since $\mathscr{P}_k$ is compact-finite, $\Lambda(K) \cap [0, \beta_{k+1})$ is finite, so $\Lambda(K)$ is countable. For each $m \in \omega$, since $\{P_\alpha : \beta_m \leqslant \alpha < \beta_\omega\}$ is a k-network for X, there is an increasing sequence $\{k_i\}_{i\in\omega}$ of natural numbers such that $K \subset \bigcup_{\beta_{k_i} \leqslant \alpha < \beta_{k_{i+1}}} P_\alpha$.

For each $n \in \omega$, the index set $\Lambda(x_n) = \{\alpha < \beta_\omega : x_n \in P_\alpha\}$ has the order type of ω and we may suppose $\alpha_n = \alpha_{x_n} : \omega \to \Lambda(x_n)$ is an order-preserving bijection. For each $i \in \omega$, since $x_n \in K$, $\alpha_n(i) < \beta_{k_{i+1}}$, so $\alpha_n(i) \in \Lambda(K) \cap [0, \beta_{k_{i+1}})$. Denote $\Lambda_i(K) = \Lambda(K) \cap [0, \beta_{k_{i+1}})$. Let $\Pi = \prod_{i\in\omega} \Lambda_i(K)$ be the product space, where each finite set $\Lambda_i(K)$ has the discrete topology. Then Π is a compact metrizable space and each $\alpha_n \in \Pi$. Thus, the sequence $\{\alpha_n\}_{n\in\omega}$ in Π has a subsequence $\{\alpha_n\}_{n\in J}$ converging to some point $\widehat{\alpha}$. For each $i \in \omega$, since $\lim\limits_{J \ni n \to \infty} \alpha_n(i) = \widehat{\alpha}(i)$, $\alpha_n(i) = \widehat{\alpha}(i)$ when n is sufficient large in J, so $x_n \in P_{\alpha_n(i)} = P_{\widehat{\alpha}(i)}$, and hence $x = \lim\limits_{J \ni n \to \infty} x_n \in \overline{P_{\widehat{\alpha}(i)}}$.

In the above, we have proved that for each $i \in \omega$, $\alpha_n(i) = \widehat{\alpha}(i)$ when n is sufficient large in J, so for each $k \in \omega$, $\bigcap_{i\leqslant k} P_{\alpha_n(i)} = \bigcap_{i \leqslant k} P_{\widehat{\alpha}(i)}$ when n is sufficient large in J. Let $\mathbf{p} = \left(\bigcap_{i\leqslant k} P_{\widehat{\alpha}(i)}\right)_{k\in\omega} \in \mathscr{P}(X)^\omega$. Since every $s(x_n) = \left(\bigcap_{i\leqslant k} P_{\alpha_n(i)}\right)_{k\in\omega}$, the sequence $\{s(x_n)\}_{n\in J}$ converges to $\mathbf{p}$ in the product space $\mathscr{P}(X)^\omega$. Below we prove $\mathbf{p} \in \mathrm{Nw}(x)$. By the regularity of X, we only need to prove that if U is a neighborhood of x in X, then there is $m \in \omega$ such that $\overline{P_{\widehat{\alpha}(m)}} \subset \overline{U}$. Since $\mathscr{P}$ is a k-network for X, there is $\gamma < \beta_\omega$ such that P_γ contains infinitely many terms of the convergent sequence $\{x_n\}_{n\in J}$ and $\overline{P_\gamma} \subset \overline{U}$. Let $J_1 = \{n \in J : x_n \in P_\gamma\}$. Then J_1 is infinite. For each $n \in J_1$, since $\gamma \in \Lambda(x_n)$, there is only one natural number $k_n \leqslant |\Lambda(K) \cap [0, \gamma]|$ such that $\gamma = \alpha_{x_n}(k_n)$. Since $\Lambda(K) \cap [0, \gamma]$ is a finite set, there is $m \in \omega$ such that $J_2 = \{n \in J_1 : k_n = m\}$ is infinite. Since $\{\alpha_n\}_{n\in J_2}$ converges to $\widehat{\alpha}$, $\widehat{\alpha}(m) = \alpha_{x_n}(m) = \alpha_{x_n}(k_n) = \gamma$ for sufficient large n in J_2, and hence $\overline{P_{\widehat{\alpha}(m)}} = \overline{P_\gamma} \subset \overline{U}$.

To sum up, there is a subsequence $\{s(x_n)\}_{n\in J}$ of $\{s(x_n)\}_{n\in\omega}$ converging to $\mathbf{p}$ in $\mathrm{Nw}(X)$. Thus, if K is a compact subset of X, then $\overline{s(K)}$ is a compact subset of $\mathrm{Nw}(X)$. We have proved Claim.

To complete the proof, it remains to prove $\pi : \mathrm{Nw}(X) \to X$ is a subproper mapping. Let $Z = s(X)$. Then $Z \subset \mathrm{Nw}(X)$ and $\pi(Z) = \pi \circ s(X) = X$. If K is a compact subset of X, then $\overline{Z \cap \pi^{-1}(K)} = \overline{s(K)}$ is a compact subset of $\mathrm{Nw}(X)$, so π is subproper. Therefore, X is k^*-metrizable. ■

Example 2.8.14 indicates that a space having a σ-discrete k-network may not be a k^*-metrizable space, and hence the regularity in Theorem 2.6.9 is important.

Corollary 2.6.10 *For every space X, the following are equivalent:*

(1) X is a Lašnev space.
(2) X is a k^*-metrizable Fréchet–Urysohn space.
(3) X is a Fréchet–Urysohn space with a σ-compact-finite family $\mathscr{P}$ such that $\overline{\mathscr{P}}$ is a k-network for X.

Proof (1) $\Rightarrow$ (2) is obtained by Theorems 2.5.8 and 2.6.9.

(2) $\Rightarrow$ (3). Suppose X is a k^*-metrizable Fréchet–Urysohn space. By using the notations in the proof of the necessity of Theorem 2.6.9. we first prove $\overline{f(Z \cap B)} \subset f(\overline{B})$ for every $B \in \mathscr{B}$. Suppose $x \in \overline{f(Z \cap B)}$. Since X is a Fréchet–Urysohn space, there is a sequence $\{b_n\}$ in $Z \cap B$ such that $\{f(b_n)\}$ converges to x. Since each $b_n \in Z \cap f^{-1}(\{x\} \cup \{f(b_n) : n \in \mathbb{N}\})$ and $\overline{Z \cap f^{-1}(\{x\} \cup \{f(b_n) : n \in \mathbb{N}\})}$ is a compact subset of M, $\{b_n\}$ has an accumulation point $b \in \overline{B}$. By the continuity of f, we have $x = f(b) \in f(\overline{B})$. So $\overline{f(Z \cap B)} \subset f(\overline{B})$.

We have proved $f(\mathscr{B}_{|Z})$ is σ-compact-finite in the proof of the necessity of Theorem 2.6.9. It remains to prove $\overline{f(\mathscr{B}_{|Z})}$ is a k-network for X. Suppose K is compact and U is open in X such that $K \subset U$. Then there is $\mathscr{B}' \in \mathscr{B}^{<\omega}$ such that

$$\overline{Z \cap f^{-1}(K)} \subset \cup\mathscr{B}' \subset \cup\overline{\mathscr{B}'} \subset f^{-1}(U),$$

so $f(\mathscr{B}'_{|Z}) \in [f(\mathscr{B}_{|Z})]^{<\omega}$ and

$$K = f(Z \cap f^{-1}(K)) \subset \cup f(\mathscr{B}'_{|Z}) \subset \cup\overline{f(\mathscr{B}'_{|Z})} \subset \cup f(\overline{\mathscr{B}'}) \subset U.$$

Thus, $\overline{f(\mathscr{B}_{|Z})}$ is a k-network for X.

(3) $\Rightarrow$ (1). Suppose the Fréchet–Urysohn space X has a σ-compact-finite family $\mathscr{P}$ such that $\overline{\mathscr{P}}$ is a k-network for X. By Lemma 2.5.5, $\overline{\mathscr{P}}$ is a σ-*HCP* closed k-network for X. Since the proof of (3) $\Rightarrow$ (1) in Theorem 2.5.8 dedicates that every Fréchet–Urysohn space with a σ-*HCP* closed k-network is a Lašnev space, X is a Lašnev space. ∎

Example 2.6.11 On the property of every compact set being metrizable and the generalized sequentiality property.

(1) A space in which every compact set is metrizable may not be a k-space. The Michael space (see Example 1.8.8) is such a space.
(2) Suppose $\mathscr{A}$ is the class of spaces in which every compact set is metrizable. We have the following conclusions in the class $\mathscr{A}$:

 (i) There is a k^*-metrizable k-space which is neither a k-metrizable space nor a Fréchet–Urysohn space. The Arens space S_2 (see Example 1.8.6) is such a space.

(ii) There is a Fréchet–Urysohn space which is not a strongly Fréchet–Urysohn space. The sequential fan S_ω (see Example 1.8.7) is such a space.
(iii) There is a strongly Fréchet–Urysohn space which is not a first countable space. The space X/K in Example 2.4.14 is such a space.
(iv) There is a first countable space which is not k^*-metrizable and does not satisfying the condition that every compact set has a countable neighborhood base. The butterfly space (see Example 1.8.3) is such a space. The butterfly space X illustrates a noteworthy fact that a space X may not be a compact-covering open image of a metric space, even if X is both a compact-covering image of a metric space and an open image of a metric space.

(3) A space in which every compact subset has a countable neighborhood base does not imply a space in which every compact set is metrizable. The Alexandroff double-arrow space (see Example 1.8.9) offers such an example.

2.7 s-Mappings

The purpose of this section is to give characterizations of various classes of s-images of metric spaces. The problem of seeking the intrinsic characterizations of quotient s-images of metric spaces was regarded as an "important and difficult" problem posed by Arhangel'skiĭ [31]. Looking for the solutions of this problem and to improve them gradually has experienced about 20 years. Until 1987, some satisfactory answers were given by using the concept of cs^*-networks. In this section, starting from the point-countable families, we describe the intrinsic characterizations of quotient s-images, pseudo-open s-images, open s-images and closed s-images of metric spaces by using the concepts of cs^*-networks, k-networks, bases and so on.

A cover $\mathscr{U}$ of a set X is said to be *precisely refining* $\mathscr{V}$ if, $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$, $\mathscr{V} = \{V_\alpha\}_{\alpha \in A}$ and $U_\alpha \subset V_\alpha$ for every $\alpha \in A$.

Proposition 2.7.1 *Let $\mathscr{K}$ and $\mathscr{P}$ be covers of a space X, where $\mathscr{K} \subset \mathscr{K}(X)$. If $\mathscr{K}$ and $\mathscr{P}$ satisfy that for each $K \in \mathscr{K}$, there is a sequence $\{\mathscr{P}_n\}$ of finite subsets of $\mathscr{P}$ such that*

(i) for each $n \in \mathbb{N}$, $\mathscr{P}_n$ is precisely refined by a finite closed cover of K;
(ii) if $\{P_n\}_{n \in \mathbb{N}}$ is a net at x in X for every $x \in K$ and $P_n \in (\mathscr{P}_n)_x$, $\forall n \in \mathbb{N}$,

then there are a metrizable space M and a mapping $f : M \to X$ such that

(1) for each $K \in \mathscr{K}$, there is $L \in \mathscr{K}(M)$ such that $f(L) = K$;
(2) for each $E \subset X$, if $(\mathscr{P})_E$ is countable, then $f^{-1}(E)$ has a countable base.

Proof Denote $\mathscr{P} = \{P_\alpha\}_{\alpha \in \Lambda}$. For each $i \in \mathbb{N}$, let Λ_i be the space obtained by giving the discrete topology to the set Λ. Define

$$M = \left\{\beta = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i : \ \{P_{\alpha_i}\} \text{ is a net at some point } x(\beta) \text{ in } X\right\}.$$

Then M is a metrizable space and for each $\beta \in M$, a point $x(\beta)$ in X can be determined by β uniquely. So we can define a function $f : M \to X$ by $f(\beta) = x(\beta)$. By (ii), f is a mapping. By the definition of f, (2) holds. We prove (1) holds below.

For each $K \in \mathscr{K}$, assume that a sequence $\{\mathscr{P}_n\}$ of finite sets of $\mathscr{P}$ satisfies (i) and (ii). For each $n \in \mathbb{N}$, there exist $\Gamma_n \in \Lambda_n^{<\omega}$ and a closed cover $\{K_\alpha : \alpha \in \Gamma_n\}$ of K such that $\mathscr{P}_n = \{P_\alpha : \alpha \in \Gamma_n\}$ and $K_\alpha \subset P_\alpha$. Let

$$L = \left\{(\alpha_i) \in \prod_{i\in\mathbb{N}} \Gamma_i : \bigcap_{i\in\mathbb{N}} K_{\alpha_i} \neq \varnothing\right\}.$$

Then $L \in \tau^c(\prod_{i\in\mathbb{N}} \Gamma_i)$. Because in fact, if $(\alpha_i) \in \prod_{i\in\mathbb{N}} \Gamma_i - L$, then $\bigcap_{i\in\mathbb{N}} K_{\alpha_i} = \varnothing$, and hence there is $i_0 \in \mathbb{N}$ such that $\bigcap_{i\leqslant i_0} K_{\alpha_i} = \varnothing$. Let

$$W = \left\{(\beta_i) \in \prod_{i\in\mathbb{N}} \Gamma_i : \beta_i = \alpha_i, i \leqslant i_0\right\}.$$

Then $(\alpha_i) \in W \in \tau(\prod_{i\in\mathbb{N}} \Gamma_i)$ and $W \cap L = \varnothing$. Therefore, $L \in \mathscr{K}(\prod_{i\in\mathbb{N}} \Gamma_i)$. For each $\alpha = (\alpha_i) \in L$, there is $x \in \bigcap_{i\in\mathbb{N}} K_{\alpha_i} \subset K \cap \bigcap_{i\in\mathbb{N}} P_{\alpha_i}$, so $\alpha \in M$ and $f(\alpha) = x$, and hence $L \subset M$ and $f(L) \subset K$. On the other hand, for every $x \in K$ and $i \in \mathbb{N}$, there is $\alpha_i \in \Gamma_i$ such that $x \in K_{\alpha_i}$. Let $\alpha = (\alpha_i)$. Then $\alpha \in L$ and $f(\alpha) = x$, so $f(L) \supset K$, thus $f(L) = K$. ■

Proposition 2.7.2 *Let $f : X \to Y$ be a sequentially quotient mapping. If $\mathscr{P}$ is a cs^*-network for X, then $f(\mathscr{P})$ is a cs^*-network for Y.*

Lemma 2.7.3 ([269]) *If $\mathscr{P}$ is a point-countable cs^*-network for X and $\mathscr{K} = \mathscr{S}(X)$, then $\mathscr{K}$ and $\mathscr{P}$ satisfy the assumptions of Proposition 2.7.1.*

Proof For each $K \in \mathscr{K}$, denote $K = \{x\} \cup \{x_n : n \in \mathbb{N}\}$, where $\{x_n\}$ is a nontrivial sequence converging to x. We first prove that for every $K \subset U \in \tau$, there is $\mathscr{F} \in \mathscr{P}^{<\omega}$ with the following property, briefly referred as $\Phi(K, U)$:

$$K \subset \cup\mathscr{F} \subset U \text{ and } \mathscr{F}_{|K} \subset \tau^c - \{\varnothing\}.$$

Let $\mathscr{P}' = \{P \in \mathscr{P} : x \in P \subset U\} = \{P_i\}_{i\in\mathbb{N}}$. If for each $k \in \mathbb{N}$, $\{x_n\}$ is not eventually in $\bigcup_{i\leqslant k} P_i$, then we can take $n_k \in \mathbb{N}$ for each $k \in \mathbb{N}$ such that $x_{n_k} \in X - \bigcup_{i\leqslant k} P_i$. Since $\mathscr{P}$ is a cs^*-network, there exist a subsequence $\{y_j\}$ of $\{x_{n_k}\}$ and $m \in \mathbb{N}$ such that the sequence $\{y_j\} \subset P_m$, a contradiction. As a consequence, there is $k \in \mathbb{N}$ such that the sequence $\{x_n\}$ is eventually in $\bigcup_{i\leqslant k} P_i$. Now it is easy to see that there exists $\mathscr{F} \in \mathscr{P}^{<\omega}$ with the property $\Phi(K, U)$.

The family $\{\mathscr{F} \in \mathscr{P}^{<\omega} : \mathscr{F}$ has the property $\Phi(K, X)\}$ is countable, and we denote it as $\{\mathscr{P}_n\}_{n\in\mathbb{N}}$. Obviously, $\{\mathscr{P}_n\}$ satisfies (i) of Proposition 2.7.1. Below we prove $\{\mathscr{P}_n\}$ satisfies (ii) of Proposition 2.7.1.

Suppose $y \in K$ and $P_n \in (\mathscr{P}_n)_y$, $\forall n \in \mathbb{N}$. Let $y \in V \in \tau$. If $y = x$, take $K_1 = V \cap K$, then there is $\mathscr{F}' \in \mathscr{P}^{<\omega}$ with the property $\Phi(K_1, V)$ and there is $\mathscr{F}'' \in \mathscr{P}^{<\omega}$, such that, $K - K_1 \subset \cup\mathscr{F}'' \subset X - K_1$, so $\mathscr{F}' \cup \mathscr{F}''$ also has the property $\Phi(K, X)$, and hence there is $i \in \mathbb{N}$ such that $\mathscr{F}' \cup \mathscr{F}'' = \mathscr{P}_i$, thus $y \in P_i \subset \cup\mathscr{F}' \subset V$. If $y \neq x$, then there is $P \in \mathscr{P}$ such that $y \in P \subset V - (K - \{y\})$, so there is $\mathscr{F}' \in \mathscr{P}^{<\omega}$ with the property $\Phi(K - \{y\}, X - \{y\})$, and hence $\mathscr{F}' \cup \{P\}$ has the property $\Phi(K, X)$, it follows that there is $j \in \mathbb{N}$ such that $\mathscr{F}' \cup \{P\} = \mathscr{P}_j$, therefore $y \in P_j = P \subset V$. Thus $\{P_n\}_{n\in\mathbb{N}}$ is a net at y. ■

Theorem 2.7.4 ([269]) *For every space X, the following are equivalent:*

(1) X has a point-countable cs-network.*
(2) X is a sequentially quotient s-image of a metric space.
(3) X is a sequence-covering s-image of a metric space.

Proof By Proposition 2.7.1 and Lemma 2.7.3, we get (1) ⇒ (3). (3) ⇒ (2) is obtained by Proposition 2.1.13. (2) ⇒ (1) follows from the Nagata–Smirnov metrization theorem and Proposition 2.7.2. ■

Corollary 2.7.5 ([167, 455]) *For every space X, the following are equivalent:*

(1) X is a sequential space with a point-countable cs-network.*
(2) X is a sequentially quotient s-image of a metric space.
(3) X is a sequence-covering quotient s-image of a metric space.
(4) X is a quotient s-image of a metric space.

Proof By Theorem 2.7.4, the mapping lemma (see Proposition 2.1.12) and Proposition 2.3.1 we get (1) ⇔ (2) ⇔ (3) and (2) ⇔ (4). ■

The following facts should be mentioned:

(1) In (1), (3) and (4) of Corollary 2.7.5, if we replace "sequential space" and "quotient mapping" with "Fréchet–Urysohn space" and "pseudo-open mapping" respectively, then the conclusion still holds.
(2) There is a k-space with a point-countable cs^*-network which is not a sequential space. The compactification $\beta\mathbb{N}$ is such a space.

Corollary 2.7.6 ([194]) *Every space with a point-countable weak base is a quotient s-image of a metric space.*

Now we turn to discuss relationships among spaces with a point-countable cs^*-network, spaces with a point-countable k-network and compact-covering quotient s-images of metric spaces.

Definition 2.7.7 Let $\mathscr{P}$ be a family of sets in a space X. For $A \subset X$ and $\mathscr{F} \subset \mathscr{P}$, we have the following definitions:

(1) $\mathscr{F}$ is called a *minimal cover* [344] (or an *irreducible cover* [119]) of A if, $\mathscr{F}$ covers A and A cannot be covered by any proper subset of $\mathscr{F}$.
(2) $\mathscr{F}$ is called a *minimal interior cover* [89] of A if, $A \subset (\cup\mathscr{F})^\circ$ and for any proper subset $\mathscr{H}$ of $\mathscr{F}$, $A \not\subset (\cup\mathscr{H})^\circ$.

Lemma 2.7.8 [344] (The Miščenko lemma) *If $\mathscr{P}$ is a point-countable cover of a space X, then any subset of X has only countably many minimal finite covers consisting of elements of $\mathscr{P}$.*

Proof Suppose A is a subset of X and $\{\mathscr{P}_\alpha\}_{\alpha\in\Lambda}$ is the set of all minimal finite covers of A consisting of elements of $\mathscr{P}$. If the lemma is not true, then there is $n \in \mathbb{N}$ such that $\Psi = \{\mathscr{P}_\alpha : \alpha \in \Lambda, |\mathscr{P}_\alpha| = n\}$ is uncountable. For each $P \in \mathscr{P}$, take $\Psi(P) = \{\mathscr{P}_\alpha \in \Psi : P \in \mathscr{P}_\alpha\}$. Let $x_1 \in A$. Then $\Psi = \cup\{\Psi(P) : x_1 \in P \in \mathscr{P}\}$. Since $\mathscr{P}$ is point-countable, there is $P_1 \in \mathscr{P}$ such that $x_1 \in P_1$ and $|\Psi(P_1)| > \aleph_0$. Then $n > 1$ and there is $x_2 \in A - P_1$. Let $\Psi(P_1, P) = \{\mathscr{P}_\alpha \in \Psi(P_1) : P \in \mathscr{P}_\alpha\}$. Then $\Psi(P_1) = \cup\{\Psi(P_1, P) : x_2 \in P \in \mathscr{P}\}$, and hence there is $P_2 \in \mathscr{P}$ such that $x_2 \in P_2 \neq P_1$ and $|\Psi(P_1, P_2)| > \aleph_0$. Repeating the above process, we can get a set $\{x_i : i \leqslant n\}$ and a family $\{P_i\}_{i\leqslant n}$ such that $x_i \in P_i \in \mathscr{P}$ and when $i \neq j \leqslant n$, $P_i \neq P_j$ and $|\Psi(P_1, \ldots, P_n)| > \aleph_0$. However $|\Psi(P_1, \ldots, P_n)| = 1$, a contradiction. ■

Miščenko [344] proved that a compact space with a point-countable base has a countable base. Although this result has a lot of improved forms (for example Lemma 3.1.6), the following Miščenko's proof is worth admiring.

Suppose $\mathscr{P}$ is a point-countable base of a compact space X. Let

$$\mathscr{H} = \cup\{\mathscr{F} \in \mathscr{P}^{<\omega} : \mathscr{F} \text{ is a minimal cover of } X\}.$$

By the Miščenko lemma, $\mathscr{H}$ is countable. We prove $\mathscr{H}$ is a base for X. Suppose $x \in U \in \tau$. Take $P \in \mathscr{P}$ such that $x \in P \subset U$. If $y \in X - P$, then there is $P_y \in \mathscr{P}$ such that $y \in P_y \subset X - \{x\}$. Since $X - P$ is compact, the cover $\{P_y : y \in X - P\}$ of $X - P$ has a finite subcover $\mathscr{P}'$. Thus, $\{P\} \cup \mathscr{P}'$ is a finite cover of X consisting of elements of $\mathscr{P}$, and hence there exists a minimal cover $\mathscr{F}$ of X such that $\mathscr{F} \subset \{P\} \cup \mathscr{P}'$. It follows that $P \in \mathscr{F} \subset \mathscr{H}$ and $x \in P \subset U$. So $\mathscr{H}$ is a countable base for X.

Lemma 2.7.9 *Let $\mathscr{P}$ be a point-countable k-network in a space X. If $K \in \mathscr{K}(X)$, then there is a sequence $\{\mathscr{P}_n\}$ of finite subfamilies of $\mathscr{P}$ covering K such that $\{\mathscr{P}_n\}$ satisfies the condition (ii) of Proposition 2.7.1.*

Proof By the Miščenko lemma, denote

$$\{\mathscr{F} \in \mathscr{P}^{<\omega} : \mathscr{F} \text{ is a minimal cover of } K\} \text{ by } \{\mathscr{P}_n\}_{n\in\mathbb{N}}.$$

For every $x \in K$ and $P_n \in (\mathscr{P}_n)_x$, $\forall n \in \mathbb{N}$, let $x \in V \in \tau(X)$. Take $W \in \tau(K)$ such that $x \in W$ and $\mathrm{cl}_K(W) \subset V$. Then there exist $\mathscr{H}_1, \mathscr{H}_2 \in \mathscr{P}^{<\omega}$ such that $\mathrm{cl}_K(W) \subset \cup\mathscr{H}_1 \subset V$ and $K - W \subset \cup\mathscr{H}_2 \subset X - \{x\}$. So $K \subset \cup(\mathscr{H}_1 \cup \mathscr{H}_2)$, and hence there is $n \in \mathbb{N}$ such that $\mathscr{P}_n \subset \mathscr{H}_1 \cup \mathscr{H}_2$, thus $x \in P_n \subset \cup\mathscr{H}_1 \subset V$. Therefore, $\{P_n\}_{n\in\mathbb{N}}$ is a net at x. ■

By Lemma 2.7.9, Propositions 2.7.1 and 2.1.12 (the mapping lemma), we have the following corollary.

Corollary 2.7.10 ([337]) *If a space X has a point-countable closed k-network, then X is a compact-covering s-image of a metric space. Suppose further that X is a k-space, then X is a compact-covering quotient s-image of a metric space.*

Lemma 2.7.11 ([262]) *Suppose $f : X \to Y$ is a quotient mapping and X is a k-space. If $\mathscr{B}$ is a k-network for X and $f(\mathscr{B})$ is a point-countable family of sets in Y, then $f(\mathscr{B})$ is a k-network for Y.*

Proof Let $\mathscr{P} = f(\mathscr{B})$. For a compact set $K \subset U \in \tau(Y)$, take

$$\mathscr{H} = \{P \in \mathscr{P} : P \cap K \neq \varnothing,\ P \subset U\}.$$

Then there is $\mathscr{F} \in \mathscr{H}^{<\omega}$ such that $K \subset \cup\mathscr{F}$. Because otherwise, for each $y \in K$, denote $(\mathscr{H})_y = \{P_i(y)\}_{i\in\mathbb{N}}$. Then there is a subset $A = \{y_n : n \in \mathbb{N}\}$ of K such that $y_n \notin P_i(y_j)$ whenever $i, j < n$. Take $a \in A^d$, and let $L = A - \{a\}$. Then $L \notin \tau^c(Y)$, so there is $C \in \mathscr{K}(X)$ such that $L \cap f(C) \notin \tau^c(Y)$ (see the proof of Proposition 2.3.1(1)), and hence $A \cap f(C)$ is an infinite set. Take $D = f^{-1}(K) \cap C$. Then there is $\mathscr{B}' \in \mathscr{B}^{<\omega}$ such that $D \subset \cup\mathscr{B}' \subset f^{-1}(U)$, and hence $f(D) \subset \cup f(\mathscr{B}') \subset U$. Since $f(D) \cap A = f(C) \cap A$, $(\cup f(\mathscr{B}')) \cap A$ is an infinite set, so there is $P \in f(\mathscr{B}') \in \mathscr{H}^{<\omega}$ containing an infinite subset of A. Let $P = P_i(y_j)$ for some $i, j \in \mathbb{N}$. Then there is $n > i, j$ such that $y_n \in P_i(y_j)$, a contradiction. Thus $\mathscr{P}$ is a k-network for Y. ■

Theorem 2.7.12 *Suppose X is a locally compact metric space. If $f : X \to Y$ is a quotient s-mapping, then f is a compact-covering mapping and there is a point-countable closed k-network for Y.*

Proof Let $\mathscr{B}$ be a σ-locally finite base for X satisfying $\overline{\mathscr{B}} \subset \mathscr{K}(X)$ and let $\mathscr{P} = f(\overline{\mathscr{B}})$. Then $\mathscr{P}$ is a point-countable family of closed sets in Y. By Lemma 2.7.11, $\mathscr{P}$ is a k-network for Y. If $K \in \mathscr{K}(Y)$, then there is $\mathscr{P}' \in \mathscr{P}^{<\omega}$ such that $K \subset \cup\mathscr{P}'$, so there is $\mathscr{F} \in \overline{\mathscr{B}}^{<\omega}$ such that $f(\mathscr{F}) = \mathscr{P}'$, and hence there is $L \in \mathscr{K}(X)$ such that $f(L) = K$. Thus f is a compact-covering mapping. ■

Nagami [369] proved that every quotient L-mapping on a locally compact paracompact space is a compact-covering mapping. Below we construct an example to show Corollary 2.7.10 is not reversible.

Lemma 2.7.13 *Let $\mathscr{P}$ be a point-countable family of sets in a strongly Fréchet–Urysohn space X. If $\mathscr{K}(X)$ refines $\mathscr{P}^F$, then $(\mathscr{P}^F)^\circ$ covers X.*

Proof Denote $(\mathscr{P})_C = \{P_i(C)\}_{i\in\mathbb{N}}$ for each countable set C in X. If there is $x \in X - \cup(\mathscr{P}^F)^\circ$, then by the Fréchet–Urysohn property, we can choose a sequence $\{C_n\}$ of countable subsets of X such that $C_1 = \{x\}$, $x \in \bigcap_{n\in\mathbb{N}} \overline{C_n}$ and $C_n \cap P_i(C_j) = \varnothing$ whenever $i, j < n$. So each element of $\mathscr{P}$ only meets finitely many C_n. By the strongly Fréchet–Urysohn property, there exist a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and $x_k \in C_{n_k}$ such that $x_k \to x$. Since $\{x\} \cup \{x_k : k \in \mathbb{N}\} \in \mathscr{K}(X)$, there is an element P in $\mathscr{P}$ containing infinitely many terms of $\{x_k\}$, and hence P meets infinitely many C_n, a contradiction. ■

Example 2.7.14 (*[437, 492]*) A compact-covering open image of a separable metric space which is a non-regular space with a countable base and has no point-countable closed k-network.

Let $X = \mathbb{R}$. Define the *pointed irrational extension topology* on X as follows: for each $x \in X$, an element of a neighborhood base of x has the form $\{x\} \cup (\mathbb{P} \cap U)$, where U is a Euclidean neighborhood of x in $\mathbb{R}$. The space X is called the pointed irrational extension topological space X of $\mathbb{R}$. Obviously, X has a countable base. Since $\mathbb{Q}$ is a closed subspace of X and $\overline{V} = X$ for any V with $\mathbb{Q} \subset V \in \tau(X)$, X is not a regular space. By Proposition 2.4.4, X is a compact-covering open image of a separable metric space.

Suppose there is a point-countable closed k-network $\mathscr{P}$ for X. Define $\mathscr{H} = \{P \in \mathscr{P} : P^\circ = \varnothing\}$. Then $\mathscr{K}(X)$ refines $\mathscr{H}^F$. In fact, for each $K \in \mathscr{K}(X)$, let $V = (K \cap \mathbb{Q}) \cup \mathbb{P}$. Since $\mathbb{Q}$ is a discrete closed subspace of X, $K \subset V \in \tau$, so there is $\mathscr{F} \in \mathscr{P}^{<\omega}$ such that $K \subset \cup\mathscr{F} \subset V$. If $(\cup\mathscr{F})^\circ \neq \varnothing$, then there is an open interval J in $\mathbb{R}$ such that $J \cap \mathbb{P} \subset \cup\mathscr{F}$, so $\mathbb{Q} \cap J \subset \mathbb{Q} \cap \overline{J \cap \mathbb{P}} \subset \mathbb{Q} \cap V = \mathbb{Q} \cap K$, and hence $\mathbb{Q} \cap J$ is a finite set, a contradiction. Thus $(\cup\mathscr{F})^\circ = \varnothing$, so $\mathscr{F} \in \mathscr{H}^{<\omega}$ and $K \subset \cup\mathscr{F}$. Hence $\mathscr{K}(X)$ refines $\mathscr{H}^F$. By Lemma 2.7.13, $X = \cup(\mathscr{H}^F)^\circ$, which contradicts the definition of $\mathscr{H}$. Consequently, X has no point-countable closed k-network.

In the second part of this section, we search for intrinsic characterizations of open s-images of metric spaces.

Lemma 2.7.15 ([89]) *If $\mathscr{P}$ is a point-countable family of sets in a Fréchet–Urysohn space X, then any subset of X has only countably many finite minimal interior covers consisting of elements of $\mathscr{P}$.*

Proof For each $\mathscr{F} \in \mathscr{P}^{<\omega}$, define

$$\mathscr{H}(\mathscr{F}) = \{H \subset X : \mathscr{F} \text{ is a finite minimal interior cover of } H\}.$$

Let $A \subset X$. If there are uncountably many $\mathscr{F} \in \mathscr{P}^{<\omega}$ such that $A \in \mathscr{H}(\mathscr{F})$, then there exist $m \in \mathbb{N}$ and an uncountable subset Ψ of $\mathscr{P}^{<\omega}$ such that $|\mathscr{F}| = m$ and $A \in \mathscr{H}(\mathscr{F})$ whenever $\mathscr{F} \in \Psi$. Suppose $\mathscr{R}$ is a maximal subset of $\mathscr{P}$ such that $\mathscr{R} \subset \mathscr{F}$ for uncountably many $\mathscr{F} \in \Psi$. Then $0 \leqslant |\mathscr{R}| < m$ and $A \not\subset (\cup\mathscr{R})^\circ$. Pick $x \in A - (\cup\mathscr{R})^\circ$. Then $x \in \overline{X - \cup\mathscr{R}}$. Since X is a Fréchet–Urysohn space, there is a countable subset L of $X - \cup\mathscr{R}$ such that $x \in \overline{L}$. Let $\Omega = \{\mathscr{F} \in \Psi : \mathscr{R} \subset \mathscr{F}\}$. If $\mathscr{F} \in \Omega$, then $x \in (\cup\mathscr{F})^\circ$, so L meets some element of $\mathscr{F}$. By the point-countability of $\mathscr{P}$ and the uncountability of Ω, there is $P \in \mathscr{P}$ such that $L \cap P \neq \varnothing$ and there are uncountably many elements of Ω containing P. Then $P \notin \mathscr{R}$ and there are uncountably many $\mathscr{F}$ in Ω (hence in Ψ) containing $\mathscr{R} \cup \{P\}$, which contradicts the maximality of $\mathscr{R}$. ■

The assumption "Fréchet–Urysohn space" in Lemma 2.7.15 can be replaced by the "space of countable tightness" weaker than the "Fréchet–Urysohn space" [89]. A space X is said to be of *countable tightness* if, for every $A \subset X$ and $x \in \overline{A} \subset X$, there is a countable subset C of A such that $x \in \overline{C}$.

Proposition 2.7.16 ([89]) *Suppose there is a point-countable family $\mathscr{P}$ of sets in a space X satisfying that for every $x \in U \in \tau$, there is $\mathscr{F} \in (\mathscr{P})_x^{<\omega}$ such that $x \in (\cup\mathscr{F})^\circ \subset \cup\mathscr{F} \subset U$. Then X has a point-countable base.*

Proof For each $\mathscr{F} \in \mathscr{P}^{<\omega}$, let

$$\mathscr{H}(\mathscr{F}) = \{H \subset X : \mathscr{F} \text{ is a finite minimal interior cover of } H\},$$
$$V(\mathscr{F}) = (\cup(\mathscr{H}(\mathscr{F}) \cap \mathscr{P}))^\circ, \text{ and } \mathscr{V} = \{V(\mathscr{F}) : \mathscr{F} \in \mathscr{P}^{<\omega}\}.$$

First, $\mathscr{V}$ is a point-countable family of sets in X. Because if $x \in V(\mathscr{F})$, then there is $A \in \mathscr{H}(\mathscr{F}) \cap \mathscr{P}$ such that $x \in A$. Since X is a first countable space, by Lemma 2.7.15, $A \in \mathscr{H}(\mathscr{F})$ for only countably many $\mathscr{F} \in \mathscr{P}^{<\omega}$. Since $(\mathscr{P})_x$ is countable, $(\mathscr{V})_x$ is also countable. Second, $\mathscr{V}$ is a base of X. Because for every $x \in U \in \tau$, there is $\mathscr{F} \in \mathscr{P}^{<\omega}$ such that $x \in (\cup\mathscr{F})^\circ \subset U$. We may assume that $\mathscr{F}$ is a minimal interior cover of $\{x\}$. Take $\mathscr{B} \in (\mathscr{P})_x^{<\omega}$ such that $x \in (\cup\mathscr{B})^\circ \subset \cup\mathscr{B} \subset (\cup\mathscr{F})^\circ$. If $B \in \mathscr{B}$, then $B \in \mathscr{H}(\mathscr{F})$, so $(\cup\mathscr{B})^\circ \subset V(\mathscr{F})$, and hence $x \in V(\mathscr{F}) \subset U$. Thus $\mathscr{V}$ is a point-countable base for X. ■

A mapping $f : X \to Y$ is said to be *countable-to-one* [167] if every $f^{-1}(y)$ is a countable subset of X.

Theorem 2.7.17 *For every space X, the following are equivalent:*

(1) X has a point-countable base.
(2) X is a compact-covering open s-image of a metric space [340]. (The Michael-Nagami theorem.)
(3) X is an open s-image of a metric space [401].
(4) X is a countable-to-one open image of a metric space [309].
(5) X is a countably bi-quotient s-image of a metric space [123].

Proof (1) ⇒ (2). Suppose X has a point-countable base. By the Miščenko lemma, every compact set in X has a countable outer base. Further by Proposition 2.4.4, X is a compact-covering open s-image of a metric space.

(2) ⇒ (3) and (4) ⇒ (5) are obvious.

(3) ⇒ (4). Let $f : M \to X$ be an open s-mapping, where M is a metric space. For each $x \in X$, let D_x be a countable dense subset of $f^{-1}(x)$. Define $D = \bigcup_{x\in X} D_x$ and $g = f_{|D} : D \to X$. Then g is a countable-to-one mapping. Below we prove g is an open mapping. Let U be an open subset of D. Then there is an open subset V of M such that $U = V \cap D$. If $g(U)$ is not open in X, then there is $x \in g(U) \cap \overline{X - g(U)}$. Since X is first countable, there is a sequence $\{x_n\}$ in $X - g(U)$ converging to x. Since $x \in f(V)$ and $f(V)$ is open in X, there is $x_m \in f(V)$, so $f^{-1}(x_m) \cap V \neq \varnothing$, and hence $D_{x_m} \cap V \neq \varnothing$. Pick $z_m \in D_{x_m} \cap V \subset U$. Then $x_m = g(z_m) \in g(U)$, a contradiction.Thus g is an open mapping, it follows that X is a countable-to-one open image of a metric space.

(5) ⇒ (1). Suppose M is a metric space and $f : M \to X$ is a countably bi-quotient s-mapping. Let $\mathscr{B}$ be a point-countable base for M. Then $f(\mathscr{B})$ is a point-countable

family of sets in X satisfying the assumption of Proposition 2.7.16, and hence X has a point-countable base. ■

Corollary 2.7.18 *For every space X, the following are equivalent:*

(1) X has a point-countable base.
(2) X is a Fréchet–Urysohn space with a point-countable weak base.
(3) X is a strongly Fréchet–Urysohn space with a point-countable cs^*-network.

Proof (1) ⇒ (2) is obvious, (2) ⇒ (3) can be obtained by Corollaries 1.6.18 and 1.6.20, and (3) ⇒ (1) follows from Corollary 2.7.5, the mapping lemma (see Proposition 2.1.12) and Theorem 2.7.17. ■

Corollary 2.7.19 [122, 123] (The Filippov theorem) *The point-countable base property is invariant under perfect mappings or countably bi-quotient s-mappings.*

Proof Suppose $f : X \to Y$ is a mapping and X has a point-countable base. By Theorems 2.7.17 and 2.6.1, every compact space with a point-countable base is metrizable. Therefore, if f is a perfect mapping, then f is a countably bi-quotient s-mapping. It was proved by (4) ⇒ (1) of Theorem 2.7.17 that the point-countable base property is invariant under countably bi-quotient s-mapping, so Y has a point-countable base. ■

Let $f : X \to Y$ be a mapping. f is called a *bi-quotient mapping* [123, 171, 332] (or limit lifting mapping [171]) if, for each $y \in Y$ and each open family $\mathscr{U}$ in X covering $f^{-1}(y)$, there is $P \in \mathscr{U}^F$ such that $y \in f(P)^\circ$. Filippov [123] proved the bi-quotient s-mapping case for Corollary 2.7.19. Obviously, every open mapping is a bi-quotient mapping, and every bi-quotient mapping is a countably bi-quotient mapping.

In 1973, Michael and Nagami [340] asked the following question: Is every quotient s-image of a metric space a compact-covering quotient s-image of a metric space? In 1999, H. Chen [101] answered this question negatively. In 2003, Chen [102] answer this question negatively once again by constructing a regular counterexample under the assumption that there exists a σ'-set. These two examples are quite complex and readers can refer to the reference papers listed.

In the third part of this section, we investigate intrinsic characterizations of closed s-images of metric spaces. We first prove a property of closed mappings of metric spaces, and Theorem 3.4.16 is a more general form of this result.

Lemma 2.7.20 ([284]) *Suppose $f : X \to Y$ is a quotient mapping, where X is a sequential space and Y contains a closed copy of S_{ω_1} (resp. S_ω). If f is a peripheral L-mapping (resp. peripherally compact mapping), then X contains a closed copy of S_2 or S_{ω_1} (resp. S_ω).*

Proof Suppose Y contains a closed copy of S_{ω_1} and f is a peripheral L-mapping. We may assume that $Y = \{b\} \cup \bigcup_{\alpha<\omega_1} Y_\alpha$ is a copy of S_{ω_1}, where every sequence Y_α in $Y - \{b\}$ converges to b and $\{Y_\alpha : \alpha < \omega_1\}$ is a family of pairwise disjoint sets. Since each Y_α is not closed in Y, $f^{-1}(Y_\alpha)$ is not closed in X. Since X is a sequential space, there is a sequence T_α in $f^{-1}(Y_\alpha)$ converging to some point $x_\alpha \in X - f^{-1}(Y_\alpha)$, so $x_\alpha \in \partial f^{-1}(b)$. Let $L = \{x_\alpha : \alpha < \omega_1\}$.

Claim. $\bigcup_{\alpha<\omega_1} F_\alpha$ is a closed discrete subset of X, where each F_α is a finite subset of T_α.

In fact, since $\bigcup_{\alpha<\omega_1} f(F_\alpha)$ is a closed discrete subset of Y, $\{f^{-1}(y) : y \in \bigcup_{\alpha<\omega_1} f(F_\alpha)\}$ is a closed discrete family in X, and hence $\bigcup_{\alpha<\omega_1} F_\alpha$ is a closed discrete subset of X.

If L is not a closed discrete subset of X, then since X is a sequential space, there is a sequence $\{x_{\alpha_n}\}$ consisting of different elements of L which converges to some point $a \in X - L$. Let

$$M = \{a\} \cup \{x_{\alpha_n} : n \in \mathbb{N}\} \cup (\cup\{T_{\alpha_n} : n \in \mathbb{N}\}).$$

By Claim, M is a sequentially closed subset of X homeomorphic to S_2, so M is closed in X. Thus, X contains a closed copy of S_2.

If L is a closed discrete subset of X, then since $\partial f^{-1}(b)$ is a Lindelöf space, L is also a Lindelöf space, so L is countable, and hence there exist ω_1 sequences T_α such that all of them converge to x_α. We may assume that $L = \{a\}$. Let

$$S = \{a\} \cup (\cup\{T_\alpha : \alpha < \omega_1\}).$$

By Claim and the assumption that X is a sequential space, S is a closed copy of S_{ω_1} in X.

For the case of S_ω, by using the fact that f is a peripherally compact mapping, similarly we can prove that X contains a closed copy of S_2 or S_ω. ■

Lemma 2.7.21 ([454]) *If $f : X \to Y$ is a closed mapping and X is a metric space, then f is a peripherally compact mapping (resp. peripheral L-mapping) if and only if Y contains no closed copy of S_ω (resp. S_{ω_1}).*

Proof Our proof is only for the peripheral L-mappings case. Since any metrizable space does not contain a closed copy of S_2 or S_{ω_1}, we get the necessity by Lemma 2.7.20.

Conversely, if f is not a peripheral L-mapping, then there is $s \in Y$ such that $\partial f^{-1}(s)$ is not a Lindelöf subspace of X, and hence there exist a discrete closed set $\{x_\alpha : \alpha < \omega_1\}$ in $\partial f^{-1}(s)$ and a discrete family $\{D_\alpha\}_{\alpha<\omega_1}$ of open sets in X such that $x_\alpha \in D_\alpha$. If $s \in V \in \tau(Y)$, then $f^{-1}(V) \cap (D_\alpha - f^{-1}(s)) \neq \varnothing$ for any $\alpha < \omega_1$, it follows that $V \cap (f(D_\alpha) - \{s\}) \neq \varnothing$, and hence $s \in \overline{f(D_\alpha) - \{s\}}$. Since Y is a Fréchet–Urysohn space, there is a subset $E_\alpha = \{y_{\alpha,n} : n \in \mathbb{N}\} \subset f(D_\alpha) - \{s\}$ such that the sequence $\{y_{\alpha,n}\}_{n\in\mathbb{N}}$ converges to the point s. Since $\{\overline{f(D_\alpha)}\}_{\alpha<\omega_1}$ is an *HCP* family of sets in Y, $\{E_\alpha \cup \{s\}\}_{\alpha<\omega_1}$ is also an *HCP* family in Y. For each $\alpha < \omega_1$, by Lemma 2.5.4, there exist $F_\alpha \in E_\alpha^{<\omega}$ and $\Lambda_\alpha \in \omega_1^{<\omega}$ such that $(E_\alpha - F_\alpha) \cap E_\beta = \varnothing$ whenever $\beta \in \omega_1 - \Lambda_\alpha$. Let $K_\alpha = E_\alpha - F_\alpha$. By the transfinite induction method, one can choose an uncountable subset Γ of ω_1 such that $\{K_\alpha\}_{\alpha\in\Gamma}$ is a disjoint family. Then the subspace $\{s\} \cup (\bigcup_{\alpha\in\Gamma} K_\alpha)$ of Y is a closed copy of S_{ω_1}. ■

A family $\mathscr{P}$ of subsets in a space X is said to be *locally countable* if every point of X has a neighborhood which only meets at most countably many elements of $\mathscr{P}$.

Theorem 2.7.22 *For every space X, the following are equivalent:*

(1) X is a closed s-image of a metric space.
(2) X is a Fréchet $\aleph$*-space [150].*
(3) X is a Lašnev space containing no closed copy of S_{ω_1} *[454].*

Proof By Theorem 2.5.8 and Example 1.8.7 we obtained (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (1) is followed from Lemmas 2.7.20 and 2.1.15. Below we prove (1) $\Rightarrow$ (2).

Suppose M is a metric space and $f : M \to X$ is a closed s-mapping. Obviously, X is a paracompact Fréchet–Urysohn space. Let $\mathscr{B}$ be a σ-locally finite base for M and let $\mathscr{P} = f(\mathscr{B})$. By Propositions 2.1.16 and 2.5.7, $\mathscr{P}$ is a σ-locally countable k-network for X. Denote $\mathscr{P} = \bigcup_{i\in\mathbb{N}} \mathscr{P}_i$, where $\mathscr{P}_i$ is a locally countable family of sets and $\mathscr{P}_i \subset \mathscr{P}_{i+1}$. For each $i \in \mathbb{N}$, since $\mathscr{P}_i$ is locally countable and X is paracompact, there is a locally finite open cover $\mathscr{U}_i$ of X such that each element of $\mathscr{U}_i$ only meets countably many sets in $\mathscr{P}_i$, and hence $\mathscr{P}_i \wedge \mathscr{U}_i$ is a σ-locally finite family of sets. We prove that $\bigcup_{i\in\mathbb{N}}(\mathscr{P}_i \wedge \mathscr{U}_i)$ is a k-network for X. Let K be a compact set and $K \subset U \in \tau(X)$. Then there exist $m \in \mathbb{N}$, $\mathscr{P}' \in \mathscr{P}_m^{<\omega}$ and $\mathscr{U}' \in \mathscr{U}_m^{<\omega}$ such that $K \subset \cup\mathscr{P}' \subset U$ and $K \subset \cup\mathscr{U}'$, so $\mathscr{P}' \wedge \mathscr{U}' \in (\mathscr{P}_m \wedge \mathscr{U}_m)^{<\omega}$ and $K \subset \cup(\mathscr{P}' \wedge \mathscr{U}') \subset U$. Thus $\bigcup_{i\in\mathbb{N}}(\mathscr{P}_i \wedge \mathscr{U}_i)$ is a k-network for X, and hence X is an $\aleph$-space. ■

A space X is said to be *bi-sequential* [336] if, whenever a filter base $\mathscr{F}$ accumulates at x in X, then there is a decreasing sequence $\{A_n\}$ in X converges to x and every A_n intersects every $F \in \mathscr{F}$. An ω-filter base is defined to be a family $\mathscr{F}$ of nonempty subsets of X such that for any countable subfamily $\mathscr{F}'$ of $\mathscr{F}$, there is $F \in \mathscr{F}$ such that $F \subset \cap\mathscr{F}'$. A space X is said to be *weakly bi-sequential* [304] if, for any ω-filter base $\mathscr{B}$ in X accumulating at some $x \in X$ there is a countable filter base $\mathscr{C}$ in X such that $\mathscr{C}$ converges to x and every $B \in \mathscr{B}$ intersects every $C \in \mathscr{C}$. Every be-sequential space is weakly be-sequential, and every weakly bi-sequential space is Fréchet–Urysohn. C. Liu [304] proved that a Lašnev space X is a closed s-image of a metric space if and only if X is weakly bi-sequential.

2.8 *ss*-Mappings

As a special case of the s-images of metric spaces, we introduce in this section the ss-images of metric spaces. In 1996, C. Liu and M. Dai [308] found for the first time a characterization of quotient s-images of locally separable metric spaces. Seeking more concise characterizations of this kind images is still an open problem [311]. The class of ss-mappings plays an exclusive role in characterizing images of locally separable metric spaces. This section consists mainly four parts. The first two parts are about images of separable metric spaces and quotient ss-images of metric spaces respectively. The third part is about pseudo-open ss-images or closed ss-images of metric spaces and about pseudo-open s-images or closed s-images of locally separable metric spaces. In the last part, we describe some kinds of additional conditions associated with s-images and ss-images by means of several examples.

Definition 2.8.1 (*[255]*) Let $f : X \to Y$ be a mapping. Then f is called an *ss-mapping* if for each $y \in Y$, there is an open neighborhood V of y in Y such that $f^{-1}(V)$ is a separable subspace of X.

First, we give a characterization of ss-images of metric spaces. Note that if a mapping $f : X \to Y$ is an ss-mapping, then X is a locally separable space.

Proposition 2.8.2 ([21, 22]) *Let $f : X \to Y$ be a mapping. If $\mathscr{P}$ is a network for X, then $f(\mathscr{P})$ is a network for Y.*

Thus, every regular image of a cosmic space is a cosmic space [21, 22].

Theorem 2.8.3 ([288]) *A space X has a locally countable network if and only if X is an ss-image of a metric space.*

Proof Let $\mathscr{K} = \{\{x\} : x \in X\}$ and let $\mathscr{P}$ be a locally countable network for X. Then by Proposition 2.7.1 we can get the necessity. Conversely, let X be the image of a metric space M under an ss-mapping f. If $\mathscr{B}$ is a σ-locally finite base in M, then $f(\mathscr{B})$ is a network in X. For each $x \in X$, there is an open neighborhood V of x such that $f^{-1}(V)$ is a separable subspace of M, so $f^{-1}(V)$ only meets countably many elements of $\mathscr{B}$, and hence V only meets countably many elements of $f(\mathscr{B})$. Thus $f(\mathscr{B})$ is a locally countable network for X. ■

Corollary 2.8.4 ([331]) *A regular space X is a cosmic space if and only if X is an image of a separable metric space.*

Next, we give characterizations of quotient ss-images of metric spaces. A family $\mathscr{U}$ of sets is said to be *star-countable* if, for each $U \in \mathscr{U}$, $(\mathscr{U})_U$ is countable.

Lemma 2.8.5 ([177]) *If $\mathscr{U}$ is a star-countable family of sets in a space X, then $\mathscr{U} = \bigcup_{\alpha\in\Lambda} \mathscr{U}_\alpha$, where each $\mathscr{U}_\alpha$ is countable and $(\cup\mathscr{U}_\alpha) \cap (\cup\mathscr{U}_\beta) = \varnothing$ whenever $\alpha, \beta \in \Lambda$ and $\alpha \neq \beta$.*

Proof For every $A, B \in \mathscr{U}$, we said that $A \sim B$ if there are finitely many sets $\{U_i\}_{i\leqslant n}$ in $\mathscr{U}$ such that $A = U_1, B = U_n$, and $U_i \cap U_{i+1} \neq \varnothing$ when $i < n$. For each $A \in \mathscr{U}$, let $\mathscr{U}_A = \{B \in \mathscr{U} : A \sim B\}$. Then $\mathscr{U}_A$ is countable, $\mathscr{U} = \cup\{\mathscr{U}_A : A \in \mathscr{U}\}$ and for each pair A, B of $\mathscr{U}$, $(\cup\mathscr{U}_A) \cap (\cup\mathscr{U}_B) \neq \varnothing$ if and only if $\mathscr{U}_A = \mathscr{U}_B$. ■

Theorem 2.8.6 ([255]) *For every space X, the following are equivalent:*

(1) X has a locally countable cs^*-network.
(2) X has a locally countable cs-network.
(3) X is a sequentially quotient ss-image of a metric space.
(4) X is a sequence-covering ss-image of a metric space.
(5) X is a compact-covering ss-image of a metric space.
If further assume that X is a regular space, then each item above is also equivalent to the following:

(6) X has a locally countable k-network.

Proof (1) $\Rightarrow$ (2). Let $\mathscr{P}$ be a locally countable cs^*-network for X. For each $x \in X$, there is an open neighborhood V_x of x such that V_x only meets countably many elements of $\mathscr{P}$. Let $\mathscr{U} = \{P \in \mathscr{P} : \text{there is some } V_x \supset P\}$. Then $\mathscr{U}$ is star-countable. By Lemma 2.8.5, we can suppose $\mathscr{U} = \bigcup_{\alpha\in\Lambda} \mathscr{U}_\alpha$, where each $\mathscr{U}_\alpha$ is countable and $(\cup\mathscr{U}_\alpha) \cap (\cup\mathscr{U}_\beta) = \varnothing$ whenever $\alpha, \beta \in \Lambda$ and $\alpha \neq \beta$. Let $\mathscr{F} = \bigcup_{\alpha\in\Lambda} \mathscr{U}_\alpha^F$. Then $\mathscr{F}$ is also locally countable. Below we prove that $\mathscr{F}$ is a cs-network for X. Suppose $\{x_n\}$ is a sequence in X converging to a point $x \in V \in \tau$. By Lemma 2.7.3, there exist $m \in \mathbb{N}$ and $\mathscr{P}' \in (\mathscr{P})_x^{<\omega}$ such that $\{x\} \cup \{x_n : n \geqslant m\} \subset \cup\mathscr{P}' \subset V \cap V_x$. Then there is only one $\alpha \in \Lambda$ such that $x \in \cup\mathscr{U}_\alpha$, so $\cup\mathscr{P}' \in \mathscr{U}_\alpha^F$, and hence $\mathscr{F}$ is a cs-network for X.

(2) $\Rightarrow$ (5). Let $\mathscr{K} = \mathscr{K}(X)$ and let $\mathscr{P}$ be a locally countable cs-network for X. We only need to prove $\mathscr{K}$ and $\mathscr{P}$ satisfy the conditions of Proposition 2.7.1. For each $K \in \mathscr{K}$, since $\mathscr{P}_{|K}$ is a countable cs-network for K, K is metrizable. Let $\mathscr{P}' = (\mathscr{P})_K$. Since

$$\{\mathscr{F} \in \mathscr{P}'^{<\omega} : \mathscr{F} \text{ is precisely refined by an finite closed cover of } K\}$$

is countable, we can denote it by $\{\mathscr{P}_n\}_{n\in\mathbb{N}}$. We prove $\{\mathscr{P}_n\}$ satisfies Proposition 2.7.1(ii). Suppose $x \in K$ and $P_n \in (\mathscr{P}_n)_x$, $\forall n \in \mathbb{N}$. If $x \in V \in \tau(X)$, then there is an open neighborhood W of x in K such that $\mathrm{cl}_K(W) \subset V$. Let $\{V_i\}_{i\in\mathbb{N}}$ be a decreasing local base of x in K. Put

$$\mathscr{P}_x = \{P \cap K : P \in \mathscr{P}',\ P \subset V \text{ and there is } i \in \mathbb{N} \text{ such that } V_i \subset P \cap K\}.$$

By the proof of Proposition 1.6.21, $\mathscr{P}_x$ is a neighborhood base of x in K. So there is $P_x \in \mathscr{P}$ such that $x \in \mathrm{int}_K(P_x \cap K) \subset P_x \subset V$, and hence there is $V_x \in \tau(K)$ such that $x \in V_x \subset \mathrm{cl}_K(V_x) \subset \mathrm{int}_K(P_x \cap K)$. Obviously, the compact set $K - V_x \subset X - \{x\} \in \tau(X)$. As proved above, for each $y \in K - V_x$, there exist $P_y \in \mathscr{P}$ and $V_y \in \tau(K)$ such that $y \in V_y \subset \mathrm{cl}_K(V_y) \subset \mathrm{int}_K(P_y \cap K) \subset P_y \subset X - \{x\}$. Now the open cover $\{V_y : y \in K - V_x\}$ of $K - V_x$ has a finite subcover $\{V_{y_k}\}_{k\leqslant m}$. Let $\mathscr{F} = \{P_x\} \cup \{P_{y_k} : k \leqslant m\}$. Then $\mathscr{F}$ is precisely refined by a finite closed cover of K, so there is $j \in \mathbb{N}$ such that $\mathscr{F} = \mathscr{P}_j$, thus $x \in P_j = P_x \subset V$. Consequently, $\{P_n\}_{n\in\mathbb{N}}$ is a net at x.

(5) $\Rightarrow$ (4) $\Rightarrow$ (3) is obvious. By Theorem 1.3.2 and Proposition 2.7.2, we get (3) $\Rightarrow$ (1). (1) $\Rightarrow$ (6) follows from Proposition 1.6.7 and Corollary 2.2.20. Finally, if a regular space X has a locally countable k-network, then X has a locally countable closed k-network, and hence X has a locally countable cs^*-network. ■

Corollary 2.8.7 ([331]) *A regular space X is an $\aleph_0$-space if and only if X is a compact-covering (resp. sequence-covering, sequentially quotient) image of a separable metric space.*

Example 2.8.8 *([465])* There is a connected $\aleph_0$-space which is not an image of a connected metric space.

Fix $p \in \beta\mathbb{R} - \mathbb{R}$ and let $X = \mathbb{R} \cup \{p\}$. Take X as a subspace of the compactification $\beta\mathbb{R}$. Then X is a connected space. By Example 2.3.19, there is no sequence in $\mathbb{R}$ converging to p, so X has a countable cs-network, and hence X is an $\aleph_0$-space. Since X is not an s-connected space, by Theorem 2.3.17, X is not an image of a connected metric space.

Lemma 2.8.9 ([259]) *Every k-space with a σ-locally countable cs^*-network is a meta-Lindelöf sequential space.*

Proof Suppose X is a k-space and $\mathscr{P}$ is a σ-locally countable cs^*-network for X. Then for each $K \in \mathscr{K}(X)$, $\mathscr{P}_{|K}$ is a countable network for K, and hence K is metrizable. By Corollary 2.3.5, X is a sequential space.

Suppose $\mathscr{P} = \bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ is closed under finite intersections, where $\mathscr{P}_n$ is locally countable and $\mathscr{P}_n \subset \mathscr{P}_{n+1}$. We first prove that for any locally countable family $\mathscr{F} = \{F_\alpha\}_{\alpha\in\Gamma}$ in X, there is a point-countable family $\mathscr{W} = \{W_\alpha\}_{\alpha\in\Gamma}$ of open sets of X such that $F_\alpha \subset W_\alpha$ (for simplicity, we say that $\mathscr{W}$ is a point-countable open expansion of $\mathscr{F}$). Let

$$\Lambda = \{\lambda : \lambda \text{ is a finite sequence consisting of elements of } \mathbb{N}\}.$$

For each $\lambda \in \Lambda$, we inductively define a locally countable family $\mathscr{F}(\lambda) = \{F_\alpha(\lambda)\}_{\alpha\in\Gamma}$ of sets in X as follows.

Let $\mathscr{F}(\varnothing) = \mathscr{F}$ and $F_\alpha(\varnothing) = F_\alpha$. Suppose a locally countable family $\mathscr{F}(\lambda)$ has been defined. For each $n \in \mathbb{N}$, denote the finite sequence consisting of members of $\mathbb{N}$ obtained by adding one term n behind λ by λn. Take

$$\begin{aligned}
&\mathscr{P}(\lambda n) = \{P \in \mathscr{P}_n : F_\alpha(\lambda) \cap P \neq \varnothing \text{ only for countably many } \alpha \in \Gamma\};\\
&F_\alpha(\lambda n) = \cup\{P \in \mathscr{P}(\lambda n) : F_\alpha(\lambda) \cap P \neq \varnothing\},\ \alpha \in \Gamma;\\
&\mathscr{F}(\lambda n) = \{F_\alpha(\lambda n) : \alpha \in \Gamma\}.
\end{aligned}$$

For each $x \in X$, there is an open neighborhood V of x such that V only meets countably many elements of $\mathscr{P}_n$. Take

$$(\mathscr{P}_n)_V \cap \mathscr{P}(\lambda n) = \{P_i\}_{i\in\mathbb{N}}.$$

For each $i \in \mathbb{N}$, there is a countable set Γ_i in Γ such that $P_i \cap F_\alpha(\lambda) = \varnothing$ whenever $\alpha \in \Gamma - \Gamma_i$. Hence

$$|\{\alpha \in \Gamma : \text{ there is } i \in \mathbb{N} \text{ such that } P_i \cap F_\alpha(\lambda) \neq \varnothing\}| \leqslant \aleph_0.$$

It means that $V \cap F_\alpha(\lambda n) \neq \varnothing$ holds only for countably many α in Γ, so $\mathscr{F}(\lambda n)$ is a locally countable family.

Let $W_\alpha = \bigcup_{\lambda\in\Lambda} F_\alpha(\lambda)$ for each $\alpha \in \Gamma$. Then $F_\alpha \subset W_\alpha$ and W_α is a sequentially open set in X. Because in fact, if a sequence $\{x_n\}$ converges to $x \in W_\alpha$, then there is $\lambda \in \Lambda$ such that $x \in F_\alpha(\lambda)$, and hence there is an open neighborhood W of x

such that $(\mathscr{F}(\lambda))_W$ is countable. By the proof of Lemma 2.7.3, there exist $k \in \mathbb{N}$ and $\mathscr{H} \in (\mathscr{P}_k)_x^{<\omega}$ such that the sequence $\{x_n\}$ is eventually in $\cup\mathscr{H} \subset W$. Since $\cup\mathscr{H} \subset F_\alpha(\lambda k) \subset W_\alpha$, $\{x_n\}$ is eventually in W_α, so W_α is a sequentially open set in X. Thus W_α is an open set in X, because X is a sequential space.

Furthermore, take $\mathscr{W} = \{W_\alpha\}_{\alpha\in\Gamma}$. Then $\mathscr{W}$ is point-countable. Because otherwise, there exist $x \in X$ and an uncountable subset Γ' of Γ such that $x \in W_\alpha$ whenever $\alpha \in \Gamma'$. For each $\alpha \in \Gamma'$, there is $\lambda_\alpha \in \Lambda$ such that $x \in F_\alpha(\lambda_\alpha)$, and hence there exist an uncountable subset Γ'' of Γ' and $\lambda \in \Lambda$ such that $\lambda_\alpha = \lambda$ for each $\alpha \in \Gamma''$, and it means $x \in F_\alpha(\lambda)$, which contradicts the point-countability of $\mathscr{F}(\lambda)$.

Finally, we prove that X is a meta-Lindelöf space. Let $\mathscr{U}$ be any open cover of X. Then $\mathscr{U}$ has a refinement $\bigcup_{i\in\mathbb{N}} \mathscr{F}_i$, where each $\mathscr{F}_i = \{F_\alpha : \alpha \in \Gamma_i\}$ is a locally countable family of sets of X. For each $i \in \mathbb{N}$, suppose $\mathscr{W}_i = \{W_\alpha : \alpha \in \Gamma_i\}$ is a point-countable open expansion of $\mathscr{F}_i$. For each $\alpha \in \Gamma_i$, choose $U_\alpha \in \mathscr{U}$ such that $F_\alpha \subset U_\alpha$. Then $\bigcup_{i\in\mathbb{N}}\{U_\alpha \cap W_\alpha : \alpha \in \Gamma_i\}$ is a point-countable open refinement of $\mathscr{U}$. Hence X is a meta-Lindelöf space. ■

Lemma 2.8.10 *Every locally separable meta-Lindelöf space is the topological sum of separable Lindelöf spaces.*

Proof If X is a locally separable meta-Lindelöf space, then X has a point-countable open cover $\mathscr{U}$ consisting of separable subspaces. Since every point-countable family of open sets in a separable space is countable, $\mathscr{U}$ is a star-countable family. By Lemma 2.8.5, X has a cover $\{X_\alpha\}_{\alpha\in\Lambda}$ consisting of disjoints open sets such that each X_α is a separable subspace of X, and hence $X = \bigoplus_{\alpha\in\Lambda} X_\alpha$. Since every separable meta-Lindelöf space is a Lindelöf space, each X_α is a separable Lindelöf space. ■

Corollary 2.8.11 ([255, 259]) *For every space X, the following are equivalent:*

(1) X is a k-space with a locally countable cs^*-network.
(2) X is the topological sum of sequential spaces with a countable cs-network.
(3) X is a compact-covering quotient (sequentially quotient or sequence-covering) ss-image of a metric space.
(4) X is a quotient ss-image of a metric space.
If we assume further that X is a regular space, then each term of the above is equivalent the following:
(5) X is a k-space with a locally countable k-network.

Proof By Theorem 2.8.6, Lemmas 2.8.9 and 2.8.10 we obtain (1) $\Leftrightarrow$ (2). (1) $\Rightarrow$ (3) follows from Theorem 2.8.6 and the mapping lemma. (3) $\Rightarrow$ (4) is obvious. By Propositions 2.3.1, 2.1.12 (the mapping lemma) and 2.7.2, (4) $\Rightarrow$ (1) holds. Under the assumption of regular spaces, (1) $\Leftrightarrow$ (5) follows from Theorem 2.8.6. ■

In the third part of this section, we give characterizations of pseudo-open ss-images or closed ss-images of metric spaces. We also give characterizations of pseudo-open s-images or closed s-images of locally separable metric spaces.

Lemma 2.8.12 *For every space X, the following are equivalent [167]:*

(1) X is a Fréchet–Urysohn space with a countable cs^*-network.

(2) X is a pseudo-open image of a separable metric space.
(3) X is both a separable space and a pseudo-open s-image of a metric space.
If further assume that X is a regular space, then each item above and each of the following are equivalent [130]:
(4) X is a Fréchet–Urysohn $\aleph_0$-space.
(5) X is a closed image of a separable metric space.

Proof By Proposition 2.7.1, Lemma 2.7.3 and the mapping lemma (see Proposition 2.1.12), we get (1) $\Rightarrow$ (2). (5) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious. By Lemma 2.7.11, (2) $\Rightarrow$ (4) holds. Below we prove (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5).

(3) $\Rightarrow$ (1). By Proposition 2.3.1, X is a Fréchet–Urysohn space. By Corollary 2.7.5, X has a point-countable cs^*-network $\mathscr{P}$. Let D be a countable dense subset of X and let $\mathscr{H} = (\mathscr{P})_D$. Then $\mathscr{H}$ is countable. We prove that $\mathscr{H}$ is a cs^*-network for X. Let $\{x_n\}$ be a sequence converging to $x \in U \in \tau$ in X. Define

$$S = \{x\} \cup \{x_n : n \in \mathbb{N}\},$$
$$\mathscr{H}' = \{H \in \mathscr{H} : x \in H \subset U\}.$$

Below we show $\mathscr{H}' \neq \varnothing$. Denote $\mathscr{H}' = \{H_i\}_{i\in\mathbb{N}}$.

We first prove there is $m \in \mathbb{N}$ such that $x \in \text{int}_S((\bigcup_{i\leqslant m} H_i) \cap S)$. Otherwise, there is a sequence $\{z_m\}$ satisfying $z_m \to x$ and $z_m \in S - \bigcup_{i\leqslant m} H_i$. Note that, $(\bigcup_{i\leqslant m} H_i) \cap S$ is a closed set in X, we get $z_m \in \overline{D - (\bigcup_{i\leqslant m} H_i) \cap S}$, and hence there is a sequence $\{z_{m,k}\}_k$ converging to z_m in $D - (\bigcup_{i\leqslant m} H_i)$. Thus $x \in \overline{\{z_{m,k} : m, k \in \mathbb{N}\}}$, it follows that there is a sequence in $\{z_{m,k} : m, k \in \mathbb{N}\}$, we denote it as $\{z_{m_j,k_j}\}_{j\in\mathbb{N}}$, converging to x when $m_j \to +\infty$. We may assume there is $P \in \mathscr{P}$ such that $\{x\} \cup \{z_{m_j,k_j} : j \in \mathbb{N}\} \subset P \subset U$, it means $P \in \mathscr{H}'$, and hence $\mathscr{H}' \neq \varnothing$. Take $i \in \mathbb{N}$ such that $P = H_i$. Pick $j \in \mathbb{N}$ such that $m_j \geqslant i$. Then $z_{m_j,k_j} \notin P$, a contradiction.

Thus, there is $m \in \mathbb{N}$ such that $x \in \text{int}_S((\bigcup_{i\leqslant m} H_i) \cap S)$, and hence there exist a subsequence $\{x_{n_j}\}$ of S and $i \in \mathbb{N}$ such that $\{x\} \cup \{x_{n_j} : j \in \mathbb{N}\} \subset H_i \subset U$. So $\mathscr{H}$ is a countable cs^*-network for X.

(4) $\Rightarrow$ (5). By Theorem 2.7.22, there exist a metric space M and a closed s-mapping $f : M \to X$. Since X is a Lindelöf space and f is a closed L-mapping, M is a Lindelöf space. Thus X is a closed image of a separable metric space. ■

Note that the meta-Lindelöf property is invariant under pseudo-open s-mappings (see Corollary A.5.3 in Appendix A). By Corollary 2.8.11, Lemmas 2.8.10 and 2.8.12, we have the following theorem.

Theorem 2.8.13 ([255, 259]) *For every space X, the following are equivalent:*

(1) X is a Fréchet–Urysohn space with a locally countable cs^*-network.
(2) X is a pseudo-open s-image of a locally separable metric space.
(3) X is a pseudo-open ss-image of a metric space.
(4) X is both a locally separable space and a pseudo-open s-image of a metric space.
If further assume that X is a regular space, then each item above and each of the following are equivalent:

(5) *X is a closed s-image of a locally separable metric space.*
(6) *X is both a locally separable space and a closed s-image of a metric space.*

Example 2.7.14 shows that the assumption of regularity in Lemma 2.8.12 and Theorem 2.8.13 is important. The next example shows that the assumption of regularity in Theorem 2.8.6 and Corollary 2.8.11 is also necessary.

Example 2.8.14 There is a space X with a locally countable and σ-discrete k-network, such that, X is neither a meta-Lindelöf space nor a k^*-metrizable space, and X has no point-countable cs^*-network.

Let X be the half-disc topological space (see Example 2.2.22(3)). It is easy to verify that X is a separable first countable space but X is not a Lindelöf space. That shows X is not a meta-Lindelöf space, so X has no point-countable base. By Corollary 2.7.18, X has no point-countable cs^*-network. Since X is not a regular space, by Corollary 2.6.10, X is not a k^*-metrizable space.

For every $x \in \mathbb{R}^2, r > 0$, let $B(x, r)$ be a spherical neighborhood of x in $\mathbb{R}^2$. Define

$$\mathscr{P} = \{\{p\} : p \in L\} \cup \{B(q, 1/n) \cap S : q \in \mathbb{Q} \times \mathbb{Q}, n \in \mathbb{N}\}.$$

Since L is a discrete closed set in X, $\mathscr{P}$ is a locally countable and σ-discrete family of sets in X. We prove $\mathscr{P}$ is a k-network for X. Assume that $K \subset U \in \tau$, where K is a compact set in X and τ is the half-disc topology. For each $x \in X$, denote $\{P \in \mathscr{P} : x \in P \subset U\} = \{P_i(x)\}_{i \in \mathbb{N}}$. Then K is covered by some finite subfamily of $\{P_i(x) : x \in K, i \in \mathbb{N}\}$. Because otherwise, there is a sequence $\{p_n\}$ in K such that $p_n \notin P_i(p_j)$ whenever $i, j < n$. So there is a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ converging to $p \in K$. Since L is discrete, we may assume all $p_{n_k} \in S$, and hence $\{p_{n_k}\}$ also converges to p in the Euclidean subspace topology τ^* on X. Since $\{B(q, 1/n) \cap X : q \in \mathbb{Q} \times \mathbb{Q}, n \in \mathbb{N}\}$ is a countable base of τ^*, there exist $q \in \mathbb{Q} \times \mathbb{Q}$ and $h, m \in \mathbb{N}$ such that $\{p\} \cup \{p_{n_k} : k \geqslant h\} \subset B(q, 1/m) \cap X \subset U$, and hence $\{p_{n_k} : k \geqslant h\} \subset B(q, 1/m) \cap S \subset U$. Thus, there exist $i, j \in \mathbb{N}$ such that $B(q, 1/m) \cap S = P_i(p_j)$, and hence we can pick $n > i, j$ such that $p_n \in P_i(p_j)$, a contradiction. Therefore, $\mathscr{P}$ is a k-network for X.

Example 2.8.15 There is a compact-covering open s-image of a metric space which is not metrizable.

It was proved in Example 1.8.5 that the Michael line X is a Lindelöf regular space with a point-countable base. By Theorem 2.7.17, X is a compact-covering open s-image of a metric space. But X is not a β-space, and hence it is neither a metrizable space nor an $\aleph_0$-space. This example shows that the assumption of the separability in Lemma 2.8.12(3) and Theorem 2.8.13(4) can not be replaced with the Lindelöf property.

Example 2.8.16 (*[167]*) There is a compact-covering finite-to-one quotient mapping $f : M \to X$ such that M is a locally compact metrizable space M, X is a separable regular space but X is not a meta-Lindelöf space.

Let

$$X = \mathbb{I} \times \mathbb{S}_1, \ \ Y = \mathbb{I} \times (\mathbb{S}_1 - \{0\}).$$

Define a topology for X as follows: Y is a Euclidean subspace of X and each element of a neighborhood base of $(t, 0) \in X$ has the form

$$\{(t, 0)\} \cup (\cup\{V(t, k) : k \geqslant n\}), \ \ n \in \mathbb{N},$$

where $V(t, k)$ is an open neighborhood of $(t, 1/k)$ in the subspace $\mathbb{I} \times \{1/k\}$. Let

$$M = (\oplus\{\mathbb{I} \times \{1/n\} : n \in \mathbb{N}\}) \oplus (\oplus\{\{t\} \times \mathbb{S}_1 : t \in \mathbb{I}\}).$$

Then M is a locally compact metrizable space. Let $f : M \to X$ be the natural mapping. Since X has the weak topology with respect to the point-finite cover $\{\mathbb{I} \times \{1/n\} : n \in \mathbb{N}\} \cup \{\{t\} \times \mathbb{S}_1 : t \in \mathbb{I}\}$, by Proposition 2.3.3, f is a finite-to-one quotient mapping. By Theorem 2.7.12, f is a compact-covering mapping.

Obviously, X is a separable regular space. Since $\mathbb{I} \times \{0\}$ is an uncountable discrete closed subspace of X, X is not a Lindelöf space, and hence X is not a meta-Lindelöf space. By Lemma 2.8.9, X has no σ-locally countable cs^*-network.

Example 2.8.16 shows the following facts:

(1) The assumption "pseudo-open mapping" in Lemma 2.8.12(3) and Theorem 2.8.13(4) cannot be reduced to "quotient mapping".
(2) A finite-to-one compact-covering quotient image of a locally compact metric space may not have a σ-locally countable cs^*-network.
(3) Not every quotient s-image of a locally separable metric space is a quotient ss-image of a metric space.

Example 2.8.17 (*[259]*) There is a regular space with a locally countable k-network which is not an $\aleph$-space.

Let $X = \omega_1 \times \mathbb{S}_1$. Define a topology for X as follows: each point of $X - (\omega_1 \times \{0\})$ is isolated in X and each element of a neighborhood base of a point $(\alpha, 0) \in X$ has the form $\{(\alpha, 0)\} \cup (\bigcup_{n \geqslant m}(V(\alpha, n) \times \{1/n\}))$, where $m \in \mathbb{N}$ and $V(\alpha, n)$ is an open neighborhood of α in ω_1 with the ordered topology. Then X is a regular space. Let

$$\mathscr{P} = \{\{x\} : x \in X\} \cup \{\{\alpha\} \times (\{0\} \cup \{1/n : n \geqslant m\}) : \alpha < \omega_1, \ m \in \mathbb{N}\}.$$

Then $\mathscr{P}$ is a locally countable family of sets in X. Suppose $K \in \mathscr{K}(X)$. Then

(17.1) for each $s \in \mathbb{S}_1$, $K \cap (\omega_1 \times \{s\})$ is a finite set;
(17.2) if $(\alpha, 0) \in \omega_1 \times \{0\} - K$, then $(\alpha, 1/n) \in K$ only for finitely many $n \in \mathbb{N}$;
(17.3) the set $K - \cup\{\{\alpha\} \times \mathbb{S}_1 : (\alpha, 0) \in K \cap (\omega_1 \times \{0\})\}$ is a finite set.

Consequently, $\mathscr{P}$ is a locally countable k-network for X.

Take $f = \pi_1 : X \to \omega_1$ and give ω_1 the ordered topology. Then f is a pseudo-open compact mapping. If X is an $\aleph$-space, then ω_1 is a subparacompact space (see Proposition 3.4.14), a contradiction. Thus X is not an $\aleph$-space.

Example 2.8.17 shows that in Lemma 2.8.9 and Corollary 2.8.11 the assumption "k-space" cannot be omitted. Sakai [416] constructed a locally countable regular space X in which every compact set is a finite set and X is not a countably metacompact space. This space has a locally countable k-network and it is not a perfect space.

2.9 π-Mappings

In the above two sections, we introduced some mappings with separable fibers. In this section, we turn to discuss π-mappings which is closely relevant to mappings with compact fibers. Our purpose is to extend the concept of weak developments, to explore the characterizations of quotient π-images, pseudo-open π-images and open π-images of metric spaces and to characterize symmetrizable spaces and semi-metrizable spaces satisfying the weak Cauchy condition by means of such images of metric spaces.

Definition 2.9.1 (*[401]*) Let (X, d) be a metric space. A mapping $f : X \to Y$ is called a *π-mapping* if $d(f^{-1}(y),\ X - f^{-1}(U)) > 0$ whenever $y \in U \in \tau(Y)$.

Obviously, every compact mapping on a metric space is a π-mapping. For each space X, let M be a space obtained by giving X the discrete topology. Then $\mathrm{id}_M : M \to X$ is a compact mapping, hence a π-image on the metric space X. Thus it is necessary for us to add certain additional conditions when discussing the π-mappings on metric spaces.

Definition 2.9.2 Let $\mathscr{P}$ be a cover of a space X.

(1) $\mathscr{P}$ is called a *cfp-cover* of X [486] if, for each $K \in \mathscr{K}(X)$, there is $\mathscr{F} \in \mathscr{P}^{<\omega}$ such that $\mathscr{F}$ is precisely refined by a finite closed cover of K.
(2) $\mathscr{P}$ is called an *fcs-cover* of X [153] if, for any sequence S converging to a point x in X, there is $\mathscr{P}' \in (\mathscr{P})_x^{<\omega}$ such that S is eventually in $\cup\mathscr{P}'$.
(3) $\mathscr{P}$ is called a *cs^*-cover* of X [243] if, for any convergent sequence S in X, there is $P \in \mathscr{P}$ such that some subsequence of S is eventually in P.

fcs-covers were called *wcs*-covers by Y. Ge [152]. It is easy to verify that for every space X, open cover $\Rightarrow$ *cfp*-cover $\Rightarrow$ *fcs*-cover $\Rightarrow$ cs^*-cover.

Lemma 2.9.3 *Let $\mathscr{P}$ be an fcs-cover of a space X. If $S \in \mathscr{S}(X)$, then there is $\mathscr{F} \in \mathscr{P}^{<\omega}$ such that $\mathscr{F}$ is precisely refined by a finite closed cover of S.*

Proof Suppose S is a sequence converging to $x \in X$. Then there is $\mathscr{P}' \in (\mathscr{P})_x^{<\omega}$ such that S is eventually in $\cup\mathscr{P}'$. Since $S - \cup\mathscr{P}'$ is a finite set and $P \cap S$ is a closed set for each $P \in \mathscr{P}'$, there is $\mathscr{F} \in \mathscr{P}^{<\omega}$ such that $\mathscr{F}$ is precisely refined by a finite closed cover of S. ■

Proposition 2.9.4 *Suppose $f : X \to Y$ is a mapping and (X, d) is a metric space. For each $n \in \mathbb{N}$, define $\mathscr{U}_n = \{f(B(x, 1/n)) : x \in X\}$.*

(1) If f is a π-mapping, then $\{\mathscr{U}_n\}$ is a point-star network for Y [297].
(2) If f is a compact-covering mapping, then $\mathscr{U}_n$ is a cfp-cover of Y [293].
(3) If f is a sequence-covering mapping, then $\mathscr{U}_n$ is an fcs-cover of Y [152].
(4) If f is a sequentially quotient mapping, then $\mathscr{U}_n$ is a cs^*-cover of Y [297].

Proof (1) For every $y \in U \in \tau(Y)$, there is $n \in \mathbb{N}$ such that $d(f^{-1}(y), X - f^{-1}(U)) \geqslant 1/n$. Take $m = 2n$. If $y \in f(B(x, 1/m))$, then $f^{-1}(y) \cap B(x, 1/m) \neq \varnothing$. If $B(x, 1/m) \not\subset f^{-1}(U)$, then $d(f^{-1}(y), X - f^{-1}(U)) < 2/m = 1/n$, a contradiction. So $B(x, 1/m) \subset f^{-1}(U)$, and hence $f(B(x, 1/m)) \subset U$, it follows that $\mathrm{st}(y, \mathscr{U}_m) \subset U$. Thus $\{\mathscr{U}_n\}$ is a point-star network for Y.
(2) For each $K \in \mathscr{K}(Y)$, there is $L \in \mathscr{K}(X)$ such that $f(L) = K$, and hence there is a finite subfamily $\mathscr{F}$ of $\{B(x, 1/n)\}_{x\in X}$ such that $\mathscr{F}$ is precisely refined by a finite closed cover $\mathscr{L}$ of L. Then the finite subfamily $f(\mathscr{F})$ of $\mathscr{U}_n$ is precisely refined by the finite closed cover $f(\mathscr{L})$ of K, and hence $\mathscr{U}_n$ is a *cfp*-cover of Y.
(3) Suppose S is a sequence converging to some point y in Y. Then there is $L \in \mathscr{K}(X)$ such that $f(L) = S \cup \{y\}$, and hence there is a finite set F in X such that $f^{-1}(y) \cap L \subset \bigcup_{x\in F} B(x, 1/n)$. Then $\mathscr{U} = \{f(B(x, 1/n))\}_{x\in F}$ is a finite subfamily of $\mathscr{U}_n$ and S is eventually in $\cup\mathscr{U}$. Because otherwise, there is a subsequence $\{y_k\}$ of S with $\{y_k\} \subset Y - \cup\mathscr{U}$. For each $k \in \mathbb{N}$, there is $x_k \in L - \bigcup_{x\in F} B(x, 1/n)$ such that $f(x_k) = y_k$. If a is an accumulation point of $\{x_k\}$, then $a \notin \bigcup_{x\in F} B(x, 1/n)$, and hence $f(a) \neq y$, a contradiction.
(4) When f is a sequentially quotient mapping, it is easy to verify that $\mathscr{U}_n$ is a cs^*-cover of Y. ■

Proposition 2.9.5 *Let $\{\mathscr{U}_n\}$ be a point-star network for a space X. Then there are a metric space (M, d) and a π-mapping $f : M \to X$ such that*

(1) if $\{\mathscr{U}_n\}$ is a sequence of open covers of X, then f is an open mapping [179];
(2) if $\{\mathscr{U}_n\}$ is a sequence of cfp-covers of X, then f is a compact-covering mapping [294];
(3) if $\{\mathscr{U}_n\}$ is a sequence of fcs-covers of X, then f is a sequence-covering mapping [152];
(4) if $\{\mathscr{U}_n\}$ is a sequence of cs^*-covers of X, then f is a sequentially quotient mapping [297];
(5) if E is a subset of X and each $(\mathscr{U}_n)_E$ is countable, then $f^{-1}(E)$ has a countable base.

Proof It is easy to verify that for each $x \in X$ with $x \in U_i \in \mathscr{U}_i$ ($\forall i \in \mathbb{N}$), $\{U_i\}_{i\in\mathbb{N}}$ is a net at x. For each $i \in \mathbb{N}$, let $\mathscr{U}_i = \{U_\alpha\}_{\alpha\in\Lambda_i}$ and let each Λ_i be a discrete space. Define

$$M = \left\{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Lambda_i : \{U_{\alpha_i}\} \text{ is a net at some point } x(\alpha) \in X\right\}.$$

Then M is a metrizable space, and the metric d defined in the following way is a compatible metric on M: for every pair $\alpha, \beta \in M$,

$$d(\alpha, \beta) = \begin{cases} 0, & \alpha = \beta, \\ \max\{1/k : \pi_k(\alpha) \neq \pi_k(\beta), k \in \mathbb{N}\}, & \alpha \neq \beta. \end{cases}$$

Define $f : M \to X$ by $f(\alpha) = x(\alpha)$. Then f is a mapping. For every $x \in U \in \tau(X)$, there is $n \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{U}_n) \subset U$. For every $\alpha \in f^{-1}(x)$ and $\beta \in M$ with $d(\alpha, \beta) < 1/n$, we have $\pi_i(\alpha) = \pi_i(\beta)$ whenever $i \leqslant n$, so $x \in U_{\pi_n(\alpha)} = U_{\pi_n(\beta)}$, and hence

$$f(\beta) \in \bigcap_{i \in \mathbb{N}} U_{\pi_i(\beta)} \subset U_{\pi_n(\beta)} \subset U.$$

As a consequence, $d(f^{-1}(x), \ M - f^{-1}(U)) \geqslant 1/n$, thus f is a π-mapping.

(1) If $\{\mathscr{U}_n\}$ is a sequence of open covers of X, then for every $n \in \mathbb{N}$ and $\alpha_i \in \Lambda_i$ $(\forall i \leqslant n)$, we have

$$f(B(\alpha_1, \ldots, \alpha_n)) = \cap\{U_{\alpha_i} : i \leqslant n\},$$

where $B(\alpha_1, \ldots, \alpha_n) = \{\beta \in M : \ \pi_i(\beta) = \alpha_i, i \leqslant n\}$. So f is an open mapping.

(2) If $\{\mathscr{U}_n\}$ is a sequence of *cfp*-covers of X, then for every $K \in \mathscr{K}(X)$ and $n \in \mathbb{N}$, there is $\mathscr{P}'_n \in \mathscr{P}_n^{<\omega}$ such that $\mathscr{P}'_n = \{P_\alpha\}_{\alpha \in \Gamma_n}$ is precisely refined by the closed cover $\{K_\alpha\}_{\alpha \in \Gamma_n}$ of K, where $\Gamma_n \subset \Lambda_n$. We may assume each K_α is not empty. Take

$$L = \left\{(\alpha_n) \in \prod_{n \in \mathbb{N}} \Gamma_n : \bigcap_{n \in \mathbb{N}} K_{\alpha_n} \neq \varnothing\right\}.$$

Similar to the proof of Proposition 2.7.1(1), we can prove that L is a compact set in M and $f(L) = K$. So f is a compact-covering mapping.

(3) If $\{\mathscr{U}_n\}$ is a sequence of *fcs*-covers of X, then for each $S \in \mathscr{S}(X)$, by Lemma 2.9.3 and the proof of (2), there is a compact set L in M such that $f(L) = S$. So f is a sequence-covering mapping.

(4) Suppose $\{\mathscr{U}_n\}$ is a sequence of cs^*-covers of X and $\{x_n\}$ is a nontrivial sequence converging to $x_0 \in X$. Since $\mathscr{U}_1$ is a cs^*-cover of X, there exist a subsequence T_1 of $\{x_n\}$ and $\alpha_1 \in \Lambda_1$ such that T_1 is eventually in U_{α_1}. By the inductive method, for each $i \in \mathbb{N}$, we can choose T_i and $\alpha_i \in \Lambda_i$ such that T_{i+1} is a subsequence of T_i and T_i is eventually in U_{α_i}, and hence $T_i \subset \bigcap_{k \leqslant i} U_{\alpha_k}$. Pick $x_{n_i} \in T_i$ and $\beta_i \in f^{-1}(x_{n_i})$ such that $n_i < n_{i+1}$ and $\pi_k(\beta_i) = \alpha_k$ whenever $k \leqslant i$. Then $\lim_{i \to \infty} \pi_k(\beta_i) = \alpha_k$. Let $\beta_0 = (\alpha_i)$. Then the sequence $\{\beta_i\}$ converges to β_0 in M, and hence f is a sequentially quotient mapping.

By the definition of f, we can obtain (5) easily. ∎

Theorem 2.9.6 *For every space X, the following are equivalent:*

(1) X is a developable space.
(2) X is a compact-covering open π-image of a metric space [300].
(3) X is an open π-image of a metric space [179].

Definition 2.9.7 (*[29]*) A symmetrizable space (X, d) is said to satisfy the *weak Cauchy condition* if each convergent sequence $\{x_n\}$ in X has a *Cauchy subsequence* $\{x_{n_i}\}$, i.e. for any $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $d(x_{n_i}, x_{n_j}) < \varepsilon$ whenever $i, j > k$.

It is easy to verify that for any symmetrizable space (X, d) and any convergent sequence $\{x_n\}$ in X, $\{x_n\}$ has a Cauchy subsequence if and only if for any $\varepsilon > 0$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $d(x_{n_i}, x_{n_j}) < \varepsilon$ for every $i, j \in \mathbb{N}$.

Lemma 2.9.8 ([78, 456]) *Let (X, d) be a symmetrizable space. Then d satisfies the weak Cauchy condition if and only if for any $F \subset X$, if F is not closed in X, then for any $\varepsilon > 0$, there is $x, y \in F$ with $x \neq y$ such that $d(x, y) < \varepsilon$. The above condition is equivalent to that X is a sequential space with a cs^*-cover point-star network.*

Proof (8.1) Suppose (X, d) is a symmetrizable space satisfying the weak Cauchy condition. By Proposition 1.6.16, X is a sequential space. For each $n \in \mathbb{N}$, take $\mathscr{U}_n = \{A \subset X : \mathrm{diam} A < 1/n\}$. Then for each $x \in X$, $\mathrm{st}(x, \mathscr{U}_n) = B(x, 1/n)$. So $\{\mathscr{U}_n\}$ is a point-star network in X. For each $n \in \mathbb{N}$ and any sequence $\{x_k\}$ converging to some point x in X, there is a Cauchy subsequence $\{x_{k_i}\}$ such that $d(x, x_{k_i}) < 1/(n+1)$ for every $i \in \mathbb{N}$, and hence there is $m \in \mathbb{N}$ such that $d(x_{k_i}, x_{k_j}) < 1/(n+1)$ whenever $i, j \geqslant m$. Let $A_n = \{x\} \cup \{x_{k_i} : i \geqslant m\}$. Then $A_n \in \mathscr{U}_n$. Thus $\mathscr{U}_n$ is a cs^*-cover of X.

(8.2) Let $\{\mathscr{U}_n\}$ be a cs^*-cover point-star network in the sequential space X, and we may assume $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$. We first prove that $\mathrm{st}(x, \mathscr{U}_n)$ is a sequential neighborhood of x for every $x \in X$ and $n \in \mathbb{N}$. Otherwise, there is a sequence $\{x_m\}$ in $X - \mathrm{st}(x, \mathscr{U}_n)$ converging to x, and hence there is a subsequence $\{x_{m_i}\}$ eventually in some $U \in \mathscr{U}_n$, so $x_{m_i} \in U \subset \mathrm{st}(x, \mathscr{U}_n)$, a contradiction. Since X is a sequential space, $\{\mathrm{st}(x, \mathscr{U}_n)\}_{n\in\mathbb{N}}$ is a weak base of x. By the sufficiency of Proposition 1.6.14, there is a symmetric d on X such that $\mathrm{st}(x, \mathscr{U}_n) = B(x, 1/2^n)$ for every $x \in X$ and $n \in \mathbb{N}$. Let $F \subset X$ and $\varepsilon > 0$. Take $m \in \mathbb{N}$ such that $1/2^m < \varepsilon$. If F is not a closed set in X, then there is a sequence $\{x_n\}$ in F converging to $x \notin F$, so there is $U \in \mathscr{U}_m$ such that some subsequence of $\{x_n\}$ is eventually in U, and hence there are $x, y \in F$ with $x \neq y$ such that $d(x, y) < 1/2^m < \varepsilon$.

(8.3) Let (X, d) be a symmetrizable space satisfying that if $F \subset X$ and F is not closed in X, then for any $\varepsilon > 0$, there is a pair x, y of F with $x \neq y$ such that $d(x, y) < \varepsilon$. Suppose $\varepsilon > 0$ and $\{x_n\}$ is a nontrivial convergent sequence in X. If every subsequence $\{y_n\}$ of $\{x_n\}$ has a subsequence $\{z_n\}$ such that $d(z_1, z_n) \geqslant \varepsilon$ whenever $n > 1$, then there is a subsequence $\{a_n\}$ of $\{y_n\}$ such that $d(a_n, a_m) \geqslant \varepsilon$ whenever $n \neq m$, a contradiction. So there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that $d(z_1, z_n) < \varepsilon$ for every subsequence $\{z_n\}$ of $\{y_n\}$. Thus, we can choose a subsequence $\{a_n\}$ of $\{x_n\}$ such that $d(a_n, a_m) < \varepsilon$, which shows that $\{x_n\}$ is a Cauchy subsequence. ■

Theorem 2.9.9 ([232]) *A space X is a quotient π-image of a metric space if and only if X is a symmetrizable space satisfying the weak Cauchy condition.*

Proof By Lemma 2.9.8, Propositions 2.1.12 (the mapping lemma) and 2.9.5, we get the sufficiency. The necessity is obtained by Proposition 2.9.4. ■

Theorem 2.9.10 ([485]) *A space X is a compact-covering quotient π-image of a metric space if and only if X has a weak development consisting of cfp-covers.*

Proof By Propositions 2.9.5 and 2.1.12 (the mapping lemma), we get the sufficiency. The necessity follows from Proposition 2.9.4. ∎

Example 2.9.11 (*[232]*) There is a symmetrizable space which does not satisfy the weak Cauchy condition.

Take $X = \mathbb{R}$. Define a symmetric d on X as follows: for every $x, y \in X$,

$$d(x, y) = \begin{cases} 1, & x, y \in \mathbb{P} \text{ and } x \neq y, \\ |x - y|, & \text{otherwise.} \end{cases}$$

Give X the symmetric topology generated by d. Then for every $x \in X$ and $n \in \mathbb{N}$,

$$B(x, 1/n) = \begin{cases} (x - 1/n,\ x + 1/n), & x \in \mathbb{Q}, \\ \{x\} \cup ((x - 1/n,\ x + 1/n) - \mathbb{P}), & x \in \mathbb{P}. \end{cases}$$

Hence,

(11.1) any convergent sequence in the subspace $\mathbb{P}$ is trivial;

(11.2) if $x \in \mathbb{Q}$, $A \subset X$ and x is an accumulation point of A with respect to the Euclidean topology τ^* on $\mathbb{R}$, then x is also an accumulation point of A in X.

For each $n \in \mathbb{N}$ and any symmetric ρ on X, define

$$D_n = \{x \in \mathbb{P} : \rho(x, \mathbb{P} - \{x\}) \geqslant 1/n\}.$$

If there is $x \in \mathbb{P} - \bigcup_{n \in \mathbb{N}} D_n$, then for each $n \in \mathbb{N}$, there is $x_n \in \mathbb{P} - \{x\}$ such that $\rho(x, x_n) < 2/n$, and hence $x_n \to x \in \mathbb{P}$, which contradicts (11.1). Thus $\mathbb{P} = \bigcup_{n \in \mathbb{N}} D_n$. Since $(\mathbb{R}, \tau^*)$ is the second category, there is $m \in \mathbb{N}$ such that $\mathrm{int}_{\tau^*}(\mathrm{cl}_{\tau^*}(D_m)) \neq \varnothing$. Pick $x \in \mathbb{Q} \cap \mathrm{cl}_{\tau^*}(D_m)$. By (11.2), $x \in \mathbb{Q} \cap \overline{D}_m$, so D_m is not a closed set in X. By Lemma 2.9.8, (X, ρ) does not satisfy the weak Cauchy condition.

For the above X, by Theorem 2.3.6, there exist a metric space M and a quotient mapping $f : M \to X$. By Theorem 2.9.9, f is not a π-mapping with respect to any compatible metric on M. However, this is not the same case for semi-metrizable spaces.

Lemma 2.9.12 ([78]) *Every semi-metrizable space has a compatible semi-metric satisfying the weak Cauchy condition.*

Proof Let (X, d) be a semi-metrizable space. For each $r \in \mathbb{R}^+$, define

$$\mathscr{B}(r) = \{B \subset X : d(x, y) \geqslant r \text{ for every } x, y \in B \text{ with } x \neq y\}.$$

For every $x, y \in X$, let

$$A(x, y) = \{z \in X : \text{ there is } B \in \mathscr{B}(d(x, y)/2) \text{ such that } x, y \in B \text{ and } z \in \overline{B}\},$$
$$\rho(x, y) = \inf\{d(x, z) + d(z, y) : z \in A(x, y)\}.$$

(12.1) ρ is a semi-metric on X.
Obviously, ρ is a symmetric on X and for every $x, y \in X$, $\rho(x, y) \leqslant d(x, y)$, and hence $B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon)$ for any $x \in X$ and $\varepsilon > 0$. If there is a sequence $\{x_n\}$ in X such that $x_n \in B_\rho(x, 1/n) - B_d(x, \varepsilon)$, take $\varepsilon_n = d(x, x_n)$, then $\varepsilon_n \geqslant \varepsilon$. Since $\rho(x, x_n) < 1/n$, there is $y_n \in A(x, x_n)$ such that $d(x, y_n) + d(y_n, x_n) < 1/n$, and hence there is $B_n \in \mathscr{B}(\varepsilon_n/2)$ such that $x, x_n \in B_n$, $y_n \in \overline{B}_n$ and $d(x, y_n) < 1/n$. Choose a sequence $\{y_{n,i}\}_i$ converging to y_n in B_n. Then there is $i \in \mathbb{N}$ such that $y_{n,i} \in B_d(x, \varepsilon/2)$, which means $d(x, y_{n,i}) < \varepsilon/2$. Since $y_{n,i} \in B_n \in \mathscr{B}(\varepsilon_n/2)$ and $x \in B_n$, $d(x, y_{n,i}) \geqslant \varepsilon_n/2 \geqslant \varepsilon/2$, a contradiction. Thus, there is $n \in \mathbb{N}$ such that $B_\rho(x, 1/n) \subset B_d(x, \varepsilon)$. So ρ is a semi-metric on X.

(12.2) (X, ρ) satisfies the weak Cauchy condition.
For any $\varepsilon > 0$, $F \subset X$, if $F \notin \tau^c(X)$, then pick $z \in \overline{F} - F$ and take

$$B = F \cap B_d(z, \varepsilon/2), \quad \varepsilon_1 = \inf\{d(x, y) : x, y \in B \text{ with } x \neq y\}.$$

Then $B \in \mathscr{B}(\varepsilon_1)$ and $\varepsilon_1 \geqslant \varepsilon$. Choose $x, y \in B$ such that $0 < d(x, y) < 2\varepsilon_1$. Then $B \in \mathscr{B}(d(x, y)/2)$ and $z \in \overline{B}$, and hence $z \in A(x, y)$. So $\rho(x, y) \leqslant d(x, z) + d(z, y) < \varepsilon$. By Lemma 2.9.8, (X, ρ) satisfies the weak Cauchy condition. ■

Example 2.9.13 (*[78]*) There is a semi-metrizable space (X, d) such that d does not satisfy the weak Cauchy condition.

Let $X = A \cup B$, where

$$A = \{(0, a) \in \mathbb{R}^2 : |a| \leqslant 1\}, \quad B = \{(b, \sin(1/b)) : 0 < b \leqslant 1\}.$$

Give X the Euclidean topology. Define a symmetric d on X as follows: let $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$; if $x \in A$ or $y \in A$, take

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2};$$

if $x, y \in B$, then $d(x, y)$ is the arc length in B between x and y. Then d is a semi-metric compatible to the topology of X. Since no Cauchy sequence in B converging to a point of A, any sequence in B converging to a point of A has no Cauchy subsequence with respect to d, and hence d does not satisfy the weak Cauchy condition. ■

Theorem 2.9.14 ([1, 78]) *For every space X, the following are equivalent:*

(1) X is a semi-metrizable space.
(2) X is a countably bi-quotient π-image of a metric space.

(3) X is a pseudo-open π-image of a metric space.

Proof By Lemma 2.9.12, Theorem 2.9.9 and the mapping lemma (see Proposition 2.1.12), we get (1) $\Rightarrow$ (2). (2) $\Rightarrow$ (3) is obvious. By Theorems 2.9.9 and 1.2.8, (3) $\Rightarrow$ (1) holds. ■

Corollary 2.9.15 *Metrizability is invariant under closed π-mappings.*

Question 2.9.16 (1) Is every semi-metrizable space a sequence-covering π-image of a metric space?
(2) Is every semi-metrizable space a compact-covering π-image of a metric space?

Example 2.9.17 (*[277]*) There exist a metric space (X, d) and a countably bi-quotient π-mapping $f : (X, d) \to \mathbb{S}_1$ such that f is not a sequence-covering mapping.

Let $\mathscr{A}$ be a maximal almost disjoint family of $\mathbb{N}$ (see Example 1.8.4) and denote the uncountable family $\mathscr{A}$ as $\{A_\alpha\}_{\alpha\in\Gamma}$. Let $B_\alpha = \{\alpha\} \cup A_\alpha$ for each $\alpha \in \Gamma$. Define a symmetric d_α on B_α as follows: for every $x, y \in B_\alpha$,

$$d_\alpha(x, y) = \begin{cases} 0, & x = y, \\ 1/y, & x \neq y,\ x = \alpha, \\ |1/x - 1/y|, & x \neq y,\ x \neq \alpha,\ y \neq \alpha. \end{cases}$$

Then (B_α, d_α) is a metric space. Suppose $X = \bigoplus_{\alpha\in\Gamma} B_\alpha$ and d is a standard topological sum metric on X. Define a function $f : X \to \mathbb{S}_1$ by

$$f(x) = \begin{cases} 0, & x \in \Gamma, \\ 1/x, & x \notin \Gamma. \end{cases}$$

(17.1) f is a mapping.
It is obvious that $f^{-1}(y) = \oplus\{1/y : 1/y \in A_\alpha\}$ is a clopen set in X for each $y \in \mathbb{S}_1 - \{0\}$. If U is a neighborhood of 0 in $\mathbb{S}_1$, then for each $\alpha \in \Gamma$, $f^{-1}(U) \cap B_\alpha$ is an open set in B_α, so $f^{-1}(U) \in \tau(X)$.

(17.2) f is a countably bi-quotient mapping.
By the mapping lemma (see Proposition 2.1.12), we only need to prove that f is a quotient mapping. Suppose $U \subset \mathbb{S}_1$ and $f^{-1}(U) \in \tau(X)$. For each $y \in U$, we may assume $y = 0$. If U is not a neighborhood of y, then there is an infinite subset M of $\mathbb{N}$ such that $1/n \notin U$ for each $n \in M$. If $M \in \mathscr{A}$, then there is $\alpha \in \Gamma$ such that $B_\alpha = \{\alpha\} \cup M$. Since $f^{-1}(U)$ is a neighborhood of α, the sequence B_α is eventually in $f^{-1}(U)$, so the sequence $\{1/n\}_{n\in M}$ is eventually in U, a contradiction. Therefore, $M \notin \mathscr{A}$, so there is $\alpha \in \Gamma$ such that $M \cap A_\alpha$ is an infinite set, and hence the sequence $\{x : x \in M \cap A_\alpha\}$ is eventually in $f^{-1}(U)$, a contradiction. Thus, U is a neighborhood of y, and hence f is a quotient mapping.

(17.3) f is a π-mapping.
Otherwise, there exist $z \in \mathbb{S}_1$ and an open neighborhood U of z such that $d(f^{-1}(z),\ X - f^{-1}(U)) = 0$, and hence there exist sequences $\{z_n\}$ and $\{x_n\}$

in X such that $z_n \in f^{-1}(z)$, $x_n \in X - f^{-1}(U)$ and $d(z_n, x_n) < 1/n$. Then $f(z_n) = z$ and $f(x_n) \notin U$ for every $n \in \mathbb{N}$. So there is $\alpha_n \in \Gamma$ such that $x_n, z_n \in B_{\alpha_n}$ and $d_{\alpha_n}(z_n, x_n) < 1/n$. Hence $|f(z_n) - f(x_n)| < 1/n$, thus $f(x_n) \to z$, a contradiction.

(17.4) f is not a sequence-covering mapping.
Otherwise, there is a compact set K in X such that $f(K) = \mathbb{S}_1$. By the compactness of K, there is $\Gamma' \in \Gamma^{<\omega}$ such that $K \subset \bigcup_{\alpha\in\Gamma'} B_\alpha$. Pick $\beta \in \Gamma - \Gamma'$. Then there is $n_0 \in A_\beta - (\bigcup_{\alpha\in\Gamma'} A_\alpha) \subset A_\beta - K$, and hence there does not exist $x_0 \in K$ such that $f(x_0) = 1/n_0$, a contradiction.

2.10 Compact Mappings

In this section, we investigate characterizations of quotient compact images, pseudo-open compact images and open compact images of metric spaces on the basis of π-mappings of metric spaces. We also characterize developable spaces and meta-compact developable spaces by means of such images.

Theorem 2.10.1 ([277]) *Every sequentially quotient compact mapping on a metric space is a sequence-covering mapping.*

Proof Suppose $f : X \to Y$ is a sequentially quotient compact mapping and X is a metric space. Let $\{y_n\}$ be a nontrivial sequence converging to a point y_0 in Y. Take

$$S_1 = \{y_0\} \cup \{y_n : n \in \mathbb{N}\},\ X_1 = f^{-1}(S_1) \text{ and } g = f_{|X_1}.$$

Then g is also a sequentially quotient compact mapping. By the mapping lemma (see Proposition 2.1.12), g is a pseudo-open mapping. Let $\{U_n\}_{n\in\mathbb{N}}$ be a decreasing neighborhood base of the compact set $g^{-1}(y_0)$ in the metric space X_1. Then for each $n \in \mathbb{N}$, $y_0 \in g(U_n)^\circ$, so there is $i_n \in \mathbb{N}$ such that for each $i \geqslant i_n$, $y_i \in g(U_n)$, and hence $g^{-1}(y_i) \cap U_n \neq \varnothing$. We may assume $1 < i_n < i_{n+1}$. For each $j \in \mathbb{N}$, take

$$x_j \in \begin{cases} f^{-1}(y_j), & j < i_1, \\ f^{-1}(y_j) \cap U_n, & i_n \leqslant j < i_{n+1}. \end{cases}$$

Let $K = g^{-1}(y_0) \cup \{x_j : j \in \mathbb{N}\}$. Then K is a compact set in X_1 and $g(K) = S_1$, so $f(K) = S_1$. Thus, f is a sequence-covering mapping. ■

Corollary 2.10.2 ([277]) *Every quotient compact mapping on a metric space is a sequence-covering mapping.*

Example 2.10.3 (*[339]*) There is a countably bi-quotient compact mapping from a separable metric space onto a metric space which is not a compact-covering mapping.

Let $Z = \mathbb{I} \times \mathbb{I}$ and $\mathscr{A} = \{A \in \mathscr{K}(Z) : \pi_1(A) = \mathbb{I}\}$. Then $|\mathscr{A}| = 2^{\aleph_0}$, and hence there is a one-to-one onto function $\mu : \mathbb{I} \to \mathscr{A}$. If $s \in \mathbb{I}$, then $s \in \pi_1(\mu(s))$, so $\mu(s) \cap \pi_1^{-1}(s) \neq \varnothing$. Pick $x_s \in \mu(s) \cap \pi_1^{-1}(s)$. Let O_s be an open interval in $\{s\} \times \mathbb{I}$ of length 1/4 containing x_s, and let $X = \{(s,t) \in Z : (s,t) \notin O_s\}$. Then X is a separable metric space. Define $f = \pi_{1|X}$. Then $f : X \to \mathbb{I}$ is a compact mapping.

(3.1) f is not a compact-covering mapping.
Otherwise, there is a compact set B in X such that $f(B) = \mathbb{I}$. Then $B \in \mathscr{A}$, so there is $s \in \mathbb{I}$ such that $B = \mu(s)$, and hence $x_s \in \mu(s) - X$, thus $B \not\subset X$, a contradiction.

(3.2) f is a countably bi-quotient mapping.
By the mapping lemma (see Proposition 2.1.12), we only need to prove that f is a sequence-covering mapping. Suppose $\{s_n\}$ is a sequence converging to some point s_0 in $\mathbb{I}$. Fix two points (s_0, t_1), (s_0, t_2) in $f^{-1}(s_0)$ such that $|t_1 - t_2| = 1/2$. Let $K = \{s_i : i \in \omega\}$ and $C = f^{-1}(K) \cap (K \times \{t_1, t_2\})$. Then C is a compact set in X and $f(C) = K$. So f is a sequence-covering mapping.

Definition 2.10.4 A sequence $\{\mathscr{U}_n\}$ of covers of a space X is called a *point-finite weak development* of X [209] (resp. *point-finite semi-development* [1], *point-finite development*) if, $\{\mathscr{U}_n\}$ is a weak development (resp. semi-development, development) of X and each $\mathscr{U}_n$ is a point-finite cover of X.

Lemma 2.10.5 ([276]) *Suppose $\{\mathscr{U}_n\}$ is a sequence of point-finite covers of a space X such that $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$. Then $\{\mathscr{U}_n\}$ is a cs^*-cover point-star network for X if and only if $\{\mathscr{U}_n\}$ is a point-star sequential neighborhood network for X.*

Proof By (8.2) of Lemma 2.9.8, if $\{\mathscr{U}_n\}$ is a cs^*-cover point-star network for X, then $\{\mathscr{U}_n\}$ is a point-star sequential neighborhood network for X. Conversely, suppose $\{\mathscr{U}_n\}$ is a point-star sequential neighborhood network for X. For each $n \in \mathbb{N}$ and any nontrivial sequence $\{x_k\}$ converging to x in X, if $m \leqslant n$ and $\mathrm{st}(x, \mathscr{U}_m) \neq \{x\}$, pick $z_m \in \mathrm{st}(x, \mathscr{U}_m) - \{x\}$, then there is $i \in \mathbb{N}$ such that

$$\mathrm{st}(x, \mathscr{U}_i) \subset X - \{z_m : m \leqslant n \text{ and } \mathrm{st}(x, \mathscr{U}_m) \neq \{x\}\}.$$

Since $\{x_k\}$ is eventually in $\mathrm{st}(x, \mathscr{U}_i)$, $i > n$ and $\{x_k\}$ is eventually in $\mathrm{st}(x, \mathscr{U}_n)$. Since $\mathscr{U}_n$ is point-finite, there is a subsequence of $\{x_k\}$ eventually in some element of $\mathscr{U}_n$. So $\mathscr{U}_n$ is a cs^*-cover of X. ■

Theorem 2.10.6 *For every space X, the following are equivalent:*

(1) X has a point-finite weak development.
(2) X is a sequentially quotient and quotient compact image of a metric space [484].
(3) X is a quotient compact image of a metric space [209, 263].
(4) X is a sequence-covering quotient compact image of a metric space [484].

Proof (1) ⇒ (2). Let $\{\mathscr{U}_n\}$ be a point-finite weak development for X. For each $i \in \mathbb{N}$, let $\mathscr{U}_i = \{U_\alpha\}_{\alpha \in \Lambda_i}$. By using the notations of Proposition 2.9.5, we can define a metric space M and a mapping $f : (M, d) \to X$. For every $x \in X$ and $i \in \mathbb{N}$, take $\Gamma_i = \{\alpha \in \Lambda_i : x \in U_\alpha\}$. Then $\prod_{i\in\mathbb{N}} \Gamma_i$ is a compact set in $\prod_{i\in\mathbb{N}} \Lambda_i$. If $\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Gamma_i$, then $x \in \bigcap_{i\in\mathbb{N}} U_{\alpha_i}$, so $\alpha \in M$ and $f(\alpha) = x$, and hence $\prod_{i\in\mathbb{N}} \Gamma_i \subset f^{-1}(x)$. If $\alpha = (\alpha_i) \in f^{-1}(x)$, then $x \in \bigcap_{i\in\mathbb{N}} U_{\alpha_i}$, so $\alpha \in \prod_{i\in\mathbb{N}} \Gamma_i$, and hence $f^{-1}(x) \subset \prod_{i\in\mathbb{N}} \Gamma_i$. Thus $f^{-1}(x) = \prod_{i\in\mathbb{N}} \Gamma_i$, it follows that f is a compact mapping. By Lemma 2.10.5 and Proposition 2.9.5, f is a sequentially quotient mapping.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follows from the mapping lemma (see Proposition 2.1.12) and Corollary 2.10.2. We prove (3) $\Rightarrow$ (1). Let X be an image of a metric space M under a quotient compact mapping f. By Theorem 1.3.5, there is a sequence $\{\mathscr{B}_i\}$ of locally finite open covers of M, such that, $\mathscr{B}_{i+1}$ refines $\mathscr{B}_i$ and $\{\mathrm{st}(K, \mathscr{B}_i)\}_{i\in\mathbb{N}}$ is a neighborhood base of K in M for each $K \in \mathscr{K}(M)$. Let $\mathscr{U}_i = f(\mathscr{B}_i)$. Then $\mathscr{U}_i$ is a point-finite cover of X. Below we prove $\{\mathscr{U}_i\}$ is a weak development for X. If $x \in U \in \tau(X)$, then there is $n \in \mathbb{N}$ such that $\mathrm{st}(f^{-1}(x), \mathscr{B}_n) \subset f^{-1}(U)$, and hence $\mathrm{st}(x, \mathscr{U}_n) \subset U$. On the other hand, if a subset U of X satisfying that for each $x \in U$, there is $n \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{U}_n) \subset U$, then for any $z \in f^{-1}(U)$, there is $n \in \mathbb{N}$ such that $\mathrm{st}(f(z), \mathscr{U}_n) \subset U$, so $\mathrm{st}(z, \mathscr{B}_n) \subset f^{-1}(U)$, and hence $f^{-1}(U) \in \tau(M)$, thus $U \in \tau(X)$. Therefore, $\{\mathscr{U}_i\}$ is a point-finite weak development for X. ■

Theorem 2.10.7 ([15]) *For every space X, the following are equivalent:*

(1) X is a sequence-covering quotient compact image of a separable metric space.
(2) X is a quotient π-image of a separable metric space.
(3) X is a g-second countable symmetric space.

Proof (1) $\Rightarrow$ (2) is obvious. By Corollary 2.8.11, Theorems 2.9.9 and 1.6.22 we get (2) $\Rightarrow$ (3).

Now suppose X is a g-second countable symmetric space. Let

$$\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x = \{P_n : n \in \mathbb{N}\}$$

is a weak base for X, where each $\mathscr{P}_x$ is a weak base of x in X. Let d be a symmetric on X. For every $m, n \in \mathbb{N}$, define

$$A_{m,n} = \{x \in X : B(x, 1/n) \subset P_m\},\ B_{m,n} = X - A_{m,n} \text{ and } \mathscr{F}_{m,n} = \{P_m, B_{m,n}\}.$$

(7.1) $\mathscr{F}_{m,n}$ is an *fcs*-cover of X.

Suppose $L = \{x_i\}$ is a sequence in X converging to a point $x \in X$. We may assume $x \notin P_m \cap B_{m,n}$. If $x \in A_{m,n}$, then $B(x, 1/n) \subset P_m$, so L is eventually in $P_m \in \mathscr{F}_{m,n}$. If $x \notin A_{m,n}$, then $x \in B_{m,n}$, so $x \notin P_m$, and hence L is eventually in $B_{m,n} \in \mathscr{F}_{m,n}$. Because otherwise, there is a subsequence $\{x_{i_k}\}$ of L such that each $x_{i_k} \in A_{m,n}$, so $x_{i_k} \in B(x_{i_k}, 1/n) \subset P_m$. Since $\{x_{i_k}\}$ converges to x, we may assume $d(x, x_{i_k}) < 1/n$ for every $k \in \mathbb{N}$, and hence $x \in B(x_{i_k}, 1/n) \subset P_m$, a contradiction.

(7.2) $\{\mathscr{F}_{m,n}\}_{m,n\in\mathbb{N}}$ is a point-star network for X.

Suppose $x \in U \in \tau(X)$. Since $\mathscr{P}_x$ is a weak base of x in X, there is $m_0 \in \mathbb{N}$ such that $P_{m_0} \in \mathscr{P}_x$ and $P_{m_0} \subset U$. If $B(x, 1/n) \not\subset P_{m_0}$ for each $n \in \mathbb{N}$, then there is a sequence $\{x_n\}$ in X such that $x_n \in B(x, 1/n) - P_{m_0}$ for each $n \in \mathbb{N}$, so $\{x_n\}$ converges to x, which contradicts the fact that P_{m_0} is a sequential neighborhood of x. Thus, there is $n_0 \in \mathbb{N}$ such that $B(x, 1/n_0) \subset P_{m_0}$, so $x \in A_{m_0,n_0}$, and hence $\mathrm{st}(x, \mathscr{F}_{m_0,n_0}) = P_{m_0} \subset U$. Consequently, $\{\mathscr{F}_{m,n}\}_{m,n\in\mathbb{N}}$ is a point-star network for X.

By using the notations in Proposition 2.9.5, from (7.1) and (7.2), we know there exist a metrizable space M and a sequence-covering mapping $f : (M, \rho) \to X$. Since each $\mathscr{F}_{m,n}$ is finite, M is a separable metrizable space. By the mapping lemma (see Proposition 2.1.12) and the proof of Theorem 2.10.6, f is a compact quotient mapping. ■

Corollary 2.10.8 *For every regular space X, the following are equivalent:*

(1) X is a (compact-covering and) quotient compact image of a separable metric space.
(2) X is a quotient π-image of a separable metric space.
(3) X is a g-second countable space.

Proof (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3) is obtained by Theorem 2.10.7.

(3) $\Rightarrow$ (1). Suppose $\mathscr{B}$ is a countable weak base for X and we may assume $\mathscr{B} \subset \tau^c(X)$. Let $\mathscr{B} = \{B_i\}_{i\in\mathbb{N}} = \bigcup_{x\in X} \mathscr{B}_x$, where $\mathscr{B}_x$ is a weak base of x. For each $i \in \mathbb{N}$, define

$$C_i = \{x \in X : B_i \notin \mathscr{B}_x\}, \ \mathscr{U}_i = \{B_i, C_i\}.$$

Then $\mathscr{U}_i$ is a *cfp*-cover of X. Because in fact, for any $K \in \mathscr{K}(X)$, take

$$K_1 = B_i \cap K \text{ and } K_2 = \overline{K - B_i}.$$

Then $K_1 \subset B_i$ and $K = K_1 \cup K_2$. Since K is metrizable, for each $x \in K_2$, there is a sequence $\{x_n\}$ converging to $x \in K$ in $K - B_i$, so $B_i \notin \mathscr{B}_x$, and hence $x \in C_i$. Thus $K_2 \subset C_i$. If $x \in X$, then

$$\mathrm{st}(x, \mathscr{U}_i) = \begin{cases} B_i, \ B_i \in \mathscr{B}_x, \\ X, \ B_i \notin \mathscr{B}_x, \ x \in B_i, \\ C_i, \ B_i \notin \mathscr{B}_x, \ x \notin B_i. \end{cases}$$

So $\{\mathrm{st}(x, \mathscr{U}_i)\}_{i\in\mathbb{N}}$ is a weak base of x, which shows $\{\mathscr{U}_i\}$ is a weak development for X. By Proposition 2.9.5 and Theorem 2.10.6, X is a compact-covering quotient compact image of a separable metric space. ■

Corollary 2.10.9 *Consider the following conditions:*

(1) X is a (compact-covering and) quotient compact ss-image of a metric space.
(2) X is a quotient π-ss-image of a metric space.
(3) X has a locally countable weak base.
(4) X is a topological sum of g-second countable spaces.

(5) X is a g-first countable space with a locally countable cs^*-network.
(6) X is a g-first countable space with a locally countable k-network.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6). If further assume that X is a regular space, then (6) $\Rightarrow$ (1).

Proof (1) $\Rightarrow$ (2) is obvious. By Corollary 2.8.11 and Theorem 2.9.9, we get (2) $\Rightarrow$ (5). (5) $\Rightarrow$ (3) follows from Corollary 2.8.11 and Proposition 1.6.21. By Corollary 2.8.11, (3) $\Rightarrow$ (4) holds. (4) $\Rightarrow$ (5) $\Rightarrow$ (6) is obvious. When X is a regular space, (6) $\Rightarrow$ (1) can be obtained by Corollaries 2.8.11 and 2.10.8. ■

Example 2.8.14 shows the assumption of regularity in the above corollary is necessary.

Example 2.10.10 There is a g-first countable space with a locally countable k-network which is not a quotient π-image of a metric space [276].

Let X be the pointed irrational extension topological space of $\mathbb{R}$ (see Example 2.7.14). We use the notations in Example 2.7.14 and first prove (X, τ) is not a countably metacompact space. Denote the Euclidean topology for X by τ^* and let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$. Take $F_n = \{r_i : i \geqslant n\}$ for each $n \in \mathbb{N}$. Since $\mathbb{Q}$ is a discrete closed set in X, $\{F_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of closed sets in X and $\bigcap_{n\in\mathbb{N}} F_n = \varnothing$. If X is a countably metacompact space, then there is a sequence $\{G_n\}_{n\in\mathbb{N}}$ of open sets in X such that $F_n \subset G_n$ and $\bigcap_{n\in\mathbb{N}} G_n = \varnothing$ (see Proposition A.2.8 in Appendix A). For every $n \in \mathbb{N}$ and $x \in F_n$, there is $U_{n,x} \in \tau^*$ such that

$$x \in U_{n,x},\ \ U_{n,x} \cap \{r_i : i < n\} = \varnothing \text{ and } \{x\} \cup (\mathbb{P} \cap U_{n,x}) \subset G_n,$$

so $U_{n,x} = (U_{n,x} \cap \mathbb{Q}) \cup (U_{n,x} \cap \mathbb{P}) \subset G_n$, and hence there is $O_n \in \tau^*$ such that $F_n \subset O_n \subset G_n$. Thus O_n is a dense open subset of $(\mathbb{R}, \tau^*)$ and $\bigcap_{n\in\mathbb{N}} O_n = \varnothing$, which contradicts the fact that τ^* is a Baire topology. Consequently, X is not a countably metacompact space, so X is not a subparacompact space either (see Proposition A.4.11 in Appendix A).

If X is a quotient π-image of a metric space, then by the mapping lemma (see Proposition 2.1.12) and Theorem 2.9.14, X is a semi-metrizable space, and hence X is a subparacompact space, a contradiction.

Question 2.10.11 (*[200]*) Is every symmetrizable space with a σ-point-finite cs-network a quotient compact image of a metric space?

Definition 2.10.12 Let $\mathscr{P}$ be a base of a space X.

(1) $\mathscr{P}$ is a *uniform base* [2] (resp. *uniform base at non-isolated points* [247]) for X, if for each (resp. each non-isolated) point $x \in X$ and each countably infinite subset $\mathscr{P}'$ of $(\mathscr{P})_x$, $\mathscr{P}'$ is a neighborhood base at x.
(2) $\mathscr{P}$ is a *point-regular base* [2] (resp. *point-regular base at non-isolated points* [247]) for X, if for each (resp. each non-isolated) point $x \in X$ and $x \in U$ with U open in X, $\{P \in (\mathscr{P})_x : P \not\subset U\}$ is finite.

Definition 2.10.13 (*[247]*) A space X is called *developable at non-isolated points* if X has a sequence $\{\mathscr{P}_n\}$ of open subsets such that $\{\text{st}(x, \mathscr{P}_n)\}_{n\in\mathbb{N}}$ is a neighborhood base at each non-isolated point $x \in X$. The sequence $\{\mathscr{P}_n\}$ is called a *development at non-isolated points* for X.

In the definitions, "at non-isolated points" means "at each non-isolated point of X". In the following, let $I(X)$ be the set of all isolated points in a space X, and $\mathscr{I}(X) = \{\{x\} : x \in I(X)\}$.

Theorem 2.10.14 ([247]) *The following are equivalent for a space X:*

(1) X is an open, peripherally compact image of a metric space.
(2) X has a uniform base at non-isolated points.
(3) X has a point-regular base at non-isolated points.
(4) X has a point-finite development at non-isolated points.

Proof (1) $\Rightarrow$ (2). Let M be a metric space and $f : M \to X$ be an open, peripherally compact mapping. By Theorem 1.3.5, we can choose a sequence $\{\mathscr{B}_i\}$ of open covers of M such that $\{\text{st}(K, \mathscr{B}_i)\}_{i\in\mathbb{N}}$ is a neighborhood base of K in M for each $K \in \mathscr{K}(M)$. Let $\mathscr{P} = \bigcup_{i\in\mathbb{N}} \mathscr{P}_i$, where each $\mathscr{P}_i = f(\mathscr{B}_i)$. Then $\mathscr{P}$ is a base for X because f is open. For each $i \in \mathbb{N}$, we can assume that $\mathscr{B}_{i+1}$ is a locally finite open refinement of $\mathscr{B}_i$. Let $x \in U - I(X)$ with U open in X, and let $\mathscr{P}'$ be a countably infinite subset of $(\mathscr{P})_x$. Then $\text{int}f^{-1}(x) = \varnothing$, thus $f^{-1}(x) = \partial f^{-1}(x)$ is compact in M. There exists $m \in \mathbb{N}$ such that $\text{st}(f^{-1}(x), \mathscr{B}_m) \subset f^{-1}(U)$, so $\text{st}(x, \mathscr{P}_m) \subset U$. For each $i \in \mathbb{N}$, $\{B \in \mathscr{B}_i : B \cap f^{-1}(x) \neq \varnothing\}$ is finite by the local finiteness of $\mathscr{B}_i$, i.e., $(\mathscr{P}_i)_x$ is finite. There exists $P \in \mathscr{P}' \cap \mathscr{P}_i$ for some $i \geqslant m$. Thus $P \subset \text{st}(x, \mathscr{P}_i) \subset \text{st}(x, \mathscr{P}_m) \subset U$. Hence, $\mathscr{P}$ is a uniform base at non-isolated points.

(2) $\Rightarrow$ (3). Let $\mathscr{P}$ be a uniform base at non-isolated points for X. If there exist a non-isolated point $x \in X$ and an open subset U in X with $x \in U$ such that $\{P \in (\mathscr{P})_x : P \not\subset U\}$ is infinite, take $\{P_n : n \in \mathbb{N}\} \subset \{P \in (\mathscr{P})_x : P \not\subset U\}$. Then $\{P_n : n \in \mathbb{N}\}$ is a neighborhood base at x, thus $P_m \subset U$ for some $m \in \mathbb{N}$, a contradiction. Therefore, $\mathscr{P}$ is a point-regular base at non-isolated points for X.

(3) $\Rightarrow$ (4). Let $\mathscr{P}$ be a point-regular base at non-isolated points for X. Obviously, $\mathscr{P}$ is also a uniform base at non-isolated points for X. We can assume that $|P| = 1$ if $P \in \mathscr{P}$ and $P \subset I(X)$.

Claim. Let x be a non-isolated point of X and $x \neq y \in X$. Then $\{H \in \mathscr{P} : \{x, y\} \subset H\}$ is finite.

In fact, $\{H \in \mathscr{P} : \{x, y\} \subset H\} \subset (\mathscr{P})_x$. If $\{H \in \mathscr{P} : \{x, y\} \subset H\}$ is infinite, then it is a local base at x, hence $y = x$, a contradiction.

(14.1) $\mathscr{P}$ is point-countable at non-isolated points in X.

Let $x \in X$ be a non-isolated point. There is a nontrivial sequence $\{x_n\}$ converging to x. By Claim, $\{P \in (\mathscr{P})_x : x_n \in P\}$ is finite for each $n \in \mathbb{N}$, then $(\mathscr{P})_x = \bigcup_{n\in\mathbb{N}}\{P \in \mathscr{P} : \{x, x_n\} \subset P\}$ is countable.

A family $\mathscr{F}$ of subsets of X is said to have the property ($\sharp$) if, for any $F \in \mathscr{F} - \mathscr{I}(X)$, then $\{H \in \mathscr{F} : F \subset H\}$ is finite.

(14.2) $\mathscr{P}$ has the property ($\sharp$).
Since $F \in \mathscr{P} - \mathscr{I}(X)$, then F contains a non-isolated point and $|F| > 1$. By Claim, $\mathscr{P}$ has the property ($\sharp$).
Put

$$\mathscr{P}^m = \{H \in \mathscr{P} : \text{ if } H \subset P \in \mathscr{P}, \text{ then } P = H\} \cup \mathscr{I}(X), \text{ and}$$
$$\mathscr{P}' = (\mathscr{P} - \mathscr{P}^m) \cup \mathscr{I}(X).$$

(14.3) $\mathscr{P}^m$ is an open cover and is point-finite at non-isolated points of X.
There exists $H_P \in \mathscr{P}^m$ such that $P \subset H_P$ for each $P \in \mathscr{P} - \mathscr{I}(X)$ by (14.2). Thus, $\mathscr{P}_m$ is an open cover of X. If $\mathscr{P}^m$ is not point-finite at some non-isolated point $x \in X$, then there exists an infinite subset $\{H_n : n \in \mathbb{N}\}$ of $(\mathscr{P}^m)_x$. For each $n \in \mathbb{N}$, $H_{n+1} \not\subset H_1$, there exists $x_n \in H_{n+1} - H_1$. Then the sequence $\{x_n\}$ converges to $x \in H_1$, a contradiction.
(14.4) $\mathscr{P}'$ is a point-regular base at non-isolated points for X.
Let $x \in U - I(X)$ with U open in X. There exist $V, W \in \mathscr{P}$ and $y \in V - \{x\}$ such that $x \in W \subset V - \{y\} \subset V \subset U$. Thus, $W \in \mathscr{P}'$. Then $\mathscr{P}'$ is a base for X, and it is a point-regular base at non-isolated points for X.
Put

$$\mathscr{P}_1 = \mathscr{P}^m \quad \text{and} \quad \mathscr{P}_{n+1} = \left[\left(\mathscr{P} - \bigcup_{i \leqslant n} \mathscr{P}_i\right) \cup \mathscr{I}(X)\right]^m, \quad n \in \mathbb{N}.$$

Then $\mathscr{P} = \bigcup_{n \in \mathbb{N}} \mathscr{P}_n$ by (14.2).
(14.5) $\{\mathscr{P}_n\}$ is a point-finite development at non-isolated points for X.
Each $\mathscr{P}_n$ is point-finite at non-isolated points by (14.3) and (14.4). If $x \in U - I(X)$ with U open in X, then $\{P \in (\mathscr{P})_x : P \not\subset U\}$ is finite, thus there is $n \in \mathbb{N}$ such that $P \subset U$ whenever $x \in P \in \mathscr{P}_n$, i.e., $\mathrm{st}(x, \mathscr{P}_n) \subset U$. So $\{\mathscr{P}_n\}$ is a development at non-isolated points.

(4) $\Rightarrow$ (1). First, a metric space M and a function $f : M \to X$ are defined as follows. Let $\{\mathscr{P}_n\}$ be a point-finite development at non-isolated points for X. For each $n \in \mathbb{N}$, assume that $\mathscr{I}(X) \subset \mathscr{P}_n$, put $\mathscr{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$, and endow Λ_n with the discrete topology. Put

$$M = \left\{\alpha = (\alpha_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{P_{\alpha_n}\}_{n \in \mathbb{N}} \text{ is a neighborhood} \right.$$
$$\left. \text{base at some point } x(\alpha) \in X\right\}.$$

Then M, which is a subspace of the product space $\prod_{n \in \mathbb{N}} \Lambda_n$, is a metric space. Define a function $f : M \to X$ by $f((\alpha_n)) = x_\alpha$. Then $f((\alpha_n)) = \bigcap_{n \in \mathbb{N}} P_{\alpha_n}$, and f is well defined. It is easy to see that f is an open mapping by Proposition 2.9.5. The following will prove that f is a peripherally compact map.

Let $x \in X$. If $x \in I(X)$, then $\partial f^{-1}(x) = \varnothing$. If $x \notin I(X)$, for each $i \in \mathbb{N}$, let $\Gamma_i = \{\alpha \in \Lambda_i : x \in P_\alpha\}$, then Γ_i is finite. Thus, $\partial f^{-1}(x) = f^{-1}(x) = \prod_{n\in\mathbb{N}} \Gamma_n$ by the proof of Theorem 2.10.6, so $\partial f^{-1}(x)$ is compact in M. ■

Corollary 2.10.15 ([247]) *Let X be a space having a uniform base at non-isolated points. Then,*

(1) X is a quasi-developable space with a σ-interior-preserving base.
(2) X is a developable space if $I(X)$ is an F_σ-set in X.

Proof By Theorem 2.10.14, we may assume that $\{\mathscr{P}\}_{n\in\mathbb{N}}$ is a point-finite development at non-isolated points for X.

(1) Put $\mathscr{P}_0 = \mathscr{I}(X)$. It is easy to check that $\{\mathscr{P}_n\}_{n\in\omega}$ is a quasi-development for X. Let $\mathscr{P} = \bigcup_{n\in\omega} \mathscr{P}_n$. Then $\mathscr{P}$ is a base for X and each $\mathscr{P}_n$ is interior-preserving. Indeed, for each $\mathscr{F} \subset \mathscr{P}_n$, if $x \in \cap\mathscr{F} - I(X)$, then $(\mathscr{P}_n)_x$ is finite, thus $\cap\mathscr{F}$ is a neighborhood of x in X. Thus $\mathscr{P}$ is a σ-interior-preserving base for X.
(2) If $I(X)$ is an F_σ-set, there exists a sequence $\{G_n\}$ of open subsets of X such that $X - I(X) = \bigcup_{n\in\mathbb{N}} G_n$. For each $n \in \mathbb{N}$, let $\mathscr{U}_n = \{G_n\} \cup \{\{x\} : x \in X - G_n\}$. Then $\{\mathscr{P}_n, \mathscr{U}_n\}_{n\in\mathbb{N}}$ is a development for X. Hence, X is a developable space. ■

Lemma 2.10.16 ([1]) *For every space X, the following are equivalent:*

(1) X has a point-finite development.
(2) X has a point-finite semi-development.
(3) X is a Fréchet–Urysohn space with a point-finite weak development.
(4) X is a metacompact developable space.

Proof (4) ⇒ (1) ⇒ (3) is obvious. By Proposition 1.6.17, we get (3) ⇒ (2). Below we prove (2) ⇒ (4). Suppose X has a point-finite semi-development. By Theorem 2.10.6 and the mapping lemma (see Proposition 2.1.12), X is a countably bi-quotient compact image of a metric space, so X is a metacompact space (see Corollary A.2.7 in Appendix A). Further by Theorems 2.7.17, 2.9.14 and 1.2.14, X is a developable space. ■

Thus, by Proposition 2.9.5 and Theorem 2.10.6, we get the following characterizations of open compact images of metric spaces.

Theorem 2.10.17 ([23, 33]) *For every space X, the following are equivalent:*

(1) X is an open compact (and compact-covering) image of a metric space.
(2) X is a pseudo-open compact image of a metric space.
(3) X is a metacompact developable space.
(4) X is a perfect, metacompact space which is an open peripherally compact image of a metric space [247].
(5) X has a uniform base.
(6) X has a point-regular base.

Proof (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Lemma 2.10.16, Proposition 2.9.5 and Theorem 2.10.6. (1) $\Leftrightarrow$ (4) by Theorem 2.10.14 and Corollary 2.10.15. (1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) by the proof of Theorem 2.10.14. ■

Michael [328] proved that every open compact mapping on a metric space is a compact-covering mapping. Example 2.10.3 shows the assumption "open mapping" cannot be weaken to "countably bi-quotient mapping". P. Yan [487] and F. Lin [249] etc. discussed the characterizations of quotient images of submetrizable spaces. The characterizations of quotient compact images of metric spaces in Theorem 2.10.6 and open compact images of metric spaces in Theorem 2.10.17 have evoked a lot of interesting work. For example, Theorem 2.10.6 has an improved form [14] that for every space X, the following are equivalent:

(1) X is a sequentially quotient compact image of a metric space.
(2) X is a sequence-covering compact image of a metric space.
(3) X has a point-star network consisting of point-finite cs^*-covers.
(4) X has a uniform cs^*-network.
(5) X has a point-regular cs^*-network.

Theorem 2.10.18 ([252]) *The following are equivalent for a space X:*

(1) X has a point-countable base, and a point-regular base at non-isolated points.
(2) X has a point-countable base which is point-regular at non-isolated points.
(3) X is an open peripherally compact, s-image of a metric space.
(4) X is an open s-image of a metric space, and is an open peripherally compact image of a metric space.
(5) X is a meta-Lindelöf space with a point-regular base at non-isolated points.

Proof (1) $\Rightarrow$ (2). Suppose that X has a point-countable base $\mathscr{B}$, and a point-regular base at non-isolated points $\mathscr{P}$. By Theorem 2.10.14, we can assume that $\mathscr{P} = \bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ satisfies the following conditions:

(i) $\mathscr{P}_n$ is an open cover and is point-finite at non-isolated points for X;
(ii) $\{\mathscr{P}_n\}$ is a development at non-isolated points for X.

For each $n \in \mathbb{N}$, put

$$\mathscr{B}' = \{B \in \mathscr{B} : B \not\subset I(X)\};$$
$$\mathscr{V}_n(B) = \{P \in \mathscr{P}_n : B \subset P\},\ B \in \mathscr{B}';$$
$$\hat{P} = \cup\{B \in \mathscr{B}' : P \in \mathscr{V}_n(B)\},\ P \in \mathscr{P}_n;$$
$$\hat{\mathscr{P}}_n = \{\hat{P} : P \in \mathscr{P}_n\}.$$

Then $\hat{\mathscr{P}}_n$ is point-countable. In fact, if $x \in \hat{P} \in \hat{\mathscr{P}}_n$, then there is $B' \in \mathscr{B}'$ such that $x \in B'$ and $P \in \mathscr{V}_n(B')$. Since $\{B \in \mathscr{B}' : x \in B\}$ is countable, and each $\mathscr{V}_n(B)$ is finite for each $B \in \mathscr{B}'$ by the condition (i), it follows that $\{P \in \mathscr{V}_n(B) : x \in B \in \mathscr{B}'\}$ is countable.

Put

$$\hat{\mathscr{P}} = \left(\bigcup_{n\in\mathbb{N}} \hat{\mathscr{P}}_n\right) \cup \mathscr{I}(X).$$

Then $\hat{\mathscr{P}}$ is point-countable. If $x \in U - I(X)$ with U open in X, then there is $m \in \mathbb{N}$ such that $x \in \mathrm{st}(x, \mathscr{P}_m) \subset U$ by the condition (ii). Take $P \in \mathscr{P}_m$ with $x \in P$, then there is $B \in \mathscr{B}'$ such that $x \in B \subset P$, thus $P \in \mathscr{V}_m(B)$, and $x \in B \subset \hat{P} \subset P \subset U$. So $\hat{\mathscr{P}}$ is a base for X. Finally, it is easy to see that $\hat{\mathscr{P}}$ is point-regular at non-isolated points by $\hat{P} \subset P$ for each $P \in \mathscr{P}$.

(2) $\Rightarrow$ (3). Let $\mathscr{P}$ be a point-countable base which is point-regular at non-isolated points for X. By Theorem 2.10.14, we can express $\mathscr{P}$ by $\bigcup_{n\in\mathbb{N}} \mathscr{P}_n$, where each $\mathscr{P}_n$ is a point-countable open cover of X, and $\{\mathscr{P}_n\}$ is a point-finite development at non-isolated points for X. Put $\mathscr{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ for each $n \in \mathbb{N}$. By the proof of (4) $\Rightarrow$ (1) of Theorem 2.10.14, there are a metric space $M \subset \prod_{n\in\mathbb{N}} \Lambda_n$ and an open peripherally compact mapping $f : M \to X$ such that $f^{-1}(x) \subset \prod_{n\in\mathbb{N}}\{\alpha \in \Lambda_n : x \in P_\alpha\}$, thus f is an s-mapping.

(3) $\Rightarrow$ (4) is obvious. (4) $\Rightarrow$ (5) by Theorem 2.10.14.

(5) $\Rightarrow$ (1). Let X be a meta-Lindelöf space with a point-regular base at non-isolated points. By Theorem 2.10.14, there is a sequence $\{\mathscr{P}_n\}$ of open covers of X such that $\{\mathscr{P}_n\}$ is a point-finite development at non-isolated points for X. For each $n \in \mathbb{N}$, let $\mathscr{B}_n$ be a point-countable open refinement of $\mathscr{P}_n$. And put

$$\mathscr{B} = \left(\bigcup_{n\in\mathbb{N}} \mathscr{B}_n\right) \cup \mathscr{I}(X).$$

Then $\mathscr{B}$ is a point-countable base for X. In fact, if a non-isolated point $x \in U$ with U open in X, then there is $n \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{P}_n) \subset U$. Take $B \in \mathscr{B}_n$ with $x \in B$, then $x \in B \subset \mathrm{st}(x, \mathscr{B}_n) \subset \mathrm{st}(x, \mathscr{P}_n) \subset U$. ■

Example 2.10.19 The Michael line (see Example 1.8.5) is an open compact image of a metacompact developable space [56].

We use the notations of Example 1.8.5, and the Michael line and the Bernstein set are denoted by X and B respectively. Let

$$\begin{aligned} H &= (\mathbb{I} \times \{0\}) \cup (B \times \mathbb{N}); \\ V(x, m) &= \{x\} \times (\{0\} \cup \{n : n \geqslant m\}),\ x \in \mathbb{I}, m \in \mathbb{N}; \\ W(J, m) &= ((J \cap (\mathbb{I} - B)) \times \{0\}) \cup \\ &\qquad ((J \cap B) \times \{n : n \geqslant m\}),\ J \subset \mathbb{I}, m \in \mathbb{N}. \end{aligned}$$

Give H the following topology: each element of a base has the forms $V(x, m)$ for every $x \in B$ and $m \in \mathbb{N}$, $W(J, m)$ for every open interval $J \subset \mathbb{I}$ and $m \in \mathbb{N}$, and $\{h\}$ for each $h \in B \times \mathbb{N}$. There is a development $\{\mathscr{U}_m\}$ for $\mathbb{I}$ with respect to Euclidean topology such that $\mathscr{U}_m$ is a finite set and $\mathscr{U}_{m+1}$ refines $\mathscr{U}_m$. For each $m \in \mathbb{N}$, let $\mathscr{P}_m$

be the family of all sets of the following forms: $V(x, m)$ (for each $x \in B$), $W(U, m)$ (for each $U \in \mathscr{U}_m$) and $\{h\}$ (for each $h \in B \times \{1, 2, \dots, m-1\}$). Then $\{\mathscr{P}_m\}_{m \geqslant 2}$ is a point-finite development for H, so H is a metacompact developable space.

It is easy to verify that $\pi_{1|H} : H \to X$ is an open compact mapping and X has a point-countable base which is uniform at non-isolated points. Since the set B is not an F_σ-set in X, X is not an open compact image of a metric space. Let X^* be a copy of the Michael line X and $f : X \to X^*$ be a homeomorphism. Put $Z = X \oplus X^*$, and let Y be the quotient space obtained from Z by identifying $\{x, f(x)\}$ to a point for each $x \in X - B$. Then Y has a point-countable base which is uniform at non-isolated points. Hence, Y is an open peripherally compact, s-image of a metric space by Theorem 2.10.18.

Because the Michael line is not a β-space, it is not a metacompact developable space either. From this we can know that the class of open compact images of metric spaces is not closed under open compact mappings. Hence, the class of spaces containing metric spaces and closed under open compact mappings is of special significance.

Definition 2.10.20 (*[31]*) The *class MOBI* is the smallest class of spaces such that

(1) every metric space is in this class;
(2) this class is closed under open compact mappings.

Obviously, the class MOBI is preserved by open compact mappings.

Theorem 2.10.21 ([56]) *A space Y is in the class MOBI if and only if there is a metric space M and finitely many open compact mappings $f_1, \dots, f_n$ such that $(f_n \circ \cdots \circ f_1)(M) = Y$.*

Proof Denote the set of classes of spaces satisfying the two conditions of Definition 2.10.20 by $\{\mathscr{H}_\alpha\}_{\alpha \in \Lambda}$. Then MOBI $= \bigcap_{\alpha \in \Lambda} \mathscr{H}_\alpha$. Let

$$\begin{aligned}\mathscr{B} = \{X :\ & \text{there exist a metric space } Z \text{ and finitely many open} \\ & \text{compact mappings } f_1, \dots, f_n, \text{ such that } (f_n \circ \cdots \circ f_1)(Z) = X\}.\end{aligned}$$

Then there is $\alpha \in \Lambda$ such that $\mathscr{B} = \mathscr{H}_\alpha$, and hence $\mathscr{B} \supset$ MOBI. If $X \in \mathscr{B}$, then there is a metric space Z and finitely many open compact mappings $f_1, \dots, f_n$ such that $(f_n \circ \cdots \circ f_1)(Z) = X$. By Definition 2.10.20, $X \in \mathscr{H}_\alpha$ for each $\alpha \in \Lambda$, so $\mathscr{B} \subset \mathscr{H}_\alpha$, and hence $\mathscr{B} \subset$ MOBI. Thus $\mathscr{B} =$ MOBI. ■

By Corollary 2.7.19 and Theorem 2.10.21, we have the following corollary.

Corollary 2.10.22 ([31]) *If $Y \in$ MOBI, then Y has a point-countable base.*

The problem whether the inverse proposition of Corollary 2.10.22 is set up affords much food for thought. Chaber [98] proved that every T_1 space with a point-countable base is an open compact image of a metacompact developable T_1 space. So in the class of T_1 spaces, the class MOBI can be characterized by the class of spaces with a point-countable base. If we denote the subclass of class MOBI consisting of spaces every open compact image of which is a T_i space, as we described in Theorem 2.10.21, by MOBI_i, where $i = 2, 3, 4$, then a major problem is whether every T_i space with a point-countable base belongs to the class MOBI_i. Example 2.10.19 shows the Michael line belongs to the class MOBI_2.

Question 2.10.23 (*[31]*) Is the class MOBI preserved by perfect mappings?

Question 2.10.24 (*[98]*) Does every space with a point-countable base belong to MOBI_2?

2.11 σ-Locally Finite Mappings

The way of investigating the relationships between spaces and mappings in the previous sections is to study characterizations of images of metric spaces under several well-known mappings. For the classes of generalized metric spaces, after defining suitable mappings, we may characterize these classes by means of images of metric spaces under such mappings. Following this train of thought, by σ-locally finite mappings and σ-mappings defined in this section, we characterize σ-spaces and $\aleph$-spaces in terms of images of metric spaces under these two mappings.

Definition 2.11.1 (*[338]*) A mapping $f : X \to Y$ is called a *σ-locally finite mapping* if, for every σ-locally finite cover $\mathscr{P}$ of X, there is a refinement $\mathscr{F}$ of $\mathscr{P}$ such that $f(\mathscr{F})$ is a σ-locally finite cover of Y.

Let $\mathscr{P}$ and $\mathscr{F}$ be families of sets in X. Then $\mathscr{F}$ is called a *B-refinement* of $\mathscr{P}$ if, $\mathscr{F}$ partially refines $\mathscr{P}$ and each element of $\mathscr{P}$ is the union of some sets in $\mathscr{F}$.

Lemma 2.11.2 ([338]) *If $\mathscr{P}$ is a locally finite family of set in a space X, then $\mathscr{P}$ has a disjoint locally finite B-refinement $\mathscr{F}$ such that $(\mathscr{P})_F$ is finite for any $F \in \mathscr{F}$.*

Proof Let

$$F(\mathscr{B}) = \cap\mathscr{B} - \cup(\mathscr{P} - \mathscr{B}),\ \mathscr{B} \subset \mathscr{P};$$
$$\mathscr{F} = \{F(\mathscr{B}) : \varnothing \neq \mathscr{B} \subset \mathscr{P}\}.$$

Then $\mathscr{P}$ is a locally finite B-refinement of $\mathscr{F}$ consisting of disjoint sets and $(\mathscr{P})_F$ is finite for any $F \in \mathscr{F}$. ■

Proposition 2.11.3 ([338]) *If $f : X \to Y$ is a mapping, then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) for the following conclusions:*

(1) Each open cover $\mathscr{U}$ of X has a refinement $\mathscr{B}$ such that $f(\mathscr{B})$ is a σ-locally finite family of sets in Y.
(2) Each σ-locally finite family of sets in X has a B-refinement $\mathscr{B}$ such that $f(\mathscr{B})$ is a σ-locally finite family of sets in Y.

(3) f is a σ-locally finite mapping.

Proof (1) $\Rightarrow$ (3). Suppose $\mathscr{P} = \bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ is a cover of X such that each $\mathscr{P}_n$ is locally finite. For each $n \in \mathbb{N}$, there is an open cover $\mathscr{U}_n$ of X such that each element of $\mathscr{U}_n$ only meets finitely many elements of $\mathscr{P}_n$, and hence $\mathscr{U}_n$ has a refinement $\mathscr{B}_n$ such that $f(\mathscr{B}_n)$ is σ-locally finite in Y. Then $\bigcup_{n\in\mathbb{N}}(\mathscr{P}_n \wedge \mathscr{B}_n)$ is a refinement of $\mathscr{P}$ and $f(\bigcup_{n\in\mathbb{N}}(\mathscr{P}_n \wedge \mathscr{B}_n))$ is a σ-locally finite cover of Y. Thus f is a σ-locally finite mapping.

(2) $\Rightarrow$ (3) is obvious, we prove (3) $\Rightarrow$ (2). We only need to prove that each locally finite family $\mathscr{P}$ of sets in X has a B-refinement $\mathscr{B}$ such that $f(\mathscr{B})$ is σ-locally finite in Y. By Lemma 2.11.2, $\mathscr{P}$ has a locally finite B-refinement $\mathscr{F}$ consisting of disjoint sets such that each element of $\mathscr{F}$ meets only finitely many elements of $\mathscr{P}$. Let

$$\mathscr{H} = \mathscr{F} \cup \{X - \cup\mathscr{F}\}.$$

Then $\mathscr{H}$ is a locally finite cove of X, and hence $\mathscr{H}$ has a refinement $\mathscr{R}$ such that $f(\mathscr{R})$ is a σ-locally finite family of sets in Y. Since $\mathscr{H}$ is a family of disjoint sets, $\mathscr{R}$ is a B-refinement of $\mathscr{H}$. Define

$$\mathscr{B} = \{B \in \mathscr{R} : B \subset \cup\mathscr{F}\}.$$

Then $\mathscr{B}$ is B-refinement of $\mathscr{F}$, so $\mathscr{B}$ is a B-refinement of $\mathscr{P}$ and $f(\mathscr{B})$ is a σ-locally finite family of sets in Y. ■

Corollary 2.11.4 ([338]) *Let $f : X \to Y$ be a mapping. If one of the following is true, then f is a σ-locally finite mapping:*

(1) There is an almost (mod k)-network $\mathscr{P}$ for X such that $f(\mathscr{P})$ is a σ-locally finite family of sets in Y.
(2) f is a closed L-mapping and Y is a subparacompact space.

Proof Suppose $\mathscr{U}$ is an open cover of X.

(1) Let
$$\mathscr{H} = \{P \in \mathscr{P} : P \subset \cup\mathscr{F},\ \mathscr{F} \in \mathscr{U}^{<\omega}\}.$$

Then $\mathscr{H}$ is a refinement of $\mathscr{U}^F$ and $f(\mathscr{H})$ is a σ-locally finite family of sets in Y, so $\mathscr{U}$ has a refinement $\mathscr{B}$ such that $f(\mathscr{B})$ is a σ-locally finite family of sets in Y. By Proposition 2.11.3, f is a σ-locally finite mapping.

(2) For each $y \in Y$, there exist a countable subfamily $\mathscr{U}_y$ of $\mathscr{U}$ covering $f^{-1}(y)$ and an open neighborhood V_y of y such that $f^{-1}(V_y) \subset \cup\mathscr{U}_y$. Let $\mathscr{V} = \{V_y\}_{y\in Y}$. Then the open cover $\mathscr{V}$ of Y has a σ-discrete closed refinement $\mathscr{F}$. For each $F \in \mathscr{F}$, there is $y(F) \in Y$ such that $F \subset V_{y(F)}$. Let

$$\mathscr{B} = \{f^{-1}(F) \cap U : F \in \mathscr{F},\ U \in \mathscr{U}_{y(F)}\}.$$

Then $\mathscr{B}$ refines $\mathscr{U}$ and $f(\mathscr{B})$ is a σ-locally finite family of sets in Y. Thus f is a σ-locally finite mapping. ■

Corollary 2.11.5 ([338]) *The following topological properties are invariant under σ-locally finite mappings:*

(1) having a σ-locally finite network;
(2) having a σ-locally finite almost (mod k)-network.

Proof Suppose $f : X \to Y$ is a σ-locally finite mapping. Let $\mathscr{P}$ be a σ-locally finite network (resp. σ-locally finite almost (mod k)-network) for X. By Proposition 2.11.3, $\mathscr{P}$ has a B-refinement $\mathscr{B}$ such that $f(\mathscr{B})$ is a σ-locally finite family of sets in Y. It is easy to verify that $f(\mathscr{B})$ is a network (resp. an almost (mod k)-network) for Y. ■

Let X be the Michael line (see Example 1.8.5). Since X has a point-countable base, there exist a metric space M and an open s-mapping $f : M \to X$. Because X is not a σ-space, by Corollary 2.11.5, f is not a σ-locally finite mapping. Hence, the condition "closed mapping" cannot be replaced with "open mapping" in Corollary 2.11.4.

The following lemma is obtained by Lemma 1.5.13 and Proposition 2.4.8.

Lemma 2.11.6 ([338]) *If $\mathscr{P}$ is a σ-locally finite closed almost (mod k)-network which is closed under finite intersections in a space Y, then there exist a metrizable space M, a σ-discrete base $\mathscr{B}$ for M and a subspace X of $Y \times M$ satisfying the following conditions: let $f = \pi_{1|X}$ and $g = \pi_{2|X}$, then*

(1) $\mathscr{P} = f(g^{-1}(\mathscr{B}))$;
(2) $g : X \to M$ is a perfect mapping.

Theorem 2.11.7 ([338]) *For every regular space X, the following are equivalent:*

(1) X is a σ-space (resp. strong Σ-space).
(2) X is a σ-locally finite image of a metric space (resp. paracompact M-space).

Proof By Corollary 2.11.5, we get (2) $\Rightarrow$ (1) (for the paracompact M-space case, we should further use Corollary 2.2.8 and Proposition 1.5.14).

(1) $\Rightarrow$ (2). We first assume that $\mathscr{P}$ is a σ-locally finite closed network which is closed under finite intersections for the σ-space X. Let $M = X$ and give M the topology generated by taking $\mathscr{P}$ as a base. Then $\mathscr{P}$ is a σ-locally finite clopen base for M, and hence M is a metric space. Let $f = \mathrm{id}_M : M \to X$. Since $\mathscr{P}$ is a network for X, f is a mapping. Further by Corollary 2.11.4, f is a σ-locally finite mapping.

Now suppose $\mathscr{P}$ is a σ-locally finite closed almost (mod k)-network which is closed under finite intersections for the strong Σ-space X. By Lemma 2.11.6, there exist a metrizable space M, a σ-discrete base $\mathscr{B}$ for M and a subspace Z of $X \times M$ satisfying the conditions that $\mathscr{P} = f(g^{-1}(\mathscr{B}))$ and $g : Z \to M$ is a perfect mapping, where $f = \pi_{1|Z}$ and $g = \pi_{2|Z}$. Then Z is a paracompact M-space and $g^{-1}(\mathscr{B})$ is a (mod k)-network for Z. By Corollary 2.11.4, f is a σ-locally finite mapping. ■

The assumption of regularity in the above theorem is necessary. The pointed irrational extension topological space X of $\mathbb{R}$ is a σ-locally finite image of a metric space (see Examples 2.7.14 and 2.10.10). Since every strong Σ-space is a subparacompact space (see Theorem 3.2.11), X is not a strong Σ-space.

For spaces with a (mod k)-network, we have the following result parallel to Corollary 2.8.4.

Corollary 2.11.8 ([338]) *A regular space X has a countable (mod k)-network if and only if there exist a separable metric space M and a perfect mapping $g : Z \to M$ such that X is an image of Z under a mapping.*

Example 2.11.9 (*[333]*) There exist a locally compact paracompact space Z and a closed mapping $f : Z \to X$ such that

(1) X is not a strong Σ-space;
(2) f is not a σ-locally finite mapping.

Suppose $h_\alpha : \omega_1 + 1 \to T_\alpha$ is a homeomorphic mapping for each $\alpha < \omega_1$. Let $Z = \bigoplus_{\alpha<\omega_1} T_\alpha$. Then Z is a locally compact paracompact space. Let $A = \{h_\alpha(\omega_1) : \alpha < \omega_1\}$. Denote the quotient space Z/A by X and let $f : Z \to X$ be the quotient mapping. Then f is a closed mapping. Assume X is a strong Σ-space. Let $\mathscr{F}$ be a σ-locally finite closed (mod k)-network with respect to $\mathscr{K}$ in X. Define $\widehat{\omega}_1 = f(h_0(\omega_1))$. For every $\alpha, \beta < \omega_1$, let $[\alpha(\beta), \widehat{\omega}_1) = f(h_\beta([\alpha, \omega_1)))$. Since $\mathscr{F}$ is σ-locally finite, for each $\beta < \omega_1$, $\{F \in \mathscr{F} : \widehat{\omega}_1 \in F$ and $F \cap [0(\beta), \widehat{\omega}_1) \neq \varnothing\}$ is countable, and hence there is $\alpha_\beta < \omega_1$ such that $\widehat{\omega}_1 \in F$ whenever $F \in \mathscr{F}$ and $F \cap [\alpha_\beta(\beta), \widehat{\omega}_1) \neq \varnothing$. For each $F \in \mathscr{F}$, define

$$\alpha(F) = \{\gamma < \omega_1 : F \cap [0(\gamma), \widehat{\omega}_1) \neq \varnothing\},$$
$$\mathscr{F}_0 = \{F \in \mathscr{F} : \widehat{\omega}_1 \in F,\ \alpha(F) < \omega_1\}.$$

Then $\mathscr{F}_0$ is countable. So there is $\beta < \omega_1$ such that $\beta > \sup\{\alpha(F) : F \in \mathscr{F}_0\}$. Thus, if $F \in \mathscr{F}$ and $F \cap [\alpha_\beta(\beta), \widehat{\omega}_1) \neq \varnothing$, then $F \cap [0(\gamma), \widehat{\omega}_1) \neq \varnothing$ for uncountably many $\gamma < \omega_1$. Let

$$\{F \in \mathscr{F} : F \cap [\alpha_\beta(\beta), \widehat{\omega}_1) \neq \varnothing\} = \{F_n\}_{n\in\mathbb{N}}.$$

Pick $K \in \mathscr{K}$ such that $K \cap [\alpha_\beta(\beta), \widehat{\omega}_1) \neq \varnothing$. Since $\{\gamma < \omega_1 : K \cap [0(\gamma), \widehat{\omega}_1) \neq \varnothing\}$ is a finite set, we can choose a sequence $\{x_n\}$ in X and a nontrivial sequence $\{\gamma_n\}$ in ω_1 such that $x_n \in (F_n - K) \cap [0(\gamma_n), \widehat{\omega}_1)$. So there is $F \in \mathscr{F}$ such that $K \subset F \subset X - \{x_n : n \in \mathbb{N}\}$, and hence there is $m \in \mathbb{N}$ such that $F = F_m$, it follows $x_m \in F$, a contradiction. Thus X is not a strong Σ-space. Moreover, by Corollary 2.11.5, f is not a σ-locally finite mapping.

In the second part of this section, we characterize $\aleph$-spaces in terms of specific images of metric spaces.

Definition 2.11.10 (*[266]*) A mapping $f : X \to Y$ is called a *σ-mapping* if there is a base $\mathscr{B}$ for X such that $f(\mathscr{B})$ is a σ-locally finite family of sets in Y.

Every mapping on a separable metric space is a σ-mapping. By Proposition 2.11.3, every σ-mapping is a σ-locally finite mapping.

Theorem 2.11.11 ([266]) *A regular space X is a σ-space if and only if X is a σ-image of a metrizable space.*

Proof Suppose X is a σ-space. Then by the proof of (1) $\Rightarrow$ (2) in Theorem 2.11.7, X is a σ-image of a metrizable space. Conversely, if a σ-image of a metric space is a regular space, then it is a σ-space, because every σ-mapping is a σ-locally finite mapping. ∎

Theorem 2.11.12 *For every regular space X, the following are equivalent:*

(1) X is an $\aleph$-space.
(2) X is a sequentially quotient σ-image of a metric space [459].
(3) X is a sequence-covering σ-image of a metric space [459].
(4) X is a compact-covering σ-image of a metric space [266].

Proof (1) $\Rightarrow$ (4). Suppose $\mathscr{P}$ is a σ-locally finite closed k-network which is closed under finite intersections for the $\aleph$-space X. Let $\mathscr{P} = \bigcup_{n\in\mathbb{N}} \mathscr{P}_n$, where each $\mathscr{P}_n = \{P_\alpha\}_{\alpha\in\Lambda_n}$ is locally finite and we may assume $X \in \mathscr{P}_n \subset \mathscr{P}_{n+1}$. Give Λ_n the discrete topology. Let

$$M = \left\{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Lambda_i : \{P_{\alpha_i}\} \text{ is a net at a point } x(\alpha) \text{ in } X\right\}.$$

Then M is a metrizable space and for each $\alpha \in M$, $x(\alpha)$ is uniquely determined. So we can define a function $f : M \to X$ by $f(\alpha) = x(\alpha)$. It is easy to verify that f is a mapping. Below we prove f is a compact-covering σ-mapping.

(12.1) f is a σ-mapping.
For every $n \in \mathbb{N}$ and $\alpha_n \in \Lambda_n$, define

$$B(\alpha_1, \dots, \alpha_n) = \{\alpha \in M : \pi_i(\alpha) = \alpha_i, i \leqslant n\},$$
$$\mathscr{B} = \{B(\alpha_1, \dots, \alpha_n) : \alpha_i \in \Lambda_i, i \leqslant n\}.$$

Then $\mathscr{B}$ is a base for M. To prove f is a σ-mapping, we only need to verify $f(B(\alpha_1, \dots, \alpha_n)) = \bigcap_{i\leqslant n} P_{\alpha_i}$. Obviously, $f(B(\alpha_1, \dots, \alpha_n)) \subset \bigcap_{i\leqslant n} P_{\alpha_i}$. If $x \in \bigcap_{i\leqslant n} P_{\alpha_i}$, then there is $\beta = (\beta_k) \in M$ such that $f(\beta) = x$. For each $k \in \mathbb{N}$, there is $\alpha_{k+n} \in \Lambda_{k+n}$ such that $P_{\alpha_{k+n}} = P_{\beta_k}$. Let $\alpha = (\alpha_k)$. Then $\alpha \in B(\alpha_1, \dots, \alpha_n)$ and $f(\alpha) = x$. So $\bigcap_{i\leqslant n} P_{\alpha_i} \subset f(B(\alpha_1, \dots, \alpha_n))$.

(12.2) f is a compact-covering mapping.
For each $K \in \mathscr{K}(X)$, by Lemma 2.7.9, there is a sequence $\{\mathscr{F}_i\}$ of finite subsets of $\mathscr{P}$ satisfying the conditions (i) and (ii) of Proposition 2.7.1. Take a strictly increasing sequence $\{n_i\}$ in $\mathbb{N}$ such that $\mathscr{F}_i \subset \mathscr{P}_{n_i}$. We may assume $n_1 = 1$. For every $i \in \mathbb{N}$ and $n_i \leqslant n < n_{i+1}$, let $\mathscr{P}'_n = \mathscr{F}_i$. Then $\mathscr{P}'_n \subset \mathscr{P}_n$, so there is $\Gamma_n \in \Lambda_n^{<\omega}$ such that $\mathscr{P}'_n = \{P_\alpha\}_{\alpha\in\Gamma_n}$. Let

$$L = \left\{ \alpha = (\alpha_n) \in \prod_{n \in \mathbb{N}} \Gamma_n : K \cap \bigcap_{n \in \mathbb{N}} P_{\alpha_n} \neq \varnothing \right\}.$$

By the proof of Proposition 2.7.1, L is a compact set in M and $f(L) = K$. Thus f is a compact-covering mapping.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) is obvious. It remains to prove (2) $\Rightarrow$ (1). Suppose $f : M \to X$ is a sequentially quotient σ-mapping and M is a metric space. Then there is a base $\mathscr{B}$ for M such that $f(\mathscr{B})$ is a σ-locally finite family of sets in X. By Propositions 2.7.2 and 1.6.7, $f(\mathscr{B})$ is a σ-locally finite k-network for X, and hence X is an $\aleph$-space. ∎

Example 2.11.13 There exist a metric space M and a σ-locally finite mapping $f : M \to X$ such that

(1) f is a compact-covering mapping and X is not an $\aleph$-space;
(2) f is a compact mapping and X is not a g-metrizable space;
(3) f is not a σ-mapping.

Let X be the V-space in Examples 1.8.1 and 2.4.16. By Example 2.4.16, there exist a metric space M and a finite-to-one compact-covering open mapping $f : M \to X$. By Corollary 2.11.4, f is a σ-locally finite mapping. By Theorem 2.5.15, X is not an $\aleph$-space. Further by Corollary 1.6.8 and Theorem 1.6.22, X is not a g-metrizable space either. By Theorem 2.11.12, f is not a σ-mapping.

The relevant results on characterizations of g-metrizable spaces by means of mappings are in Theorem 3.9.13.

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