Chapter 2
Introductory Concepts of Integral Equations on Time Scales

A delta integral equation (or in short integral equation) is the equation in which the unknown function $\phi(x)$ appears inside a delta integral sign. The subject of integral equations is one of the most useful mathematical tools in pure and applied mathematics. Many initial and boundary value problems associated with dynamic equations on time scales can be transformed into problems of solving some approximate integral equations.
Suppose that $\mathscr{T}$ is a time scale and let σ, ρ and Δ denote the forward jump operator, the backward jump operator and the delta differentiation operator, respectively,break on $\mathscr{T}$. A standard type of integral equation in $\phi(x)$ is of the following form

$$\phi(x) = u(x) + \lambda \int_{f(x)}^{g(x)} K(x, y)\phi(y)\Delta y, \tag{2.1}$$

where $f, g : \mathscr{T} \longmapsto \mathscr{R}$ are the limits of integration, λ is a constant parameter, and $K : \mathscr{T} \times \mathscr{T} \longmapsto \mathscr{R}$ is a known function of the variables x and y, $u : \mathscr{T} \longmapsto \mathscr{R}$ is a known function. The function $K(x, y)$ in (2.1) is called the kernel or the nucleus of the integral equation. In (2.1) the unknown function $\phi(x)$ appears the integral sign. There are many other cases in which the unknown function $\phi(x)$ appears inside and outside the integral sign. The functions $u(x)$ and $K(x, y)$ are given in advance. Note that the limits of integration $f(x)$ and $g(x)$ may be both constants, variables or mixed. If the limits of integration are fixed, the Eq. (2.1) is called a generalized Fredholm integral equation given in the form

$$\phi(x) = u(x) + \lambda \int_{a}^{b} K(x, y)\phi(y)\Delta y, \tag{2.2}$$

where a and b are constants. If at least one limit is variable, the equation is called a generalized Volterra integral equation given in the form

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$$\phi(x) = u(x) + \lambda \int_a^x K(x, y)\phi(y)\Delta y. \tag{2.3}$$

Example 1 Let $\mathscr{T} = 2^{\mathscr{N}_0}$. Then the equation

$$\phi(x) = \sinh_f(x, 1) + \lambda \int_1^x (x - y)\phi(y)\Delta y,$$

where $f \in \mathscr{R}_1$, is an example for a generalized Volterra integral equation.

If the unknown function $\phi(x)$ appears only under the integral sign of generalized Fredholm or generalized Volterra equation, the integral equation is called a first kind generalized Fredholm or generalized Volterra integral equation, respectively.

Example 2 Let $\mathscr{T} = \mathscr{Z}$. The equation

$$\cos_f(x, 0) = \int_1^2 y^2\phi(y)\Delta y,$$

where $f \in \mathscr{R}_1$, is a first kind generalized Fredholm integral equation.

If the unknown function $\phi(x)$ appears both inside and outside the integral sign of generalized Fredholm integral equation or generalized Volterra integral equation, the integral equation is called a second kind generalized Fredholm or generalized Volterra integral equation, respectively.

Example 3 Let $\mathscr{T} = 2^{\mathscr{N}_0}$. The equation

$$\phi(x) = x^2 - 1 + 2\int_1^4 (x - y)\phi(y)\Delta y$$

is a second kind generalized Fredholm integral equation.

If in the Eq. (2.2) or (2.3) the function $u(x)$ is identically zero, the resulting equation

$$\phi(x) = \lambda \int_a^b K(x, y)\phi(y)\Delta y$$

or

$$\phi(x) = \lambda \int_a^x K(x, y)\phi(y)\Delta y$$

is called homogeneous generalized Fredholm or homogeneous generalized Volterra integral equation, respectively. Any equation that includes both (delta-)integrals and (delta-)derivatives of the unknown function $\phi(x)$ is called delta-integro-delta-differential equation (or in short integro-differential equation). The Fredholm integro-differential equation is of the form

$$\phi^{\Delta^k}(x) = u(x) + \lambda \int_a^b K(x, y)\phi(y)\Delta y$$

However, the Volterra integro-differential equation is of the form

$$\phi^{\Delta^k}(x) = u(x) + \lambda \int_a^x K(x, y)\phi(y)\Delta y.$$

The equation

$$u(x) = \int_0^x \frac{1}{(x-y)^\alpha}\phi(y)\Delta y, \quad 0 < \alpha < 1,$$

is called generalized Abel's integral equation. The equation

$$\phi(x) = u(x) + \int_0^x \frac{1}{(x-y)^\alpha}\phi(y)\Delta y, \quad 0 < \alpha < 1,$$

is called generalized weakly singular integral equation. If the unknown function $\phi(x)$ inside the integral sign is one, the integral equation or the integro-differential equation is called linear.

Example 4 Let $\mathscr{T} = \mathscr{Z}$. The equation

$$x = \int_1^3 (x - 2y)\phi(y)\Delta y$$

is a linear equation.

If the equation contains nonlinear function of the unknown function $\phi(x)$, the integral equation or the integro-differential equation is called nonlinear.

Example 5 Let $\mathscr{T} = 2^{\mathscr{N}_0}$. Then

$$u(x) = x^2 - \int_1^x (x-y)^2 u^2(y)\Delta y$$

is a nonlinear equation.

The main objective of this text is to determine the unknown function $\phi(x)$ that will satisfy (2.1) using a number of solution techniques. We shall explain these methods to find solutions of the unknown function.

2.1 Reducing Double Integrals to Single Integrals

It will be seen later that we can convert initial value problems and other problems to integral equations. It is useful to outline the formula that will reduce double integrals to single integrals.

Theorem 1 *Let* $f : \mathscr{T} \longmapsto \mathscr{R}$ *be integrable and* $a \in \mathscr{T}$. *Then*

$$\int_a^x \int_a^{x_1} f(t)\Delta t \Delta x_1 = \int_a^x (x - \sigma(t))f(t)\Delta t \quad for \quad x \in \mathscr{T}. \tag{2.4}$$

Proof Using integration by parts, we have

$$\begin{aligned}
\int_a^x \int_a^{x_1} f(t)\Delta t \Delta x_1 &= \int_a^x (x_1 - a)^{\Delta} \int_a^{x_1} f(t)\Delta t \Delta x_1 \\
&= (x_1 - a) \int_a^{x_1} f(t)\Delta t \Big|_{x_1=a}^{x_1=x} - \int_a^x (\sigma(x_1) - a)f(x_1)\Delta x_1 \\
&= (x - a) \int_a^x f(t)\Delta t - \int_a^x (\sigma(t) - a)f(t)\Delta t \\
&= \int_a^x (x - \sigma(t))f(t)\Delta t.
\end{aligned}$$

Example 6 Let $\mathscr{T} = \mathscr{Z}$. Then $\sigma(t) = t + 1$, $t \in \mathscr{T}$, and

$$\begin{aligned}
\int_0^x \int_0^{x_1} t^2 \Delta t \Delta x_1 &= \int_0^x (x - \sigma(t))t^2 \Delta t \\
&= \int_0^x (x - t - 1)t^2 \Delta t.
\end{aligned}$$

Example 7 Let $\mathscr{T} = 2^{\mathscr{N}_0}$. Then $\sigma(t) = 2t$, $t \in \mathscr{T}$, and

$$\begin{aligned}
\int_1^x \int_1^{x_1} e_t(t, 1)\Delta t \Delta x_1 &= \int_1^x (x - \sigma(t))e_t(t, 1)\Delta t \\
&= \int_1^x (x - 2t)e_t(t, 1)\Delta t.
\end{aligned}$$

Example 8 Let $\mathscr{T} = 3\mathscr{Z}$. Then $\sigma(t) = t + 3$, $t \in \mathscr{T}$, and

$$\int_3^x \int_3^{x_1} (\sin_t(t,0) + t^3)\Delta t \Delta x_1 = \int_3^x (x - \sigma(t))(\sin_t(t,0) + t^3)\Delta t$$
$$= \int_3^x (x - t - 3)(\sin_t(t,0) + t^3)\Delta t.$$

Exercise 1 Convert the following double integrals to single integrals.

1. $\displaystyle\int_0^x \int_0^{x_1} (t^2 - t)\Delta t \Delta x_1$, $\mathscr{T} = \mathscr{Z}$.
2. $\displaystyle\int_1^x \int_1^{x_1} \frac{t^2+1}{t^4+1}\Delta t \Delta x_1$, $\mathscr{T} = 2^{\mathscr{N}_0}$.
3. $\displaystyle\int_2^x \int_2^{x_1} (3t - 2)\Delta t \Delta x_1$, $\mathscr{T} = 2\mathscr{N}$.

Answer

1. $\displaystyle\int_0^x (x - t - 1)(t^2 - t)\Delta t$,
2. $\displaystyle\int_1^x (x - 2t)\frac{t^2+1}{t^4+1}\Delta t$,
3. $\displaystyle\int_2^x (x - t - 2)(3t - 2)\Delta t$.

As a result to (2.11) we can show the following corollary.

Corollary 1 *Let $f : \mathscr{T} \longmapsto \mathscr{R}$ be integrable and $a \in \mathscr{T}$. Then*

$$\int_a^x \int_a^{x_1} (x - \sigma(t))f(t)\Delta t \Delta x_1 = \int_a^x (x - \sigma(t))^2 f(t)\Delta t \quad \textit{for} \quad x \in \mathscr{T}.$$

Proof By Theorem 1, we have

$$\int_a^x \int_a^{x_1} (x - \sigma(t))f(t)\Delta t \Delta x_1 = x\int_a^x \int_a^{x_1} f(t)\Delta t \Delta x_1 - \int_a^x \int_a^{x_1} \sigma(t)f(t)\Delta t \Delta x_1$$
$$= x\int_a^x (x - \sigma(t))f(t)\Delta t - \int_a^x (x - \sigma(t))\sigma(t)f(t)\Delta t$$
$$= \int_a^x (x - \sigma(t))^2 f(t)\Delta t.$$

Example 9 Let $\mathscr{T} = 2\mathscr{Z}$. Then $\sigma(t) = t + 2$, $t \in \mathscr{T}$, and

$$\int_{-2}^x \int_{-2}^{x_1} (x - t - 2)(t^3 + 1)\Delta t \Delta x_1 = \int_{-2}^x (x - t - 2)^2 (t^3 + 1)\Delta t.$$

Example 10 Let $\mathscr{T} = 4^{\mathscr{N}_0}$. Then $\sigma(t) = 4t$, $t \in \mathscr{T}$, and

$$\int_{16}^{x}\int_{16}^{x_1}(x-4t)(t^2+t)\Delta t\Delta x_1 = \int_{16}^{x}(x-4t)^2(t^2+t)\Delta t.$$

Example 11 Let $\mathscr{T} = \mathscr{N}_0^2$. Then $\sigma(t) = (\sqrt{t}+1)^2$, $t \in \mathscr{T}$. Then

$$\int_{0}^{x}\int_{0}^{x_1}(x-(\sqrt{t}+1)^2)\sqrt{t}\Delta t\Delta x_1 = \int_{0}^{x}(x-(\sqrt{t}+1)^2)^2\sqrt{t}\Delta t.$$

Exercise 2 Convert the following double integrals to single integrals.

1. $\int_{0}^{x}\int_{0}^{x_1}(x-t-2)t^2\Delta t\Delta x_1$, $\mathscr{T} = 2^{\mathscr{N}_0}\cup\{0\}$,
2. $\int_{0}^{x}\int_{0}^{x_1}(x-5t)(t^2+2t+3)\Delta t\Delta x_1$, $\mathscr{T} = 5^{\mathscr{N}_0}\cup\{0\}$.
3. $\int_{1}^{x}\int_{1}^{x_1}(x-t-1)t\Delta t\Delta x_1$, $\mathscr{T} = \mathscr{N}_0$,

where for $q > 1$ with $q^{\mathscr{N}_0}\cup\{0\}$ we will denote in all places in this book the set

$$0, \ldots, \frac{1}{q^k}, \frac{1}{q^{k-1}}, \ldots, 1, q, q^2, \ldots.$$

Answer

1. $\int_{0}^{x}(x-t-2)^2t^2\Delta t$,
2. $\int_{0}^{x}(x-5t)^2(t^2+2t+3)\Delta t$,
3. $\int_{1}^{x}(x-t-1)^2t\Delta t$.

2.2 Converting IVP to Generalized Volterra Integral Equations

In this section we will convert an initial value problem (IVP) to an equivalent generalized Volterra integral equation and generalized Volterra integro-differential equation. We will apply this process to a first order IVP

$$z^{\Delta}(x) + a(x)z(x) = u(x) \tag{2.5}$$

subject to the initial condition

$$z(x_0) = z_0, \tag{2.6}$$

where $x_0 \in \mathscr{T}$, $z_0 \in \mathscr{R}$, $a, u \in \mathscr{C}_{rd}(\mathscr{T})$, and to a second order IVP

$$z^{\Delta^2}(x) + a(x)z^{\Delta}(x) + b(x)z(x) = u(x) \tag{2.7}$$

subject to the initial conditions

$$z(x_0) = z_0, \quad z^{\Delta}(x_0) = z_0^{\Delta}, \tag{2.8}$$

where $x_0 \in \mathscr{T}$, $z_0, z_0^{\Delta} \in \mathscr{R}$, and $a, b, u \in \mathscr{C}_{rd}(\mathscr{T})$.
Firstly, we will consider the problem (2.5), (2.6). Let

$$\phi(x) = z^{\Delta}(x). \tag{2.9}$$

Now, using (2.6), we get

$$\begin{aligned}\int_{x_0}^{x} \phi(y)\Delta y &= \int_{x_0}^{x} z^{\Delta}(y)\Delta y \\ &= z(y)\Big|_{y=x_0}^{y=x} \\ &= z(x) - z(x_0) \\ &= z(x) - z_0,\end{aligned}$$

i.e.,

$$z(x) = z_0 + \int_{x_0}^{x} \phi(y)\Delta y. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.5) yields the following generalized Volterra integral equation

$$\phi(x) + a(x)\left(z_0 + \int_{x_0}^{x} \phi(y)\Delta y\right) = u(x).$$

The last equation can be written as standard generalized Volterra integral equation in the following way

$$\phi(x) = u(x) - z_0 a(x) - \int_{x_0}^{x} a(x)\phi(y)\Delta y.$$

Example 12 Let $\mathscr{T} = \mathscr{N}$. Consider populations with a fixed interval between generations or possibly a fixed interval between measurements. With x_0 we will denote the initial population size and with $x(t)$ we will denote the population size at time t.

Suppose the population changes only through births and deaths and suppose further that the birth and death rates are constants b and d, respectively. Then

$$x^{\Delta}(t) = (b-d)x(t), \quad t \in \mathscr{T}, \tag{2.11}$$

$$x(0) = x_0 \tag{2.12}$$

determines the population size in each generation. We integrate the Eq. (2.11) from 0 to t, and using (2.12), we get the integral equation

$$x(t) = x_0 + (b-d)\int_0^t x(s)\Delta s, \quad t \in \mathscr{T}.$$

Example 13 (*Verhulst difference equation*) Let $\mathscr{T} = \mathscr{N}$. The dynamic equation

$$x^{\Delta}(t) = \frac{(r-A)x(t) - x^2(t)}{A + x(t)}, \quad t \in \mathscr{T}, \tag{2.13}$$

describes a population that die out completely in each generation and has birth rates that saturate for large population sizes. Here A and r are positive constants. If we suppose that $x(0) = x_0$, then the Eq. (2.13) can be converted to a generalized Volterra integral equation

$$x(t) = x_0 + \int_0^t \frac{(r-A)x(s) - x^2(s)}{A + x(s)}\Delta s.$$

Exercise 3 Let $\mathscr{T} = \mathscr{H}$. Consider a simple electric circuit. The total charge $Q(t)$ on the capacitor at $t \in \mathscr{H}$ is given by the equation

$$Q^{\Delta}(t) = bQ(t), \quad b = \text{const}.$$

Reduce it to an integral equation if $Q(t_0) = Q_0$ for some $t_0 \in \mathscr{H}$ and some real constant Q_0.

Example 14 Let $\mathscr{T} = \mathscr{Z}$. Let us consider the IVP

$$z^{\Delta}(x) + 2xz(x) = 0, \quad z(0) = 1.$$

Here

$$a(x) = 2x, \quad u(x) = 0, \quad z_0 = 1.$$

Then we get the integral equation

$$\phi(x) = -2x - 2x\int_0^x \phi(y)\Delta y.$$

Example 15 Let $\mathscr{T} = \mathscr{N}$. Let us consider the IVP

$$z^{\Delta}(x) + x^2 z(x) = x, \quad z(1) = 1.$$

Here

$$x_0 = 1, \quad a(x) = x^2, \quad u(x) = x, \quad z_0 = 1.$$

Then we obtain the following integral equation

$$\phi(x) = x - x^2 - \int_1^x x^2 \phi(y) \Delta y.$$

Example 16 Let $\mathscr{T} = 2^{\mathscr{N}_0} \cup \{0\}$. Let us consider the IVP

$$z^{\Delta}(x) + e_x(x, 1) z(x) = \sinh_x(x, 1), \quad z(0) = 1.$$

Here

$$x_0 = 0, \quad z_0 = 1, \quad a(x) = e_x(x, 1). \quad u(x) = \sinh_x(x, 1).$$

Then we get the following integral equation

$$\phi(x) = \sinh_x(x, 1) - e_x(x, 1) - \int_0^x e_x(x, 1) \phi(y) \Delta y.$$

Exercise 4 Convert the following IVPs to integral equations.

1.
$$\begin{cases} z^{\Delta}(x) + (x^2 + 2x - 1) z(x) = 3 \\ z(0) = 2, \quad \mathscr{T} = \mathscr{Z}, \end{cases}$$

2.
$$\begin{cases} z^{\Delta}(x) + e_x(x, 0) z(x) = 3x^2 \\ z(0) = 1, \quad \mathscr{T} = \mathscr{N}_0, \end{cases}$$

3.
$$\begin{cases} z^{\Delta}(x) - \cos_x(x, 1) z(x) = -x \\ z(0) = 0, \quad \mathscr{T} = 2^{\mathscr{N}_0} \cup \{0\}. \end{cases}$$

Answer

1. $\phi(x) = 3 - 2(x^2 + 2x - 1) - \int_0^x (x^2 + 2x - 1) \phi(y) \Delta y,$
2. $\phi(x) = 3x^2 - e_x(x, 0) - \int_0^x e_x(x, 0) \phi(y) \Delta y,$
3. $\phi(x) = -x + \int_0^x \cos_x(x, 1) \phi(y) \Delta y.$

Now we consider the problem (2.7), (2.8). Set

$$z^{\Delta^2}(x) = \phi(x). \tag{2.14}$$

Then, using (2.8), we get

$$z^{\Delta}(x) - z^{\Delta}(x_0) = \int_{x_0}^{x} \phi(y)\Delta y$$

or

$$z^{\Delta}(x) = z_0^{\Delta} + \int_{x_0}^{x} \phi(y)\Delta y, \tag{2.15}$$

whereupon

$$z(x) - z(x_0) = \int_{x_0}^{x} z_0^{\Delta}\Delta y + \int_{x_0}^{x}\int_{x_0}^{x_1} \phi(y)\Delta y \Delta x_1,$$

or

$$z(x) = z_0 + z_0^{\Delta}(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{x_1} \phi(y)\Delta y \Delta x_1.$$

Hence, applying Theorem 1, we find

$$z(x) = z_0 + z_0^{\Delta}(x - x_0) + \int_{x_0}^{x} (x - \sigma(y))\phi(y)\Delta y. \tag{2.16}$$

Substituting (2.14), (2.15) and (2.16) in (2.7), we get

$$\begin{aligned}&\phi(x) + a(x)\left(z_0^{\Delta} + \int_{x_0}^{x} \phi(y)\Delta y\right)\\ &+ b(x)\left(z_0 + z_0^{\Delta}(x - x_0) + \int_{x_0}^{x} (x - \sigma(y))\phi(y)\Delta y\right) = u(x)\end{aligned}$$

or

$$\begin{aligned}\phi(x) &= u(x) - a(x)z_0^{\Delta} - b(x)z_0 - b(x)z_0^{\Delta}(x - x_0)\\ &- a(x)\int_{x_0}^{x} \phi(y)\Delta y - b(x)\int_{x_0}^{x} (x - \sigma(y))\phi(y)\Delta y,\end{aligned}$$

i.e.,

$$\phi(x) = u(x) - a(x)z_0^{\Delta} - b(x)z_0 - b(x)z_0^{\Delta}(x - x_0) - \int_{x_0}^{x} \left[a(x) + b(x)(x - \sigma(y))\right]\phi(y)\Delta y. \tag{2.17}$$

Example 17 Let $\mathscr{T} = 2^{\mathscr{N}_0}$. Consider the IVP

$$z^{\Delta^2}(x) + x^2 z^{\Delta}(x) + xz(x) = x - 1,$$

$$z(1) = 1, \quad z^{\Delta}(1) = 2.$$

Here

$$\sigma(x) = 2x, \quad a(x) = x^2, \quad b(x) = x, \quad u(x) = x - 1,$$

$$z_0 = 1, \quad z_0^{\Delta} = 2, \quad x_0 = 1.$$

Then, using (2.17), we get the following integral equation

$$\phi(x) = x - 1 - 2x^2 - x - 2x(x-1) - \int_1^x \left[x^2 + x(x - 2y)\right] \phi(y) \Delta y$$

$$= -4x^2 + 2x - 1 - 2\int_1^x (x^2 - xy)\phi(y)\Delta y.$$

Example 18 Let $\mathscr{T} = 2\mathscr{Z}$. Consider the IVP

$$z^{\Delta^2}(x) + xz^{\Delta}(x) - x^2 z(x) = x,$$

$$z(0) = 0, \quad z^{\Delta}(0) = 1.$$

Here

$$\sigma(x) = x + 2, \quad a(x) = x, \quad b(x) = -x^2, \quad u(x) = x,$$

$$z_0 = 0, \quad z_0^{\Delta} = 1, \quad x_0 = 0.$$

Then, using (2.17), we get the following integral equation

$$\phi(x) = x - x + x^3 - \int_0^x \left[x - x^2(x - y - 2)\right] \phi(y) \Delta y$$

$$= x^3 - \int_0^x (-x^3 + 2x^2 + x + x^2 y)\phi(y)\Delta y.$$

Example 19 Let $\mathscr{T} = \mathscr{N}_0^3$. Consider the IVP

$$z^{\Delta^2}(x) - 2e_x(x, 0)z(x) = \sinh_x(x, 0),$$

$$z(0) = 0, \quad z^{\Delta}(0) = 1.$$

Here

$$\sigma(x) = (\sqrt[3]{x}+1)^3, \quad a(x) = 0, \quad b(x) = -2e_x(x,0), \quad u(x) = \sinh_x(x,0),$$

$$z_0 = 0, \quad z_0^{\Delta} = 1, \quad x_0 = 0.$$

Then, using (2.17), we get the following integral equation

$$\phi(x) = \sinh_x(x,0) + 2xe_x(x,0) + 2\int_0^x e_x(x,0)\left[x - (\sqrt[3]{y}+1)^3\right]\phi(y)\Delta y.$$

Exercise 5 Convert the following IVPs to integral equations.

1.
$$\begin{cases} z^{\Delta^2}(x) - x^2 z^{\Delta}(x) = 0, \\ z(0) = 0, \quad z^{\Delta}(0) = 0, \quad \mathscr{T} = \mathscr{N}_0, \end{cases}$$

2.
$$\begin{cases} z^{\Delta^2}(x) + z^{\Delta}(x) + z(x) = 1, \\ z(0) = 0, \quad z^{\Delta}(0) = 1, \quad \mathscr{T} = 2^{\mathscr{N}_0} \cup \{0\}, \end{cases}$$

3.
$$\begin{cases} z^{\Delta^2}(x) + e_x(x,1) z^{\Delta}(x) = 0, \\ z(1) = 1, \quad z^{\Delta}(1) = 2, \quad \mathscr{T} = \mathscr{N}. \end{cases}$$

Answer

1. $\phi(x) = \int_0^x x^2\phi(y)\Delta y,$
2. $\phi(x) = -x - \int_0^x (1+x-2y)\phi(y)\Delta y,$
3. $\phi(x) = -2e_x(x,1) - \int_1^x e_x(x,1)\phi(y)\Delta y.$

2.3 Converting Generalized Volterra Integral Equations to IVP

A method for solving generalized Volterra integral and Volterra integro-differential equation converts these equations to equivalent initial value problems. This method is achieved by differentiating both sides of generalized Volterra equations with respect to x as many times as we need to get rid of the integral sign and obtain a differential equation. The conversion of generalized Volterra equations requires to use Leibnitz rule for differentiating the integral at the right hand side. The initial conditions are obtained by substituting $x = a$ into $u(x)$ and its derivatives. For instance, after we differentiate (2.3) with respect to x we get

$$\phi^{\Delta}(x) = u^{\Delta}(x) + \lambda \int_a^x K_x^{\Delta}(x, y)\phi(y)\Delta y + \lambda K(\sigma(x), x)\phi(x) \tag{2.18}$$

and substituting $x = a$ in (2.3) we find

$$\phi(a) = u(a).$$

If there is an integral sign in (2.18), then we differentiate it with respect to x and so on.

Example 20 Let $\mathscr{T} = \mathscr{Z}$. Consider the equation

$$\phi(x) = x^2 + \int_0^x \phi(y)\Delta y. \tag{2.19}$$

We have

$$\sigma(x) = x + 1, \quad x \in \mathscr{T},$$

and

$$(x^2)^{\Delta} = \sigma(x) + x = x + 1 + x = 2x + 1.$$

Hence, differentiating (2.19) with respect to x, we get

$$\begin{aligned}\phi^{\Delta}(x) &= (x^2)^{\Delta} + \phi(x)\\ &= 2x + 1 + \phi(x).\end{aligned}$$

Substituting $x = 0$ in (2.19), we find $\phi(0) = 0$.
In this way, we get the following IVP

$$\begin{cases}\phi^{\Delta}(x) - \phi(x) = 2x + 1\\ \phi(0) = 0.\end{cases}$$

Example 21 Let $\mathscr{T} = 2^{\mathscr{N}_0}$. Consider the equation

$$\phi(x) = x^3 + \int_1^x (x - y)\phi(y)\Delta y. \tag{2.20}$$

We have

$$\sigma(x) = 2x, \quad x \in \mathscr{T},$$
$$(x^3)^{\Delta} = \sigma^2(x) + x\sigma(x) + x^2$$

$$
\begin{aligned}
&= 4x^2 + 2x^2 + x^2 \\
&= 7x^2, \\
(x-y)_x^{\Delta} &= 1.
\end{aligned}
$$

We differentiate with respect to x the Eq. (2.20) and we get

$$
\begin{aligned}
\phi^{\Delta}(x) &= (x^3)^{\Delta} + \left(\int_1^x (x-y)\phi(y)\Delta y\right)^{\Delta} \\
&= 7x^2 + \int_1^x \phi(y)\Delta y + (\sigma(x) - x)\phi(x) \\
&= 7x^2 + x\phi(x) + \int_1^x \phi(y)\Delta y,
\end{aligned}
$$

i.e.,

$$\phi^{\Delta}(x) = 7x^2 + x\phi(x) + \int_1^x \phi(y)\Delta y. \tag{2.21}$$

Now we differentiate (2.21) with respect to x and we find

$$
\begin{aligned}
\phi^{\Delta^2}(x) &= (7x^2)^{\Delta} + (x\phi(x))^{\Delta} + \left(\int_1^x \phi(y)\Delta y\right)^{\Delta} \\
&= 7(\sigma(x) + x) + \phi(x) + \sigma(x)\phi^{\Delta}(x) + \phi(x) \\
&= 21x + 2\phi(x) + 2x\phi^{\Delta}(x)
\end{aligned}
$$

or

$$\phi^{\Delta^2}(x) - 2x\phi^{\Delta}(x) - 2\phi(x) = 21x.$$

We put $x = 1$ in (2.20) and we get $\phi(1) = 1$.
We substitute $x = 1$ in (2.21) and we find

$$\phi^{\Delta}(1) = 7 + \phi(1) = 8.$$

In this way we go to the following IVP

$$
\begin{cases}
\phi^{\Delta^2}(x) - 2x\phi^{\Delta}(x) - 2\phi(x) = 21x, \\
\phi(1) = 1, \quad \phi^{\Delta}(1) = 8.
\end{cases}
$$

Example 22 Let $\mathscr{T} = 3\mathscr{Z}$. Consider the equation

$$\phi(x) = e_x(x, 1) + \int_1^x (x + 2y)\phi(y)\Delta y. \tag{2.22}$$

Here

$$\sigma(x) = x + 3, \quad x \in \mathscr{T}.$$

Then, differentiating (2.22) with respect to x, we get

$$\begin{aligned}\phi^{\Delta}(x) &= e_x^{\Delta}(x, 1) + \left(\int_1^x (x + 2y)\phi(y)\Delta y\right)^{\Delta} \\ &= xe_x(x, 1) + \int_1^x \phi(y)\Delta y + (\sigma(x) + 2x)\phi(x) \\ &= xe_x(x, 1) + (3x + 3)\phi(x) + \int_1^x \phi(y)\Delta y,\end{aligned}$$

i.e.,

$$\phi^{\Delta}(x) = xe_x(x, 1) + 3(x + 1)\phi(x) + \int_1^x \phi(y)\Delta y. \tag{2.23}$$

Now we differentiate (2.23) with respect to x and we find

$$\begin{aligned}\phi^{\Delta^2}(x) &= (xe_x(x, 1))^{\Delta} + 3((x + 1)\phi(x))^{\Delta} + \left(\int_1^x \phi(y)\Delta y\right)^{\Delta} \\ &= e_x(x, 1) + \sigma(x)xe_x(x, 1) + 3\phi(x) + 3(\sigma(x) + 1)\phi^{\Delta}(x) + \phi(x) \\ &= (x^2 + 3x + 1)e_x(x, 1) + 4\phi(x) + 3(x + 4)\phi^{\Delta}(x),\end{aligned}$$

i.e.,

$$\phi^{\Delta^2}(x) - 3(x + 4)\phi^{\Delta}(x) - 4\phi(x) = (x^2 + 3x + 1)e_x(x, 1).$$

We put $x = 1$ in (2.22) and we find $\phi(1) = e_1(1, 1) = 1$.
We substitute $x = 1$ in (2.23) and we get

$$\phi^{\Delta}(1) = e_1(1, 1) + 6\phi(1) = 7e_1(1, 1) = 7.$$

In this way we get the following IVP

$$\begin{cases}\phi^{\Delta^2}(x) - 3(x + 4)\phi^{\Delta}(x) - 4\phi(x) = (x^2 + 3x + 1)e_x(x, 1), \\ \phi(1) = 1, \quad \phi^{\Delta}(1) = 7.\end{cases}$$

Exercise 6 Convert the following generalized Volterra integral equations to IVPs.

1. $\phi(x) = 2x^2 - 1 + \int_0^x \phi(y)\Delta y, \ \mathscr{T} = \mathscr{N}_0,$
2. $\phi(x) = x + 3 + x\int_1^x \phi(y)\Delta y, \ \mathscr{T} = 3^{\mathscr{N}_0},$
3. $\phi(x) = 2x + \int_0^x (x - y)\phi(y)\Delta y, \ \mathscr{T} = \mathscr{N}_0^2.$

Answer

1.
$$\begin{cases} \phi^{\Delta}(x) - \phi(x) = 4x + 2, \\ \phi(0) = -1, \end{cases}$$

2.
$$\begin{cases} \phi^{\Delta^2}(x) - 9x\phi^{\Delta}(x) - 4\phi(x) = 0, \\ \phi(1) = 4, \quad \phi^{\Delta}(1) = 13, \end{cases}$$

3.
$$\begin{cases} \phi^{\Delta^2}(x) - (3 + 2\sqrt{x})\phi^{\Delta}(x) - \frac{3+2\sqrt{x}}{1+2\sqrt{x}}\phi(x) = 0, \\ \phi(0) = 0, \quad \phi^{\Delta}(0) = 2. \end{cases}$$

Example 23 Let $\mathscr{T} = 2^{\mathscr{N}_0}$. Consider the generalized Volterra integro-differential equation

$$\phi(x) = \phi^{\Delta}(x) + x + \int_1^x (x + y)\phi(y)\Delta y. \tag{2.24}$$

We have $\sigma(x) = 2x$, $x \in \mathscr{T}$.
Then we differentiate with respect to x the Eq. (2.24) and we get

$$\begin{aligned} \phi^{\Delta}(x) &= \phi^{\Delta^2}(x) + 1 + \int_1^x \phi(y)\Delta y + (\sigma(x) + x)\phi(x) \\ &= \phi^{\Delta^2}(x) + 3x\phi(x) + 1 + \int_1^x \phi(y)\Delta y. \end{aligned}$$

i.e.,

$$\phi^{\Delta}(x) = \phi^{\Delta^2}(x) + 3x\phi(x) + 1 + \int_1^x \phi(y)\Delta y. \tag{2.25}$$

Now we differentiate (2.25) with respect to x and we find

$$\begin{aligned} \phi^{\Delta^2}(x) &= \phi^{\Delta^3}(x) + 3\phi(x) + 3\sigma(x)\phi^{\Delta}(x) + \phi(x) \\ &= \phi^{\Delta^3}(x) + 4\phi(x) + 6x\phi^{\Delta}(x) \end{aligned}$$

or

$$\phi^{\Delta^3}(x) - \phi^{\Delta^2}(x) + 6x\phi^{\Delta}(x) + 4\phi(x) = 0.$$

We put $x = 1$ in (2.24) and we get

$$\phi(1) = \phi^{\Delta}(1) + 1.$$

Now we substitute $x = 1$ in (2.25) and we find

$$\phi^{\Delta}(1) = \phi^{\Delta^2}(1) + 3\phi(1) + 1$$

$$= \phi^{\Delta^2}(1) + 3\phi^{\Delta}(1) + 3 + 1$$

$$= \phi^{\Delta^2}(1) + 3\phi^{\Delta}(1) + 4$$

or

$$\phi^{\Delta^2}(1) + 2\phi^{\Delta}(1) + 4 = 0.$$

In this way we go to the following problem

$$\begin{cases} \phi^{\Delta^3}(x) - \phi^{\Delta^2}(x) + 6x\phi^{\Delta}(x) + 4\phi(x) = 0, \\ \phi^{\Delta}(1) - \phi(1) + 1 = 0, \quad \phi^{\Delta^2}(1) + 2\phi^{\Delta}(1) + 4 = 0. \end{cases}$$

Example 24 Let $\mathscr{T} = 3\mathscr{N}_0$. Consider the generalized Volterra integro-differential equation

$$\phi(x) = \phi^{\Delta}(x) + \int_1^x y^2\phi(y)\Delta y. \tag{2.26}$$

Then, differentiating (2.26) with respect to x, we get

$$\phi^{\Delta}(x) = \phi^{\Delta^2}(x) + x^2\phi(x)$$

or

$$\phi^{\Delta^2}(x) - \phi^{\Delta}(x) + x^2\phi(x) = 0.$$

We put $x = 1$ in (2.26) and we find

$$\phi(1) = \phi^{\Delta}(1).$$

Therefore we obtain the following problem

$$\begin{cases} \phi^{\Delta^2}(x) - \phi^{\Delta}(x) + x^2\phi(x) = 0 \\ \phi(1) = \phi^{\Delta}(1). \end{cases}$$

Example 25 Let $\mathscr{T} = 5^{\mathscr{N}_0} \cup \{0\}$. Consider the generalized Volterra integro-differential equation

$$\phi^{\Delta}(x) = \phi(x) - x \int_0^x \phi(y)\Delta y. \tag{2.27}$$

Here $\sigma(x) = 5x$, $x \in \mathscr{T}$. Then, differentiating (2.27) with respect to x, we get

$$\begin{aligned}\phi^{\Delta^2}(x) &= \phi^{\Delta}(x) - \int_0^x \phi(y)\Delta y - \sigma(x)\phi(x)\\ &= \phi^{\Delta}(x) - \int_0^x \phi(y)\Delta y - 5x\phi(x),\end{aligned}$$

i.e.,

$$\phi^{\Delta^2}(x) - \phi^{\Delta}(x) + \int_0^x \phi(y)\Delta y + 5x\phi(x) = 0. \tag{2.28}$$

Hence,

$$\phi^{\Delta^3}(x) - \phi^{\Delta^2}(x) + \phi(x) + 5\phi(x) + 5\sigma(x)\phi^{\Delta}(x) = 0$$

or

$$\phi^{\Delta^3}(x) - \phi^{\Delta^2}(x) + 25x\phi^{\Delta}(x) + 6\phi(x) = 0.$$

We substitute $x = 0$ in (2.27) and (2.28) and we obtain

$$\phi(0) = \phi^{\Delta}(0) = \phi^{\Delta^2}(0).$$

Consequently we get the following problem

$$\begin{cases}\phi^{\Delta^3}(x) - \phi^{\Delta^2}(x) + 25x\phi^{\Delta}(x) + 6\phi(x) = 0\\ \phi(0) = \phi^{\Delta}(0) = \phi^{\Delta^2}(0).\end{cases}$$

Exercise 7 Convert the following generalized Volterra integro-differential equations to IVPs.

1. $\phi^{\Delta^2}(x) = \phi(x) + x^2 + \int_1^x \sinh_y(y, 1)\phi(y)\Delta y, \quad \mathscr{T} = \mathscr{N}_0,$
2. $\phi^{\Delta}(x) = \phi(x) + x^3 + \int_1^x (x + y)e_y(y, 1)\phi(y)\Delta y, \quad \mathscr{T} = 2^{\mathscr{N}_0},$
3. $\phi(x) = \phi^{\Delta}(x) + x^4 - \int_0^x (2x + y)\phi(y)\Delta y, \quad \mathscr{T} = 2\mathscr{Z}.$

Answer

1.
$$\begin{cases} \phi^{\Delta^3}(x) - \phi^{\Delta}(x) - \sinh_x(x, 1)\phi(x) = 2x + 1, \\ \phi^{\Delta^2}(1) = \phi(1) + 1, \end{cases}$$

2.
$$\begin{cases} \phi^{\Delta^3}(x) - \phi^{\Delta^2}(x) - 6xe_{2x}(2x, 1)\phi^{\Delta}(x) - (4 + 6x^2)e_x(x, 1)\phi(x) = 21x, \\ \phi^{\Delta}(1) = \phi(1) + 1, \quad \phi^{\Delta^2}(1) = \phi^{\Delta}(1) + 3\phi(1) + 7, \end{cases}$$

3.
$$\begin{cases} \phi^{\Delta^3}(x) - \phi^{\Delta^2}(x) - (3x + 10)\phi^{\Delta}(x) - 5\phi(x) = -12x^2 - 48x - 56, \\ \phi(0) = \phi^{\Delta}(0), \quad \phi^{\Delta}(0) = \phi^{\Delta^2}(0) - 4\phi(0) + 8. \end{cases}$$

2.4 Converting BVP to Generalized Fredholm Integral Equation

In this section we will represent a method for converting boundary value problems to generalized Fredholm integral equation. This method is similar to the method for converting of IVP to generalized Volterra integral equation. Here boundary conditions will be used instead of initial conditions. In this case we will determine another initial condition that is not given in the problem.
We consider the following boundary value problem

$$z^{\Delta^2}(x) + f(x)z(x) = g(x), \quad x_0 < x < x_1, \quad x_0, x_1 \in \mathscr{T}, \tag{2.29}$$

$$z(x_0) = z_0, \quad z(x_1) = z_1. \tag{2.30}$$

We set

$$\phi(x) = z^{\Delta^2}(x). \tag{2.31}$$

Integrating both sides of (2.31) from x_0 to x we obtain

$$\int_{x_0}^{x} z^{\Delta^2}(t)\Delta t = \int_{x_0}^{x} \phi(t)\Delta t$$

or

$$z^{\Delta}(x) - z^{\Delta}(x_0) = \int_{x_0}^{x} \phi(t)\Delta t,$$

or

$$z^{\Delta}(x) = z^{\Delta}(x_0) + \int_{x_0}^{x} \phi(t)\Delta t.$$

We integrate the last equation from x_0 to x and we find

$$\int_{x_0}^{x} z^{\Delta}(y)\Delta y = \int_{x_0}^{x} z^{\Delta}(x_0)\Delta y + \int_{x_0}^{x}\int_{x_0}^{x_2} \phi(t)\Delta t\Delta x_2$$

or

$$z(x) - z(x_0) = z^{\Delta}(x_0)(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{x_2} \phi(t)\Delta t\Delta x_2,$$

or

$$z(x) = z_0 + z^{\Delta}(x_0)(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{x_2} \phi(t)\Delta t\Delta x_2.$$

Applying Theorem 1 we find

$$z(x) = z_0 + z^{\Delta}(x_0)(x - x_0) + \int_{x_0}^{x} (x - \sigma(t))\Delta t. \tag{2.32}$$

We substitute $x = x_1$ in the last equation and using that $z(x_1) = z_1$ we go to

$$z_1 = z_0 + z^{\Delta}(x_0)(x_1 - x_0) + \int_{x_0}^{x_1} (x_1 - \sigma(t))\phi(t)\Delta t$$

or

$$z^{\Delta}(x_0)(x_1 - x_0) = z_1 - z_0 - \int_{x_0}^{x_1} (x_1 - \sigma(t))\phi(t)\Delta t,$$

or

$$z^{\Delta}(x_0) = \frac{z_1 - z_0}{x_1 - x_0} - \frac{1}{x_1 - x_0}\int_{x_0}^{x_1} (x_1 - \sigma(t))\phi(t)\Delta t.$$

We substitute the last expression in (2.32) and we find

$$\begin{aligned} z(x) = z_0 &+ \frac{z_1 - z_0}{x_1 - x_0}(x - x_0) - \frac{x - x_0}{x_1 - x_0}\int_{x_0}^{x_1} (x_1 - \sigma(t))\phi(t)\Delta t \\ &+ \int_{x_0}^{x} (x - \sigma(t))\phi(t)\Delta t. \end{aligned}$$

The last expression and (2.31) we put in (2.29). Then

$$\phi(x) + f(x)z_0 + \frac{z_1 - z_0}{x_1 - x_0} f(x)(x - x_0) - f(x)\frac{x - x_0}{x_1 - x_0}\int_{x_0}^{x_1}(x_1 - \sigma(t))\phi(t)\Delta t$$

$$+ f(x)\int_{x_0}^{x}(x - \sigma(t))\phi(t)\Delta t = g(x)$$

or

$$\begin{aligned}
\phi(x) &= g(x) - f(x)z_0 - \frac{z_1 - z_0}{x_1 - x_0} f(x)(x - x_0) + f(x)\frac{x - x_0}{x_1 - x_0}\int_{x_0}^{x_1}(x_1 - \sigma(t))\phi(t)\Delta t \\
&\quad - f(x)\int_{x_0}^{x}(x - \sigma(t))\phi(t)\Delta(t) \\
&= g(x) - f(x)z_0 - \frac{z_1 - z_0}{x_1 - x_0} f(x)(x - x_0) + f(x)\frac{x - x_0}{x_1 - x_0}\int_{x_0}^{x}(x_1 - \sigma(t))\phi(t)\Delta t \\
&\quad + f(x)\frac{x - x_0}{x_1 - x_0}\int_{x}^{x_1}(x_1 - \sigma(t))\phi(t)\Delta t - f(x)\int_{x_0}^{x}(x - \sigma(t))\phi(t)\Delta t \\
&= g(x) - f(x)z_0 - \frac{z_1 - z_0}{x_1 - x_0} f(x)(x - x_0) \\
&\quad + \int_{x_0}^{x} f(x)\left(x_1\frac{x - x_0}{x_1 - x_0} - \sigma(t)\frac{x - x_0}{x_1 - x_0} - x + \sigma(t)\right)\phi(t)\Delta t \\
&\quad + f(x)\frac{x - x_0}{x_1 - x_0}\int_{x}^{x_1}(x_1 - \sigma(t))\phi(t)\Delta t \\
&= g(x) - f(x)z_0 - \frac{z_1 - z_0}{x_1 - x_0} f(x)(x - x_0) \\
&\quad + \int_{x_0}^{x} f(x)\left(-\frac{(x_1 - x)x_0}{x_1 - x_0} - \frac{x - x_1}{x_1 - x_0}\sigma(t)\right)\phi(t)\Delta t \\
&\quad + f(x)\frac{x - x_0}{x_1 - x_0}\int_{x}^{x_1}(x_1 - \sigma(t))\phi(t)\Delta t,
\end{aligned}$$

i.e.,

$$\phi(x) = g(x) - f(x)z_0 - \frac{z_1 - z_0}{x_1 - x_0} f(x)(x - x_0)$$

$$+ \int_{x_0}^{x} f(x)\left(-\frac{(x_1 - x)x_0}{x_1 - x_0} - \frac{x - x_1}{x_1 - x_0}\sigma(t)\right)\phi(t)\Delta t$$

$$+f(x)\frac{x-x_0}{x_1-x_0}\int_x^{x_1}(x_1-\sigma(t))\phi(t)\Delta t.$$

Let

$$K(x,t)=\begin{cases} f(x)\left(-\frac{(x_1-x)x_0}{x_1-x_0}-\frac{x-x_1}{x_1-x_0}\sigma(t)\right) & \textit{for} \quad x_0\le t\le x \\ f(x)(x_1-\sigma(t))\frac{x-x_0}{x_1-x_0} & \textit{for} \quad x\le t\le x_1 \end{cases}$$

and

$$h(x)=g(x)-f(x)z_0-\frac{z_1-z_0}{x_1-x_0}f(x)(x-x_0) \quad \textit{for} \quad x_0\le x\le x_1.$$

Consequently we obtain the following generalized Fredholm integral equation

$$\phi(x)=h(x)+\int_{x_0}^{x_1}K(x,t)\phi(t)\Delta t. \tag{2.33}$$

Example 26 Let $\mathscr{T}=\mathscr{N}_0^3$. Consider the following BVP

$$z^{\Delta^2}(x)+x^2z(x)=x, \quad 0<x<8,$$

$$z(0)=0, \quad z(8)=2.$$

Here

$$\sigma(t)=(\sqrt[3]{t}+1)^3, \quad t\in\mathscr{T}, \quad f(x)=x^2, \quad g(x)=x, \quad x\in[0,8],$$

$$x_0=0, \quad x_1=8, \quad z_0=0, \quad z_1=2.$$

From here we obtain

$$h(x)=x-2x^3.$$

Substituting this in (2.33) gives the following generalized Fredholm integral equation

$$\phi(x)=x-\frac{1}{4}x^3+\int_0^8 K(x,t)\phi(t)\Delta t,$$

where

$$K(x,t)=\begin{cases} -\frac{1}{8}x^2(x-8)(\sqrt[3]{t}+1)^3 & \textit{for} \quad 0\le t\le x \\ \frac{1}{8}x^3\left(8-(\sqrt[3]{t}+1)^3\right) & \textit{for} \quad x\le t\le 8. \end{cases}$$

Example 27 Let $\mathscr{T} = 4^{\mathscr{N}_0} \cup \{0\}$. Consider the following BVP

$$z^{\Delta^2}(x) + \cosh_x(x, 1)z(x) = x, \quad 1 < x < 16,$$

$$z(1) = 0, \quad z(16) = -3.$$

Here

$$\sigma(x) = 4x, \quad x \in \mathscr{T}, \quad f(x) = \cosh_x(x, 1), \quad g(x) = x,$$

$$x_0 = 1, \quad x_1 = 16, \quad z_0 = 0, \quad z_1 = -3.$$

Hence, we find

$$h(x) = x + \frac{1}{15}\cosh_x(x, 1)(x - 1).$$

Substituting this in (2.33) gives the following generalized Fredholm integral equation

$$\phi(x) = x + \frac{1}{15}\cosh_x(x, 1)(x - 1) + \int_1^{16} K(x, t)\phi(t)\Delta t,$$

where

$$K(x, t) = \begin{cases} -\frac{1}{15}\cosh_x(x, 1)(16 - x)(1 - 4t) & \textit{for} \quad 1 \le x \le t \\ \frac{4}{15}(x - 1)(4 - t)\cosh_x(x, 1) & \textit{for} \quad t \le x \le 16. \end{cases}$$

Example 28 Let $\mathscr{T} = 3\mathscr{N}_0$. Consider the following BVP

$$z^{\Delta^2}(x) + e_x(x, 1)z(x) = x^2, \quad 0 < x < 6,$$

$$z(0) = 1, \quad z(6) = 1.$$

Here

$$\sigma(x) = x + 3, \quad x \in \mathscr{T}, \quad f(x) = e_x(x, 1), \quad g(x) = x^2,$$

$$x_0 = 0, \quad x_1 = 6, \quad z_0 = z_1 = 1.$$

Then

$$h(x) = x^2 - e_x(x, 1).$$

Substituting this in (2.33) we get

$$\phi(x) = x^2 - e_x(x, 1) + \int_0^6 K(x, t)\phi(t)\Delta t,$$

where

$$K(x,t)=\begin{cases}-\frac{1}{6}e_x(x,1)(x-6)(t+3) & \textit{for} \quad 0\le t\le x\\ \\ \frac{1}{6}e_x(x,1)x(3-t) & \textit{for} \quad x\le t\le 6.\end{cases}$$

Exercise 8 Convert the following BVPs to generalized Fredholm integral equations.

1.

$$\begin{cases} z^{\Delta^2}(x)-z(x)=x^2, \quad 0<x<2,\\ \\ z(0)=z(2)=0, \quad \mathscr{T}=\mathscr{Z},\end{cases}$$

2.

$$\begin{cases} z^{\Delta^2}(x)+2z(x)=x, \quad 1<x<5,\\ \\ z(1)=0, \quad z(5)=1, \quad \mathscr{T}=\mathscr{N},\end{cases}$$

3.

$$\begin{cases} z^{\Delta^2}(x)-(2x+1)z(x)=1, \quad 0<x<9,\\ \\ z(0)=0, \quad z(9)=1, \quad \mathscr{T}=\mathscr{N}_0^2.\end{cases}$$

Answer

1. $h(x)=x^2$,

$$K(x,t)=\begin{cases}\frac{1}{2}(x-2)(t+1) & \textit{for} \quad 0\le x\le t\\ \\ -\frac{1}{2}x(1-t) & \textit{for} \quad t\le x\le 2,\end{cases}$$

2. $h(x)=\dfrac{1}{2}(x+1)$,

$$K(x,t)=\begin{cases}-\frac{1}{2}t(x-5) & \textit{for} \quad 1\le x\le t\\ \\ \frac{1}{2}(x-1)(4-t) & \textit{for} \quad t\le x\le 5,\end{cases}$$

3. $h(x)=1+\dfrac{1}{9}x(2x+1)$,

$$K(x,t)=\begin{cases}\frac{1}{9}(2x+1)(x-9)(\sqrt{t}+1)^2 & \textit{for} \quad 0\le x\le t\\ \\ -\frac{1}{9}x(2x+1)\left(9-(\sqrt{t}+1)^2\right) & \textit{for} \quad x\le t\le 9.\end{cases}$$

We next consider the following boundary value problem for the Eq. (2.29) with boundary conditions

$$z(x_0) = z_2, \quad z^{\Delta}(x_1) = z_3. \tag{2.34}$$

Again we set $z^{\Delta^2}(x) = \phi(x)$. Integrating both sides of (2.31) from x_0 to x we get

$$z^{\Delta}(x) = z^{\Delta}(x_0) + \int_{x_0}^{x} \phi(t)\Delta t. \tag{2.35}$$

We put $x = x_1$ in the last expression and we find

$$z^{\Delta}(x_1) = z^{\Delta}(x_0) + \int_{x_0}^{x_1} \phi(t)\Delta t$$

or

$$z_3 = z^{\Delta}(x_0) + \int_{x_0}^{x_1} \phi(t)\Delta t,$$

whereupon

$$z^{\Delta}(x_0) = z_3 - \int_{x_0}^{x_1} \phi(t)\Delta t.$$

We substitute the last expression in (2.35) and we obtain

$$z^{\Delta}(x) = z_3 - \int_{x_0}^{x_1} \phi(t)\Delta t + \int_{x_0}^{x} \phi(t)\Delta t,$$

which we integrate from x_0 to x_1 and we get

$$\int_{x_0}^{x} z^{\Delta}(t)\Delta t = \int_{x_0}^{x} \left(z_3 - \int_{x_0}^{x_1} \phi(t)\Delta t \right) \Delta y + \int_{x_0}^{x} \int_{x_0}^{x_2} \phi(t)\Delta t \Delta x_2,$$

or

$$z(x) - z(x_0) = \left(z_3 - \int_{x_0}^{x_1} \phi(t)\Delta t \right) (x - x_0) + \int_{x_0}^{x} \int_{x_0}^{x_2} \phi(t)\Delta t \Delta x_2,$$

or

$$z(x) = z_2 + \left(z_3 - \int_{x_0}^{x_1} \phi(t)\Delta t \right) (x - x_0) + \int_{x_0}^{x} \int_{x_0}^{x_2} \phi(t)\Delta t \Delta x_2.$$

Applying Theorem 1 we get

$$z(x) = z_2 + \left(z_3 - \int_{x_0}^{x_1} \phi(t)\Delta t \right) (x - x_0) + \int_{x_0}^{x} (x - \sigma(t))\phi(t)\Delta t. \tag{2.36}$$

We substitute (2.31) and (2.36) in (2.29) and we find

$$\phi(x) + z_2 f(x) + f(x)\left(z_3 - \int_{x_0}^{x_1} \phi(t)\Delta t\right)(x - x_0)$$

$$+f(x)\int_{x_0}^{x}(x - \sigma(t))\Delta t = g(x)$$

or

$$\begin{aligned}
\phi(x) = \; & g(x) - z_2 f(x) - z_3(x - x_0)f(x) + f(x)(x - x_0)\int_{x_0}^{x_1} \phi(t)\Delta t \\
& - f(x)\int_{x_0}^{x}(x - \sigma(t))\phi(t)\Delta t \\
= \; & g(x) - z_2 f(x) - z_3(x - x_0)f(x) + f(x)(x - x_0)\int_{x_0}^{x} \phi(t)\Delta t \\
& + f(x)(x - x_0)\int_{x}^{x_1} \phi(t)\Delta t - f(x)\int_{x_0}^{x}(x - \sigma(t))\phi(t)\Delta t \\
= \; & g(x) - z_2 f(x) - z_3(x - x_0)f(x) \\
& + \int_{x_0}^{x} \left(f(x)(x - x_0) - f(x)(x - \sigma(t))\right)\phi(t)\Delta t \\
& + f(x)(x - x_0)\int_{x}^{x_1} \phi(t)\Delta t \\
= \; & g(x) - z_2 f(x) - z_3(x - x_0)f(x) + \int_{x_0}^{x} f(x)(-x_0 + \sigma(t))\phi(t)\Delta t \\
& + \int_{x}^{x_1} f(x)(x - x_0)\phi(t)\Delta t.
\end{aligned}$$

Let

$$h_1(x) = g(x) - z_2 f(x) - z_3(x - x_0)f(x),$$

$$K_1(x,t) = \begin{cases} f(x)(-x_0 + \sigma(t)) & \textit{for} \quad x_0 \le t \le x \\ f(x)(x - x_0) & \textit{for} \quad x \le t \le x_1. \end{cases}$$

Then we get the following generalized Fredholm integral equation

$$\phi(x) = h_1(x) + \int_{x_0}^{x_1} K_1(x,t)\phi(t)\Delta t. \tag{2.37}$$

Example 29 Let $\mathscr{T} = \mathscr{N}_0^4$. Consider the following BVP

$$z^{\Delta^2}(x) + x^2 z(x) = \cos_x(x, 1) + \sinh_x(x, 2), \quad 0 < x < 81,$$

$$z(0) = 0, \quad z^{\Delta}(81) = 1.$$

Here

$$\sigma(x) = (\sqrt[4]{x} + 1)^4, \quad x \in \mathscr{T},$$

$$f(x) = x^2, \quad g(x) = \cos_x(x, 1) + \sinh_x(x, 2),$$

$$x_0 = 0, \quad x_1 = 81, \quad z_2 = 0, \quad z_3 = 1.$$

Then

$$h_1(x) = -x^3 + \cos_x(x, 1) + \sinh_x(x, 2).$$

Substituting this in (2.37) gives the following generalized Fredholm integral equation

$$\phi(x) = -x^3 + \cos_x(x, 1) + \sinh_x(x, 2) + \int_0^{81} K_1(x, t)\phi(t)\Delta t,$$

where

$$K_1(x, t) = \begin{cases} x^2(\sqrt[4]{t} + 1)^4 & for \quad 0 \le t \le x \\ \\ x^3 & for \quad x \le t \le 81. \end{cases}$$

Example 30 Let $\mathscr{T} = 3^{\mathscr{N}_0} \cup \{0\}$. Consider the BVP

$$z^{\Delta^2}(x) + (2x - 3)z^{\Delta}(x) = x^2 - 1, \quad 0 < x < 27,$$

$$z(0) = 1, \quad z^{\Delta}(1) = -3.$$

Here

$$\sigma(x) = 3x, \quad x \in \mathscr{T},$$
$$f(x) = 2x - 3, \quad g(x) = x^2 - 1,$$
$$x_0 = 0, \quad x_1 = 1, \quad z_2 = 1, \quad z_3 = -3.$$

Then

$$\begin{aligned} h_1(x) &= g(x) - z_2 f(x) - z_3(x - x_0)f(x) \\ &= x^2 - 1 - (2x - 3) - (-3)x(2x - 3) \\ &= x^2 - 1 - 2x + 3 + 6x^2 - 9x \\ &= 7x^2 - 11x + 2. \end{aligned}$$

Substituting this is (2.37) gives the following generalized Fredholm integral equation

$$\phi(x) = 7x^2 - 11x + 2 + \int_0^{27} K_1(x,t)\phi(t)\Delta t,$$

where

$$K_1(x,t) = \begin{cases} 3(2x-3)t & for \quad 0 \le t \le x \\ 2x^2 - 3x & for \quad x \le t \le 27. \end{cases}$$

Example 31 Let $\mathscr{T} = 2\mathscr{Z}$. Consider the BVP

$$z^{\Delta^2}(x) - xz^{\Delta}(x) = 1 + x, \quad -2 < x < 6,$$

$$z(-2) = 0, \quad z^{\Delta}(6) = 1.$$

Here

$$\sigma(x) = x + 2, \quad x \in \mathscr{T},$$

$$f(x) = -x, \quad g(x) = 1 + x,$$

$$x_0 = -2, \quad x_1 = 6, \quad z_2 = 0, \quad z_3 = 1.$$

Then

$$\begin{aligned} h_1(x) &= g(x) - z_2 f(x) - z_3(x - x_0)f(x) \\ &= 1 + x - (x+2)(-x) \\ &= 1 + x + x^2 + 2x \\ &= x^2 + 3x + 1. \end{aligned}$$

Substituting this in (2.37) gives the following generalized Fredholm integral equation

$$\phi(x) = x^2 + 3x + 1 + \int_{-2}^{6} K_1(x,t)\phi(t)\Delta t,$$

where

$$K_1(x,t) = \begin{cases} -x(t+4) & for \quad -2 \le t \le x \\ -x^2 - 2x & for \quad x \le t \le 6. \end{cases}$$

Exercise 9 Convert the following BVPs to generalized Fredholm integral equations.

1.
$$\begin{cases} z^{\Delta^2}(x) + \sin x z(x) = \cos x, & 0 < x < 4, \\ z(0) = 0, \quad z^{\Delta}(4) = 3, & \mathscr{T} = \mathscr{Z}, \end{cases}$$

2.
$$\begin{cases} z^{\Delta^2}(x) - 3z(x) = x^2 + 1, & -1 < x < 4, \\ z(-1) = 1, \quad z(4) = 0, & \mathscr{T} = \mathscr{Z}, \end{cases}$$

3.
$$\begin{cases} z^{\Delta^2}(x) + \frac{1}{1+x^2} z(x) = 1, & 1 < x < 27, \\ z(1) = 0, \quad z^{\Delta}(27) = 1, & \mathscr{T} = 3^{\mathscr{N}_0}. \end{cases}$$

Answer

1. $h_1(x) = \cos x - 3x \sin x$,

$$K_1(x, t) = \begin{cases} (t+1) \sin x & for \quad 0 \le t \le x, \\ x \sin x & for \quad x \le t \le 4, \end{cases}$$

2. $h_1(x) = x^2 + 4$,

$$K_1(x, t) = \begin{cases} -3(t+2) & for \quad -1 \le t \le x \\ -3(x+1) & for \quad x \le t \le 4, \end{cases}$$

3. $h_1(x) = \dfrac{x^2 - x + 2}{x^2 + 1}$,

$$K_1(x, t) = \begin{cases} \frac{1}{1+x^2}(-1 + 3t) & for \quad 1 \le t \le x \\ \frac{x-1}{1+x^2} & for \quad x \le t \le 27. \end{cases}$$

2.5 Converting Generalized Fredholm Integral Equation to BVP

In the previous sections, we have represented a technique to convert Volterra integral equations to equivalent initial value problems. In a similar manner, we will represent a technique that converts Fredholm integral equations to equivalent boundary value problems.

We first consider the generalized Fredholm integral equation

$$\phi(x) = g(x) + \int_{x_0}^{x_1} K(x,t)\phi(t)\Delta t, \tag{2.38}$$

where

$$K(x,t) = \begin{cases} f(x)(x_1 - x)(\sigma(t) - x_0) & \textit{for} \quad x_0 \le t \le x \\ f(x)(x_1 - \sigma(t))(x - x_0) & \textit{for} \quad x \le t \le x_1. \end{cases}$$

The Eq. (2.38) we can rewrite in the following form

$$\begin{aligned} \phi(x) &= g(x) + \textstyle\int_{x_0}^{x} f(x)(x_1 - x)(\sigma(t) - x_0)\phi(t)\Delta t \\ &+ \textstyle\int_{x}^{x_1} f(x)(x_1 - \sigma(t))(x - x_0)\phi(t)\Delta t. \end{aligned} \tag{2.39}$$

For simplicity reason, we may assume that $f(x) = a$, where a is a constant. Then (2.39) takes the form

$$\phi(x) = g(x) + a(x_1 - x)\int_{x_0}^{x} (\sigma(t) - x_0)\phi(t)\Delta t + a(x - x_0)\int_{x}^{x_1} (x_1 - \sigma(t))\phi(t)\Delta t. \tag{2.40}$$

We differentiate (2.40) with respect to x and we get

$$\begin{aligned} \phi^{\Delta}(x) &= g^{\Delta}(x) - a\int_{x_0}^{x} (\sigma(t) - x_0)\phi(t)\Delta t \\ &\quad + a(x_1 - \sigma(x))(\sigma(x) - x_0)\phi(x) + a\int_{x}^{x_1} (x_1 - \sigma(t))\phi(t)\Delta t \\ &\quad - a(\sigma(x) - x_0)(x_1 - \sigma(x))\phi(x) \\ &= g^{\Delta}(x) - a\int_{x_0}^{x} (\sigma(t) - x_0)\phi(t)\Delta t + a\int_{x}^{x_1} (x_1 - \sigma(t))\phi(t)\Delta t. \end{aligned}$$

Again we differentiate with respect to x and we find

$$\begin{aligned} \phi^{\Delta^2}(x) &= g^{\Delta^2}(x) - a(\sigma(x) - x_0)\phi(x) - a(x_1 - \sigma(x))\phi(x) \\ &= g^{\Delta^2}(x) - a(x_1 - x_0)\phi(x), \end{aligned}$$

i.e.,

$$\phi^{\Delta^2}(x) + a(x_1 - x_0)\phi(x) = g^{\Delta^2}(x). \tag{2.41}$$

By substituting $x = x_0$ and $x = x_1$ in (2.40) we find that

$$\phi(x_0) = g(x_0) \quad and \quad \phi(x_1) = g(x_1). \tag{2.42}$$

Combining (2.41) and (2.42) gives the following boundary value problem

$$\begin{cases} \phi^{\Delta^2}(x) + a(x_1 - x_0)\phi(x) = g^{\Delta^2}(x), \quad x_0 < x < x_1, \\ \\ \phi(x_0) = g(x_0), \quad \phi(x_1) = g(x_1). \end{cases} \tag{2.43}$$

Example 32 Let $\mathscr{T} = 2^{\mathscr{N}_0} \cup \{0\}$. We consider the following generalized Fredholm integral equation

$$\phi(x) = 2x + 3 + \int_0^4 K(x,t)\phi(t)\Delta t,$$

where

$$K(x,t) = \begin{cases} 2t(4-x) \quad for \quad 0 \le t \le x \\ \\ 2x(2-t) \quad for \quad x \le t \le 4. \end{cases}$$

Here $\sigma(x) = 2x,\ x \in \mathscr{T},\ a = 1,\ g(x) = 2x + 3.$ Then $g^{\Delta}(x) = 2,\ g^{\Delta^2}(x) = 0.$ Hence, using (2.43), we get the following BVP

$$\begin{cases} \phi^{\Delta^2}(x) + 4\phi(x) = 0, \quad 0 < x < 4, \\ \\ \phi(0) = 3, \quad \phi(4) = 11. \end{cases}$$

Example 33 Let $\mathscr{T} = \mathscr{N}_0^3 \cup \{0\}$. Consider the following generalized Fredholm integral equation

$$\phi(x) = x^2 + 2x + \int_0^{27} K(x,t)\phi(t)\Delta t,$$

where

$$K(x,t) = \begin{cases} 2(27-x)(\sqrt[3]{t}+1)^3 \quad for \quad 0 \le t \le x \\ \\ 2x(27 - (\sqrt[3]{t}+1)^3) \quad for \quad x \le t \le 27. \end{cases}$$

Here

$$\sigma(x) = (\sqrt[3]{x}+1)^3, \quad x \in \mathscr{T},$$

$$a = 2, \quad g(x) = x^2 + 2x, \quad x_0 = 0, \quad x_1 = 27.$$

Then

$$\begin{aligned} g^{\Delta}(x) &= \sigma(x) + x + 2 \\ &= (\sqrt[3]{x} + 1)^3 + x + 2 \\ &= x + 3\sqrt[3]{x^2} + 3\sqrt[3]{x} + 1 + x + 2 \\ &= 2x + 3\sqrt[3]{x^2} + 3\sqrt[3]{x} + 3, \end{aligned}$$

$$\begin{aligned} g^{\Delta^2}(x) &= 2 + 3\frac{\sqrt[3]{\sigma^2(x)} - \sqrt[3]{x^2}}{\sigma(x) - x} + 3\frac{\sqrt[3]{\sigma(x)} - \sqrt[3]{x}}{\sigma(x) - x} \\ &= 2 + 3\frac{(\sqrt[3]{\sigma(x)} - \sqrt[3]{x})(\sqrt[3]{\sigma(x)} + \sqrt[3]{x})}{(\sqrt[3]{\sigma(x)} - \sqrt[3]{x})(\sqrt[3]{\sigma^2(x)} + \sqrt[3]{x}\sqrt[3]{\sigma(x)} + \sqrt[3]{x^2})} \\ &\quad + 3\frac{\sqrt[3]{\sigma(x)} - \sqrt[3]{x}}{(\sqrt[3]{\sigma(x)} - \sqrt[3]{x})(\sqrt[3]{\sigma^2(x)} + \sqrt[3]{x}\sqrt[3]{\sigma(x)} + \sqrt[3]{x^2})} \\ &= 2 + 3\frac{\sqrt[3]{\sigma(x)} + \sqrt[3]{x} + 1}{\sqrt[3]{\sigma^2(x)} + \sqrt[3]{x}\sqrt[3]{\sigma(x)} + \sqrt[3]{x^2}} \\ &= 2 + 3\frac{\sqrt[3]{x} + 1 + \sqrt[3]{x} + 1}{(\sqrt[3]{x} + 1)^2 + \sqrt[3]{x}(\sqrt[3]{x} + 1) + \sqrt[3]{x^2}} \\ &= 2 + 6\frac{\sqrt[3]{x} + 1}{\sqrt[3]{x^2} + 2\sqrt[3]{x} + 1 + \sqrt[3]{x^2} + \sqrt[3]{x} + \sqrt[3]{x^2}} \\ &= 2 + 6\frac{\sqrt[3]{x} + 1}{3\sqrt[3]{x^2} + 3\sqrt[3]{x} + 1}. \end{aligned}$$

Also,

$$g(0) = 0, \quad g(27) = 783.$$

Hence, using (2.43), we get the following boundary value problem

$$\begin{cases} \phi^{\Delta^2}(x) + 54\phi(x) = 2 + 6\frac{\sqrt[3]{x}+1}{3\sqrt[3]{x^2}+3\sqrt[3]{x}+1}, \quad 0 < x < 27, \\ \phi(0) = 0, \quad \phi(27) = 783. \end{cases}$$

Example 34 Let $\mathscr{T} = 3^{\mathscr{N}_0}$. Consider the following generalized Fredholm integral equation

$$\phi(x) = x^3 - 2x^2 + \int_1^9 K(x,t)\phi(t)\Delta t,$$

where

$$K(x,t) = \begin{cases} (9-x)(3t-1) & \textit{for} \quad 1 \le t \le x \\ (9-3t)(x-1) & \textit{for} \quad x \le t \le 9. \end{cases}$$

Here

$$a = 1, \quad \sigma(x) = 3x, \quad x \in \mathscr{T},$$

$$x_0 = 1, \quad x_1 = 9, \quad g(x) = x^3 - 2x^2.$$

Then

$$\begin{aligned} g^{\Delta}(x) &= \sigma^2(x) + x\sigma(x) + x^2 - 2(x + \sigma(x)) \\ &= 9x^2 + 3x^2 + x^2 - 8x \\ &= 13x^2 - 8x, \\ g^{\Delta^2}(x) &= 13(\sigma(x) + x) - 8 \\ &= 52x - 8, \\ g(1) &= -1, \quad g(9) = 567. \end{aligned}$$

Hence, using (2.43), we get the following boundary value problem

$$\begin{cases} \phi^{\Delta^2}(x) + 8\phi(x) = 52x - 8, \quad 1 < x < 9, \\ \phi(1) = -1, \quad \phi(9) = 567. \end{cases}$$

Exercise 10 Convert the following generalized Fredholm integral equations to BVPs.

1.

$$\phi(x) = x - 2 + \int_0^{10} K(x,t)\phi(t)\Delta t,$$

where

$$K(x,t) = \begin{cases} (10-x)(t+1) & \textit{for} \quad 0 \le t \le x \\ (9-t)x & \textit{for} \quad x \le t \le 10, \end{cases}$$

$\mathscr{T} = \mathscr{Z}$,

2.

$$\phi(x) = x^2 + \int_{-1}^{2} K(x,t)\phi(t)\Delta t,$$

where

$$K(x,t) = \begin{cases} 2(2-x)(t+2) & \textit{for} \quad -1 \le t \le x \\ 2(1-t)(x+1) & \textit{for} \quad x \le t \le 2, \end{cases}$$

$\mathscr{T} = \mathscr{Z}$,

3.

$$\phi(x) = x^3 + \int_{1}^{4} K(x,t)\phi(t)\Delta t,$$

where

$$K(x,t) = \begin{cases} (4-x)(2t-1) & \textit{for} \quad 1 \le t \le x \\ 2(2-t)(x-1) & \textit{for} \quad x \le t \le 4, \end{cases}$$

$\mathscr{T} = 2^{\mathscr{N}_0}$.

Answer

1.

$$\begin{cases} \phi^{\Delta^2}(x) + 10\phi(x) = 0, \quad 0 < x < 10, \\ \phi(0) = -2, \quad \phi(10) = 8, \end{cases}$$

2.

$$\begin{cases} \phi^{\Delta^2}(x) + 6\phi(x) = 2, \quad -1 < x < 2, \\ \phi(-1) = 1, \quad \phi(2) = 4, \end{cases}$$

3.

$$\begin{cases} \phi^{\Delta^2}(x) + 3\phi(x) = 21x, \quad 1 < x < 4, \\ \phi(1) = 1, \quad \phi(4) = 64. \end{cases}$$

Next we consider the following generalized Fredholm integral equation

$$\phi(x) = g(x) + \int_{x_0}^{x_1} K(x,t)\phi(t)\Delta t, \tag{2.44}$$

where

$$K(x,t)=\begin{cases} f(x)(-x_0+\sigma(t)) & for \quad x_0\le t\le x \\ \\ f(x)(x-x_0) & for \quad x\le t\le x_1. \end{cases}$$

The Eq. (2.44) we can rewrite in the following form

$$\begin{aligned} \phi(x) &= g(x)+\int_{x_0}^{x} K(x,t)\phi(t)\Delta t+\int_{x}^{x_1} K(x,t)\phi(t)\Delta t \\ &= g(x)+\int_{x_0}^{x} f(x)(-x_0+\sigma(t))\phi(t)\Delta t \\ &\quad +\int_{x}^{x_1} f(x)(x-x_0)\phi(t)\Delta t. \end{aligned}$$

For simplicity reasons, we may assume that $f(x)=b$, where b is a real constant. Then the Eq. (2.44) takes the form

$$\phi(x)=g(x)+b\int_{x_0}^{x}(-x_0+\sigma(t))\phi(t)\Delta t+b\int_{x}^{x_1}(x-x_0)\phi(t)\Delta t. \tag{2.45}$$

We differentiate the last equation with respect to x and we find

$$\begin{aligned} \phi^{\Delta}(x) &= g^{\Delta}(x)+b(-x_0+\sigma(x))\phi(x)+b\int_{x}^{x_1}\phi(t)\Delta t \\ &\quad -b(\sigma(x)-x_0)\phi(x) \\ &= g^{\Delta}(x)+b\int_{x}^{x_1}\phi(t)\Delta t, \end{aligned}$$

i.e.,

$$\phi^{\Delta}(x)=g^{\Delta}(x)+b\int_{x}^{x_1}\phi(t)\Delta t. \tag{2.46}$$

We differentiate with respect to x the Eq. (2.46) and we find

$$\phi^{\Delta^2}(x)=g^{\Delta^2}(x)-b\phi(x)$$

or

$$\phi^{\Delta^2}(x)+b\phi(x)=g^{\Delta^2}(x). \tag{2.47}$$

We substitute $x=x_0$ and $x=x_1$ in (2.45) and (2.46), respectively. We find

$$\phi(x_0)=g(x_0), \quad \phi^{\Delta}(x_1)=g^{\Delta}(x_1). \tag{2.48}$$

Combining (2.47) and (2.48) we get the following boundary value problem

$$\begin{cases} \phi^{\Delta^2}(x) + b\phi(x) = g^{\Delta^2}(x), \quad x_0 < x < x_1, \\ \\ \phi(x_0) = g(x_0), \quad \phi^{\Delta}(x_1) = g^{\Delta}(x_1). \end{cases} \tag{2.49}$$

Example 35 Let $\mathscr{T} = \mathscr{Z}$. Consider the generalized Fredholm integral equation

$$\phi(x) = x^4 + 3x^2 + 3x + \int_0^4 K(x,t)\phi(t)\Delta t,$$

where

$$K(x,t) = \begin{cases} t+1 \quad for \quad 0 \le x \le t \\ x \quad for \quad x \le t \le 4. \end{cases}$$

Here

$$\begin{aligned} &\sigma(x) = x+1, \quad x \in \mathscr{T}, \quad b = 1, \quad x_0 = 0, \quad x_1 = 4, \\ &g(x) = x^4 + 3x^2 + 3x. \end{aligned}$$

Then

$$\begin{aligned} g^{\Delta}(x) &= \sigma^3(x) + x\sigma^2(x) + x^2\sigma(x) + x^3 + 3(\sigma(x) + x) + 3 \\ &= (x+1)^3 + x(x+1)^2 + x^2(x+1) + x^3 + 3(x+1+x) + 3 \\ &= x^3 + 3x^2 + 3x + 1 + x^3 + 2x^2 + x + x^3 + x^2 \\ &\quad + x^3 + 6x + 6 \\ &= 4x^3 + 6x^2 + 10x + 7, \\ g^{\Delta^2}(x) &= 4(\sigma^2(x) + x\sigma(x) + x^2) + 6(\sigma(x) + x) + 10 \\ &= 4(x+1)^2 + 4x(x+1) + 4x^2 + 6(x+1+x) + 10 \\ &= 4x^2 + 8x + 4 + 4x^2 + 4x + 4x^2 + 12x + 16 \\ &= 12x^2 + 24x + 20, \\ g(0) &= 0, \quad g^{\Delta}(4) = 399. \end{aligned}$$

Hence and (2.49) we get the following BVP

$$\begin{cases} \phi^{\Delta^2}(x) + \phi(x) = 12x^2 + 24x + 20, \quad 0 < x < 4, \\ \phi(0) = 0, \quad \phi^{\Delta}(4) = 399. \end{cases}$$

Example 36 Let $\mathscr{T} = 2^{\mathscr{N}_0} \cup \{0\}$. Consider the following generalized Fredholm integral equation

$$\phi(x) = x^3 - 7x^2 + 2x + \int_0^4 K(x,t)\phi(t)\Delta t,$$

where

$$K(x,t) = \begin{cases} 4t \quad for \quad 0 \le x \le t \\ 2x \quad for \quad x \le t \le 4. \end{cases}$$

Here

$$\sigma(x) = 2x, \quad x \in \mathscr{T}, \quad x_0 = 0, \quad x_1 = 4, \quad b = 2,$$

$$g(x) = x^3 - 7x^2 + 2x.$$

Then

$$\begin{aligned} g^{\Delta}(x) &= \sigma^2(x) + x\sigma(x) + x^2 - 7(\sigma(x) + x) + 2 \\ &= 4x^2 + 2x^2 + x^2 - 21x + 2 \\ &= 7x^2 - 21x + 2, \\ g^{\Delta^2}(x) &= 7(\sigma(x) + x) - 21 \\ &= 21x - 21, \\ g(0) &= 0, \quad g^{\Delta}(4) = 30. \end{aligned}$$

Hence, using (2.49), we get the following BVP

$$\begin{cases} \phi^{\Delta^2}(x) + 2\phi(x) = 21x - 21, \quad 0 < x < 4, \\ \phi(0) = 0, \quad \phi^{\Delta}(4) = 30. \end{cases}$$

Example 37 Let $\mathscr{T} = 4\mathscr{Z}$. Consider the following generalized Fredholm integral equation

$$\phi(x) = 2x^3 - x^2 + 4x + 2 + \int_{-4}^{8} K(x,t)\phi(t)\Delta t,$$

where

$$K(x,t) = \begin{cases} -t-8 \quad for \quad -4 \le t \le x \\ -x-4 \quad for \quad x \le t \le 8. \end{cases}$$

Here

$$\sigma(x) = x + 4, \quad x \in \mathscr{T}, \quad x_0 = -4, \quad x_1 = 8, \quad b = -1,$$

$$g(x) = 2x^3 - x^2 + 4x + 2.$$

Then

$$\begin{aligned} g^{\Delta}(x) &= 2(\sigma^2(x) + x\sigma(x) + x^2) - (\sigma(x) + x) + 4 \\ &= 2((x+4)^2 + x(x+4) + x^2) - (x + 4 + x) + 4 \\ &= 2(x^2 + 8x + 16 + x^2 + 4x + x^2) - 2x \\ &= 2(3x^2 + 12x + 16) - 2x \\ &= 6x^2 + 24x + 32 - 2x \\ &= 6x^2 + 22x + 32, \\ g^{\Delta^2}(x) &= 6(\sigma(x) + x) + 22 \\ &= 6(x + 4 + x) + 22 \\ &= 6(2x + 4) + 22 \\ &= 12x + 24 + 22 \\ &= 12x + 46, \\ g(-4) &= -158, \quad g^{\Delta}(8) = 592. \end{aligned}$$

Hence, using (2.49), we get the following BVP

$$\begin{cases} \phi^{\Delta^2}(x) - \phi(x) = 12x + 46, \quad -4 < x < 8, \\ \phi(-4) = -158, \quad \phi^{\Delta}(8) = 592. \end{cases}$$

Exercise 11 Convert the following generalized Fredholm integral equations to BVPs.

1. $\phi(x) = x + 10 + \int_{-2}^{4} K(x,t)\phi(t)\Delta t$, $\mathscr{T} = \mathscr{Z}$, where

$$K(x,t) = \begin{cases} t+3 \quad \textit{for} \quad -2 \le t \le x \\ x+2 \quad \textit{for} \quad x \le t \le 4, \end{cases}$$

2. $\phi(x) = x^3 + 1 + \int_{0}^{6} K(x,t)\phi(t)\Delta t$, $\mathscr{T} = 2\mathscr{Z}$, where

$$K(x,t) = \begin{cases} -2(t+2) \quad \textit{for} \quad 0 \le t \le x \\ -2x \quad \textit{for} \quad x \le t \le 6, \end{cases}$$

3. $\phi(x) = x^2 + 3x + \int_{0}^{4} K(x,t)\phi(t)\Delta t$, $\mathscr{T} = 2^{\mathscr{N}_0} \cup \{0\}$, where

$$K(x,t) = \begin{cases} 2t \quad \textit{for} \quad 0 \le t \le x \\ x \quad \textit{for} \quad x \le t \le 4. \end{cases}$$

Answer

1.

$$\begin{cases} \phi^{\Delta^2}(x) + \phi(x) = 0, \quad -2 < x < 4, \\ \phi(-2) = 8, \quad \phi^{\Delta}(4) = 1, \end{cases}$$

2.

$$\begin{cases} \phi^{\Delta^2}(x) - 2\phi(x) = 6x + 12, \quad 0 < x < 6, \\ \phi(0) = 1, \quad \phi^{\Delta}(6) = 148, \end{cases}$$

3.

$$\begin{cases} \phi^{\Delta^2}(x) + \phi(x) = 3, \quad 0 < x < 4, \\ \phi(0) = 0, \quad \phi^{\Delta}(4) = 15. \end{cases}$$

2.6 Solutions of Generalized Integral Equations and Generalized Integro-Differential Equations

Definition 1 A solution of a generalized integral equation or generalized integro-differential equation is a function $\phi(x)$ that satisfies the given equation. In other words, the solution $\phi(x)$ must satisfy both sides of the examined equation.

Definition 2 The solution is called exact if it can be represented in a closed form, such as a polynomial, exponential function, trigonometric function or the combination of two or more of these elementary functions.

The following examples will be examined to explain the meaning of a solution.

Example 38 Some examples of exact solutions are as follows:

$$\phi(x) = x^2 + x + e^x + \cos x,$$

$$\phi(x) = x - 2e_x(x, 3),$$

$$\phi(x) = 1 + \cosh_x(x, 1) + \cos x.$$

Example 39 Let $\mathscr{T} = \mathscr{Z}$. Consider the equation

$$\phi(x) = x^2 + 4x + 4 - \int_{-1}^{x} \phi(t)\Delta t.$$

We will prove that $\phi(x) = 2x + 3$ is its solution. Indeed,

$$\begin{aligned}
\int_{-1}^{x} \phi(t)\Delta t &= \int_{-1}^{x} (2t+3)\Delta t \\
&= 2\int_{-1}^{x} t\Delta t + 3\int_{-1}^{x} \Delta t \\
&= 2\int_{-1}^{x} \left(\frac{1}{2}(t^2)^{\Delta} - \frac{1}{2}\right) \Delta t + 3(x+1) \\
&= \int_{-1}^{x} (t^2)^{\Delta}\Delta t - \int_{-1}^{x} \Delta t + 3x + 3 \\
&= x^2 - 1 - (x+1) + 3x + 3 \\
&= x^2 - 1 - x - 1 + 3x + 3 \\
&= x^2 + 2x + 1.
\end{aligned}$$

Hence,

$$
\begin{aligned}
x^2 + 4x + 4 - \int_{-1}^{x} \phi(t)\Delta t &= x^2 + 4x + 4 - (x^2 + 2x + 1) \\
&= 2x + 3 \\
&= \phi(x).
\end{aligned}
$$

Example 40 Let $\mathscr{T} = 2\mathscr{Z}$. Consider the equation

$$
\phi(x) = \frac{1}{3}x^4 - \frac{1}{2}x^3 - \frac{1}{3}x^2 + x + 1 - x\int_0^x t\phi(t)\Delta t.
$$

We will prove that $\phi(x) = x + 1$ is its solution. Indeed, we have that $\sigma(x) = x + 2$ and

$$
\begin{aligned}
\int_0^x t\phi(t)\Delta t &= \int_0^x t(t+1)\Delta t \\
&= \int_0^x (t^2 + t)\Delta t \\
&= \int_0^x \left(\frac{1}{3}(t^3)^{\Delta} - (t^2)^{\Delta} + \frac{2}{3} + \frac{1}{2}(t^2)^{\Delta} - 1\right)\Delta t \\
&= \int_0^x \left(\frac{1}{3}(t^3)^{\Delta} - \frac{1}{2}(t^2)^{\Delta} - \frac{1}{3}\right)\Delta t \\
&= \frac{1}{3}\int_0^x (t^3)^{\Delta}\Delta t - \frac{1}{2}\int_0^x (t^2)^{\Delta}\Delta t - \frac{1}{3}\int_0^x \Delta t \\
&= \frac{1}{3}x^3 - \frac{1}{2}x^2 - \frac{1}{3}x.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{1}{3}x^4 - \frac{1}{2}x^3 - \frac{1}{3}x^2 + x + 1 - x\int_0^x t\phi(t)\Delta t \\
&= \frac{1}{3}x^4 - \frac{1}{2}x^3 - \frac{1}{3}x^2 + x + 1 - x\left(\frac{1}{3}x^3 - \frac{1}{2}x^2 - \frac{1}{3}x\right) \\
&= \frac{1}{3}x^4 - \frac{1}{2}x^3 - \frac{1}{3}x^2 + x + 1 - \frac{1}{3}x^4 + \frac{1}{2}x^3 + \frac{1}{3}x^2 \\
&= x + 1 \\
&= \phi(x).
\end{aligned}
$$

Example 41 Let $\mathscr{T} = 2^{\mathscr{N}_0}$. Consider the integral equation

$$\phi(x) = -\frac{1}{3}x^2 + x + \frac{4}{3} + (x-1)e_x(x,1) + \int_1^x \phi(t)\Delta t.$$

We will prove that $\phi(x) = e_x(x,1) + x$ is its solution. Really, we have that $\sigma(x) = 2x$ and

$$\begin{aligned}\int_1^x \phi(t)\Delta t &= \int_1^x (te_t(t,1) + t)\Delta t \\ &= \int_1^x te_t(t,1)\Delta t + \int_1^x t\Delta t \\ &= e_t(t,1)\Big|_{t=1}^{t=x} + \frac{1}{3}\int_1^x (t^2)^{\Delta}\Delta t \\ &= e_x(x,1) - e_1(1,1) + \frac{1}{3}t^2\Big|_{t=1}^{t=x} \\ &= e_x(x,1) + \frac{1}{3}x^2 - \frac{4}{3}.\end{aligned}$$

Hence,

$$\begin{aligned}&-\frac{1}{3}x^2 + x + \frac{4}{3} + (x-1)e_x(x,1) + \int_1^x \phi(t)\Delta t \\ &= -\frac{1}{3}x^2 + x + \frac{4}{3} + (x-1)e_x(x,1) + e_x(x,1) - e_1(1,1) + \frac{1}{3}x^2 - \frac{1}{3} \\ &= xe_x(x,1) + x \\ &= \phi(x).\end{aligned}$$

Exercise 12 Show that the given function is a solution of the corresponding generalized Volterra integral equation.

1. $\phi(x) = x^2, \quad \mathscr{T} = 2\mathscr{Z}$,

$$\phi(x) = -\frac{1}{5}x^6 + x^5 - \frac{4}{3}x^4 + x^2 + \frac{8}{15}x + x\int_0^x t^2\phi(t)\Delta t,$$

2. $\phi(x) = x + 2, \quad \mathscr{T} = \mathscr{Z}$,

$$\phi(x) = -\frac{1}{3}x^4 - \frac{1}{6}x^3 + \frac{4}{3}x^2 + \frac{1}{6}x + 2 + (x-1)\int_0^x t\phi(t)\Delta t,$$

3. $\phi(x) = 2\sqrt{x} + 2x + 1, \quad \mathscr{T} = \mathscr{N}_0^2,$

$$\phi(x) = -x^3 + 2x + 2\sqrt{x} + 1 + x \int_1^x \phi(t) \Delta t.$$

Example 42 Let $\mathscr{T} = \mathscr{Z}$. Consider the equation

$$\phi(x) = x^2 - 4x + x \int_0^4 \phi(t) \Delta t.$$

We will prove that $\phi(x) = x^2 - 2x$ is its solution. Indeed, we have that $\sigma(x) = x + 1$ and

$$x^2 - 4x + x \int_0^4 \phi(t) \Delta t = x^2 - 4x + x \int_0^4 (t^2 - 2t) \Delta t$$

$$= x^2 - 4x + x \int_0^4 \left(\frac{1}{3}(t^3)^\Delta - \frac{1}{2}(t^2)^\Delta + \frac{1}{6} - 2\left(\frac{1}{2}(t^2)^\Delta - \frac{1}{2} \right) \right) \Delta t$$

$$= x^2 - 4x + x \int_0^4 \left(\frac{1}{3}(t^3)^\Delta - \frac{1}{2}(t^2)^\Delta + \frac{1}{6} - (t^2)^\Delta + 1 \right) \Delta t$$

$$= x^2 - 4x + x \int_0^4 \left(\frac{1}{3}(t^3)^\Delta - \frac{3}{2}(t^2)^\Delta + \frac{7}{6} \right) \Delta t$$

$$= x^2 - 4x + x \left(\frac{1}{3}t^3 \Big|_{t=0}^{t=4} - \frac{3}{2}t^2 \Big|_{t=0}^{t=4} + \frac{7}{6}t \Big|_{t=0}^{t=4} \right)$$

$$= x^2 - 4x + x \left(\frac{64}{3} - 24 + \frac{14}{3} \right)$$

$$= x^2 - 4x + 2x$$

$$= x^2 - 2x$$

$$= \phi(x).$$

Example 43 Let $\mathscr{T} = 3^{\mathscr{N}_0}$. Consider the equation

$$\phi(x) = \frac{1}{164}x^2 \int_1^9 t\phi(t)\Delta t.$$

We will prove that $\phi(x) = x^2$ is its solution. Really, we have that $\sigma(x) = 3x$ and

$$\begin{aligned}\frac{1}{164}x^2 \int_1^9 t\phi(t)\Delta t &= \frac{1}{164}x^2 \int_1^9 t^3 \Delta t \\ &= \frac{1}{164}x^2 \frac{1}{40} \int_1^9 (t^4)^{\Delta} \Delta t \\ &= \frac{1}{6560}x^2 t^4 \Big|_{t=1}^{t=9} \\ &= x^2 \\ &= \phi(x).\end{aligned}$$

Example 44 Let $\mathscr{T} = \mathscr{N}_0^2 \cup \{0\}$. Consider the equation

$$\phi(x) = 12x\sqrt{x} + 6x - 2 + \frac{3}{32}x^2 \int_0^4 \phi(t)\Delta t.$$

We will prove that

$$\phi(x) = 6x^2 + 12x\sqrt{x} + 6x - 2$$

is its solution. Really, we have that $\sigma(x) = (\sqrt{x} + 1)^2$ and

$$\begin{aligned}&12x\sqrt{x} + 6x - 2 + \frac{3}{32}x^2 \int_0^4 \phi(t)\Delta t \\ &= 12x\sqrt{x} + 6x - 2 + \frac{3}{32}x^2 \int_0^4 (6x^2 + 12x\sqrt{x} + 6x - 2)\Delta x \\ &= 12x\sqrt{x} + 6x - 2 \\ &+ \frac{3}{32}x^2 \int_0^4 \big(2(3x^2 + 6x\sqrt{x} + 7x + 4\sqrt{x} + 1) - 4(2x + 2\sqrt{x} + 1)\big)\, \Delta x\end{aligned}$$

$$= 12x\sqrt{x} + 6x - 2 + \frac{3}{32}x^2 \int_0^4 \left(2(x^3)^{\Delta} - 4(x^2)^{\Delta}\right) \Delta x$$

$$= 12x\sqrt{x} + 6x - 2 + \frac{3}{32}\left(2x^3\Big|_{x=0}^{x=4} - 4x^2\Big|_{x=0}^{x=4}\right)$$

$$= 12x\sqrt{x} + 6x - 2 + \frac{3}{32}x^2(128 - 64)$$

$$= 12x\sqrt{x} + 6x - 2 + 6x^2$$

$$= \phi(x).$$

Exercise 13 Show that the given function is a solution of the corresponding generalized Fredholm integral equation.

1. $\phi(x) = x^2 + 2x + 1, \quad \mathscr{T} = \mathscr{Z},$

$$\phi(x) = x^2 - 4x + 1 + x\int_{-2}^{2} \phi(t)\Delta t,$$

2. $\phi(x) = x - 4, \quad \mathscr{T} = 3\mathscr{Z},$

$$\phi(x) = 16x - 4 + x\int_0^6 \phi(t)\Delta t,$$

3. $\phi(x) = x - 1, \quad \mathscr{T} = 2^{\mathscr{N}_0} \cup \{0\},$

$$\phi(x) = -\frac{1}{3}x - 1 + x\int_0^4 \phi(t)\Delta t.$$

2.7 Advanced Practical Exercises

Problem 1 Convert the following multiple integrals to single integrals.

1. $\int_0^x \int_0^{x_1} e_{t^2}(t, 1)\Delta t \Delta x_1, \ \mathscr{T} = 3\mathscr{Z}.$
2. $\int_0^x \int_0^{x_1} e_{t \ominus t^2}(t, 1)\Delta t \Delta x_1, \ \mathscr{T} = 3^{\mathscr{N}_0} \cup \{0\}.$
3. $\int_0^x \int_0^{x_1} \sinh_t(t, 2)\Delta t \Delta x_1, \ \mathscr{T} = 2^{\mathscr{N}_0} \cup \{0\}.$

Answer

1. $\int_0^x (x-t-3)e_{t^2}(t,1)\Delta t,$
2. $\int_0^x (x-3t)e_{t\ominus t^2}(t,1)\Delta t,$
3. $\int_0^x (x-2t)\sinh_t(t,2)\Delta t.$

Problem 2 Convert the following multiple integrals to single integrals.

1. $\int_0^x \int_0^{x_1} (x_1-t)t^4 \Delta t \Delta x_1,\ \mathscr{T}=\mathscr{R},$
2. $\int_0^x \int_0^{x_1} (x_1-3t)e_t(t,1)\Delta t \Delta x_1,\ \mathscr{T}=3^{\mathscr{N}_0}\cup\{0\},$
3. $\int_2^x \int_2^{x_1} (x_1-2t)\sinh_t(t,2)\Delta t \Delta x_1,\ \mathscr{T}=2^{\mathscr{N}_0}.$

Answer

1. $\int_0^x (x-t)^2 t^4 \Delta t,$
2. $\int_0^x (x-3t)^2 e_t(t,1)\Delta t,$
3. $\int_2^x (x-2t)^2\sinh_t(t,2)\Delta t.$

Problem 3 Convert the following IVPs to integral equations.

1.
$$\begin{cases} z^{\Delta}(x) - \frac{x^2+1}{x^2+3}z(x) = -1 \\ z(0)=0, \quad \mathscr{T}=\mathscr{Z}, \end{cases}$$

2.
$$\begin{cases} z^{\Delta}(x) - (2x^2+1)z(x) = 2 \\ z(0)=1, \quad \mathscr{T}=\mathscr{N}_0, \end{cases}$$

3.
$$\begin{cases} z^{\Delta}(x) - z(x) = -1 \\ z(2)=10, \quad \mathscr{T}=\mathscr{N}. \end{cases}$$

Answer

1. $\phi(x) = -x + \int_0^x \frac{y^2+1}{y^2+3}\phi(y)\Delta y,$
2. $\phi(x) = 2x+1+\int_0^x (2y^2+1)\phi(y)\Delta y,$

3. $\phi(x) = 12 - x + \int_2^x \phi(y)\Delta y.$

Problem 4 Convert the following IVPs to integral equations.

1.
$$\begin{cases} z^{\Delta^2}(x) - \sin_x(x,0)z(x) = 1, \\ z(0) = 0, \quad z^{\Delta}(0) = 1, \quad \mathscr{T} = 2\mathscr{Z}, \end{cases}$$

2.
$$\begin{cases} z^{\Delta^2}(x) + 2z^{\Delta}(x) + z(x) = 0, \\ z(1) = 1, \quad z^{\Delta}(1) = 2, \quad \mathscr{T} = \mathscr{N}, \end{cases}$$

3.
$$\begin{cases} z^{\Delta^2}(x) + z^{\Delta}(x) + z(x) = e_x(x,2), \\ z(0) = 1, \quad z^{\Delta}(0) = 2, \quad \mathscr{T} = \mathscr{N}_0. \end{cases}$$

Answer

1. $\phi(x) = \frac{1}{2}x^2 + \int_0^x \sin_y(y,0)(x-2-y)\phi(y)\Delta y,$
2. $\phi(x) = -3 + 4x - \int_1^x (1+x-y)\phi(y)\Delta y,$
3. $\phi(x) = 3x + 1 - \int_0^x (x-y)\phi(y)\Delta y + \int_0^x (x-y-1)e_y(y,2)\Delta y.$

Problem 5 Convert the following generalized Volterra integral equations to IVPs.

1. $\phi(x) = \sinh_x(x,2) + \int_2^x (1-2y)\phi(y)\Delta y,\ \mathscr{T} = \mathscr{Z},$
2. $\phi(x) = x^2 + x + \int_1^x y^2\phi(y)\Delta y,\ \mathscr{T} = \mathscr{N}_0^3,$
3. $\phi(x) = x^2 - 2x + 2 + \int_1^x (x+y)\phi(y)\Delta y,\ \mathscr{T} = 4^{\mathscr{N}_0}.$

Answer

1.
$$\begin{cases} \phi^{\Delta}(x) - (1-2x)\phi(x) = x\cosh_x(x,2), \\ \phi(2) = 0, \end{cases}$$

2.
$$\begin{cases} \phi^{\Delta}(x) - x^2\phi(x) = 2x + 3\sqrt[3]{x^2} + 3\sqrt[3]{x} + 2, \\ \phi(1) = 2, \end{cases}$$

3.

$$\begin{cases} \phi^{\Delta^2}(x) - 20x\phi^{\Delta}(x) - 6\phi(x) = 5, \\ \phi(1) = 1, \quad \phi^{\Delta}(1) = 8. \end{cases}$$

Problem 6 Convert the following generalized Volterra integro-differential equations to IVPs.

1. $\phi(x) = \phi^{\Delta}(x) + \int_0^x y\phi(y)\Delta y, \quad \mathscr{T} = \mathscr{N}_0^2,$
2. $\phi^{\Delta}(x) = \phi(x) + x^2 + \int_0^x \phi(y)\Delta y, \quad \mathscr{T} = \mathscr{N}_0^2,$
3. $\phi^{\Delta}(x) = \phi(x) + x^3 - 2x + \int_0^x x\phi(y)\Delta y, \quad \mathscr{T} = 3^{\mathscr{N}_0} \cup \{0\}.$

Answer

1.

$$\begin{cases} \phi^{\Delta^2}(x) - \phi^{\Delta}(x) + x\phi(x) = 0, \\ \phi(0) = \phi^{\Delta}(0), \end{cases}$$

2.

$$\begin{cases} \phi^{\Delta^2}(x) - \phi^{\Delta}(x) - \phi(x) = 2x + 2\sqrt{x} + 1, \\ \phi^{\Delta}(0) = \phi(0), \end{cases}$$

3.

$$\begin{cases} \phi^{\Delta^3}(x) - \phi^{\Delta^2}(x) - 9x\phi^{\Delta}(x) - 4\phi(x) = 52x, \\ \phi^{\Delta}(0) = \phi(0), \quad \phi^{\Delta^2}(0) = \phi^{\Delta}(0) - 2. \end{cases}$$

Problem 7 Convert the following BVPs to generalized Fredholm integral equations.

1.

$$\begin{cases} z^{\Delta^2}(x) + z(x) = 1, \quad 0 < x < 9, \\ z(0) = z(9) = 0, \quad \mathscr{T} = \mathscr{Z}, \end{cases}$$

2.

$$\begin{cases} z^{\Delta^2}(x) + xz(x) = 1, \quad 1 < x < 16, \\ z(1) = z(16) = 0, \quad \mathscr{T} = \mathscr{N}_0^2, \end{cases}$$

3.

$$\begin{cases} z^{\Delta^2}(x) + x^2 z(x) = 1 - x, \quad 1 < x < 8, \\ z(1) = 0, \quad z(8) = 1, \quad \mathscr{T} = 2^{\mathscr{N}_0}. \end{cases}$$

Answer

1. $h(x) = 1$,

$$K(x,t) = \begin{cases} -\frac{1}{9}(x-9)(t+1) \quad for \quad 0 \le t \le x \\ \\ \frac{x}{9}(8-t) \quad for \quad x \le t \le 9, \end{cases}$$

2. $h(x) = 1$,

$$K(x,t) = \begin{cases} -\frac{1}{15}x(x-16)(t+2\sqrt{t}) \quad for \quad 1 \le t \le x \\ \\ \frac{1}{15}x(x-1)(15-t-2\sqrt{t}) \quad for \quad x \le t \le 16, \end{cases}$$

3. $h(x) = -(x-1)\left(\frac{1}{7}x^2+1\right)$,

$$K(x,t) = \begin{cases} -\frac{1}{7}(x-8)x^2(-1+2t) \quad for \quad 1 \le t \le x \\ \\ \frac{2}{7}x^2(x-1)(4-t) \quad for \quad x \le t \le 8. \end{cases}$$

Problem 8 Convert the following BVPs to generalized Fredholm integral equations.

1.

$$\begin{cases} z^{\Delta^2}(x) - (3x+7)z(x) = 1 + \cos x, \quad 0 < x < 4, \\ \\ z(0) = 0, \quad z^{\Delta}(4) = 1, \quad \mathscr{T} = \mathscr{Z}, \end{cases}$$

2.

$$\begin{cases} z^{\Delta^2}(x) - z(x) = 1, \quad -1 < x < 9, \\ \\ z(-1) = z^{\Delta}(9) = 0, \quad \mathscr{T} = \mathscr{Z}, \end{cases}$$

3.

$$\begin{cases} z^{\Delta^2}(x) + xz(x) = x^2 + 2x, \quad 1 < x < 27, \\ \\ z(1) = 0, \quad z^{\Delta}(27) = 1, \quad \mathscr{T} = 3^{\mathscr{N}_0}. \end{cases}$$

Answer

1. $h_1(x) = 1 + \cos x + 3x^2 + 7x$,

$$K_1(x,t) = \begin{cases} -(3x+7)(t+1) \quad for \quad 0 \le t \le x \\ \\ -x(3x+7) \quad for \quad x \le t \le 4, \end{cases}$$

2. $h_1(x) = 1$,

$$K_1(x,t) = \begin{cases} -t-2 & \textit{for} \quad -1 \le t \le x \\ -x-1 & \textit{for} \quad x \le t \le 9, \end{cases}$$

3. $h_1(x) = 3x$,

$$K_1(x,t) = \begin{cases} x(-1+3t) & \textit{for} \quad 1 \le t \le x \\ x(x-1) & \textit{for} \quad x \le t \le 27. \end{cases}$$

Problem 9 Convert the following generalized Fredholm integral equations to BVPs

1.
$$\phi(x) = x^2 + 2x + 4 + \int_0^4 K(x,t)\phi(t)\Delta t,$$

where
$$K(x,t) = \begin{cases} 3(4-x)(t+1) & \textit{for} \quad 0 \le t \le x \\ 3x(3-t) & \textit{for} \quad x \le t \le 4, \end{cases}$$

$\mathscr{T} = \mathscr{Z}$,

2.
$$\phi(x) = x^3 - 3x^2 + \int_{-1}^2 K(x,t)\phi(t)\Delta t,$$

where
$$K(x,t) = \begin{cases} 2(2-x)(t+2) & \textit{for} \quad -1 \le t \le x \\ 2(1-t)(x+1) & \textit{for} \quad x \le t \le 2, \end{cases}$$

$\mathscr{T} = \mathscr{Z}$,

3.
$$\phi(x) = x^4 + x + \int_0^3 K(x,t)\phi(t)\Delta t,$$

where
$$K(x,t) = \begin{cases} 3t(3-x) & \textit{for} \quad 0 \le t \le x \\ 3x(1-t) & \textit{for} \quad x \le t \le 3, \end{cases}$$

$\mathscr{T} = 3^{\mathscr{N}_0} \cup \{0\}$.

Answer

1.

$$\begin{cases} \phi^{\Delta^2}(x) + 12\phi(x) = 2 \\ \phi(0) = 4, \quad \phi(4) = 28, \end{cases}$$

2.

$$\begin{cases} \phi^{\Delta^2}(x) + 6\phi(x) = 6x \\ \phi(-1) = -4, \quad \phi(2) = -4, \end{cases}$$

3.

$$\begin{cases} \phi^{\Delta^2}(x) + 3\phi(x) = 520x^2 \\ \phi(0) = 0, \quad \phi(3) = 84. \end{cases}$$

Problem 10 Convert the following generalized Fredholm integral equations to BVPs.

1. $\phi(x) = x^2 - 10x + 5 + \int_0^8 K(x,t)\phi(t)\Delta t,\ \mathscr{T} = \mathscr{Z}$, where

$$K(x,t) = \begin{cases} t+1 \quad \mathit{for} \quad 0 \le t \le x \\ x+2 \quad \mathit{for} \quad x \le t \le 8, \end{cases}$$

2. $\phi(x) = x^4 + \int_{-2}^8 K(x,t)\phi(t)\Delta t,\ \mathscr{T} = 2\mathscr{Z}$, where

$$K(x,t) = \begin{cases} t+4 \quad \mathit{for} \quad -2 \le t \le x \\ x+2 \quad \mathit{for} \quad x \le t \le 8, \end{cases}$$

3. $\phi(x) = x^2 + \int_0^4 K(x,t)\phi(t)\Delta t,\ \mathscr{T} = \mathscr{N}_0^2 \cup \{0\}$, where

$$K(x,t) = \begin{cases} 2(\sqrt{t}+1)^2 \quad \mathit{for} \quad 0 \le t \le x \\ 2x \quad \mathit{for} \quad x \le t \le 4. \end{cases}$$

Answer

1.
$$\begin{cases} \phi^{\Delta^2}(x) + \phi(x) = 2, \quad 0 < x < 8, \\ \phi(0) = 5, \quad \phi^{\Delta}(8) = 7, \end{cases}$$

2.
$$\begin{cases} \phi^{\Delta^2}(x) + \phi(x) = 12x^2 + 48x + 56, \quad -2 < x < 8, \\ \phi(-2) = 16, \quad \phi^{\Delta}(8) = 2952, \end{cases}$$

3.
$$\begin{cases} \phi^{\Delta^2}(x) + 2\phi(x) = 2 + \frac{2}{1+2\sqrt{x}}, \quad 0 < x < 4, \\ \phi(0) = 0, \quad \phi^{\Delta}(4) = 13. \end{cases}$$

Problem 11 Show that the given function is a solution of corresponding generalized Volterra integral equation.

1. $\phi(x) = x^2, \quad \mathscr{T} = \mathscr{Z},$

$$\phi(x) = -\frac{1}{5}x^6 + \frac{1}{4}x^5 + \frac{1}{6}x^4 - \frac{1}{4}x^3 + \frac{1}{30}x + x^2 + x\int_0^x (t^2 + t)\phi(t)\Delta t,$$

2. $\phi(x) = x^2 + \sin x, \quad \mathscr{T} = \mathscr{R},$

$$\phi(x) = -\frac{x^5}{4} + \frac{x^4}{3} + \frac{7}{12}x - (x-1)\sin x + (x^2 - x)\cos x + x^2$$
$$+ 2x\cos 1 - x\sin 1 + x\int_{-1}^x (t-1)\phi(t)\Delta t,$$

3. $\phi(x) = x, \quad \mathscr{T} = 3^{\mathscr{N}_0},$

$$\phi(x) = -\frac{1}{13}x(x^3 - 14) + x\int_1^x t\phi(t)\Delta t.$$

Problem 12 Show that the given function is a solution of the corresponding generalized Fredholm integral equation.

1. $\phi(x) = x^3 + x, \quad \mathscr{T} = \mathscr{Z},$

$$\phi(x) = x^3 - 23x + 2x\int_0^3 \phi(t)\Delta t,$$

2. $\phi(x) = x^2 + 2x - 4, \quad \mathscr{T} = 2\mathscr{Z},$

$$\phi(x) = x^2 - 13x - 4 + x\int_0^5 \phi(t)\Delta t,$$

3. $\phi(x) = 2x + 2\sqrt{x} + 1, \quad \mathscr{T} = \mathscr{N}_0^2,$

$$\phi(x) = 2x + 1 + \frac{1}{8}\sqrt{x}\int_0^4 \phi(t)\Delta t.$$

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